S-modules and symmetric spectra

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Abstract: We study a symmetric monoidal adjoint functor pair between the category of S-modules and the category of symmetric spectra. The functors induce equivalences between the respective homotopy categories of spectra, module spectra and ring spectra.

1 Introduction

Stable homotopy theory studies the homotopy category of spectra. This category has a symmetric monoidal smash product which allows the definition of ring spectra 'up to homotopy'. In recent years there was an increasing interest in more refined notions of ring spectra which are associative (and possibly commutative) up to coherent homotopy, and a complex machinery developed around this issue. The coherence questions can be avoided if there is a model for the category of spectra (not just its homotopy category) which admits a symmetric monoidal smash product. For a long time no such category was known, and there was even evidence that it might not exist [L].

Then at approximately the same time, two categories of spectra with nice smash products were discovered. Elmendorf, Kriz, Mandell and May constructed the category of *S-modules* [EKMM], and Jeff Smith introduced *symmetric spectra* [HSS]. Both categories are Quillen model categories and have associated notions of ring and module spectra. However these two categories arise in completely different ways. And even though the homotopy categories are equivalent, it is not a priori clear if both frameworks give rise to the same homotopy theory of rings and modules. Both categories have their merits, described in detail in the introductions of [EKMM] and [HSS], and it is desirable to be able to translate results obtained in one category into conclusions valid in the other. The present paper describes an easy mechanism which facilitates such comparisons.

Below we define a lax symmetric monoidal functor $\Phi : \mathcal{M}_S \longrightarrow Sp^\Sigma$ from the category of S-modules to the category of symmetric spectra. The functor Φ preserves homotopy groups and has a strong symmetric monoidal left adjoint. We show that the two functors induce inverse equivalences of the homotopy categories of spectra, ring spectra, commutative ring spectra and module spectra:

Main Theorem. *The functor Φ from the category $\mathcal{M}_S$ of S-modules to the category Sp^Σ of symmetric spectra passes to a symmetric monoidal equivalence of homotopy categories*

$$\mathrm{Ho}\,(\mathcal{M}_S) \xrightarrow{\cong} \mathrm{Ho}\,(Sp^\Sigma)\ .$$

Furthermore, Φ induces equivalences of homotopy categories

$$\mathrm{Ho}\,(S\text{-algebras}) \xrightarrow{\cong} \mathrm{Ho}\,(\text{symmetric ring spectra})\ ,$$
$$\mathrm{Ho}\,(\text{com. } S\text{-algebras}) \xrightarrow{\cong} \mathrm{Ho}\,(\text{com. symmetric ring spectra})\ ,$$

and

$$\mathrm{Ho}\,(\mathcal{M}_R) \xrightarrow{\cong} \mathrm{Ho}\,(\Phi(R)\text{-mod})$$

for any S-algebra R.

The functor Φ and its left adjoint are not quite a Quillen equivalence with respect to the standard stable model category structure of symmetric spectra [HSS, 3.4.4]. However, there is a slight variation which leads to a Quillen equivalence, see Remark 2.2.

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We assume a certain familiarity with the category of symmetric spectra, but for most of this paper Sections 1-3 of [HSS] suffice. On the other hand, no deep knowledge about the internal structure of the category of S-modules is required, and we will mainly rely on the following formal properties: (i) the category $\mathcal{M}_S$ is a simplicial and monoidal model category [HSS, 5.3.2]; (ii) the model category $\mathcal{M}_S$ is stable in the sense that homotopy fibre sequences and homotopy cofibre sequences coincide; (iii) for any cofibrant approximation $S_c \longrightarrow S$ of the sphere S-module and any S-module M the induced map $S_c \wedge M \longrightarrow S \wedge M \cong M$ is a weak equivalence (this follows from [EKMM, III 3.8] since every cofibrant S-module is a retract of a cell S-module). In fact our comparison method generalizes from S-modules to model categories sharing these formal properties, see Remark 2.3. When using the terms 'fibrations' or 'cofibrations' in the context of S-modules, we refer the model category structure of [EKMM, VII 4.6], where those notions are called 'q-fibrations' and 'q-cofibrations'. In the context of symmetric spectra, we work with the stable model category structure of [HSS, 3.4.4], unless otherwise stated. The unit of the smash product (i.e., the sphere spectrum) is denoted 'S' in both [EKMM] and [HSS]. In order to distiguish between these two objects we use the notation S_Σ for the symmetric sphere spectrum.

The present paper could not have been written without an ongoing collaboration with Mike Mandell, Peter May and Brooke Shipley. It is a spin-off of our joint effort to understand the relationship between various spectra categories. I learned a lot of what I know about S-modules and symmetric spectra through our extensive discussions. I would also like to thank Jeff Smith for always keeping me informed about the progress of his work on commutative symmetric ring spectra, and Mike Hopkins and Charles Rezk for explaining the operad trick of Lemma 3.1 to me.

2 Formal and homotopical properties

Since we are mainly working simplicially, we briefly recall how the topological model category of S-modules [EKMM, VII 4.2] can be regarded as a simplicial model category [Q, II.2 Def. 2]. For every pointed simplicial set K and every S-module M we can use the topological enrichment and define the smash product $K \wedge M$ to be the smash product $|K| \wedge M$ of the geometric realization of K with M. Similarly we can define the function S-module (or cotensor) M^K of K with M to be the function S-module $F_{\mathcal{M}_S}(|K|, M)$. Finally, for two S-modules M and N we define the homomorphism simplicial set $\hom_{\mathcal{M}_S}(M, N)$ to be the singular complex of the topological mapping space $\mathcal{M}_S(M, N)$. An application of the singular complex functor to the topological adjunction isomorphisms [EKMM, VII 2.2] gives adjunction isomorphism of simplicial hom sets

$$\hom_{\mathcal{M}_S}(K \wedge M, N) \;\cong\; \hom_{\mathcal{M}_S}(M, N^K) \;\cong\; \hom_{\mathcal{S}_*}(K, \hom_{\mathcal{M}_S}(M, N)) \; .$$

Here $\hom_{\mathcal{S}_*}$ denotes the simplicial hom set in the category $\mathcal{S}_*$ of pointed simplicial sets. Since the singular complex functor takes Serre fibrations of spaces to Kan fibrations of simplicial sets, Quillen's compatibility axiom SM7 of [Q, II.2 Def. 2] follows from the corresponding topological axiom [EKMM, VII 4.3]. For the purpose of this paper we can now forget about the topological enrichment of the category of S-modules and simply treat it as a simplicial model category.

In order to define the functor Φ from the category $\mathcal{M}_S$ of S-modules to the category Sp^Σ of symmetric spectra we start by choosing a desuspension of the sphere S-module. By definition, such a desuspension consists of a cofibrant S-module S^{-1}_c together with a weak equivalence $S^1 \wedge S^{-1}_c \longrightarrow S$, where $S^1 = \Delta[1]/\partial\Delta[1]$ denotes the simplicial circle. The functor Φ is then given by

$$\Phi(M)_n \;=\; \hom_{\mathcal{M}_S}(\underbrace{S^{-1}_c \wedge \ldots \wedge S^{-1}_c}_{n}, M) \; .$$

The symmetric group acts on the mapping space by permuting the smash factors of the source. The 0-th smash power is taken to be S, the unit of the smash product. The m-fold smash power of the desuspension map $S^1 \wedge S_c^{-1} \longrightarrow S$ induces a map

$$\hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge n}, M) \longrightarrow \hom_{\mathcal{M}_S}(S^m \wedge (S_c^{-1})^{\wedge(m+n)}, M) \cong \hom_{\mathcal{S}_*}(S^m, \hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge(m+n)}, M))$$

whose adjoint

$$S^m \wedge \hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge n}, M) \longrightarrow \hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge(m+n)}, M)$$

makes $\Phi(M)$ into a symmetric spectrum. For $n \geq 1$, the S-module $(S_c^{-1})^{\wedge n}$ is a cofibrant model of the $(-n)$-sphere. So the functor Φ takes weak equivalences of S-modules to maps which are level equivalences above level 0, and the i-th homotopy group of the space $\Phi(M)_n$ is isomorphic to the $(i-n)$-th homotopy group of the S-module M. In particular there is a natural isomorphism of stable homotopy groups $\pi_*\Phi(M) \cong \pi_* M$, and Φ takes equivalences of S-modules to stable homotopy equivalences of symmetric spectra.

The functor Φ is lax symmetric monoidal: smashing maps induces

$$\hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge m}, M) \wedge \hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge n}, N) \longrightarrow \hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge(m+n)}, M \wedge N)$$

which assemble into a natural map $\Phi(M) \wedge \Phi(N) \longrightarrow \Phi(M \wedge N)$. The unit map $S_\Sigma \longrightarrow \Phi(S)$ comes from the identity map of S which is a point in $\Phi(S)_0$. (We remind the reader that we use the notation S_Σ for the symmetric sphere spectrum).

The functor Φ has a left adjoint functor $\Lambda : Sp^\Sigma \longrightarrow \mathcal{M}_S$ which is strong symmetric monoidal. To construct Λ we first define a functor $\lambda : \mathcal{S}_*^\Sigma \longrightarrow \mathcal{M}_S$ from the category of symmetric sequences [HSS, 2.1.1] to the category of S-modules by the formula

$$\lambda(X) = \bigvee_{n \geq 0} X_n \wedge_{\Sigma_n} (S_c^{-1})^{\wedge n} .$$

By inspection of definitions λ takes the unit of the tensor product of symmetric sequences [HSS, 2.1] to the sphere S-module and there is a natural, unital, associative and commutative isomorphism $\lambda(X) \wedge \lambda(Y) \cong \lambda(X \otimes Y)$. In other words, λ is a strong symmetric monoidal functor. Furthermore, λ is left adjoint to the composite of Φ with the forgetful functor from symmetric spectra to symmetric sequences, and the adjunction maps are monoidal transformations. Since the symmetric sphere spectrum S_Σ is a commutative monoid in the category of symmetric sequences, the S-module $\lambda(S_\Sigma)$ is a commutative S-algebra, and λ induces a strong symmetric monoidal functor

$$\lambda_\Sigma : Sp^\Sigma = (S_\Sigma\text{-modules in } \mathcal{S}_*^\Sigma) \longrightarrow \mathcal{M}_{\lambda(S_\Sigma)}$$

from the category of symmetric spectra to the category of $\lambda(S_\Sigma)$-modules.

The smash powers of the desuspension map $S^1 \wedge S_c^{-1} \longrightarrow S$ assemble into a homomorphism of commutative S-algebras

$$\lambda(S_\Sigma) = \bigvee_{n \geq 0} S^n \wedge_{\Sigma_n} (S_c^{-1})^{\wedge n} \longrightarrow S .$$

So we can define the desired functor Λ by extension of scalars as $\Lambda(A) = S \wedge_{\lambda(S_\Sigma)} \lambda_\Sigma(A)$. The resulting functor Λ is left adjoint to Φ. Furthermore, Λ is strong symmetric monoidal and the adjunction maps are monoidal transformations, basically because the same was true for λ. We can also read off the values of Λ for free symmetric spectra and for spectra induced from symmetric sequences: if X is a symmetric sequence, then $\Lambda(S_\Sigma \otimes X)$ is isomorphic to $\lambda(X)$. In particular for a free symmetric spectrum $F_n K$ [HSS, 2.2.5] we obtain an isomorphism $\Lambda(F_n K) \cong K \wedge (S_c^{-1})^{\wedge n}$.

The functor Φ preserves and reflects weak equivalences, so it passes to a functor on homotopy categories. By the following theorem the unit of the adjunction $A \longrightarrow \Phi(\Lambda(A))$ is a stable equivalence for all cofibrant symmetric spectra. So applying Λ to a functorial cofibrant replacement gives a functor from symmetric spectra to S-modules which on the level of homotopy categories is inverse to Φ. This proves the first statement of the main theorem.

Theorem 2.1 *For every cofibrant symmetric spectrum A, the unit of the adjunction $A \longrightarrow \Phi(\Lambda(A))$ is a stable equivalence of symmetric spectra.*

Proof: Our first claim is that for every cofibration $A \longrightarrow B$ of symmetric spectra which is an isomorphism at level 0, the map $\Lambda(A) \longrightarrow \Lambda(B)$ is a cofibration of S-modules. This is equivalent to showing that for every acyclic fibration of S-modules $X \longrightarrow Y$, the map $\Phi(X) \longrightarrow \Phi(Y)$ has the right lifting property with respect to the map $A \longrightarrow B$. In this situation the map $\Phi(X) \longrightarrow \Phi(Y)$ is a level acyclic fibration above level 0, and since $A \longrightarrow B$ is an isomorphism at level 0 and a cofibration above, any lifting problem has a solution.

As a consequence we obtain the following two-out-of-three property: suppose $A \longrightarrow B$ is a cofibration of symmetric spectra which is an isomorphism at level 0, and assume further that the unit of the adjunction is a stable equivalence for two of the three symmetric spectra A, B and B/A. Then the unit map is also a stable equivalence for the third spectrum. Since $A \longrightarrow B \longrightarrow B/A$ is a cofibre sequence of symmetric spectra and symmetric spectra are left proper [HSS, 5.4.2], it suffices to show that the sequence $\Phi(\Lambda(A)) \longrightarrow \Phi(\Lambda(B)) \longrightarrow \Phi(\Lambda(B/A))$ is also a homotopy cofibre sequence. Since Φ preserves homotopy groups, this would follow from knowing that the sequence of S-modules $\Lambda(A) \longrightarrow \Lambda(B) \longrightarrow \Lambda(B/A)$ induces a long exact sequence of homotopy groups. This in turn is a consequence of the fact that the category of S-modules is also left proper and stable. Unfortunately, properness is not stated explicitly in [EKMM], hence we argue as follows: by [EKMM, I 6.4] there is a long exact sequence of homotopy groups when $\Lambda(B/A)$ is substituted by the mapping cone of the map $\Lambda(A) \longrightarrow \Lambda(B)$ (note that the forgetful functor from S-modules to $\mathbb{L}$-spectra preserves mapping cones). Since $\Lambda(A) \longrightarrow \Lambda(B)$ is a cofibration of S-modules, the map

$$(\Lambda(A) \wedge \Delta[1]_+) \cup_{\Lambda(A)} \Lambda(B) \longrightarrow \Lambda(B) \wedge \Delta[1]_+$$

is an acyclic cofibration. So the map $\mathrm{Cone}(\Lambda(A)) \cup_{\Lambda(A)} \Lambda(B) \longrightarrow (\Lambda(B) \wedge \Delta[1]_+)/\Lambda(A)$ obtained by cobase change is also an acyclic cofibration. The target of the last map is homotopy equivalent to the cofibre $\Lambda(B/A)$.

Now we claim that if the unit map is a stable equivalence for every member of a family $\{A_i\}_{i\in I}$ of cofibrant symmetric spectra, then it is a stable equivalence for the coproduct of the family. To see this we factor the unit map for the coproduct as the composite

$$\bigvee_{i\in I} A_i \longrightarrow \bigvee_{i\in I} \Phi(\Lambda(A_i)) \longrightarrow \Phi(\bigvee_{i\in I} \Lambda(A_i)) \cong \Phi(\Lambda(\bigvee_{i\in I} A_i)) .$$

A wedge of stable equivalences is again a stable equivalence. Furthermore, for every family $\{B_i\}_{i\in I}$ of symmetric spectra, the natural map

$$\bigoplus_{i\in I} \pi_* B_i \longrightarrow \pi_* \left(\bigvee_{i\in I} B_i \right)$$

is an isomorphism. The same is true for any family of cofibrant S-modules, so the second map in the factorization is a stable homotopy equivalence. Hence the unit map for the wedge is a stable equivalence.

Now we prove the statement of the theorem for suspension spectra, i.e, symmetric spectra of the form $A = \Sigma^\infty K = K \wedge S_\Sigma$ for some pointed simplicial set K. The functor Λ takes this symmetric suspension spectrum to the S-module suspension spectrum $K \wedge S$. We claim that for every cofibrant approximation of the sphere S-module $S_c \longrightarrow S$ the simplicial set $\hom_{\mathcal{M}_S}(S_c, K \wedge S)$ is weakly equivalent to the singular complex of the infinite loop space $\Omega^\infty\Sigma^\infty|K|$ in such a way that the composite map

$$K \longrightarrow \hom_{\mathcal{M}_S}(S, K \wedge S) \longrightarrow \hom_{\mathcal{M}_S}(S_c, K \wedge S)$$

corresponds to the inclusion $K \longrightarrow \mathrm{Sing}(\Omega^\infty\Sigma^\infty|K|)$. It suffices to verify this for a single choice of cofibrant approximation. If we use the model S^0_S of [EKMM, II 1.7], then by the various defining adjunctions, the simplicial set $\hom_{\mathcal{M}_S}(S^0_S, K \wedge S)$ is isomorphic to the singular complex of the infinite loop space of the underlying coordinate free spectrum of $K \wedge S$, which proves the claim. Now we examine the adjunction map $\Sigma^\infty K \longrightarrow \Phi(K \wedge S)$ at positive level n. We consider the composite

$$\begin{aligned} \Sigma^n K \;=\; (\Sigma^\infty K)_n \;&\longrightarrow\; \Phi(K \wedge S)_n \;=\; \hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge n}, K \wedge S) \\ &\xrightarrow{\;\sim\;} \hom_{\mathcal{M}_S}((S_c^{-1})^{\wedge n}, \Omega^n\Sigma^n(K \wedge S)) \\ &\xrightarrow{\;\cong\;} \hom_{\mathcal{M}_S}((S^1 \wedge S_c^{-1})^{\wedge n}, (\Sigma^n K) \wedge S)\ . \end{aligned}$$

The second map is a weak equivalence since for every S-module M the map $M \longrightarrow \Omega^n\Sigma^n M$ is a weak equivalence between fibrant objects [EKMM, I 6.3]. Since the S-module $(S^1 \wedge S_c^{-1})^{\wedge n}$ is a cofibrant replacement of the sphere S-module, the composite map is weakly equivalent to the $(2n-1)$-connected inclusion $\Sigma^n K \longrightarrow \mathrm{Sing}(\Omega^\infty\Sigma^\infty|\Sigma^n K|)$. The map of symmetric spectra $\Sigma^\infty K \longrightarrow \Phi(K \wedge S)$ is thus a stable homotopy equivalence.

The next case is that of a free symmetric spectrum $F_n K$ for a pointed simplicial set K and some $n \geq 1$. Since this symmetric spectrum is cofibrant and trivial at level 0 it suffices (by the two-out-of-three property) to show that the adjunction map for the n-fold suspension of $F_n K$ is a stable equivalence. This suspension admits a stable equivalence $\Sigma^n F_n K \longrightarrow \Sigma^\infty K$ which is the identity at level n. The functor Λ takes this stable equivalence to the weak equivalence of S-modules

$$\Lambda(\Sigma^n F_n K) \;\cong\; K \wedge S^n \wedge (S_c^{-1})^{\wedge n} \;\xrightarrow{\;\sim\;}\; K \wedge S \;\cong\; \Lambda(\Sigma^\infty K)\ .$$

So the adjunction map for $F_n K$ is a stable equivalence because the one for $\Sigma^\infty K$ is.

Now we consider *reduced* cofibrant symmetric spectra, i.e., those cofibrant A for which A_0 is a one point simplicial set. We let I_{red} denote the set of maps of symmetric spectra of the form $F_n\partial\Delta[r]_+ \longrightarrow F_n\Delta[r]_+$ for $n \geq 1$ and $r \geq 0$. This is precisely the subset of the generating stable cofibrations FI_∂ [HSS, 3.2.3] with reduced sources and targets. We can factor the cofibration $* \longrightarrow A$ as an I_{red}-cofibration followed by an I_{red}-injective map [HSS, 3.2.11], by a countable number of attachment steps. This yields an I_{red}-injective map $Z \longrightarrow A$, which is thus a level acyclic fibration above level 0. Since the map is also an isomorphism at level 0, it is a trivial fibration of symmetric spectra and A is a retract of Z. We are reduced to showing that the unit map of Z is a stable equivalence.

By construction the spectrum Z is the colimit $Z = \mathrm{colim}_{n\geq 0} Z_n$ of a countable sequence of cofibrations starting with the trivial symmetric spectrum. Furthermore every subquotient Z_n/Z_{n-1} is a wedge of spectra of the form $F_n(\Delta[r]/\partial\Delta[r])$ for which we already proved the theorem. So with the two-out-of-three property we conclude inductively that the unit map is a stable equivalence for every stage Z_n in the sequence. In every simplicial model category the colimit of a sequence of cofibrations between cofibrant objects admits a weak equivalence from the simplicial mapping telescope. We apply this to the sequence $\{Z_n\}_{n\geq 0}$ of symmetric spectra and the sequence $\{\Lambda(Z_n)\}_{n\geq 0}$ of S-modules. Then in the commutative diagram

$$\begin{array}{ccccc}
\mathrm{tel}_n Z_n & \longrightarrow & \Phi(\Lambda(\mathrm{tel}_n Z_n)) & \xleftarrow{\cong} & \Phi(\mathrm{tel}_n \Lambda(Z_n)) \\
{\scriptstyle\sim}\downarrow & & \downarrow & & \downarrow{\scriptstyle\sim} \\
Z & \longrightarrow & \Phi(\Lambda(Z)) & \xleftarrow[\cong]{} & \Phi(\mathrm{colim}_{n\geq 0} \Lambda(Z_n))
\end{array}$$

the left and right vertical maps are stable equivalences. The mapping telescope is part of a cofibre sequence

$$\bigvee_{n\in\mathbb{N}} Z_n \longrightarrow \mathrm{tel}_n Z_n \longrightarrow \left(\bigvee_{n\in\mathbb{N}} \Sigma Z_n\right) .$$

By the two-out-of-three property the unit map is thus a stable equivalence for the mapping telescope, hence also for Z.

We have now shown that the conclusion of the theorem holds for all cofibrant reduced symmetric spectra. If A is an arbitrary cofibrant symmetric spectrum, then map $\Sigma^\infty A_0 \longrightarrow A$ which is the identity at level 0 is a cofibration [HSS, 5.2.2]. Since we know that the unit map for $\Sigma^\infty A_0$ and for the reduced cofibrant spectrum $A/(\Sigma^\infty A_0)$ are equivalences, it is also one for A. □

Remark 2.2 (Quillen equivalences) There is no strong symmetric monoidal functor from the category of symmetric spectra to the category of S-modules which is also a left Quillen functor. The reason for this is quite simple: any strong monoidal functor has to take the cofibrant symmetric sphere S_Σ to the non-cofibrant S-module sphere. However Φ and Λ can be made into a Quillen equivalence of model categories by slightly restricting the class of cofibrations of symmetric spectra. One can keep the stable equivalences as weak equivalences, but take the cofibrations to be those stable cofibrations which are an isomorphism at level 0. The proof that this in fact defines another model category structure for symmetric spectra proceeds along the lines of [HSS, Sec. 3.4]. As the proof of Theorem 2.1 shows, the functor Λ preserves this restricted notion of cofibration and acyclic cofibration. So Λ and Φ become a Quillen equivalence. The price to pay is that the symmetric sphere spectrum is no longer cofibrant in the restricted sense.

Remark 2.3 For the definitions of the functors Φ and Λ and for the proof of Theorem 2.1 we mainly relied on formal properties of the model category of S-modules. In fact the constructions make sense in any cocomplete symmetric monoidal category $\mathcal{C}$ which is enriched over the category of pointed simplicial sets. We denote the monoidal product of $\mathcal{C}$ by $\wedge$ and the unit object by I. If we then fix an object X of $\mathcal{C}$ and a morphism $S^1 \wedge X \longrightarrow I$, we can use X in place of S_c^{-1} to define an adjoint symmetric monoidal functor pair Φ and Λ between $\mathcal{C}$ and the category of symmetric spectra. (In fact, this classifies all strong symmetric monoidal functors L from the category of symmetric spectra to $\mathcal{C}$ which preserve simplicially enriched colimits: for any such L one obtains an object $X = L(F_1 S^0)$ together with a map $S^1 \wedge X \cong L(S^1 \wedge F_1 S^0) \longrightarrow L(S_\Sigma) \cong I$.)

Now suppose $\mathcal{C}$ also has a compatible model category structure ('monoidal model category' in the sense of [HSS, 5.3.2]) and satisfies the following additional property: for any cofibrant replacement of the unit object $I_c \longrightarrow I$ and for all objects Y the map $I_c \wedge Y \longrightarrow I \wedge Y \cong Y$ is a weak equivalence. Assume further that the unit object desuspends, i.e., we can choose a cofibrant object I_c^{-1} and a weak equivalence $S^1 \wedge I_c^{-1} \xrightarrow{\sim} I$. Then smashing with I_c^{-1} is a functor which is inverse to suspension on the level of homotopy categories. So $\mathcal{C}$ is a stable model category. Using I_c^{-1} we obtain adjoint functors Φ and Λ which are a Quillen pair if and only if the unit I is cofibrant. We refrain from axiomatizing the conditions under which these functors induce equivalences of homotopy categories.

3 Ring, module and algebra spectra

It is now relatively easy to lift the comparison of S-modules and symmetric spectra to a comparison of the module and ring categories. This discussion will prove the remaining parts of the main theorem. Since the functors Φ and Λ are symmetric monoidal, and the adjunction maps are monoidal transformations, they pass to adjoint functors between the categories of (commutative) S-algebras and (commutative) symmetric ring spectra, as well as to module categories.

Ring spectra. The categories of S-algebras and symmetric ring spectra are model categories with the weak equivalences defined on underlying spectra ([EKMM, VII 4.8] and [HSS, 5.5.3]). So Φ preserves and reflects weak equivalences between ring spectra. Furthermore, every cofibrant symmetric ring spectrum is also cofibrant as a symmetric spectrum [HSS, 5.5.3]. So Lemma 2.1 implies that for every cofibrant symmetric ring spectrum F, the unit of the adjunction $F \longrightarrow \Phi(\Lambda(F))$ is a stable equivalence of symmetric ring spectra. Since symmetric ring spectra can functorially be replaced by cofibrant ones, the functors Φ and Λ induce inverse equivalences of homotopy categories

$$\mathrm{Ho}\,(S\text{-algebras}) \;\cong\; \mathrm{Ho}\,(\text{symmetric ring spectra})\ .$$

Algebras over simplicial operads. It will be useful to extend our discussion from the case of associative or commutative algebras to algebras over operads [May]. By definition, a *simplicial operad* is an operad in the category of simplicial sets, with cartesian product as the monoidal product. We obtain strong symmetric monoidal functors from the category of simplicial sets to either the category of S-modules or the category of symmetric spectra by forming the suspension spectrum of a simplicial set with an extra basepoint. This allows us to view all simplicial operads as operads in the category of S-modules or symmetric spectra. If $\mathcal{A}$ is a simplicial operad we denote by $\mathcal{M}_S[\mathcal{A}]$ and $Sp^\Sigma[\mathcal{A}]$ the categories of $\mathcal{A}$-algebras in the category of S-modules and symmetric spectra respectively. In the context of S-modules this means that the simplicial operad $\mathcal{A}$ acts via its geometric realization which is a topological operad. Since the functors Φ and Λ are symmetric monoidal and simplicial, they pass to an adjoint functor pair

$$\mathcal{M}_S[\mathcal{A}] \quad \underset{\Lambda}{\overset{\Phi}{\rightleftarrows}} \quad Sp^\Sigma[\mathcal{A}]$$

for every simplicial operad $\mathcal{A}$. This includes the case of ring spectra and commutative ring spectra which correspond to the associative and commutative operad. The comparison of ring spectra indicates what the crucial condition on an operad $\mathcal{A}$ is so that the functor Φ passes to an equivalences of homotopy categories of the $\mathcal{A}$-algebras: if we knew that every algebra in $Sp^\Sigma[\mathcal{A}]$ can be replaced by a stably equivalent algebra whose underlying symmetric spectrum is cofibrant, then Theorem 2.1 would imply the equivalence of homotopy categories. This can in fact be arranged for *cofibrant* operads.

We call a morphism of simplicial operads a *fibration* (resp. *weak equivalence*) if it is termwise a fibration (resp. weak equivalence) of simplicial sets. A simplicial operad is the same as a simplicial object of operads over the category of sets. Since free operads exist, this latter category has a set of small projective generators. Quillen's Theorem [Q, II.4 Thm. 4] shows that the termwise fibrations and weak equivalences are part of a simplicial model category structure for simplicial operads.

Lemma 3.1 (C. Rezk [R]) *Let $\mathcal{A}$ be a cofibrant simplicial operad. Then there exists a functor c : $Sp^\Sigma[\mathcal{A}] \longrightarrow Sp^\Sigma[\mathcal{A}]$ and a natural stable equivalence $cX \xrightarrow{\sim} X$ of $\mathcal{A}$-algebras such that the underlying symmetric spectrum of cX is cofibrant.*

Proof: Morally the functor c is a cofibrant approximation for $\mathcal{A}$-algebras. However we are not assuming that the category $Sp^\Sigma[\mathcal{A}]$ has a model category structure, so we manufacture the desired object directly using the small object argument. We denote by $I_\mathcal{A}$ the set of $\mathcal{A}$-algebra maps freely generated by the generating stable cofibrations of symmetric spectra FI_∂ of [HSS, 3.3.2]. Then cX is defined by applying the $\mathcal{A}$-algebra analog of the factorization lemma [HSS, 3.2.11], with respect to the set $I_\mathcal{A}$, to the map from the initial $\mathcal{A}$-algebra to X. This yields an $I_\mathcal{A}$-injective $\mathcal{A}$-algebra map $cX \longrightarrow X$, which in particular is an acyclic fibration of underlying symmetric spectra.

To show that the underlying spectrum of cX is cofibrant we choose an acyclic fibration $f : Y \longrightarrow cX$ in the category of symmetric spectra such that Y is a cofibrant symmetric spectrum. We claim that Y can be given an $\mathcal{A}$-algebra structure in such a way that f becomes a morphism of $\mathcal{A}$-algebras. This will finish the proof: by construction cX has the left lifting property for all $\mathcal{A}$-algebra maps which are also acyclic fibrations of underlying spectra. So cX is a retract of Y, hence cofibrant as a symmetric spectrum.

It remains to put an $\mathcal{A}$-algebra structure on Y, and here we will use that $\mathcal{A}$ is cofibrant as an operad. We denote by $\mathcal{E}(cX)$ the endomorphism operad of cX [May, Ex. 5]. This is a simplicial operad with n-th space equal to

$$\mathcal{E}(cX)_n \;=\; \hom_{Sp^\Sigma}((cX)^{\wedge n}, cX)\ .$$

The map $f : Y \longrightarrow cX$ also has an endomorphism operad $\mathcal{E}(f)$ with n-th space

$$\mathcal{E}(f)_n \;=\; \hom_{Sp^\Sigma}(Y^{\wedge n}, Y) \times_{\hom_{Sp^\Sigma}(Y^{\wedge n}, cX)} \hom_{Sp^\Sigma}((cX)^{\wedge n}, cX)\ .$$

The pullback is formed using the composition with f on the left and composition with $f^{\wedge n}$ on the right. The given data correspond to a chain of operad morphisms

$$\mathcal{A} \longrightarrow \mathcal{E}(cX) \overset{\sim}{\twoheadleftarrow} \mathcal{E}(f) \longrightarrow \mathcal{E}(Y)$$

The left morphism is given by the $\mathcal{A}$-algebra structure on cX, and the other two morphisms are projections. Since Y is cofibrant as a symmetric spectrum, so is its n-fold smash power for all $n \geq 0$. Since in addition the map f is an acyclic fibration, the projection $\mathcal{E}(f)_n \longrightarrow \mathcal{E}(cX)_n$ is an acyclic fibration. So $\mathcal{E}(f) \longrightarrow \mathcal{E}(cX)$ is an acyclic fibration of simplicial operads. Since $\mathcal{A}$ is cofibrant, there exists an operad morphism $\mathcal{A} \longrightarrow \mathcal{E}(f)$ covering the $\mathcal{A}$-algebra structure on cX. Such a lift corresponds precisely to an $\mathcal{A}$-algebra structure on Y for which the map f is a homomorphism. □

As a combination of Theorem 2.1 and Lemma 3.1 we obtain

Corollary 3.2 *Let $\mathcal{A}$ be a cofibrant simplicial operad and c the functor of Lemma 3.1. Then the functors $\Phi : \mathcal{M}_S[\mathcal{A}] \longrightarrow Sp^\Sigma[\mathcal{A}]$ and the composite functor $\Lambda\, c : Sp^\Sigma[\mathcal{A}] \longrightarrow \mathcal{M}_S[\mathcal{A}]$ pass to inverse equivalences of homotopy categories.*

Commutative ring spectra. Since the functors Φ and Λ are symmetric monoidal, they pass to a point set level adjoint functor pair between the category of commutative S-algebras and the category of commutative symmetric ring spectra. The right adjoint functor Φ still preserves and reflects weak equivalences and so it passes to a functor on homotopy categories. However, the commutative operad is not cofibrant and it is not clear if every commutative symmetric ring spectrum can be replaced by a stably equivalent one with cofibrant underlying spectrum. The proof that commutative S-algebras and commutative symmetric ring spectra have equivalent homotopy categories thus passes through E_∞-ring spectra.

We let $\mathcal{E}$ be a cofibrant replacement of the commutative operad in the model category of simplicial operads. Any such $\mathcal{E}$ will be an E_∞-operad in the sense that its n-th space is a free and (non-equivariantly) weakly contractible Σ_n-simplicial set. By a theorem of M. Mandell [Man, 12.2] the inclusion (pullback) functor from the category of commutative S-algebras to the category $\mathcal{M}_S[\mathcal{E}]$ of $\mathcal{E}$-algebras induces an equivalence of homotopy categories. By an unpublished theorem of J. Smith the same is true for symmetric spectra: the inclusion of the category of commutative symmetric ring spectra into the category of $\mathcal{E}$-algebras induces an equivalence of homotopy categories.

So in the commutative diagram

$$\begin{array}{ccc} (\text{com. } S\text{-algebras}) & \xrightarrow{\Phi} & (\text{com. symmetric ring spectra}) \\ \Big\downarrow {\scriptstyle \text{incl.}} & & \Big\downarrow {\scriptstyle \text{incl.}} \\ \mathcal{M}_S[\mathcal{E}] & \xrightarrow[\Phi]{} & Sp^{\Sigma}[\mathcal{E}] \end{array}$$

the vertical and the lower horizontal functors pass to equivalences of homotopy categories (see Lemma 3.1). Consequently, the functor Φ identifies the homotopy category of commutative S-algebras with the homotopy category of commutative symmetric ring spectra

$$\mathrm{Ho}\,(\text{com. } S\text{-algebras}) \;\cong\; \mathrm{Ho}\,(\text{com. symmetric ring spectra})\ .$$

Remark 3.3 According to J. Smith there is a model category structure for commutative symmetric ring spectra in which the weak equivalences are the stable equivalences of underlying symmetric spectra. Furthermore, in this model category the fibrant objects are those commutative ring spectra whose underlying symmetric spectra are Ω-spectra above level 0. Similarly any map between fibrant objects which is a level fibration (resp. level acyclic fibration) above level 0 is a model category fibration (resp. acyclic fibration). This means that with respect to this model structure Φ is a right Quillen functor from the category of commutative S-algebras to the category of commutative symmetric ring spectra. It is quite conceivable that there is a more direct proof, avoiding E_∞-operads, that Φ and Λ are in fact a Quillen equivalence.

Remark 3.4 The argument just used to compare the commutative ring spectra should generalize to arbitrary simplicial operads as follows. For every simplicial operad $\mathcal{A}$, the category $\mathcal{M}_S[\mathcal{A}]$ is a model category with weak equivalences and fibrations defined on underlying S-modules ([Man, 12.1], [EKMM, VII 4.7]). If $\mathcal{A} \longrightarrow \mathcal{B}$ is a (termwise and non-equivariant) weak equivalence of simplicial operads, then by [Man, 12.2] the pullback functor $\mathcal{M}_S[\mathcal{B}] \longrightarrow \mathcal{M}_S[\mathcal{A}]$ is the right adjoint of a Quillen equivalence of model categories. In particular it induces an equivalence of homotopy categories. We also expect that the homotopy categories of $\mathcal{A}$-algebras and $\mathcal{B}$-algebras over symmetric spectra are equivalent. Every simplicial operad is equivalent to a cofibrant one, so together with Lemma 3.1 this would imply that the functor Φ induces an equivalence of homotopy categories of algebras over any simplicial operad.

Module spectra. Modules over an S-algebra or a symmetric ring spectrum are in general not algebras over a simplicial operad, but they behave quite similarly. We first consider a cofibrant symmetric ring spectrum F and look at the category of left F-modules. For every F-module M the S-module $\Lambda(M)$ is naturally a module over the S-algebra $\Lambda(F)$. Conversely, for every $\Lambda(F)$-module N, the symmetric spectrum $\Phi(N)$ is naturally a $\Phi(\Lambda(F))$-module, which we regard as an F-module by restriction of scalars along the morphism $F \longrightarrow \Phi(\Lambda(F))$ of symmetric ring spectra. Moreover, the functors Φ and Λ are again adjoint when considered as functors between the categories of F-modules and the category of $\Lambda(F)$-modules. The categories of F-modules and $\Lambda(F)$-modules are model categories with weak equivalences and fibrations defined on underlying spectra. Since F is a cofibrant symmetric ring spectrum, every cofibrant F-module will also be cofibrant as a symmetric spectrum (by an argument as in Lemma 3.1). So by Theorem 2.1 the unit map $M \longrightarrow \Phi(\Lambda(M))$ is a stable equivalence for every cofibrant F-module M. So Φ and Λ pass to inverse equivalence of homotopy categories

$$\mathrm{Ho}\,(\mathcal{M}_{\Lambda(F)}) \;\cong\; \mathrm{Ho}\,(F\text{-mod})\ .$$

To prove the last statement in the main theorem, we start with an S-algebra R and choose a cofibrant replacement $\Phi(R)_c \longrightarrow \Phi(R)$ in the model category of symmetric ring spectra. We consider the commutative diagram of module categories and functors

$$\begin{array}{ccc} \mathcal{M}_R & \xrightarrow{\ \Phi\ } & \Phi(R)\text{-mod} \\ {\scriptstyle \text{restr.}}\downarrow & & \downarrow{\scriptstyle \text{restr.}} \\ \mathcal{M}_{\Lambda(\Phi(R)_c)} & \xrightarrow[\ \Phi\ ]{} & \Phi(R)_c\text{-mod}. \end{array}$$

A left adjoint to the upper horizontal functor is given by $M \mapsto R \wedge_{\Lambda(\Phi(R))} \Lambda(M)$. The vertical functors are restriction of scalars along the weak equivalences of ring spectra $\Lambda(\Phi(R)_c) \longrightarrow R$ and $\Phi(R)_c \longrightarrow \Phi(R)$, so they pass to equivalences of homotopy categories (compare [HSS, Thm. 5.5.9]). The lower horizontal functor induces an equivalence of homotopy categories by the previous paragraph, hence so does the upper functor.

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