ALGEBRAIC STRUCTURE OF THE LOOP SPACE BOCKSTEIN SPECTRAL SEQUENCE

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Abstract. Let X be a finite, n-dimensional, r-connected CW complex. We prove the following theorem:

If $p \geq n/r$ is an odd prime, then the loop space homology Bockstein spectral sequence modulo p is a spectral sequence of universal enveloping algebras over differential graded Lie algebras.

Introduction

Let ΩX be the Moore loop space on a pointed topological space X. If $R \subseteq \mathbf{Q}$ is a principal ideal domain, then $H_*(\Omega X; R)$ has a natural Hopf algebra structure via composition of loops, as long as there is no torsion. The submodule $P \subset H_*(\Omega X; R)$ of primitive elements is a graded Lie subalgebra; in [5], Milnor and Moore showed that if $R = \mathbf{Q}$ and X is simply connected then $H_*(\Omega X; \mathbf{Q})$ is the universal enveloping algebra of P. In [4], Halperin established the same conclusion for $R \subset \mathbf{Q}$ when X is a finite, simply-connected CW complex, provided that $H_*(\Omega X; R)$ is torsion-free and the least non-invertible prime in R is sufficiently large.

In the presence of torsion, the loop space homology algebra does not have a natural Hopf algebra structure. However, in [2] Browder showed that the Bockstein spectral sequence

$$H_*(\Omega X; \mathbf{F}_p) \Rightarrow (H_*(\Omega X; \mathbf{Z})/\text{torsion}) \otimes \mathbf{F}_p$$

is a spectral sequence of Hopf algebras. Halperin also proved in [4] that for large enough primes, $H_*(\Omega X; \mathbf{F}_p)$ is the universal enveloping algebra of a graded Lie algebra. The present article establishes this for every term in the Bockstein spectral sequence.

Theorem 1. *Let X be a finite, n-dimensional, q-connected CW complex ($q \geq 1$). If p is an odd prime and $p \geq n/q$, then each term in the mod p homology Bockstein spectral sequence for ΩX is the universal enveloping algebra of a differential graded Lie algebra (L^r, β^r).*

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In [1], under the hypotheses of Theorem 1, Anick associates to X a differential graded Lie algebra L_X over $\mathbf{Z}_{(p)}$ and a natural quasi-isomorphism $UL_X \xrightarrow{\cong} C_*(\Omega X; \mathbf{Z}_{(p)})$ of graded Hopf algebras. The inclusion $\iota_X : L_X \hookrightarrow UL_X$ then induces a transformation of Bockstein spectral sequences $E^r(\iota_X) : E^r(L_X) \to E^r(\Omega X)$.

Theorem 2. *The image of each $E^r(\iota_X)$ is contained in L^r.*

Theorems 1 and 2 follow from the work of Anick in [1] and the following:

Theorem 3. *Let (L, ∂) be a differential graded Lie algebra over $\mathbf{Z}_{(p)}$ which is connected, free as a graded module, and of finite type. The mod p homology Bockstein spectral sequence of $U(L, \partial)$ is a sequence of universal enveloping algebras, $E^r(UL) = U(L^r, \beta^r)$. Furthermore, if $\iota : L \hookrightarrow UL$ is the inclusion, then the image of $E^r(\iota)$ is contained in L^r.*

The proof of Theorem 3 utilizes in a fundamental way the divided powers structure of the dual of a universal enveloping algebra.

The structure of the article is as follows.

Section 1. Notation and review of graded Lie algebras, divided powers algebras, Bockstein spectral sequences, acyclic closures and minimal models.

Section 2. In [4], Halperin showed that for a differential graded Lie algebra (L, ∂) over $\mathbf{F}_p$, $H(UL) = UE$ for a graded Lie algebra E. We show that the inclusion $\iota : (L, \partial) \hookrightarrow U(L, \partial)$ satisfies $\operatorname{im} H(\iota) \subset E$.

Section 3. Proof of Theorem 3.

Section 4. We show that a Hopf algebra morphism $UL_1 \to UL_2$ is of the form $U(\varphi)$ if and only if its dual respects divided powers.

Section 5. Two examples. The first gives a differential graded Lie algebra whose Bockstein spectral sequence collapses after the first term, while the spectral sequence of its universal enveloping algebra never does. The second shows that the sequence of Lie algebras given by Theorem 3 is not natural.

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1. Preliminaries

Let R be a commutative ring in which 2 is invertible. All objects are graded by the integers unless otherwise stated. Fix an odd prime p.

The ring of integers localized at p is denoted $\mathbf{Z}_{(p)}$ while the prime field is denoted $\mathbf{F}_p$. Differential graded modules, algebras, coalgebras, and Hopf algebras are shortened to DGM, DGA, DGC, and DGH, respectively; a comprehensive treatment of these objects is given in [3].

1.1. **Graded modules.** Let M be a graded module over R. If $x \in M_k$ then we say that x has degree k, and write $|x| = k$. A free graded module M is of *finite type* if each M_k is of finite rank. We raise and lower degrees by the convention $M^k = M_{-k}$. We denote by sM the suspension of M: $(sM)_i = M_{i-1}$. The dual of M is the graded module $M^\sharp = \mathrm{Hom}(M, R)$. If M is finite type and $N = (sM)^\sharp$, then $M = (sN)^\sharp$ via $x(sf) = -f(sx)$, for $x \in M$, $f \in N$.

If V is a graded module over R, then we denote by TV and ΛV the tensor algebra and free commutative algebra on V, respectively. The tensor coalgebra on V is denoted by $T_C V$. The shuffle product ([4], Appendix) makes $T_C V$ into a graded commutative (not cocommutative) Hopf algebra. Note that $TV = \oplus_{k\geq 0} T^k V$, $\Lambda V = \oplus_{k\geq 0}\Lambda^k V$ and $T_C V = \oplus_{k\geq 0} T_C^k V$, with $T^k V$, $\Lambda^k V$, and $T_C^k V$ consisting of words in V of length k. Elements of $T_C^k V$ are denoted $[v_1|\cdots|v_k]$.

The symmetric group S_k acts on $T^k V$ via $\sigma \cdot (x_1 \otimes \cdots \otimes x_k) = \pm\, x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$, where the sign is determined by the rule $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$.

1.2. **Graded Lie algebras.** A *graded Lie algebra* is a graded R-module $L = \oplus_{k\geq 0} L_k$ along with a degree-zero linear map $[\,,] : L \otimes L \to L$, called the *Lie bracket*, satisfying graded anti-commutativity, the graded Jacobi identity, and the further condition $[x, [x, x]] = 0$ if $x \in L_{\mathrm{odd}}$.

For example, any non-negatively graded associative algebra A is a graded Lie algebra via the graded commutator bracket $[a, b] = ab - (-1)^{|a||b|} ba$, for $a, b \in A$.

A graded Lie algebra is *connected* if it is concentrated in strictly positive degrees.

The *graded abelian Lie algebra* on $\{x_j\}$, denoted $L_{\mathrm{ab}}(x_j)$, is the free graded module on the basis $\{x_j\}$, with the trivial Lie bracket.

Let L be a graded Lie algebra, and denote by $L^\flat$ the underlying graded module. The *universal enveloping algebra* of L is the associative algebra $UL = (TL^\flat)/I$, where I is the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$, for $x, y \in L$. UL has the natural structure of a graded Hopf algebra; the comultiplication is defined by declaring the elements of L to be primitive and then using the universal property.

A *Lie derivation* on a graded Lie algebra L is a linear operator θ on L of degree k such that for $x, y \in L$, $\theta([x, y]) = [\theta(x), y] + (-1)^{k|x|}[x, \theta(y)]$.

A *differential graded Lie algebra* (DGL for short) is a pair (L, ∂), where L is a graded Lie algebra, and ∂ is a Lie derivation on L of degree -1 satisfying $\partial\partial = 0$. If (L, ∂) is a DGL, then ∂ extends to a derivation on UL, making $U(L, \partial)$ into a DGA.

1.3. **Divided powers algebras.** Divided powers algebras arise here as the duals of universal enveloping algebras.

Definition 1. A *divided powers algebra*, or Γ-algebra, is a commutative graded algebra A, satisfying either $A = A^{\geq 0}$ or $A = A^{\leq 0}$, equipped with set maps $\gamma^k : A^{2n} \to A^{2nk}$ for $k \geq 0$ and $n \neq 0$ satisfying the following list of conditions.

1. $\gamma^0(a) = 1; \gamma^1(a) = a$ for $a \in A$;
2. $\gamma^k(a+b) = \sum_{j=0}^{k} \gamma^j(a)\gamma^{k-j}(b)$ for $a, b \in A^{2n}$;
3. $\gamma^j(a)\gamma^k(a) = \begin{pmatrix} j+k \\ j \end{pmatrix} \gamma^{j+k}(a)$ for $a \in A^{2n}$;
4. $\gamma^j(\gamma^k(a)) = \dfrac{(jk)!}{j!^k k!} \gamma^{j+k}(a)$ for $a \in A^{2n}$;
5. $\gamma^k(ab) = \begin{cases} a^k \gamma^k(b) & \text{if } |a| \text{ and } |b| \text{ even, } |b| \neq 0, \\ 0 & \text{if } |a| \text{ and } |b| \text{ odd.} \end{cases}$

A Γ-*morphism* is an algebra morphism which respects the divided powers operations. A Γ-*derivation* on a Γ-algebra A is a derivation θ on A satisfying $\theta(\gamma^k(a)) = \theta(a)\gamma^{k-1}(a)$ for $a \in A^{2n}$, $k \geq 1$. A differential graded Γ-algebra, or Γ-DGA, is a pair (A, ∂), where A is a Γ-algebra, and ∂ is a Γ-derivation of degree -1 satisfying $\partial\partial = 0$.

Let V be a free graded R-module. Let $\Gamma^k(V)$ be the graded submodule of $T_C^k V$ of elements fixed by the action of the symmetric group S_k. Then $\Gamma(V) = \oplus_k \Gamma^k(V)$ is a Hopf subalgebra of $T_C(V)$, called the *free* Γ-*algebra* on V. Divided powers are defined on $\Gamma(V)$ by

1. $\gamma^0(v) = 1$, $\gamma^1(v) = v$ for $v \in V$,
2. $\gamma^k(v) = \underbrace{[v|\cdots|v]}_{k \text{ times}}$ for $v \in V^{2n}$

and then extending via conditions (4) and (5) of Definition 1. If $f : V \to A$ is any linear map of degree zero from V into a Γ-algebra A, then f extends to a unique Γ-morphism $\bar{f} : \Gamma(V) \to A$. If V is R-free on a countable, well-ordered basis $\{v_i\}$, then $\Gamma(V)$ is R-free, with basis consisting of elements $\gamma^{k_1}(v_1) \cdots \gamma^{k_s}(v_s)$ where $k_j \geq 0$ and $k_j = 0$ or 1 if $|v_j|$ is odd.

If $V \otimes W \xrightarrow{\langle,\rangle} R$ is a pairing, then there is an induced pairing

$$TV \otimes T_C W \to R \tag{1}$$

given by $\langle T^j V, T^k_C W\rangle = 0$ if $j \neq k$, and

$$\langle v_1 \otimes \cdots \otimes v_k, [w_1|\cdots|w_k]\rangle = \pm\langle v_1, w_1\rangle \cdots \langle v_k, w_k\rangle \tag{2}$$

where $\pm$ is the sign of the permutation

$$v_1, \dots, v_k, w_1, \dots, w_k \mapsto v_1, w_1, \dots, v_k, w_k.$$

The pairing (1) in turn induces a pairing

$$\Lambda V \otimes \Gamma W \to R. \tag{3}$$

Suppose that V is R-free of finite type, $V = V_{<0}$ or $V = V_{>0}$, and $W = V^\sharp$. Then (1) and (3) induce Hopf algebra isomorphisms $T_C(V^\sharp) \cong (TV)^\sharp$ and $\Gamma(V^\sharp) \cong (\Lambda V)^\sharp$.

1.4. **The Cartan–Chevalley–Eilenberg–Cartan complex.** Denote by $B(A)$ the bar construction on the augmented DGA (A, ∂) ([4], Section 1); recall that the underlying coalgebra of $B(A)$ is $T_C(s\bar{A})$, where $\bar{A}$ is the augmentation ideal. Let (L, ∂) be a DGL. Then $\Gamma(sL) \subset \Gamma(s\overline{UL}) \subset B(UL)$ and $(\Gamma(sL), \partial_0 + \partial_1)$ is a sub-DGC of $B(UL)$, denoted by $C_*(L, \partial)$, called the *chains on* (L, ∂).

The *Cartan–Chevalley–Eilenberg–Cartan complex* on (L, ∂) is the commutative cochain algebra $C^*(L, \partial) = (\Lambda V, d)$, dual to $C_*(L, \partial)$, where $V = (sL)^\sharp$, and the differential d is the sum of derivations d_0 and d_1. The *linear part* d_0 preserves word length and is dual to ∂ in that $\langle d_0 v, sx\rangle = (-1)^{|v|}\langle v, s\partial x\rangle$ for $v \in V$, $x \in L$. The *quadratic part* d_1 increases word length by one and is dual to the Lie bracket in L:

$$\langle d_1 v, sx \cdot sy\rangle = (-1)^{|sy|}\langle v, s[x, y]\rangle \tag{4}$$

where the pairing is (3) above with $W = sL = V^\sharp$. We will usually refer to the Cartan–Chevalley–Eilenberg–Cartan complex as the *cochains on* (L, ∂).

1.5. **Bockstein spectral sequences.** Fix a prime p. Let C be a free chain complex over $\mathbf{Z}_{(p)}$. Applying $C \otimes -$ to the short exact sequence of coefficient modules

$$0 \to \mathbf{Z}_{(p)} \overset{\times p}{\to} \mathbf{Z}_{(p)} \to \mathbf{F}_p \to 0$$

leads to a long exact sequence in homology which may be wrapped into the exact couple

$$\begin{array}{ccc} H_*(C) & \longrightarrow & H_*(C) \\ & \nwarrow \quad \swarrow & \\ & H_*(C; \mathbf{F}_p) & \end{array}$$

from which we get the *homology Bockstein spectral sequence modulo p* of C, $(E^r(C), \beta^r)$, mod p BSS for short [2]. If $C = C_*(X)$ is the normalized singular chain complex of a space X, then we refer to the homology BSS mod p of $C_*(X)$ as the mod p homology BSS of X, denoted $(E^r(X), \beta^r)$.

There is the corresponding notion of *cohomology Bockstein spectral sequence* defined in the obvious manner, using the functor $\mathrm{Hom}(C, -)$ rather than $C \otimes -$.

The mod p BSS of C measures p-torsion in $H_*(C)$: if $x, y \in E^r$, $x \neq 0$, satisfy $\beta^r(y) = x$, then x represents a torsion element of order p^r in $H_*(C)$.

Notation. If $c \in C$ is such that $[\bar{c}] \in E^1$ lives until the E^r term then we will denote the corresponding element of E^r by $[c]_r$.

1.6. **Acyclic closures and minimal models.** (Reference: [4], Sections 2 and 7.) Consider the graded algebra $\Lambda V \otimes \Gamma(sV)$ over R. Extend the divided powers operations on $\Gamma(sV)$ to $R \oplus \Lambda V \otimes \Gamma^+(sV)$ via rule 5 of Definition 1.

Definition 2. ([4], Section 2) An acyclic closure of the DGA $(\Lambda V, d)$ is a DGA of the form $C = (\Lambda V \otimes \Gamma(sV), D)$ in which D is a Γ-derivation restricting to d in ΛV and $H(C) = H^0(C) = R$.

Let (L, ∂) be a connected DGL over R which is R-free of finite type. Then $C^*(L) = (\Lambda V, d)$ where $V = (sL)^\sharp$. Let C be an acyclic closure for $C^*(L)$, and set $(\Gamma(sV), \bar{D}) = R \otimes_{C^*(L)} C$. By the work of Halperin in [4], we identify $H(UL) = H([\Gamma(sV), \bar{D}]^\sharp)$ and $UL = (\Gamma(sV), \bar{D})^\sharp$.

Let $R = \mathbf{Z}_{(p)}$ or $R = \mathbf{F}_p$, and consider a commutative algebra of the form $(\Lambda W, d)$ over R, where $W = W^{\geq 2}$ is R-free and of finite type. We may write the differential as a sum $d = \sum_{j \geq 0} d_j$ where d_j raises wordlength by j.

Definition 3. If $R = \mathbf{Z}_{(p)}$, the DGA $(\Lambda W, d)$ above is $\mathbf{Z}_{(p)}$*-minimal* if $d_0 : W \to pW$. If $R = \mathbf{F}_p$, $(\Lambda W, d)$ is $\mathbf{F}_p$*-minimal* if $d_0 = 0$.

Suppose (A, ∂) is a cochain algebra satisfying $H^0(A) = R$, $H^1(A) = 0$, $H^2(A)$ is R-free, and $H^*(A)$ is of finite type. Then by [4], Theorem 7.1, there exists a quasi-isomorphism $m : (\Lambda W, d) \xrightarrow{\simeq} (A, \partial)$ from an R-minimal algebra. This quasi-isomorphism is called a *minimal model.*

Associated to an $\mathbf{F}_p$-minimal model $m : (\Lambda W, d) \xrightarrow{\simeq} (A, \partial)$ is its *homotopy Lie algebra*, E. As a graded vector space, $E = (sW)^\sharp$; the bracket is defined by the relation

$$\langle w, s[x, y] \rangle = (-1)^{|sy|} \langle d_1 w, sx \cdot sy \rangle$$

for $w \in W$, $x, y \in E$.

2. The image of $H(L) \to H(UL)$

Let (L, ∂) be a connected DGL over $\mathbf{F}_p$ of finite type. By [4], the choice of minimal model $m : (\Lambda W, d) \xrightarrow{\simeq} C^*(L)$ determines an isomorphism of graded Hopf algebras, $H(UL) \cong UE$, where E is the homotopy Lie algebra of m.

Proposition 4. *With the notation above, the image of $H(\iota) : H(L) \to H(UL)$ lies in E.*

Proof. It suffices to prove that the following diagram commutes.

$$\begin{array}{ccc} H(L,\partial) & \xrightarrow{H(\iota_L)} & H(UL) \\ \Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle\cong} \\ E & \xrightarrow{\iota_E} & UE \end{array} \tag{5}$$

Recall that $C^*(L, \partial) = (\Lambda V, d)$, where $V = (sL)^\sharp$ and $d = d_0 + d_1$. Recall further that the minimality condition on $(\Lambda W, d)$ implies that the linear part of its differential vanishes. The *linear part* of m is the linear map $m_0 : (W, 0) \to (V, d_0)$ defined by the condition $m - m_0 : W \to \Lambda^{\geq 2} V$. Recall that $E = (sW)^\sharp$ and $UE = \Gamma(sW)^\sharp$ ([4], Theorem 6.2).

The model m extends to a morphism of constructible acyclic closures ([4], Section 2) $\hat{m} : (\Lambda W \otimes \Gamma(sW), D) \to (\Lambda V \otimes \Gamma(sV), D)$ by Proposition 2.7 of [4]. Since $(\Lambda W, d)$ is $\mathbf{F}_p$-minimal, $d_0 = 0$. By Corollary 2.6 of [4], $d_0 = 0$ is equivalent to $\bar{D} = 0$ in $(\Gamma(sW), \bar{D})$. Apply $\mathbf{F}_p \otimes_m -$ to $\hat{m}$ to get a Γ-morphism $\bar{m} : (\Gamma(sW), 0) \to (\Gamma(sV), \bar{D})$.

Let $\pi_L : (\Gamma(sV), \bar{D}) \twoheadrightarrow s(V, d_0)$ and $\pi_E : (\Gamma(sW), 0) \twoheadrightarrow s(W, 0)$ be the projections. The maps π_L and π_E fit into the diagram

$$\begin{array}{ccc} (\Gamma(sW), 0) & \xrightarrow{\pi_E} & s(W, 0) \\ \Big\downarrow{\scriptstyle\bar{m}} & & \Big\downarrow{\scriptstyle sm_0} \\ (\Gamma(sV), \bar{D}) & \xrightarrow{\pi_L} & s(V, d_0) \end{array} \tag{6}$$

For $w \in W$, Proposition 2.7 of [4] states that $\hat{m}(1 \otimes sw) - 1 \otimes sm_0 w$ has total wordlength at least two. It follows that $\bar{m}(sw) - sm_0 w$ has $\Gamma(sV)$-wordlength at least two, so $\pi_L(\bar{m}(sw)) = sm_0 w = sm_0(\pi_E(sw))$, so Diagram (6) commutes. Dualize and pass to homology to get (5). ☐

3. Bockstein spectral sequence of a universal enveloping algebra

In this section, we prove the main algebraic result, Theorem 3. Unless otherwise stated, our ground ring will be $\mathbf{Z}_{(p)}$, the integers localized at p.

Let $(\Lambda W, d)$ be a minimal Sullivan algebra over $\mathbf{Z}_{(p)}$. Let $C = (\Lambda W \otimes \Gamma(sW), D)$ be a constructible acyclic closure for $(\Lambda W, d)$ ([4], Section 2). Let $(\Gamma(sW), \bar{D})$ be the quotient $\mathbf{Z}_{(p)} \otimes_{(\Lambda W, d)} C$. $C \otimes \mathbf{F}_p$ is a constructible acyclic closure for $(\Lambda W, d) \otimes \mathbf{F}_p$. Since $(\Lambda W, d)$ is $\mathbf{Z}_{(p)}$-minimal, $p|d_0$, so the linear part of the differential vanishes in $(\Lambda W, d) \otimes \mathbf{F}_p$. It follows by Corollary 2.6 of [4] that the differential in $(\Gamma(sW), \bar{D}) \otimes \mathbf{F}_p$ is null, so that $p|\bar{D}$. Set $E^r = E^r([\Gamma(sW), \bar{D}]^\sharp)$ and $E_r = E_r(\Gamma(sW), \bar{D})$. Let $\rho : \Gamma(sW) \to \Gamma(sW) \otimes \mathbf{F}_p = E_1$ be the reduction homomorphism.

Proposition 5. *With the hypotheses and notation above, for $r \geq 1$, the following statements hold.*

1. *E_r is isomorphic to a free divided powers algebra,*
2. *there is a Γ-morphism $g_r : E_r \to E_1$ such that if $g(z) = \rho(a)$ for some $z \in E_r$, $a \in \Gamma(sW)$, then $z = [a]_r$.*
3. *there is a graded Lie algebra L^r such that $(E^r, \beta^r) = U(L^r, \beta^r)$.*

Lemma 6. *Let (UL, ∂) be a* DGH *over $\mathbf{F}_p$ of finite type, so $(UL)^\sharp = \Gamma V$ as an algebra. If $\partial^\sharp$ is a Γ-derivation, then $\partial(L) \subset L$.*

Proof of Lemma 6. It suffices to prove the dual statement, namely that $\partial^\sharp : \Gamma V \to \Gamma V$ factors over the surjection $\pi : \Gamma V \to V$ to induce a differential in V. But $\ker(\pi)$ consists of products along with elements of the form $\gamma^{p^k}(v)$ for $v \in V$, $k \geq 1$. Since $\partial^\sharp$ is a Γ-derivation, $\partial^\sharp(\gamma^{p^k}(v)) = \partial^\sharp(v)\gamma^{p^k-1}(v)$ is a product. It follows that $\partial^\sharp(\ker(\pi)) \subset \ker(\pi)$, completing the proof. □

Proof of Proposition 5. We proceed by induction. For $r = 1$, let $W_1 = W \otimes \mathbf{F}_p$. Since $p|\bar{D}$, $E_1 = \Gamma(sW_1)$, establishing the first statement. For the second statement we may take g_1 to be the identity map. Let L^1 be the homotopy Lie algebra of $(\Lambda W, d) \otimes \mathbf{F}_p$. Apply Theorems 6.2 and 6.3 of [4] to the minimal algebra $(\Lambda W, d) \otimes \mathbf{F}_p$ to get a graded Hopf algebra isomorphism $E^1 = (\Gamma(sW_1))^\sharp \cong UL^1$.

Since $p|\bar{D}$ in $(\Gamma(sW), \bar{D})$, $\beta_1 = \bar{D}/p$ (reduced modulo p). Thus because $\bar{D}$ is a Γ-derivation, so is β_1. Since $\Gamma(sW_1)$ is the Γ-algebra dual to UL^1 it follows by Lemma 6 that $\beta^1 : L^1 \to L^1$ and so $E^1 = U(L^1, \beta^1)$.

Now suppose the three statements are established for $r-1$. Let $C(r-1) = (\Lambda W_{r-1} \otimes \Gamma(sW_{r-1}), D)$ be a constructible acyclic closure for

$C^*(L^{r-1}, \beta^{r-1}) = (\Lambda W_{r-1}, d)$. By Lemma 5.4 of [4], there is a chain isomorphism $\gamma_{r-1} : (E^{r-1}, \beta^{r-1}) = U(L^{r-1}, \beta^{r-1}) \xrightarrow{\cong} (\Gamma(sW_{r-1}), \bar{D})^\sharp$. Fix a well-ordered basis $\{x_j\}$ of L^{r-1}; this determines a dual basis $\{sw_j\}$ of sW_{r-1}. The isomorphism γ_{r-1} identifies the Poincaré–Birkhoff–Witt basis element $x_1^{k_1} \cdots x_j^{k_j}$ of UL^{r-1} as a dual basis to the basis element $\gamma^{k_1}(sw_1) \cdots \gamma^{k_j}(sw_j)$ of $\Gamma(sW_{r-1})$. It follows that γ_{r-1} factors as the composition of DGC isomorphisms $UL^r \xrightarrow{\cong} \Lambda(L^r)^\flat \xrightarrow{\cong} (\Gamma(sW_{r-1}))^\sharp$. Since L^{r-1} and W_{r-1} are finite type, we dualize to obtain the DGA isomorphism $\alpha_{r-1} : (\Gamma(sW_{r-1}), \bar{D}) \xrightarrow{\cong} (E_{r-1}, \beta_{r-1})$.

Let $m_r : (\Lambda W_r, d) \xrightarrow{\simeq} C^*(L^{r-1}, \beta^{r-1})$ be a minimal model. Let $C'(r)$ be a constructible acyclic closure of $(\Lambda W_r, d)$ ([4], Section 2). Since $(\Lambda W_r, d)$ is $\mathbf{F}_p$-minimal, $d_0 = 0$, so by Corollary 2.6 of [4], the differential in $(\Gamma(sW_r), \bar{D}_{(r)}) = \mathbf{F}_p \otimes_{\Lambda W} C'(r)$ is zero. By [4], Proposition 2.7, m_r induces a Γ-morphism $\bar{m}_r : (\Gamma(sW_r), 0) \xrightarrow{\simeq} (\Gamma(sW_{r-1}), \bar{D})$. Since $\mathbf{F}_p$ is a field, by Lemma 3.3 of [4], we may identify $H(\bar{m}_r)$ with $\mathrm{Tor}^{m_r}(\mathbf{F}_p, \mathbf{F}_p)$, where Tor is the differential torsion functor [3]. Therefore since m_r is a quasi-isomorphism, $H(\bar{m}_r)$ is an isomorphism. Let $\alpha_r : \Gamma(sW_r) \xrightarrow{\cong} E_r$ be the composition of algebra isomorphisms

$$\Gamma(sW_r) \xrightarrow{H(\bar{m}_r)} H(\Gamma(sW_{r-1}), \beta_{r-1}) \xrightarrow{H(\alpha_{r-1})} E_r$$

to establish the first statement. Note that α_r will be the isomorphism dual to γ_r in the next stage of the induction.

Setting $f = \alpha_{r-1}\bar{m}_r\alpha_r^{-1}$, we get the commutative diagram

$$\begin{array}{ccc} (\Gamma(sW_r), 0) & \xrightarrow{\bar{m}_r} & (\Gamma(sW_{r-1}), \bar{D}) \\ {\scriptstyle \alpha_r}\downarrow{\scriptstyle \cong} & & {\scriptstyle \alpha_{r-1}}\downarrow{\scriptstyle \cong} \\ (E_r, 0) & \xrightarrow{f} & (E_{r-1}, \beta_{r-1}) \end{array}$$

and it follows from the definitions that $H(f)$ is the identity on E_r.

By the inductive hypothesis, there exists a Γ-morphism $g_{r-1} : E_{r-1} \to E_1$ such that $z = [a]_{r-1}$ whenever $z \in E_{r-1}$, $a \in \Gamma(sW)$ satisfy $g(z) = \rho(a)$. We now show that $g_r := g_{r-1}f$ satisfies statement 2. For $u \in E^r$ choose $a \in \Gamma(sW)$ so that $g_{r-1}(f(u)) = \rho(a)$. Then $f(u) = [a]_{r-1}$, hence $\beta_{r-1}[a]_{r-1} = 0$ and $[a]_r \in E^r$ is defined. Since f induces the identity in homology, $f([a]_r) = [a]_{r-1} + \beta_{r-1}(v)$ for some $v \in E^{r-1}$. Thus $f(u - [a]_r) = \beta_{r-1}(v)$, so $u - [a]_r$ is a boundary in $(E_r, 0)$, whence $u = [a]_r$. This establishes the second statement.

The model m_r determines an isomorphism $E^r = H(U(L^{r-1}, \beta^{r-1})) \cong UL^r$, where L^r is the homotopy Lie algebra for the model m_r. As a graded vector space, $L^r = (sW_r)^\sharp$.

Let $u \in E_r$, and suppose for some $a \in \Gamma(sW)$ that $\rho(a) = g_r(u)$. Then $u = [a]_r$, so $\bar{D}a = p^r b$ for some $b \in \Gamma(sW)$. Thus $\beta_r(u) = [b]_r$. Since g_r and ρ are Γ-morphisms, $\rho(\gamma^j(a)) = g_r(\gamma^j(u))$ so $\gamma^j(u) = [\gamma^j(a)]_r$. Furthermore, $\bar{D}(\gamma^k(a)) = p^r b \cdot \gamma^{k-1}(a)$ so

$$\beta_r \gamma^k(u) = \beta_r[\gamma^k(a)]_r = [b \cdot \gamma^{k-1}(a)]_r = [b]_r[\gamma^{k-1}(a)]_r = \beta_r(u) \cdot \gamma^{k-1}(u).$$

By Lemma 6, this establishes the third statement, completing the inductive step and the proof. □

Proof of Theorem 3. Let $m : (\Lambda W, d) \xrightarrow{\simeq} C^*(L, \partial)$ be a minimal model. Recall that the underlying algebra of $C^*(L, \partial)$ is ΛV, where $V = (sL)^\sharp$. Let $(\Lambda W \otimes \Gamma(sW), D)$ and $(\Lambda V \otimes \Gamma(sV), D)$ be constructible acyclic closures for $(\Lambda W, d)$ and $C^*(L, \partial)$, respectively. The model m determines a Γ-morphism $\bar{m} : (\Gamma(sW), \bar{D}) \to (\Gamma(sV), \bar{D})$ where $H(\bar{m}^\sharp)$ is an isomorphism. The composition

$$U(L, \partial) \xrightarrow{\cong} (\Gamma(sV), \bar{D})^\sharp \xrightarrow{\simeq} (\Gamma(sW), \bar{D})^\sharp$$

induces an isomorphism of Bockstein spectral sequences, establishing the first statement.

The reduced minimal model $m \otimes \mathbf{F}_p : (\Lambda W, d) \otimes \mathbf{F}_p \xrightarrow{\simeq} C^*(L, \partial) \otimes \mathbf{F}_p$ has homotopy Lie algebra L^1, so by Proposition 4, $\operatorname{im} E^1(\iota) \subset L^1$. Suppose that $\operatorname{im} E^{r-1}(\iota) \subset L^{r-1}$. Let $\iota^{(r-1)} : L^{r-1} \hookrightarrow UL^{r-1}$ be the inclusion. Then $\operatorname{im} E^r(\iota) \subset \operatorname{im} H(\iota^{(r-1)})$. The homotopy Lie algebra of the minimal model $m_r : (\Lambda W_r, d) \xrightarrow{\simeq} C^*(L^{r-1}, \beta^{r-1})$ is L^r, so Proposition 4 states that $\operatorname{im} H(\iota^{(r-1)}) \subset L^r$, completing the induction and the proof. □

Proof of Theorems 1 and 2. Anick in [1] proves that there is a DGL L_X and a DGH quasi-isomorphism $UL_X \to C_*(\Omega X; \mathbf{Z}_{(p)})$. Thus as Hopf algebras, for $r \geq 1$, $E^r(UL_X) = E^r(\Omega X)$ and $E_r(UL_X) = E_r(\Omega X)$. The result follows by applying Theorem 3 to the DGL L_X. □

4. Morphisms of universal enveloping algebras

Let R be a commutative ring, L_1 and L_2 connected, graded Lie algebras over R which are R-free of finite type. Let $\varphi : UL_1 \to UL_2$ be a morphism of Hopf algebras. The purpose of this section is to prove

Proposition 7. *$\varphi(L_1) \subset L_2$ if and only if $\varphi^\sharp : (UL_2)^\sharp \to (UL_1)^\sharp$ is a Γ-morphism.*

Proof. Observe that, for $j = 1, 2$, the sequence of functors

$$L_j \rightsquigarrow C^*(L_j) \rightsquigarrow \Gamma V_j = R \otimes_{C^*(L_j)} (C^*(L_j) \otimes \Gamma V_j) \rightsquigarrow (\Gamma V_j)^\sharp = UL_j$$

identifies UL_j as the natural dual of the free Γ-algebra ΓV_j, with L_j naturally dual to V_j, and the inclusion $L_j \hookrightarrow UL_j$ naturally dual to the projection $\Gamma V_j \overset{\pi_j}{\twoheadrightarrow} V_j$. Therefore if $\varphi|_{L_1} : L_1 \to L_2$, then $\varphi^\sharp$ is a Γ-morphism. Conversely, if $\varphi^\sharp$ is a Γ-morphism, then $\varphi^\sharp(\ker \pi_2) \subset \ker \pi_1$, so $\varphi(L_1) \subset L_2$. □

5. Examples

We begin with a proposition to be used in both examples.

Proposition 8. *Define a* DGL *over* $\mathbf{F}_p$ *by* $(L, \partial) = (L_{ab}(e, f), \partial f = e)$, *where* $|f| = 2n$. *Then* $C^*(L, \partial) = (\Lambda(x, y), d)$ *with* $dx = y$ *and* $|x| = 2n$. *A minimal model* $m : (\Lambda(x_1, y_1), 0) \overset{\cong}{\to} C^*(L, \partial)$, *given by* $x_1 \mapsto x^p$ *and* $y_1 \mapsto x^{p-1}y$, *induces isomorphisms* $\Gamma(sx_1, sy_1) \overset{\cong}{\to} H([UL]^\sharp)$ *and* $H(UL) \overset{\cong}{\to} UL_{ab}(e_1, f_1)$ *with* $|e_1| = |sx_1| = 2np - 1$, $|f_1| = |sy_1| = 2np$.

Proof. Straightforward. □

Example 1. Let $L = L_{\mathrm{ab}}(e, f)$ over $\mathbf{Z}_{(p)}$ on generators e and f of degrees $2n-1$ and $2n$, respectively. Set $\partial f = pe$. Applying Proposition 8 recursively, we have $E_r(UL) = \Gamma(sx_r, sy_r)$ and $E^r(UL) = UL_{\mathrm{ab}}(e_r, f_r)$, with $|e_r| = |sx_r| = 2np^r - 1$, $|f_r| = |sy_r| = 2np^r$, $\beta_r(sx_r) = sy_r$, and $\beta^r(f_r) = e_r$, while the sequence $E^r(L)$ collapses after the first term.

Example 2. Define a DGL (L, ∂) over $\mathbf{Z}_{(p)}$ by $L = L_{\mathrm{ab}}(e, f, g)$, where $|e| = 2n-1$, $|f| = |g| = 2$, and $\partial(f) = pe$. Then $L^1 = L_{\mathrm{ab}}(e, f, g)$ (over $\mathbf{F}_p$), with $\beta^1(f) = e$, and $C^*(L^1, \beta^1) = (\Lambda(x, y), dx = y) \otimes (\Lambda(z), 0)$. Recall the model m from Proposition 8. Define DGA morphisms $i, j : (\Lambda(z), 0) \to C^*(L^1, \beta^1)$ by $i(z) = z$, $j(z) = z + y$. Then $\varphi = m \otimes i$ and $\psi = m \otimes j$ are minimal models, both with homotopy Lie algebra $L^2 = L_{\mathrm{ab}}(a, b, c)$, $|a| = 2np - 1$,$|b| = 2np$, and $|c| = 2n$. The two models determine Hopf algebra isomorphisms $\varphi^*, \psi^* : H(UL^1) \to UL^2$, given by $\varphi^*[ef^{p-1}] = \psi^*[ef^{p-1}] = a$, $\varphi^*[g] = \psi^*[g] = c$, $\varphi^*[f^p] = b$, and $\psi^*[f^p] = b + c^p$. The algebra isomorphism $\psi^*(\varphi^*)^{-1} : UL_{\mathrm{ab}}(a, b, c) \to UL_{\mathrm{ab}}(a, b, c)$ is not of the form $U\theta$ for any Lie algebra morphism $\theta : L_{\mathrm{ab}}(a, b, c) \to L_{\mathrm{ab}}(a, b, c)$. Therefore the construction involved in Theorem 3 is not natural.

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