# The homotopy groups of the $L_2$-localized mod 3 Moore spectrum

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## §1. Introduction

Let $\mathcal{S}_p$ denote the category of $p$-local spectra for each prime number $p$ and $BP$ the Brown-Peterson spectrum at $p$. Then we have the Bousfield localization functor $L_n : \mathcal{S}_p \to \mathcal{S}_p$ with respect to $v_n^{-1}BP$ for the generator $v_n$ of $BP_* = \boldsymbol{Z}_{(p)}[v_i : i > 0]$. The category $L_n\mathcal{S}_p$ is easier to be understood than $\mathcal{S}_p$ itself and reflects some properties of it. $L_n\mathcal{S}_p$ is, in a sense, generated by the $L_n$-localized sphere spectrum $L_nS^0$, because $L_nX = X \wedge L_nS^0$ for any spectrum $X$ by the smash product theorem [10, Th. 7.5.6]. Besides, we have the chromatic convergence theorem due to Hopkins and Ravenel [10, Th. 7.5.7], which says that $\operatorname{holim}_{\overleftarrow{n}} L_nX = X$ for a finite spectrum $X$. Therefore it is very important to compute the homotopy groups $\pi_*(L_nS^0)$. So far we know the homotopy groups $\pi_*(L_nS^0)$ for $n < 2$ given in [8] and for $n = 2$ and $p > 3$ in [12]. The next place to study is the case where $n = 2$ and $p = 3$. They are computed by using the Bockstein spectral sequences $\pi_*(L_nV(k)) \Rightarrow \pi_*(L_nV(k-1))$, where $V(n)$ denotes the Toda-Smith spectrum, and is known to exist if $n < 4$ and $p > 2n$ (*cf.* [9]). (For $L_nV(k)$, we have some other existence theorems in [13] and [14].) Note that $V(-1) = S^0$ and $V(0)$ is the mod $p$ Moore spectrum. On the other hand, $\pi_*(L_nV(n-1))$ is computed by Ravenel (*cf.* [9]) in case of $n < 4$ and $n < p-1$, by Mahowald [5] in case of $n = p-1 = 1$, and by the author [11] and Henn and Mahowald [4] in case of $n = p-1 = 2$. In this paper we study the Bockstein spectral sequence $\pi_*(L_2V(1)) \Rightarrow \pi_*(L_2V(0))$ and determine $\pi_*(L_2V(0))$ at the

*Part of this work was done during the stay at Max-Plank-Institut für Mathematik prime number 3. Our main tool is the Adams-Novikov spectral sequence. In [6] Miller, Ravenel and Wilson introduced the chromatic spectral sequence converging to the $E_2$-term of the Adams-Novikov spectral sequence for computing the homotopy groups $\pi_*(V(n))$. We use here the modified chromatic spectral sequence which converges to the $E_2$-term of the Adams-Novikov spectral sequence $E_2^{*,*}(L_2V(0)) \Rightarrow \pi_*(L_2V(0))$ based on $E(2)$ with $E_1$-terms $H^*M_1^1$ and $H^*M_1^0$. Here $E(2)$ denotes the Johnson-Wilson spectrum with coefficient $E(2)_* = \boldsymbol{Z}_{(3)}[v_1, v_2^{\pm 1}]$, $M_1^0 = v_1^{-1}E(2)_*/(3)$ and $M_1^1 = E(2)_*/(3, v_1^\infty)$ are the $E(2)_*E(2)$-comodules and $H^*M = \mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*, M)$. $H^*M_1^0$ was determined by Ravenel [7], and so it suffices to determine $H^*M_1^1$ for the $E_2$-term $E_2^{*,*}(L_2V(0))$. In the first half of this paper, we actually determine $H^*M_1^1$ by using the Bockstein spectral sequence $H^*K(2)_* \Rightarrow H^*M_1^1$, where $K(2)_* = E(2)_*/(3, v_1)$ and $H^*K(2)_*$ is determined by Ravenel [7]. The structure of $H^*M_1^1$ is stated in Theorem 2.5, and obtain the $E_2$-term $E_2^{*,*}(L_2V(0))$ in Theorem 2.6. In [6] $H^0M_1^1$ is determined, and we studied $H^1M_1^1$ in [1]. Unfortunately Theorem 4.4 of [1] is incorrect, and so are Proposition 5.2 and Theorem 1.1 consequently. Here we replace it by Lemma 4.2, which is proved in §7, and obtain $H^1M_1^1$. In the second half of this paper, we determine the Adams-Novikov differentials $d_r$ on $E_r^{*,*}(L_2W)$ with $E_2^{*,*}(L_2W) = H^*M_1^1$, and then the homotopy groups $\pi_*(L_2W)$ which is described in Theorem 2.8. Here $W$ denotes a cofiber of the localization map $V(0) \to \underset{\overrightarrow{\alpha}}{\mathrm{holim}}\, V(0)$ for the Adams map $\alpha : \Sigma^4V(0) \to V(0)$. The homotopy groups $\pi_*(L_2V(0))$, which is our main result, are obtained in Theorem 2.11 as a corollary of Theorem 2.8. The results would have applications. Here we treat the $\beta$-family of the homotopy groups of $\pi_*(L_2S^0)$ at the prime 3. We note that though the result of [2] depends on a result of [1], it remains correct since the proof does not require the incorrect part.

This paper is organized as follows: In the next section, we state our results. Then we prove Theorem 2.5 in §3 assuming the behavior of the connecting homomorphisms $\delta_s : H^sM_1^1 \to H^{s+1}K(2)_*$ which will be studied in the following sections. In §4, assuming the behavior of the differential of the cobar complex $\Omega^*E(2)_*$ which will be studied in §§6 and 7, we prove Proposition 3.4 which determines the differentials of the Bockstein spectral sequence and is the key lemma to determine the $E_2$-term $E_2^{*,*}(L_2V(0))$. In order to study the differential of the cobar complex $\Omega^*E(2)_*$, we need some relations in $E(2)_*E(2)$, which is given in §5. In §8, we compute the differentials of the

Adams-Novikov spectral sequence, and prove Theorem 2.8. The last section is devoted to applications for $\beta$-elements.

## §2. Statement of results

Throughout this paper everything is localized at the prime 3. Let $V(0)$ denotes the mod 3 Moore spectrum and $W$ be the cofiber of the localization map $V(0) \to L_1V(0)$. Since $L_1V(0) = \underset{\overrightarrow{\alpha}}{\text{holim}}\, V(0)$ for the Adams map $\alpha : \Sigma^4V(0) \to V(0)$, we can define $W$ as follows: Let $V(1)_j$ denote a cofiber of $\alpha^j : \Sigma^{4j}V(0) \to V(0)$. In particular, $V(1)_1 = V(1)$, the Toda-Smith spectrum. Then we have canonical maps $\pi_j : V(1)_j \to V(1)_{j-1}$ and $\iota_j : V(1)_j \to V(1)_{j+1}$. We define $W = \underset{\overrightarrow{\iota_j}}{\text{holim}}\, V(1)_j$. By definition, we have cofiber sequences $V(1) \overset{v_1^{j-1}}{\to} V(1)_j \overset{\pi_j}{\to} V(1)_{j-1}$, whose homotopy colimit yields another one

$$V(1) \xrightarrow{i} W \xrightarrow{v_1} W. \tag{2.1}$$

Apply the Johnson-Wilson homology $E(2)_*(-)$ with coefficient $E(2)_* = \boldsymbol{Z}_{(3)}[v_1, v_2^{\pm 1}]$ to the cofiber sequence we have a short exact sequence

$$0 \longrightarrow K(2)_* \xrightarrow{i_*} M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0. \tag{2.2}$$

where $K(2)_* = (\boldsymbol{Z}/3)[v_2^{\pm 1}]$, $M_1^1 = E(2)_*/(3, v_1^\infty)$ and $i_*(x) = x/v_1$. Note that $K(2)_*$ is the coefficient ring of the second Morava $K$-theory $K(2)_*(-)$. Apply the functor $H^*- = \text{Ext}^*_{E(2)_*E(2)}(E(2)_*, -)$ to the exact sequence (2.2), and we obtain the Bockstein spectral sequence

$$E_2^{*,*}(L_2V(1)) = H^*K(2)_* \Longrightarrow H^*M_1^1 = E_2^{*,*}(L_2W).$$

Here $E_2^{*,*}(X)$ denotes the $E_2$-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$.

In [3], Henn computed the $E_2$-term $H^*K(2)_*$ as follows:

THEOREM 2.3. (*cf.* [11, Th. 5.8, Prop. 5.9].) *The $E_2$-term $E_2^{*,*}(L_2V(1)) = H^*K(2)_*$ is isomorphic to the $K(2)_*[b_{10}]$-module*

$$F \otimes K(2)_*[b_{10}] \otimes \Lambda(\zeta_2).$$

*Here* $F = (\boldsymbol{Z}/3)\{1, h_{10}, h_{11}, b_{11}, \xi, \psi_0, \psi_1, b_{11}\xi\}$. *Besides, we have relations:*

$$\begin{array}{l} h_{10}h_{11} = 0,\ h_{10}\xi = 0,\ h_{11}\xi = 0, \\ v_2^2h_{10}b_{10} = h_{11}b_{11},\ v_2h_{11}b_{10} = -h_{10}b_{11}, \\ b_{11}\xi = v_2h_{10}\psi_1 = v_2h_{11}\psi_0,\ b_{10}\xi = -h_{10}\psi_0 = v_2^{-1}h_{11}\psi_1, \\ v_2^3b_{10}^2 = -b_{11}^2,\ b_{10}\psi_1 = -v_2^{-1}b_{11}\psi_0,\ \textit{and}\ b_{10}\psi_0 = v_2^{-2}b_{11}\psi_1. \end{array}$$

The bidegrees of these generators are as follows:

$$\begin{array}{l} ||v_2|| = (0,16),\ ||h_{10}|| = (1,4),\ ||h_{11}|| = (1,12),\ ||b_{10}|| = (2,12), \\ ||b_{11}|| = (2,36),\ ||\xi|| = (2,8),\ ||\psi_0|| = (3,16),\ ||\psi_1|| = (3,24). \end{array}$$

In this paper, we first compute the Adams-Novikov $E_2$-term $E_2^{*,*}(L_2W)$ from Theorem 2.3 by using the Bockstein spectral sequence. In order to state the $E_2$-term, consider the algebra $k(1)_* = (\boldsymbol{Z}/3)[v_1]$ and $k(1)_*$-modules

$$(2.4)\qquad \begin{array}{rcl} F & = & E(2,1)_*\{v_2^{\pm 1}/v_1,\ v_2h_{10}/v_1^2,\ v_2^2h_{11}/v_1^2,\ v_2^{\pm 1}b_{11}/v_1\} \\ F^* & = & E(2,1)_*\{\xi/v_1^2,\ v_2^{2\pm 1}\psi_0/v_1,\ v_2^{\pm 1}\psi_1/v_1,\ b_{11}\xi/v_1^2\} \\ F_n & = & E(2,n+2)_*\{v_2^{\pm 3^{n+1}}/v_1^{4\cdot 3^n-1},\ v_2^{3^{n+1}}h_{10}/v_1^{6\cdot 3^n+1}, \\ & & v_2^{8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n+1},\ v_2^{3^n(5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}\}. \end{array}$$

Here $||v_1|| = (0,4)$, $E(2,n)_* = (\boldsymbol{Z}/3)[v_1, v_2^{\pm 3^n}]$, and $k(1)_*$ acts on an element of the form $x/v_1^j$ by the equation $v_1^i(x/v_1^j) = x/v_1^{j-i}$ if $j > i$, and $= 0$ otherwise. We also use the notation $K(1)_* = v_1^{-1}k(1)_*$. Then

THEOREM 2.5. *The $E_2$-term $E_2^{*,*}(L_2W) = H^*M_1^1$ of the Adams-Novikov spectral sequence converging to $\pi_*(L_2W)$ is isomorphic to the direct sum of $k(1)_*$-modules $(K(1)_*/k(1)_*)\otimes\Lambda(h_{10},\zeta_2)$,*

$$\sum_{n\geq 0} F_n\otimes\Lambda(\zeta_2), \quad \textit{and} \quad (F\oplus F^*)\otimes(\boldsymbol{Z}/3)[b_{10}]\otimes\Lambda(\zeta_2).$$

The short exact sequence associated to the cofiber sequence $V(0) \to L_1V(0) \to W$ yields the long exact sequence

$$H^*E(2)_*/(3) \longrightarrow H^*M_1^0 \longrightarrow H^*M_1^1 \xrightarrow{\partial} H^{*+1}E(2)_*/(3),$$

in which $M_1^0 = (\boldsymbol{Z}/3)[v_1^{\pm 1}, v_2^{\pm 1}]$ and the structures of $H^*M_1^0$ is determined to be $K(1)_*\otimes\Lambda(h_{10})$ by Ravenel [7]. Let $E_2^{*,*}(L_2V(0))$ denote the Adams-Novikov $E_2$-term converging to the homotopy groups $\pi_*(L_2V(0))$. Observing the exact sequence, we obtain the $E_2$-term by Theorem 2.5:

THEOREM 2.6. *The $E_2$-term $E_2^{*,*}(L_2V(0)) = H^*E(2)_*/(3)$ is isomorphic to the direct sum of the $k(1)_*$-modules $K(1)_*/k(1)_*\{\partial(\zeta_2)\}\otimes\Lambda(h_{10})$,*

$$k(1)_*\otimes\Lambda(h_{10}) \quad \text{and} \quad \Lambda(\zeta_2)\otimes\partial(\sum_{n\geq 0} F_n\oplus(F\oplus F^*)\otimes(\boldsymbol{Z}/3)[b_{10}])$$

In order to state the homotopy groups $\pi_*(L_2V(0))$, we prepare more notations:

$$\begin{aligned}
\widetilde{F}_0 &= E(2,2)_*\{v_2^3/v_1^2, v_2^{-3}/v_1^3, v_2^3h_{10}/v_1^6, v_2^8h_{10}/v_1^{11}, \\
&\qquad v_2^8\xi/v_1^4, v_2^2\xi/v_1^3\} \\
\widetilde{F}_1 &= E(2,3)_*\{v_2^{\pm 9}/v_1^{11}, v_2^9h_{10}/v_1^{18}, v_2^{24}h_{10}/v_1^{31}, v_2^{16\pm 9}\xi/v_1^{11}\} \\
\widetilde{F}_n &= E(2,n+2)_*\{v_2^{\pm 3^{n+1}}/v_1^{4\cdot 3^n-1}, v_2^{3^{n+1}}h_{10}/v_1^{2\cdot 3^{n+1}}, \\
&\qquad v_2^{8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n}, v_2^{3^n(5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n-1}\} \ (n\geq 2) \\
\widetilde{F} &= B_5(2,2)_*\{v_2/v_1, v_2h_{10}/v_1^2\} \\
&\qquad \oplus B_4(2,2)_*\{v_2^5h_{11}/v_1^2, v_2^4b_{11}/v_1\} \\
&\qquad \oplus B_3(2,2)_*\{v_2^2/v_1, v_2^5/v_1, v_2^7h_{10}/v_1\} \\
&\qquad \oplus B_2(2,2)_*\{v_2h_{10}/v_1, v_2^2h_{11}/v_1, v_2^5h_{11}/v_1, \\
&\qquad v_2^5b_{11}/v_1, v_2^{-1}b_{11}/v_1\} \\
\widetilde{F}^* &= B_5(2,2)_*\{v_2^7\psi_1/v_1\} \\
&\qquad \oplus B_4(2,2)_*\{v_2^3\xi/v_1^2, v_2^3\psi_0/v_1, v_2^6b_{11}\xi/v_1^2\} \\
&\qquad \oplus B_3(2,2)_*\{v_2^{-1}\psi_1/v_1\} \\
&\qquad \oplus B_2(2,2)_*\{\xi/v_1, v_2^2\psi_1/v_1, v_2^4\psi_0/v_1, v_2^3b_{11}\xi/v_1, \\
&\qquad v_2^7\psi_0/v_1, v_2^6b_{11}\xi/v_1\} \\
&\qquad \oplus \textstyle\sum_{n\geq 1}(B_3(2,n+2)_*\{v_2^{9u+3}\xi/v_1 \mid u\in\boldsymbol{Z}-I(n)\} \\
&\qquad \oplus B_2(2,n+2)_*\{v_2^{9u+3}\xi/v_1 \mid u\in I(n)\}),
\end{aligned} \tag{2.7}$$

where $B_k(2,n)_* = (\boldsymbol{Z}/3)[v_1, v_2^{\pm 3^n}, b_{10}]/(b_{10}^k)$, and

$$I(n) = \{x\in\boldsymbol{Z} \mid x = (3^{n-1}-1)/2 \ \text{ or } \ x = 5\cdot 3^{n-2}+(3^{n-2}-1)/2\}.$$

Studying the Adams-Novikov differentials $d_5$ and $d_9$ by results of [4] and [11], we obtain the following

THEOREM 2.8. *The homotopy groups $\pi_*(L_2W)$ are isomorphic to the tensor product of the exterior algebra $\Lambda(\zeta_2)$ and the direct sum of $k(1)_*$-modules $(K(1)_*/k(1)_*)\otimes\Lambda(h_{10})$, $\sum_{n\geq 0}\widetilde{F}_n$ and $\widetilde{F}\oplus\widetilde{F}^*$.*

For describing more homotopy groups, we further introduce the $k(1)_*$-modules:

$$
\begin{array}{rcl}
\overline{F_0} & = & E(2,2)_*\{v_1v_2^2h_{10}, v_2^{-4}h_{10}, v_1v_2^2\xi, v_2^5\xi, v_2^6\psi_1, v_1\psi_1\} \\
\overline{F_1} & = & E(2,3)_*\{v_2^{\pm 9-3}h_{10}, v_1v_2^8\xi, v_2^{16}\xi, v_1v_2^{12\pm 9}\psi_1\} \\
\overline{F_n} & = & E(2,n+2)_*\{v_2^{\pm 3^{n+1}-3^n}h_{10}, v_1v_2^{(3^{n+1}-1)/2}\xi, \\
& & \quad v_1v_2^{5\cdot 3^n+(3^n-1)/2}\xi, v_1v_2^{3^{n+1}(1\pm 1)+3(3^n-1)/2}\psi_1\}\ (n \geq 2) \\
\overline{F} & = & B_5(2,2)_*\{h_{11}, b_{10}\}\bigoplus B_4(2,2)_*\{v_2^3b_{11}, v_2^3h_{11}b_{11}\} \\
& & \quad \bigoplus B_3(2,2)_*\{v_2h_{11}, v_2^4h_{11}, v_1v_2^6b_{10}\} \\
& & \quad \bigoplus B_2(2,2)_*\{v_1b_{10}, v_1b_{11}, v_1v_2^3b_{11}, v_2^4h_{11}b_{11}, v_2^{-2}h_{11}b_{11}\} \\
\overline{F^*} & = & B_5(2,2)_*\{v_2^7b_{10}\xi\}\bigoplus B_4(2,2)_*\{v_2^2\psi_0, b_{11}\xi, v_2^6\psi_1b_{10}\} \\
& & \quad \bigoplus B_3(2,2)_*\{v_2^{-1}b_{10}\xi\} \\
& & \bigoplus B_2(2,2)_*\{v_1v_2^{-1}\psi_0, v_2^2b_{10}\xi, v_2^2b_{11}\xi, v_1v_2^3\psi_1b_{10}^2, \\
& & \qquad v_2^5b_{11}\xi, v_1v_2^6\psi_1b_{10}\} \\
& & \bigoplus \sum_{n\geq 1}(B_3(2,n+2)_*\{v_1v_2^{9u+2}\psi_0 \mid u \in \boldsymbol{Z} - I(n)\} \\
& & \qquad \bigoplus B_2(2,n+2)_*\{v_1v_2^{9u+2}\psi_0 \mid u \in I(n)\}),
\end{array}
\tag{2.9}
$$

where $\overline{M}$ is isomorphic to $\widetilde{M}$ for $M = F_n, F, F^*$ as $k(1)_*$-modules while there is one dimension shift. Furthermore, put $k(1)_*^\wedge = \lim\limits_{\leftarrow j} k(1)_*/(v_1^j)$. Since $\operatorname{holim}\limits_{\leftarrow j} L_2V(1)_j = L_{K(2)}V(0)$ by observing $K(2)_*$ homology, the above theorem implies

THEOREM 2.10. *The homotopy groups $\pi_*(L_{K(2)}V(0))$ are isomorphic to the tensor product of the exterior algebra $\Lambda(\zeta_2)$ and the direct sum of $k(1)_*$-modules $(k(1)_*^\wedge)\bigotimes\Lambda(h_{10})$, $\sum_{n\geq 0}\overline{F_n}$ and $\overline{F}\bigoplus\overline{F^*}$.*

Observing the cofibration $L_2V(0) \to L_1V(0) \to L_2W$ and $\pi_*(L_1V(0)) = K(1)_*\bigotimes\Lambda(h_{10})$, we have

THEOREM 2.11. *The homotopy groups $\pi_*(L_2V(0))$ are isomorphic to the direct sum of $k(1)_*\bigotimes\Lambda(h_{10})$, $(K(1)_*/k(1)_*)\partial\zeta_2\bigotimes\Lambda(h_{10})$, $\sum_{n\geq 0}\overline{F_n}\bigotimes\Lambda(\zeta_2)$ and $(\overline{F}\bigoplus\overline{F^*})\bigotimes\Lambda(\zeta_2)$.*

Here recall the conjecture due to Ravenel on the $\beta$-elements: $\beta_s \in \pi_*(S^0)$ if and only if $s \equiv 0, 1, 2, 3, 5, 6 \bmod 9$. (See §9 for the definition of $\beta$-elements.)

'Only if' part is shown in [11], in which we also show that $\beta_s \in \pi_*(L_2S^0)$ if $s \equiv 0, 1, 5 \bmod 9$. On this conjecture, we have a supporting evidence:

THEOREM 2.12. *$\beta_s \in \pi_*(L_2S^0)$ if and only if $s \equiv 0, 1, 2, 3, 5, 6 \bmod 9$.*

In the $E_2$-term, the $\beta$-elements of the form $\beta_{a/b}$ are defined [6] for integers $a$, $b > 0$ such that $b \le 3^{\nu(a)}$ if $\nu(a) \le 1$ and $b < 4 \cdot 3^{\nu(a)-1}$ otherwise, where the integer $\nu(a)$ denotes the maximal power of 3 that divides $a$. Then we have homotopy $\beta$-elements:

THEOREM 2.13. *In the Adams-Novikov spectral sequence $E_2^*(L_2S^0) \Rightarrow \pi_*(L_2S^0)$, we have*

(a) *The element $\beta_a$ with $\nu(a) = 0$ is permanent if $a \equiv 1, 2, 5 \bmod 9$,*

(b) *The element $\beta_{a/b}$ with $\nu(a) = 1$ is permanent if $9 \mid (a-3)$ and $b < 3$, or if $9 \mid (a+3)$ and $b \le 3$.*

(c) *Every element $\beta_{a/b}$ with $\nu(a) \ge 2$ is permanent.*

## §3. Proof of Theorem 2.5

The proof of Theorem 2.5 is based on the following lemma due to [6, Remark 3.11]. To state the lemma, we set up notations: Let $K$ denote a $\boldsymbol{Z}/3$-basis of the submodule $F \otimes K(2)_*[b_{10}]$ of $H^*K(2)_*$ given in Theorem 2.3, and $\overline{x}/v_1^j$ denote an element of $H^*M_1^1$ such that $v_1^{j-1}(\overline{x}/v_1^j) = x/v_1$ for an element $x \in K$ and an integer $j > 0$. Consider the maps $i_*$ and $\delta_s$ in the long exact sequence associated to the short one (2.2)

$$(3.1) \qquad \cdots \longrightarrow H^sK(2)_* \xrightarrow{i_*} H^sM_1^1 \xrightarrow{v_1} H^sM_1^1 \xrightarrow{\delta_s} H^{s+1}K(2)_* \longrightarrow \cdots.$$

Note that $i_*(x) = x/v_1$. For each base $x \in K \subset H^sK(2)_*$, define an integer $j(x)$ by $j(x) = j$ if $\delta_s(\overline{x}/v_1^j) \ne 0$, and $j(x) = \infty$ otherwise. Define a $k(1)_*$-submodule $B$ of $H^*M_1^1$ by

$$B = k(1)_*\{\overline{x}/v_1^{j(x)} \mid x \in K \textit{ and } i_*(x) \ne 0 \in H^*M_1^1\}.$$

LEMMA 3.2. *For the submodule $B$ defined above, $H^*M_1^1 = B\bigotimes\Lambda(\zeta_2)$ if $B$ satisfies the condition that the set $\{\delta(\overline{x}/v_1^{j(x)})|\ x\in K, j(x)<\infty\}$ is linearly independent.*

Therefore, we will study the connecting homomorphism $\delta_s : H^sM_1^1 \to H^{s+1}K(2)_*$ to find $j(x)$ for each $x\in K$. Note that if $x\notin \mathrm{Im}\ \delta_s$, then $i_*(x)\neq 0$ in $H^{s+1}M_1^1$. Thus a computation of $\delta_s$ shows us all information that we need. We will not distinguish $x$ and $\overline{x}$ in the sequel. The following is our key lemma:

LEMMA 3.3. *The connecting homomorphism $\delta_s : H^sM_1^1 \to H^{s+1}K(2)_*$ acts as follows:*

1. *On $F_n$ $(n\geq 0)$,*

$$\begin{aligned}
\delta_0(v_2^{3^{n+1}}/v_1^{4\cdot 3^n-1}) &= v_2^{2\cdot 3^n}h_{10}\\
\delta_1(v_2^{3^{n+1}}h_{10}/v_1^{6\cdot 3^n+1}) &= (-1)^n v_2^{(3^{n+1}-1)/2}\xi\\
\delta_1(v_2^{8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n+1}) &= -v_2^{5\cdot 3^n+(3^n-1)/2}\xi\\
\delta_2(v_2^{3^n(5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}) &= v_2^{3^{n+1}(1\pm 1)+3(3^n-1)/2}\psi_1+\cdots
\end{aligned}$$

*Here $+\cdots$ denots an element of $K(2)_*\{\xi\zeta_2\}$.*

2. *On $F$,*

$$\begin{aligned}
\delta_0(v_2/v_1) &= h_{11}\\
\delta_1(v_2h_{10}/v_1^2) &= b_{10}\\
\delta_1(v_2^2h_{11}/v_1^2) &= b_{11}\\
\delta_2(v_2b_{11}/v_1) &= h_{11}b_{11}
\end{aligned}$$

3. *On $F^*$,*

$$\begin{aligned}
\delta_2(\xi/v_1^2) &= -v_2^{-1}\psi_0\\
\delta_3(v_2(v_2^{-1}\psi_0)/v_1) &= v_2^{-2}b_{11}\xi\\
\delta_3(v_2\psi_1/v_1) &= -v_2b_{10}\xi\\
\delta_4(b_{11}\xi/v_1^2) &= -\psi_1b_{10}
\end{aligned}$$

This gives rise to all the differentials of the Bockstein spectral sequence. In fact, suppose that $\delta_s(x/v_1^j) = y$ in the above lemma. Then for an element $a \in H^t E(2)_*/(3, v_1^j)$, we have $\delta_{s+t}(ax/v_1^j) = ay$. Take $a$ to be an element of $E(2,n)_*/(v_1) = (\boldsymbol{Z}/3)[v_2^{\pm 3^n}]$ or $b_{10}^t\zeta_2^\varepsilon$ for $t \geq 0$ and $\varepsilon = 0, 1$. Then we have

$$\delta_{s+2t+\varepsilon}(v_2^{3^n s} x b_{10}^t \zeta_2^\varepsilon / v_1^j) = v_2^{3^n s} y b_{10}^t \zeta_2^\varepsilon,$$

for integers $s$ and $n$ such that $3^n \geq j$. Therefore, we see that

Proposition 3.4. *The connecting homomorphism* $\delta_s : H^s M_1^1 \to H^{s+1} K(2)_*$ *acts as follows*:

$$\begin{array}{rcl}
\delta_0(v_2^t/v_1) & = & t v_2^{t-1} h_{11} \\
\delta_0(v_2^{3^n t}/v_1^{4\cdot 3^{n-1}-1}) & = & -t v_2^{3^n t - 3^{n-1}} h_{10} (n > 0); \\
\delta_1(v_2^{3t+1} h_{10}/v_1^2) & = & v_2^{3t} b_{10} \\
\delta_1(v_2^{3^n(3t+1)} h_{10}/v_1^{2\cdot 3^n+1}) & = & v_2^{3^{n+1}t + (3^n-1)/2}\xi \quad (n > 0) \\
\delta_1(v_2^{3^n(9t+8)} h_{10}/v_1^{10\cdot 3^n+1}) & = & -v_2^{3^{n+2}t + 5\cdot 3^n + (3^n-1)/2}\xi \quad (n \geq 0) \\
\delta_1(v_2^{3t+2} h_{11}/v_1^2) & = & v_2^{3t} b_{11}; \\
\delta_2(v_2^{3t\pm 1} b_{10}/v_1) & = & \pm v_2^{3t-1\pm 1} h_{11} b_{10} \\
\delta_2(v_2^{3t\pm 1} b_{11}/v_1) & = & \pm v_2^{3t-1\pm 1} h_{11} b_{11} \;=\; \pm v_2^{3t+1\pm 1} h_{10} b_{10} \\
\delta_2(v_2^{3t}\xi/v_1^2) & = & v_2^{3t-1}\psi_0 \\
\delta_2(v_2^{3^n(9t+5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}) & = & v_2^{3^{n+1}(3t+1\pm 1)+3(3^n-1)/2}\psi_1 + \cdots \quad (n \geq 0); \\
\delta_{2s+3}(v_2^{3t+2} h_{11} b_{10}^{s+1}/v_1^2) & = & v_2^{3t} b_{11} b_{10}^{s+1} \\
\delta_{2s+3}(v_2^{3t+1} h_{10} b_{10}^{s+1}/v_1^2) & = & \pm v_2^{3t} b_{10}^{s+2} \\
\delta_{2s+3}(v_2^{3t\pm 1}(v_2^{-1}\psi_0) b_{10}^s/v_1) & = & \pm v_2^{3t\pm 1}(v_2^{-3} b_{11}\xi b_{10}^s) \\
\delta_{2s+3}(v_2^{3t\pm 1}\psi_1 b_{10}^s/v_1) & = & v_2^{3t\pm 1}\xi b_{10}^{s+1}; \\
\delta_{2s+4}(v_2^{3t\pm 1} b_{10}^{s+2}/v_1) & = & \pm v_2^{3t-1\pm 1} h_{11} b_{10}^{s+2} \\
\delta_{2s+4}(v_2^{3t\pm 1} b_{11} b_{10}^{s+1}/v_1) & = & \pm v_2^{3t+1\pm 1} h_{10} b_{10}^{s+2} \\
\delta_{2s+4}(v_2^{3t}\xi b_{10}^{s+1}/v_1^2) & = & v_2^{3t-1}\psi_0 b_{10}^{s+1} \\
\delta_{2s+4}(v_2^{3t} b_{11}\xi b_{10}^s/v_1^2) & = & -v_2^{3t}\psi_1 b_{10}^{s+1}
\end{array}$$

*for* $n, s \geq 0$ *and* $t \in \boldsymbol{Z}$, *where* $+\cdots$ *denots an element of* $K(2)_*\{\xi\zeta_2\}$.

Corollary 3.5. *The map* $i_* : H^s K(2)_* \to H^s M_1^1$ *sends each of the* *following elements in $K$ to a non-zero element*:

$$h_{10}, \quad v_2^{3^k s}h_{10}, \quad v_2^{3t-1}h_{11};$$

*for $s, t \in \boldsymbol{Z}$ with $s \equiv 1$ (3) or $s \equiv 8$ (9).*

$$v_2^{3t\pm1}b_{10}, \quad v_2^{3t\pm1}b_{11}, \quad v_2^u\xi,$$

*for $t, u \in \boldsymbol{Z}$ with $u \in 3\boldsymbol{Z}$ or $u = 3^n(9t+5\pm3)+(3^n-1)/2$.*

$$v_2^{3t-1}h_{11}b_{10}^{s+1}, \quad v_2^{3t+1}h_{10}b_{10}^{s+1}, \quad v_2^{3t-1\pm1}\psi_0 b_{10}^s, \quad v_2^{3t\pm1}\psi_1 b_{10}^s$$

*for $s, t \in \boldsymbol{Z}$ with $s \geq 0$.*

These elements in Corollary 3.5 form the set $B$, and Lemma 3.2 show Theorem 2.5.

## §4. Computation of the connecting homomorphism

In this section, we will prove Lemma 3.3 by assuming some results on the cobar complex $\Omega^* E(2)_*$ which will be shown in the next sections.

Let $(E(2)_*, E(2)_*E(2))$ denotes the Hopf algebroid associated to the Johnson-Wilson spectrum. For an $E(2)_*E(2)$-comodule $M$ with coaction $\psi : M \to M \bigotimes_{E(2)_*} E(2)_*E(2)$, $H^*M = \mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*, M)$ is given as the cohomology of the cobar complex $\Omega^*M$ with $\Omega^s M = M \bigotimes_{E(2)_*} E(2)_*E(2)^{\otimes s}$ and the differential $d_s : \Omega^s M \to \Omega^{s+1}M$ defined by $d_s(x \otimes y) = \psi(x) \otimes y + \sum_{i=1}^s (-1)^s x \otimes y_1 \otimes \cdots \otimes \Delta(y_i) \otimes \cdots \otimes y_s - (-1)^s x \otimes y \otimes 1$ for $x \in M$ and $y = y_1 \otimes \cdots \otimes y_s \in E(2)_*E(2)^{\otimes s}$. Consider the connecting homomorphism $\delta_s : H^s M_1^1 \to H^{s+1}K(2)_*$ associated to (2.2). By definition, we see that

(4.1) If $d_s(x) \equiv v_1^j y \bmod (3, v_1^{j+1})$ in $\Omega^{s+1}E(2)_*$, then $\delta_s([x]/v_1^j) = [y]$.

Now we state several lemmas:

Lemma 4.2. * *There exists a cochain $x(8 \cdot 3^n) \in \Omega^1 E(2)_*$ for each $n \geq 0$ such that $x(8 \cdot 3^n) \equiv v_2^{8 \cdot 3^n} t_1 \bmod (3, v_1)$ and*

$$d_1(x(8 \cdot 3^n)) \equiv -v_1^{10\cdot3^n+1} v_2^{5\cdot3^n+(3^n-1)/2} X \mod (3, v_1^{10\cdot3^n+2}).$$

*This is the correction of the last congruence in [1, Prop. 5.2].

*Here $X$ denotes a cocycle that represents $\xi$.*

Lemma 4.3. *There exist cochains $X(2)$ and $X(8) \in \Omega^2 E(2)_*$ such that $X(n) \equiv v_2^n X \bmod (3, v_1)$ and*

$$\begin{aligned} d_2(X(2)) &\equiv v_1^4 z^3 \otimes X^3 - v_1^4 v_2^{-3} {f_1}^3 \quad \bmod (3, v_1^5) \quad \text{and} \\ d_2(X(8)) &\equiv -v_1^4 v_2^3 z^3 \otimes X^9 + v_1^4 v_2^{-6} {f_1}^9 \quad \bmod (3, v_1^5). \end{aligned}$$

*Here $z$ and $f_1$ represent $\zeta_2$ and $\psi_1$, respectively.*

Lemma 4.4. *In the cobar complex $\Omega^3 E(2)_*$ we have a cochain $X'$ such that $X' \equiv X \bmod (3, v_1)$ and*

$$d_2(X') \equiv -v_1^2 v_2^{-1} f_0 \quad \bmod (3, v_1^3),$$

*for a cocycle $f_0$ representing $\psi_0$.*

Lemma 4.5. *In the cobar complex $\Omega^3 E(2)_*$, we have cochains $f_i$ $(i = 0, 1)$ such that $f_i$ represents $\psi_i$ in $E_2^*(L_2 V(1))$ and*

$$\begin{aligned} d_3(f_0) &\equiv v_1 v_2^{-2} b_{11} \otimes X \quad \bmod (3, v_1^2), \\ d_3(f_1) &\equiv 0 \quad \bmod (3, v_1^2). \end{aligned}$$

Note that $f_i$'s of Lemmas 4.3 and 4.4 are the same as those of Lemma 4.5, which is seen by the proofs in §6.

Assuming these lemmas we will prove Lemma 3.3 by which we obtain Proposition 3.4.

Proof of Lemma 3.3. In [6, Prop. 5.4], it is shown that $d_0(v_2) \equiv v_1 t_1^3 \bmod (3, v_1^3)$ and $d_0(v_2^{3^n}) \equiv v_1^{4\cdot 3^{n-1}-1} v_2^{2\cdot 3^{n-1}} t_1 \bmod (3, v_1^{4\cdot 3^{n-1}})$, which implies the first equations in the parts 1 and 2 of Lemma 3.3. In fact, $h_{1i} = [t_1^{3^i}]$. Besides, we see that $d_2(v_2 b_{11}) \equiv v_1 t_1^3 \otimes b_{11} \bmod (3, v_1^3)$, since $d_2(b_{11}) \equiv 0$ and $d_0(v_2) \equiv v_1 t_1^3$. Therefore the fourth one in the part 2 follows.

In [1, Prop. 5.2] and [1, Prop. 5.3], we show that

$$\begin{aligned} &d_1(x(1)) \equiv v_1^2 b_{10} \quad \bmod (3, v_1^3), \quad d_1(y(2)) \equiv v_1^2 b_{11} \quad \bmod (3, v_1^3), \\ &d_1(x(3^n)) \equiv -(-1)^n v_1^{6\cdot 3^{n-1}+1} v_2^{(3^n-1)/2} X \quad \bmod (3, v_1^{6\cdot 3^{n-1}+2}) \ (n > 0), \end{aligned}$$

where $x(n)$ and $y(n)$ are elements such that

$$x(n) \equiv v_2^n t_1 \quad \text{and} \quad y(n) \equiv v_2^n t_1^3 \mod (3, v_1).$$

These shows the second equation in the part 1 and the second and the third ones in the part 2. The third one in the part 1 follows from Lemma 4.2. Since $f_1^{3^n} \equiv v_2^{3(3^n-1)/2} f_1$ and $X^{3^n} \equiv v_2^{(3^n-1)/2} X \mod (3, v_1)$ up to homology by Theorem 2.3, we see the fourth one of the part 1 from Lemma 4.3.

Now turn to the part 3. By Lemma 4.4 we obtain the first one. The forth one also follows from it, since $b_{10}\psi_1 = -v_2 b_{11}\psi_0$ by Theorem 2.3. The second one follows immediately from Lemma 4.5. For the third, since $d_0(v_2) \equiv v_1 t_1^3$ and $d_3(f_1) \equiv 0 \mod (3, v_1^2)$ by Lemma 4.5, we see that $d_3(v_2 f_1) \equiv v_1 t_1^3 \otimes f_1 \mod (3, v_1^2)$, which is homologous to $v_1 v_2 b_{10} \otimes X$ by a relation of Theorem 2.3. q.e.d.

## §5. Some relations in $E(2)_*E(2)$

Note that in $E(2)_*E(2)$ we have the following relations (*cf.* [1, (3.8),(3.9)]):

$$\begin{aligned} t_1^9 &= v_2^{-1} t_1 \eta_R(v_2^3) - v_1 v_2^{-1} t_2^3 - v_1^2 v_2^{-1} V + v_1^9 v_2^{-1} t_2 \\ &\equiv v_2^2 t_1 - v_1 v_2^{-1} t_2^3 + v_1^2 v_2 t_1^3 \\ &\quad + v_1^3 (v_2^{-1} t_1^{10} + t_1^6) - v_1^4 v_2 t_1 + v_1^5 t_1 \mod (3, v_1^5) \\ t_n^9 &\equiv v_2^{3^n-1} t_n - v_1 v_2^{-1} t_{n+1}^3 \mod (3, v_1^2), \end{aligned} \tag{5.1}$$

in which $\eta_R(v_2^3) = v_2^3 + v_1^3 t_1^9 - v_1^9 t_1^3$ and

$$V = -v_2^2 t_1^3 - v_1 v_2 t_1^6 + v_1^2 v_2^2 t_1 - v_1^3 v_2 t_1^4 + v_1^4 t_1^7 - v_1^5 v_2 t_1^2 - v_1^6 t_1^5. \tag{5.2}$$

Therefore we see the following relation in the cobar complex $\Omega^* E(2)_*$:

$$c^9 \equiv v_2^{2t} c \mod (3, v_1), \tag{5.3}$$

for a cochain $c \in \Omega^{*,4t} E(2)_*$.

Note that $d_0 : E(2)_* \to E(2)_*E(2)$ is computed by $d_0(x) = \eta_R(x) - x$. Since $\eta_R(v_2) \equiv v_2 + v_1 t_1^3 - v_1^3 t_1 \mod (3)$ by Landweber's formula, and so we compute the followings mod $(3, v_1^8)$,

$$
(5.4)\qquad
\begin{array}{rcl}
d_0(v_2^2) & \equiv & -v_1v_2t_1^3 + v_1^2t_1^6 + v_1^3v_2t_1 + v_1^4t_1^4 + v_1^6t_1^2 \\
d_0(v_2^4) & \equiv & v_1v_2^3t_1^3 + v_1^4\tau^3 + v_1^5v_2^2t_1^3 + v_1^6v_2t_1^6 - v_1^7v_2^2t_1 \\
d_0(v_2^5) & \equiv & -v_1v_2^4t_1^3 + v_1^2v_2^3t_1^6 - v_1^3v_2^2t_1^9 + v_1^4v_2t_2^3 \\
& & \quad +v_1^5(-v_2^3t_1^3 + t_1^3t_2^3 + t_1^{15}) + v_1^6(v_2^2t_1^6 + v_2^3t_1^2) + v_1^7t_1^{13} \\
d_0(v_2^7) & \equiv & v_1v_2^6t_1^3 + v_1^3v_2^4t_1^9 - v_1^4v_2^3(t_2^3 + t_1^{12}) + v_1^5v_2^5t_1^3 \\
& & \quad +v_1^6v_2^4t_1^6 + v_1^7t_1^{21} \\
d_0(v_2^8) & \equiv & -v_1v_2^7t_1^3 + v_1^2v_2^6t_1^6 \quad \bmod (3, v_1^4),
\end{array}
$$

where $\tau = t_1^4 - t_2$.

Moreover, $d_1 : E(2)_*E(2) \to E(2)_*E(2)^{\otimes 2}$ is given by $d_1(x) = 1 \otimes x - \Delta(x) + x \otimes 1$, and we have

$$
(5.5)\qquad
\begin{array}{rcl}
d_1(t_1) & = & 0 \\
d_1(t_2) & = & -t_1 \otimes t_1^3 - v_1b_{10}
\end{array}
$$

for $b_{10} = -t_1 \otimes t_1^2 - t_1^2 \otimes t_1$.

Lemma 5.6. *There exist cochains $T', T$, and $\overline{T}$ in $\Omega^1E(2)_*$ such that*

$$
\begin{array}{rcl}
d_1(T') & \equiv & -t_1^9 \otimes z^9 - b_{11} - v_2^{-9}g_0^9 \quad \bmod (3, v_1^3), \\
d_1(T) & \equiv & -v_2t_1^9 \otimes z^9 - v_2b_{11} - v_2^{-8}g_0^9 + v_1v_2^{-27}t_1^3 \otimes (t_3^9 - t_1^{81}t_2^9) \quad \bmod (3, v_1^3), \\
d_1(\overline{T}) & \equiv & -v_2z^9 \otimes t_1^9 - v_2b_{11} - v_2^{-26}g_0'^9 \\
& & \quad +v_1v_2^{-27}t_1^3 \otimes (t_3^9 - t_1^9t_2^{27}) - v_1^3v_2^{-27}t_1 \otimes (t_3^9 - t_1^9t_2^{27}) \quad \bmod (3, v_1^6).
\end{array}
$$

*Here*

$$
\begin{array}{l}
z = v_2^{-1}t_2 + v_2^{-3}t_2^3 - v_2^{-3}t_1^{12},\ b_{11} = -t_1^3 \otimes t_1^6 - t_1^6 \otimes t_1^3, \\
g_0 = t_1 \otimes t_2 - t_1^2 \otimes t_1^3 \text{ and } g_0' = t_2^3 \otimes t_1 - t_1^3 \otimes t_1^{10}.
\end{array}
$$

Proof. First consider the cochains $\widehat{t}_3 = t_3 - t_1^9t_2$ and $\widetilde{t}_3 = t_3 - t_1t_2^3$. Then we see

$$
\begin{array}{rcl}
d_1(\widehat{t}_3) & \equiv & -v_2^3t_1 \otimes z - v_2b_{11} - v_2^2g_0 \quad \bmod (3, v_1) \quad \text{and} \\
d_1(\widetilde{t}_3) & \equiv & -v_2^3z \otimes t_1 - v_2b_{11} - g_0' \quad \bmod (3, v_1),
\end{array}
$$

by computing

$$\begin{aligned}
d_1(t_3) &\equiv -t_1\otimes t_2^3 - t_2\otimes t_1^9 - v_2b_{11},\\
&\equiv -t_1\otimes t_2^3 - v_2^2t_2\otimes t_1 - v_2b_{11}\\
d_1(-t_1^9t_2) &\equiv t_1^{10}\otimes t_1^3 + t_1\otimes t_1^{12} + t_1^9\otimes t_2 + t_2\otimes t_1^9\\
&\equiv -v_2^2g_0 + t_1\otimes t_1^{12} - v_2^2t_1\otimes t_2 + t_2\otimes t_1^9.\\
d_1(-t_1t_2^3) &\equiv t_1^4\otimes t_1^9 + v_1^3\otimes t_1^{10} + t_1\otimes t_2^3 + t_2^3\otimes t_1\\
&\equiv t_1^{12}\otimes t_1 + t_1\otimes t_2^3 - g_0' - t_2^3\otimes t_1
\end{aligned}$$

mod $(3, v_1)$.

Now put $T' = v_2^{-27}\widehat{t}_3^9$, and 9-th power of $d_1(\widehat{t}_3)$ yields $d_1(T')$. In fact, by (5.3) we see that $b_{11}^9 \equiv v_2^{18}b_{11}$ mod $(3, v_1^6)$.

We define $T = v_2T'$ and compute $d_1(T) \equiv d_1(v_2T') \equiv v_1t_1^3\otimes T' - v_2^{-26}(v_2^{27}t_1^9\otimes z^9 + v_2^9b_{11}^9 + v_2^{18}g_0^9)$ mod $(3, v_1^3)$.

In the same manner we verify that the element $\overline{T} = v_2\overline{T'}$ for $\overline{T'} = v_2^{-27}\widehat{t}_3^9$ satisfies the last congruence. q.e.d.

## §6. Proofs of Lemmas 4.3, 4.4 and 4.5

Let $x$ denote a cochain that represents $\xi$ in $H^2M_2^0$, and define $X \in \Omega^2E(2)_*$ by

$$X = v_2^{-4}x^9.$$

LEMMA 6.1. *The element $X$ is a cocycle of $\Omega^*M_2^0$ that represents $\xi$ in $H^2M_2^0$, and satisfies the following in the cobar complex $\Omega^3E(2)_*$:*

$$d_2X \equiv -v_1v_2^{-1}t_1^3\otimes X - v_1^4v_2^{-4}\tau^3\otimes X - v_1^5v_2^2t_1^3\otimes X \quad \text{mod } (3, v_1^6).$$

PROOF. By definition with (5.4) we see that

$$\begin{aligned}
0 &\equiv d_2(x^9) \equiv d_2(v_2^4X)\\
&\equiv v_1v_2^3t_1^3\otimes X + v_1^4\tau^3\otimes X + v_1^5v_2^2t_1^3\otimes X\\
&\qquad + v_1^6v_2t_1^6\otimes X + v_2^4d_2(X) \quad \text{mod } (3, v_1^7).
\end{aligned}$$

q.e.d.

As we noted in [1, (5.1)], we have an element $w$ such that $d_1(w) = x^3 + v_2x$ in $\Omega^*K(2)_*$. Since $x^9 = v_2^4x$ by (5.3), we obtain that $d_1(w^3 - v_2^3w) = 0$, that is $w^3 - v_2^3 w$ is a cocycle. Theorem 2.3 shows that $w^3 - v_2^3 w$ is bounded, but nothing bounds it by degree reason. Therefore, $w$ satisfies

$$(6.2) \qquad d_1(w) = x^3 + v_2 x \quad \text{and} \quad w^3 = v_2^3 w \quad \text{in } \Omega^* M_2^0.$$

By this, we have $d_1(w^9) \equiv x^{27} + v_2^9 x^9 \equiv v_2^{12} X^3 + v_2^{13} X \bmod (3, v_1^9)$, and so by (5.4),

$$(6.3) \qquad d_1(v_2^{-12} w^9) \equiv -v_1^3 v_2^{-15} t_1^9 \otimes w^9 + v_1^6 v_2^{-18} t_1^{18} \otimes w^9 + X^3 + v_2 X$$

mod $(3, v_1^9)$ since $d_0(v_2^{-12}) \equiv v_2^{-18}(d_0(v_2^2))^3$.

Since $h_{10}\xi = 0$ in $H^3K(2)_*$ by Theorem 2.3, we have cocycles $y_0$ such that $d_2(y_0) = t_1 \otimes x$ in $\Omega^3 K(2)_*$. Define $y_1 = -v_2^{-1}(y_0^3 + t_1^3 \otimes w)$. Then $d_2(y_1) = -v_2^{-1}(t_1^3 \otimes x^3 - t_1^3 \otimes (x^3 + v_2 x)) = t_1^3 \otimes x$.

Put $Y_0 = v_2^{-6} y_0^9$ and $Y_1 = v_2^{-10} y_1^9$. Then we have

Lemma 6.4. *$Y_i \equiv y_i \bmod (3, v_1)$ $(i = 0, 1)$ and $Y_0^3 \equiv -v_2 Y_1 - v_2^{-18} t_1^{27} \otimes w^9$ mod $(3, v_1^9)$. Besides,*

$$\begin{aligned} d_2(Y_0) &\equiv t_1 \otimes X + v_1 v_2^{-3} \tau^3 \otimes X \\ &\qquad + v_1^2 v_2^{-1} t_1^3 \otimes X \mod (3, v_1^3), \quad \textit{and} \\ d_2(Y_1) &\equiv t_1^3 \otimes X - v_1 v_2^{-1}(t_1^3 \otimes Y_1 - t_1^6 \otimes X) \\ &\qquad + v_1^3 (v_2^{-3} t_1^9 \otimes Y_1 - v_2^{-6} t_2^9 \otimes X) \mod (3, v_1^4). \end{aligned}$$

Proof. The first one follows from (5.3). By definition, $Y_0^3 = v_2^{-18} y_0^{27} \equiv -v_2^{-9} y_1^9 - v_2^{-18} t_1^{27} \otimes w^9 = -v_2 Y_1 - v_2^{-18} t_1^{27} \otimes w^9 \bmod (3, v_1^9)$.

A direct computation with (5.1) shows the following,

$$d_2(v_2^{-6} y_0^9) \equiv v_2^{-6} t_1^9 \otimes x^9 \equiv v_2^{-4}(t_1 - v_1 v_2^{-3} t_2^3 + v_1^2 v_2^{-1} t_1^3) \otimes v_2^4 X \mod (3, v_1^3),$$

and $\eta_R(v_2^4) = v_2^4 + d_0(v_2^4)$ is read off from (5.4).

For $Y_1$, noticing that $d_0(v_2^{-10}) \equiv v_2^{-18} d_0(v_2^8) \bmod (3, v_1^4)$, we compute with (5.4),

$$\begin{aligned} d_2(v_2^{-10} y_1^9) &\equiv -v_1 v_2^{-11} t_1^3 \otimes y_1^9 + v_1^2 v_2^{-12} t_1^6 \otimes y_1^9 + v_2^{-10} t_1^{27} \otimes x^9 \\ &\equiv -v_1 v_2^{-11} t_1^3 \otimes v_2^{10} Y_1 + v_1^2 v_2^{-12} t_1^6 \otimes v_2^{10} Y_1 \\ &\qquad + v_2^{-4} t_1^3 \otimes v_2^4 X - v_1^3 v_2^{-13} t_2^9 \otimes v_2^4 X \\ &\equiv t_1^3 \otimes X - v_1 v_2^{-1}(t_1^3 \otimes Y_1 - t_1^6 \otimes X) \\ &\qquad + v_1^3 v_2^{-3} t_1^9 \otimes Y_1 - v_1^3 v_2^{-9} t_2^9 \otimes X \mod (3, v_1^4). \end{aligned}$$

q.e.d.

Proof of Lemma 4.5. Define first $f_0' = t_1^2 \otimes X + t_1 \otimes Y_0$ and $f_1' = t_1 \otimes Y_1 - t_2 \otimes X$. Then Lemma 6.4 shows that $f_i'$ represents $\psi_i = \langle h_{10}, h_{1i}, \xi\rangle$ for each $i = 0, 1$.

By Lemmas 5.6, 6.1 and 6.4 with (5.3) we compute

$$\begin{aligned}
d_3(f_0') &\equiv -v_1(v_2^{-3}t_1 \otimes \tau^3 \otimes X - v_2^{-1}t_1^2 \otimes t_1^3 \otimes X) \\
&\equiv v_1(t_1 \otimes z \otimes X - v_2^{-1}g_0 \otimes X) \\
d_3(-v_1v_2^{-3}T \otimes X) &\equiv v_1v_2^{-3}(v_2^3t_1 \otimes z + v_2b_{11} + v_2^2g_0) \otimes X \\
d_3(v_1z \otimes Y_0) &\equiv -v_1z \otimes t_1 \otimes X \\
d_3(-v_1zt_1 \otimes X) &\equiv v_1z \otimes t_1 \otimes X + v_1t_1 \otimes z \otimes X
\end{aligned}$$

mod $(3, v_1^2)$. Then we have the first one by putting $f_0 = f_0' - v_1v_2^{-3}T \otimes X + v_1z \otimes Y_0 - v_1zt_1 \otimes X$.

Similarly, we compute

$$\begin{aligned}
d_3(f_1') &\equiv -t_1 \otimes t_1^3 \otimes X + v_1v_2^{-1}t_1 \otimes (t_1^3 \otimes Y_1 - t_1^6 \otimes X) \\
&\qquad +t_1 \otimes t_1^3 \otimes X + v_1b_{10} \otimes X - v_1v_2^{-1}t_2 \otimes t_1^3 \otimes X \\
d_3(v_1v_2^{-1}t_2 \otimes Y_1) &\equiv -v_1v_2^{-1}t_1 \otimes t_1^3 \otimes Y_1 - v_1v_2^{-1}t_2 \otimes t_1^3 \otimes X \\
d_3(v_1v_2^{-9}\overline{T}^3 \otimes X) &\equiv v_1v_2^{-9}(-v_2^9z \otimes t_1^3 - v_2^9b_{10} - g_0'^3) \otimes X \\
d_3(-v_1z \otimes Y_1) &\equiv v_1z \otimes t_1^3 \otimes X
\end{aligned}$$

mod $(3, v_1^2)$. Notice that $v_1v_2^{-1}(t_2 \otimes t_1^3 - t_1 \otimes t_1^6) = v_1v_2^{-9}g_0'^3$, and we obtain the second one by setting $f_1 = f_1' + v_1v_2^{-1}t_2 \otimes Y_1 + v_1v_2^{-9}\overline{T}^3 \otimes X - v_1z \otimes Y_1$.

q.e.d.

Proof of Lemma 4.3. Put $X(2) = v_2X^3 - v_1Y_0^3 - v_1^3v_2^{-2}Y_1^3$ and note that $f_1 \equiv t_1 \otimes Y_1 - t_2 \otimes X$ mod $(3, v_1)$ for $f_1$ in the proof of Lemma 4.5. We then obtain $d_2(X(2))$ from computation

$$\begin{aligned}
d_2(v_2X^3) &\equiv v_1t_1^3 \otimes X^3 - v_1^3t_1 \otimes X^3 - v_1^3v_2^{-2}t_1^9 \otimes X^3 \\
&\equiv v_1t_1^3 \otimes X^3 + v_1^3(t_1 + v_1v_2^{-3}t_2^3) \otimes X^3 \\
d_2(-v_1Y_0^3) &\equiv -v_1t_1^3 \otimes X^3 - v_1^4v_2^{-9}\tau^9 \otimes X^3 \\
d_2(-v_1^3v_2^{-2}Y_1^3) &\equiv -v_1^4v_2^{-3}t_1^3 \otimes Y_1^3 - v_1^3v_2^{-2}t_1^9 \otimes X^3 \\
&\equiv -v_1^4v_2^{-3}t_1^3 \otimes Y_1^3 - v_1^3(t_1 - v_1v_2^{-3}t_2^3) \otimes X^3
\end{aligned}$$

mod $(3, v_1^5)$ by Lemmas 6.1 and 6.4.

By defining $X(8) = v_2^4 X^9 - v_1 v_2^{-3} Y_1^9$, we compute that

$$\begin{aligned} d(v_2^4 X^9) &\equiv v_1 v_2^3 t_1^3 \otimes X^9 + v_1^4 \tau^3 \otimes X^9 \\ d(-v_1 v_2^{-3} Y_1^9) &\equiv v_1^4 v_2^{-6} t_1^9 \otimes Y_1^9 - v_1(v_2^3 t_1^3 - v_1^3 v_2^{-6} t_2^9) \otimes X^9. \end{aligned}$$

Since $\tau^3 + v_2^{-6} t_2^9 \equiv -v_2^3 z^3 - v_2^{-6} t_2^9$, we have the result. q.e.d.

Proof of Lemma 4.4. Set $\overline{X}' = X + v_1 v_2^{-1} Y_1$ and $\overline{f_0} = v_2^{-1}(t_1^3 \otimes Y_1 + t_1^6 \otimes X)$, and we obtain

$$d_2(\overline{X}') \equiv v_1^2 v_2^{-1} \overline{f_0} \quad \text{mod } (3, v_1^3)$$

by the computation:

$$\begin{aligned} d_2(X) &\equiv -v_1 v_2^{-1} t_1^3 \otimes X \\ d_2(v_1 v_2^{-1} Y_1) &\equiv -v_1^2 v_2^{-2} t_1^3 \otimes Y_1 + v_1 v_2^{-1} t_1^3 \otimes X - v_1^2 v_2^{-2}(t_1^3 \otimes Y_1 - t_1^6 \otimes X) \end{aligned}$$

mod $(3, v_1^3)$. Indeed, these are seen by using Lemmas 6.1 and 6.4 and the congruence $d_0(v_2^{-1}) \equiv -v_1 v_2^{-2} t_1^3$ mod $(3, v_1^2)$ seen by (5.4).

For a while, we argue in the $E_2$-term $E_2^{*,*}(L_2 V(1))$. Notice that $\overline{f_0}$ represents an element in $v_2^{-1}\langle h_{11}, h_{11}, \xi\rangle$, and $\psi_0$ is an element of $\langle h_{10}, h_{10}, \xi\rangle$. By a relation of the Massey products, we see that $h_{11}\langle h_{11}, h_{11}, \xi\rangle = \langle h_{11}, h_{11}, h_{11}\rangle \xi$ and $\langle h_{11}, h_{11}, h_{11}\rangle = -b_{11}$. Therefore, $h_{11}[\overline{f_0}] = -v_2^{-1} b_{11} \xi = -h_{11}\psi_0$ by a relation in Theorem 2.3, and so $[\overline{f_0}] = -\psi_0$ mod Ker $h_{11}$. Theorem 2.3 also shows us that Ker $h_{11} \subset E_2^{3,16}(L_2 V(1))$ is generated by $h_{10} b_{10}$. We have an integer $k \in \boldsymbol{Z}/3$ such that $[\overline{f_0}] = -\psi_0 + k h_{10} b_{10}$ and a cochain $e_0$ such that $d_2(e_0) = \overline{f_0} + f_0 - k t_1 \otimes b_{10}$ for $f_0$ given in the proof of Lemma 4.5. Furthermore the relation $v_2^2 h_{10} b_{10} = h_{11} b_{11}$ certifies the existence of a cochain $B$ such that $d_1(B) = v_2^2 t_1 \otimes b_{10} - t_1^3 \otimes b_{11}$.

Putting $X' = \overline{X}' + k v_1 v_2^{-1} b_{11} - v_1^2(v_2^{-1} e_0 + k v_2^{-3} B)$ leads us to the desired congruence. q.e.d.

## §7. Proof of Lemma 4.2

In this section, we correct [1, Th. 4.4], whose $X$ should be replaced by our $x(8)$.

Lemma 7.1. *There exists an element $x(7)$ such that $x(7) \equiv v_2^7 t_1 \mod (3, v_1)$, and*

$$d_1(x(7)) \equiv v_1^2 b_{11}^3 + v_1^7 v_2^5 X \mod (3, v_1^8),$$

*for a cocycle $X$ that represents $\xi$.*

Proof. Put $x(7)' = v_2^5 t_1^9 - v_1 v_2^4 t_2^3 + v_1^3 v_2^2 t_1^{18} - v_1^2 T'^3 - v_1 v_2^7 z^9 - v_1^4 v_2^4 t_1^9 z^9 + v_1^4 v_2^3 \overline{T} - v_1^5 v_2^3 t_2^3 - v_1^6 v_2^{-4} T^3$ for $T'$, $T$ and $\overline{T}$. Using (5.4), (5.5) and Lemma 5.6, we compute the followings mod $(3, v_1^8)$:

$$\begin{aligned}
d_1(v_2^5 t_1^9) &\equiv (-\underline{v_1 v_2^4 t_1^3}_1 + \underline{v_1^2 v_2^3 t_1^6}_{(a1)} - \underline{v_1^3 v_2^2 t_1^9}_2 + \underline{v_1^4 v_2 t_2^3}_c \\
&\quad + v_1^5(-\underline{v_2^3 t_1^3}_6 + t_1^3 t_2^3 + t_1^{15}) + v_1^6(\underline{v_2^2 t_1^6}_e + \underline{v_2^3 t_1^2)}_d + v_1^7 t_1^{13})) \otimes t_1^9 \\
d_1(-v_1 v_2^4 t_2^3) &\equiv -v_1(\underline{v_1 v_2^3 t_1^3}_{(a1)} + v_1^4 \tau^3 + \underline{v_1^5 v_2^2 t_1^3}_e + v_1^6 v_2 t_1^6) \otimes t_2^3 \\
&\quad + \underline{v_1 v_2^4 t_1^3 \otimes t_1^9}_1 + \underline{v_1^4 v_2^4 b_{11}}_3 \\
d_1(v_1^3 v_2^2 t_1^{18}) &\equiv -\underline{v_1^4 v_2 t_1^3 \otimes t_1^{18}}_c + v_1^5 t_1^6 \otimes t_1^{18} + \underline{v_1^6 v_2 t_1 \otimes t_1^{18}}_d \\
&\quad + v_1^7 t_1^4 \otimes t_1^{18} + \underline{v_1^3 v_2^2 t_1^9 \otimes t_1^9}_2 \\
d_1(-v_1^2 T'^3) &\equiv \underline{v_1^2 t_1^{27} \otimes z^9}_b + v_1^2 b_{11}^3 + \underline{v_1^2 v_2^{-27} g_0^{27}}_a \\
d_1(-v_1 v_2^7 z^9) &\equiv -\underline{v_1^2 v_2^6 t_1^3 \otimes z^9}_b - \underline{v_1^4 v_2^4 t_1^9 \otimes z^9}_4 + v_1^5 v_2^3 t_2^3 \otimes z^9 \\
&\quad + v_1^5 v_2^3 t_1^{12} \otimes z^9 - \underline{v_1^6 v_2^5 t_1^3 \otimes z^9}_7 - v_1^7 v_2^4 t_1^6 \otimes z^9 \\
d_1(-v_1^4 v_2^4 t_1^9 z^9) &\equiv -v_1^5 v_2^3 t_1^3 \otimes t_1^9 z^9 + \underline{v_1^4 v_2^4 t_1^9 \otimes z^9}_4 + \underline{v_1^4 v_2^4 z^9 \otimes t_1^9}_5 \\
d_1(v_1^4 v_2^3 \overline{T}) &\equiv v_1^7 t_1^9 \otimes \overline{T} + v_1^4 v_2^3(-v_2 z^9 \otimes t_1^9 - v_2 b_{11} - v_2^{-26} g_0'^9 \\
&\quad + v_1 v_2^{-27} t_1^3 \otimes (t_3^9 - t_1^9 t_2^{27}) - v_1^3 v_2^{-27} t_1 \otimes (t_3^9 - t_1^9 t_2^{27})) \\
&\equiv v_1^4 v_2^3(-\underline{v_2 z^9 \otimes t_1^9}_5 - \underline{v_2 b_{11}}_3 - \underline{v_2^{-26} g_0'^9}_c \\
&\quad + v_1 v_2^{-27} t_1^3 \otimes (t_3^9 - t_1^9 t_2^{27})) \\
d_1(-v_1^5 v_2^3 t_2^3) &\equiv \underline{v_1^5 v_2^3 t_1^3 \otimes t_1^9}_6 \\
d_1(-v_1^6 v_2^{-4} T^3) &\equiv \underline{v_1^6 v_2^5 t_1^3 \otimes z^9}_7 + \underline{v_1^6 v_2 b_{10}}_d + \underline{v_1^6 v_2^2 g_0^3}_e + v_1^7 v_2^{-5} t_1^3 \otimes T^3.
\end{aligned}$$

The elements underlined with the same number are cancelled each other. Since the sum of the elements underlined with $(a1)$ is $-v_1^2 v_2^3 g_0^3$, and $g_0^9 \equiv v_2^{10} g_0 - v_1(v_2^9 t_1^4 \otimes t_2 + v_2 t_1 \otimes t_3^3 + v_2^7(t_2^3 \otimes t_2 + t_1 t_2^3 \otimes t_1^3))$, the sum of the elements with $(a1)$ and $a$ is:

$$v_1^2 v_2^{-27} g_0^{27} - v_1^2 v_2^3 g_0^3 \equiv -v_1^5(t_1^{12} \otimes t_2^3 + v_2^{-24} t_1^3 \otimes t_3^9 + v_2^{-6}(t_2^9 \otimes t_2^3 + t_1^3 t_2^9 \otimes t_1^9)).$$

Besides, the underlined parts with $b$, $c$ and $d$ are computed as follows:

$$\begin{array}{c} v_1^2 t_1^{27} \otimes z^9 - v_1^2 v_2^6 t_1^3 \otimes z^9 \equiv -v_1^5 v_2^{-3} t_2^9 \otimes z^9 \\ v_1^4 v_2 t_2^3 \otimes t_1^9 - v_1^4 v_2 t_1^3 \otimes t_1^{18} - v_1^4 v_2^{-23} g_0'^9 \equiv v_1^7 (v_2^2 t_3 \otimes t_1 - v_2^4 t_2 \otimes t_1^2), \quad \text{and} \\ v_1^6 (v_2^3 t_1^2 \otimes t_1^9 + v_2 t_1 \otimes t_1^{18}) + v_1^6 v_2^5 b_{10} \equiv -v_1^7 v_2^2 (t_1^2 \otimes t_2^3 - t_1 \otimes t_1 t_2^3) \end{array}$$

using (5.1). Now we obtain

$$d_1(x(7)') \equiv v_1^2 b_{11}^3 + v_1^5 Z + v_1^7 x'$$

for

$$\begin{aligned} Z \;=\; & \underline{(t_1^3 t_2^3 + t_1^{15}) \otimes t_1^9}_A - \underline{\tau^3 \otimes t_2^3}_C + t_1^6 \otimes t_1^{18} + \underline{v_2^3 t_2^3 \otimes z^9}_D \\ & + \underline{v_2^3 t_1^{12} \otimes z^9}_D - \underline{v_2^3 t_1^3 \otimes t_1^9 z^9}_B + v_2^{-24} t_1^3 \otimes (\underline{t_3^9}_1 - \underline{t_1^9 t_2^{27}}_B) \\ & -(\underline{t_1^{12} \otimes t_2^3}_C + \underline{v_2^{-24} t_1^3 \otimes t_3^9}_1 + v_2^{-6}(\underline{t_2^9 \otimes t_2^3}_C + \underline{t_1^3 t_2^9 \otimes t_1^9}_A)) - \underline{v_2^{-3} t_2^9 \otimes z^9}_D \\ x' \;=\; & t_1^{13} \otimes t_1^9 + t_1^4 \otimes t_1^{18} - v_2^4 t_1^6 \otimes z^9 + v_2^{-5} t_1^3 \otimes T^3 \\ & + v_2^2 t_3 \otimes t_1 - v_2^4 t_2 \otimes t_1^2 - v_2^2 (t_1^2 \otimes t_2^3 - t_1 \otimes t_1 t_2^3). \end{aligned}$$

We introduce an element $w = -z - v_2^{-1} t_2 = v_2^{-1} t_1^4 + v_2^{-1} t_2 - v_2^{-3} t_2^3$. Notice that $z^9 \equiv z^3 \bmod (3, v_1^3)$. Then the parts underlined with $A$, $B$, $C$ and $D$ amount to $v_2^3 t_1^3 \otimes t_1^9 (w \otimes 1)$, $v_2^3 t_1^3 \otimes t_1^9 (1 \otimes w)$, $v_2^3 w^3 \otimes t_2^3$ and $v_2^6 w^3 \otimes z^3$ mod $(3, v_1^3)$, respectively, and so we have

$$Z \equiv v_2^3 t_1^3 \otimes t_1^9 (w^3 \otimes 1 + 1 \otimes w^3) + t_1^6 \otimes t_1^{18} - v_2^6 w^3 \otimes w^3.$$

Since we have $d_1(v_2 w) \equiv t_1 \otimes t_1^3 \bmod (3, v_1)$ by (5.5), we obtain

$$d_1(v_2^6 w^6) \equiv -(t_1^3 \otimes t_1^9 (w^3 \otimes 1 + 1 \otimes w^3) + t_1^6 \otimes t_1^{18} - v_2^6 w^3 \otimes w^3)$$

mod $(3, v_1^3)$. Therefore the cochain $x(7) = x(7)' + v_1^5 w^6$ satisfies the desired congruence by putting $x' = v_2^5 X$. In fact, $\xi$ is represented by a cocycle whose leading term is $v_2^{-3} t_3 \otimes t_1 + v_2^{-10} t_1^3 \otimes t_3^3$, and $T$ is congruent to $t_3$ mod $(3, v_1)$. q.e.d.

Proof of Lemma 4.2. Put $x(8) = V^3 + v_1^4 x(7)$ for $V$ in (5.2). Since $d_1(V) \equiv v_1^2 b_{11} \bmod (3, v_1^8)$ by [1, (3.7)], Lemma 7.1 implies the lemma for $n = 1$. For large $n$, use (6.3) to obtain the lemma. q.e.d.

## §8. The Adams-Novikov differentials on $E_2^{*,*}(L_2W)$

In this section, we compute the Adams-Novikov differential $d_r : E_r^{s,t}(L_2W) \to E_r^{s+r,t+r-1}(L_2W)$ for $r \geq 2$. Note that $E_2^*(L_2W)$ is given in Theorem 2.5 and that $d_r = 0$ unless $r \equiv 1 \bmod 4$ by degree reason.

Proposition 8.1. *For all $r \geq 2$, $d_r = 0$ on $K(1)_*/k(1)_* \bigotimes \Lambda(h_{10}, \zeta_2)$.*

Proof. Suppose that there are elements $x \in \Lambda(h_{10}, \zeta_2)$ and $y \in E_r^{s+r}(L_2W)$ with filt $x = s$ for integers $0 \leq s \leq 2$, $r > 4$ and $j > 0$, such that $d_u(x/v_1^j) = 0$ for $u < r$ and

$$d_r(x/v_1^j) = y \neq 0.$$

Then $d_{r'}(x/v_1^{j+1}) = y' \neq 0$ for some $r' \leq r$ and $y' \in E_{r'}^{s+r'}(L_2W)$. Since $r$ is finite, we may assume that $d_u(x/v_1^{j+k}) = 0$ for $u < r$ and

$$d_r(x/v_1^{j+k}) = y/v_1^k \neq 0 \in E_r^{s+r}(L_2W)$$

from the beginning. Thus $y$ generates a module isomorphic to $K(1)_*/k(1)_*$ in $E_r^{s+r}(L_2W)$. On the other hand, Theorem 2.5 shows that $E_2^{s+r}(L_2W) = H^{s+r}M_1^1$ does not contain such a module since $s + r \geq r > 4$. This is a contradiction. q.e.d.

In the following, an equation $d_r(x) = y$ means not only the indicated one but also $d_s(x) = 0$ for $s < r$.

Lemma 8.2. *Suppose that $d_r(x) = y$ on a element $x$ of $F_n$ or $F \bigoplus F^*$, then we get*

$$d_r(xb_{10}^s\zeta_2^\varepsilon) = yb_{10}^s\zeta_2^\varepsilon$$

*for $s \geq 0$, $\varepsilon = 0, 1$ and $xb_{10}^s\zeta_2^\varepsilon \in E_2^{*,*}(L_2W)$.*

Proof. Since $b_{10}$ represents the homotopy element $\beta_1$, the relation $d_r(x) = y$ implies $d_r(xb_{10}^s) = yb_{10}$.

The same proof that shows $d_r(x) = y$ in the spectral sequence for $\pi_*(L_2W)$ works to show $d_r(x\zeta_2) = y\zeta_2$ in it, since the proof depends on the result of the differentials of the spectral sequence for $\pi_*(L_2V(1))$ in which it is shown in [11] that $d_r(x) = y$ if and only if $d_r(x\zeta_2) = y\zeta_2$ q.e.d.

For the other differentials, we study the exact sequence

$$\cdots \longrightarrow E_2^{s,*}(L_2V(1)) \xrightarrow{i_*} E_2^{s,*}(L_2W) \xrightarrow{v_1} E_2^{s,*}(L_2W) \xrightarrow{\delta_s} E_2^{s+1,*}(L_2V(1)) \longrightarrow \cdots$$

associated to the cofiber sequence (2.1) in order to use the results on $E_2^{*,*}(L_2V(1))$:

(8.3) ([11, Prop.s 8.4, 9.13], [4]) *The differential $d_5$ of the sepectral sequence for $\pi_*(L_2V(1))$ acts as follows*:

$$\begin{array}{ll}
d_5(v_2^{3t+1}) = \pm t v_2^{3t-1}h_{11}b_{10}^2, & d_5(v_2^{3t+1}b_{11}) = \pm(1-t)v_2^{3t+1}h_{10}b_{10}^3,\\
d_5(v_2^{3t+3}\psi_0) = \pm t v_2^{3t}b_{11}\xi b_{10}^2, & d_5(v_2^{3t+1}\psi_1) = \pm(1+t)v_2^{3t}\xi b_{10}^3;\\
d_5(v_2^{3t-1}) = \pm(1+t)v_2^{3t-3}h_{11}b_{10}^2, & d_5(v_2^{3t-1}b_{11}) = \pm t v_2^{3t-1}h_{10}b_{10}^3,\\
d_5(v_2^{3t+1}\psi_0) = \pm(1+t)v_2^{3t-2}b_{11}\xi b_{10}^2 & d_5(v_2^{3t-1}\psi_1) = \pm(1-t)v_2^{3t-2}\xi b_{10}^3;\\
d_5(v_2^{3t}b_{10}) = \pm t v_2^{3t-2}h_{11}b_{10}^3, & d_5(v_2^{3t}b_{11}) = \pm(1-t)v_2^{3t}h_{10}b_{10}^3,\\
d_5(v_2^{3t-1}\psi_0) = \pm(1-t)v_2^{3t-4}b_{11}\xi b_{10}^2, & d_5(v_2^{3t}\psi_1 b_{10}) = \pm(1+t)v_2^{3t-1}\xi b_{10}^3.
\end{array}$$

*The differential $d_5$ of the sepectral sequence for $\pi_*(L_2V(1))$ acts as follows*:

Here note that the undetermined integer $k$ of [11] is shown to be 1 by [4] and the results of (8.3) follow.

LEMMA 8.4. *Let $x$ be an element of $F_n$ or $F \bigoplus F^*$. Then we see the following*:

(1) *If $x = i_*(\overline{x})$ and $d_r(\overline{x}) = \overline{y}$ in $E_r^{*,*}(L_2V(1))$, then*

$$d_r(x) = i_*(\overline{y}) \quad \text{in } E_r^{*,*}(L_2V(1)).$$

(2) *If $d_r(\delta_s(x)) = \delta_{s+r}(y)$, then*

$$d_r(x) = y + \cdots$$

*Here $\cdots$ denotes an element of $J$ given by*

$$J = E(2,1)_*\{v_2h_{10}/v_1, v_2^2h_{11}/v_1, \xi/v_1, b_{11}\xi/v_1\} \bigotimes (\boldsymbol{Z}/3)[b_{10}] \bigotimes \Lambda(\zeta_2).$$

*Furthermore the generators of $J$ have the bidegrees*:

$$||v_2^{3s+1}h_{10}/v_1|| = (1, 16(3s+1)), \quad ||v_2^{3s+2}h_{11}/v_1|| = (1, 16(3s+2)+8),$$
$$||v_2^{3s}\xi/v_1|| = (2, 48s+4), \quad ||v_2^{3s}b_{11}\xi/v_1|| = (4, 48s+40),$$
$$||b_{10}|| = (2, 12) \quad \text{and} \quad ||\zeta_2|| = (1, 0).$$

Proof. Part 1) follows from the naturality of the differential $d_r$.

The hypothesis $d_r(\delta_s(x)) = \delta_{s+r}(y)$ of 2) implies $d_r(x) \equiv y \bmod \mathrm{Ker}\, \delta_{s+r} = \mathrm{Im}\, v_1$ by naturality. Besides, $d_r(x) \in E_r^{s,*}$ for $s \geq 5$ and $\bigoplus_{s\geq 5} E_r^{s,*} \subset G = (F \bigoplus F^*) \bigotimes (\boldsymbol{Z}/3)[b_{10}] \bigotimes \Lambda(\zeta_2)$. Therefore $d_r(x) \equiv y \bmod J = \mathrm{Im}\,(v_1 : G \to G)$. The structure of $J$ follows from Theorem 2.5. q.e.d.

Proposition 8.5. *The differentials on $(F \bigoplus F^*) \bigotimes (\boldsymbol{Z}/3)[b_{10}] \bigotimes \Lambda(\zeta_2)$ are read off from the following results on $F \bigoplus F^*$ (by Lemma 8.2):*

(a) $d_5(v_2^{3t+1}/v_1) = \pm t v_2^{3t-1} h_{11} b_{10}^2/v_1$

(a') $d_5(v_2^{3t-1}/v_1) = 0$

(b) $d_5(v_2^{3t+1} h_{10}/v_1^2) = \pm t v_2^{3t-1} b_{10}^3/v_1$

(c) $d_5(v_2^{3t+2} h_{11}/v_1^2) = \pm(1-t) v_2^{3t-1} b_{11} b_{10}^2/v_1 + k v_2^{3t+1} h_{10} b_{10}^2 \zeta_2/v_1$
*for some $k \in \boldsymbol{Z}/3$*

(d) $d_5(v_2^{3t+1} b_{11}/v_1) = \pm(1-t) v_2^{3t+1} h_{10} b_{10}^3/v_1$

(d') $d_5(v_2^{3t-1} b_{11}/v_1) = 0$

(a)* $d_5(v_2^{3t+3} \psi_0/v_1) = \pm t v_2^{3t} b_{11} \xi b_{10}^2/v_1$

(a')* $d_5(v_2^{3t+1} \psi_0/v_1) = 0$

(b)* $d_5(v_2^{3t} \xi/v_1^2) = \pm(1-t) v_2^{3t-2} \psi_0 b_{10}^2/v_1$

(c)* $d_5(v_2^{3t} b_{11} \xi/v_1^2) = \pm(1+t) v_2^{3t-1} \psi_1 b_{10}^3/v_1$

(d)* $d_5(v_2^{3t+1} \psi_1/v_1) = \pm(1+t) v_2^{3t} \xi b_{10}^3/v_1$

(d')* $d_5(v_2^{3t-1} \psi_0/v_1) = 0.$

Proof. The first four equations of (8.3) give rise to (a), (d), (a)* and (d)* by Lemma 8.4 (1).

Proposition 3.4 shows the following:

$$v_2^{3t-3} h_{11} b_{10}^2 = \delta_4(v_2^{3t-2} b_{10}^2/v_1), \quad v_2^{3t-1} h_{10} b_{10}^3 = \delta_6(v_2^{3t-2} b_{11} b_{10}^2/v_1),$$
$$v_2^{3t-2} b_{11} \xi b_{10}^2 = \delta_8(v_2^{3t} \psi_0 b_{10}^2/v_1), \quad v_2^{3t-2} \xi b_{10}^3 = \delta_7(v_2^{3t-2} \psi_1 b_{10}^2/v_1).$$

Since $i_*\delta_s(y)=0$, Lemma 8.4 (1) implies that the second four equations of (8.3) yield (a'), (d'), (a')$^*$, and (d')$^*$.

Furthermore, Proposition 3.4 shows the following:

$$\begin{array}{l}\delta_1(v_2^{3t+1}h_{10}/v_1^2)=v_2^{3t}b_{10},\quad \delta_1(v_2^{3t+2}h_{11}/v_1^2)=v_2^{3t}b_{11},\\ \delta_2(v_2^{3t}\xi/v_1^2)=v_2^{3t-1}\psi_0\quad \delta_4(v_2^{3t}b_{11}\xi/v_1^2)=-v_2^{3t}\psi_1b_{10};\quad \text{and}\\ v_2^{3t-2}h_{11}b_{10}^3=-\delta_6(v_2^{3t-1}b_{10}^3/v_1),\quad v_2^{3t}h_{10}b_{10}^3=-\delta_6(v_2^{3t-1}b_{11}b_{10}^2/v_1),\\ v_2^{3t-4}b_{11}\xi b_{10}^2=-\delta_7(v_2^{3t-2}\psi_0b_{10}^2/v_1),\quad v_2^{3t-1}\xi b_{10}^3=\delta_9(v_2^{3t-1}\psi_1b_{10}^3/v_1).\end{array}$$

Therefore, we apply Lemma 8.4 (2) to show

$$\begin{array}{rcl}d_5(v_2^{3t+1}h_{10}/v_1^2)&=&\pm tv_2^{3t-1}b_{10}^3/v_1+\cdots\\ d_5(v_2^{3t+2}h_{11}/v_1^2)&=&\pm(1-t)v_2^{3t-1}b_{11}b_{10}^2/v_1+\cdots\\ d_5(v_2^{3t}\xi/v_1^2)&=&\pm(1-t)v_2^{3t-2}\psi_0b_{10}^2/v_1+\cdots\\ d_5(v_2^{3t}b_{11}\xi/v_1^2)&=&\pm(1+t)v_2^{3t-1}\psi_1b_{10}^3/v_1+\cdots\end{array}$$

by using the last four equations of (8.3). Let $(J)^{s,t}$ denotes the submodule of $J$ with bidegree $(s,t)$. Then we see that $(J)^{s,u}=0$ if $(s,u)=(6,16(3t-1)+32)$, $(7,16(3t-2)+36)$ or $(9,16(3t-1)+56)$, and $=(\boldsymbol{Z}/3)\{v_2^{3t+1}h_{10}b_{10}^2\zeta_2/v_1\}$ if $(s,u)=(6,16(3t-1)+56)$. Therefore we obtain (b), (c), (b)$^*$ and (c)$^*$.

q.e.d.

We here recall the folklore lemma which will be used later:

LEMMA 8.6. *Let* $X\overset{f}{\to}Y\overset{g}{\to}Z\overset{h}{\to}\Sigma X$ *be a cofiber sequence with* $E(2)_*(f)=0$. *Then in the exact sequence* $E_2^*(L_2X)\overset{f_*}{\to}E_2^*(L_2Y)\overset{g_*}{\to}E_2^*(L_2Z)\overset{\delta}{\to}E_2^{*+1}(X)$, *we have the followings*:

(1) *If we have a chart*

$$\begin{array}{ccccc} y & \overset{g_*}{\to} & z' & & \\ & & & \nwarrow d_5 & \\ & & & & z \overset{\delta}{\to} x,\end{array}$$

*and* $x$ *is a permanent cycle, then* $f_*([x])=[y]$, *where* $[\cdot]$ *denotes a homotopy class.*

(2) *If we have a chart*

$$\begin{array}{ccccccc} x' & \xrightarrow{f_*} & y' & & & & \\ & & \nwarrow d_5 & & & & \\ & & y & \xrightarrow{g_*} & z' & & \\ & & & & \nwarrow d_5 & & \\ & & & & z & \xrightarrow{\delta} & x, \end{array}$$

*then* $d_9(x) = x'$.

(3) *If we have a chart*

$$\begin{array}{ccccc} & x' & \xrightarrow{f_*} & y' & & \\ & & & \nwarrow d_5 & & \\ & & \nwarrow d_9 & & y & \xrightarrow{g_*} & z' \\ z & \xrightarrow{\delta} & x & & & \end{array}$$

*then* $d_5(z) = z'$.

Corollary 8.7. *Consider the cofiber sequence* $V(1) \xrightarrow{i} \Sigma^4 W \xrightarrow{v_1} W \xrightarrow{j} \Sigma V(1)$. *Then the induced map* $i_* : \pi_*(L_2V(1)) \to \pi_{*-4}(L_2W)$ *acts as follows*:

(a) $i_*(v_2^{9t+2}h_{10}) \equiv v_2^{9t+1}h_{10}b_{10}^2/v_1^2 \bmod \operatorname{Ker} \beta_1$

(a') $i_*(v_2^{9t+2}h_{10}\zeta_2) \equiv v_2^{9t+1}h_{10}b_{10}^2\zeta_2/v_1^2 \bmod \operatorname{Ker} \beta_1$

(b) $i_*(v_2^{3u+1}\xi) \equiv \pm(1+u)v_2^{3u}\xi b_{10}^2/v_1^2 + \cdots \bmod \operatorname{Ker} \beta_1 \quad (u \not\equiv 0\ (3))$.

(b') $i_*(v_2^{3u+1}\xi\zeta_2) \equiv \pm(1+u)v_2^{3u}\xi b_{10}^2\zeta_2/v_1^2 + \cdots \bmod \operatorname{Ker} \beta_1 \quad (u \not\equiv 0\ (3))$.

Proof. We obtain the following charts (up to signs)

$$\begin{array}{ccccc} v_2h_{10}b_{10}^3/v_1^2 & \xrightarrow{v_1} & v_2h_{10}b_{10}^3/v_1 & & \\ & & \nwarrow d_5 & & \\ & & v_2b_{11}/v_1 & \xrightarrow{\delta} & v_2^2h_{10}b_{10} \end{array}$$

and

$$\begin{array}{ccccc} (1+u)v_2^{3u}\xi b_{10}^3/v_1^2 & \xrightarrow{v_1} & (1+u)v_2^{3u}\xi b_{10}^3/v_1 & & \\ & & \nwarrow d_5 & & \\ & & v_2^{3u+1}\psi_1/v_1 & \xrightarrow{\delta} & v_2^{3u+1}\xi b_{10} \end{array}$$

from Propositions 3.4 and 8.5. Since $v_2^{9t+2}h_{10}$ and $v_2^{3u+1}\xi$ with $u \not\equiv 0$ mod 3 are permanent cycles by [11, Th. A], $i_*(v_2^{9t+2}h_{10}b_{10}) = v_2^{9t+1}h_{10}b_{10}^3/v_1^2$ and $i_*(v_2^{3u+1}\xi b_{10}) = (1+u)v_2^{3u}\xi b_{10}^3/v_1^2$ by Lemma 8.6. Now divide it by $b_{10}$ which represents $\beta_1 \in \pi_*(S^0)$, we obtain (a) and (b).

In the same way, we obtain (a') and (b') by Lemma 8.2. q.e.d.

PROPOSITION 8.8. *The differentials on $F_n \otimes \Lambda(\zeta_2)$ are read off from the following relation on $F_n$ (by Lemma 8.2):*

(a) $d_5(v_2^{3^{n+2}t+3^{n+1}}/v_1^{4\cdot 3^n-1}) = \begin{cases} \pm v_2^{9t+1}h_{10}b_{10}^2/v_1 & n=0 \\ 0 & n>0 \end{cases}$

(a') $d_5(v_2^{3^{n+2}t-3^{n+1}}/v_1^{4\cdot 3^n-1}) = 0$

(b) $d_5(v_2^{3^{n+2}t+3^{n+1}}h_{10}/v_1^{2\cdot 3^{n+1}+1}) = \pm v_2^{3^{n+2}t+3(3^n-1)/2}\xi b_{10}^2/v_1$ $(n \geq 0)$

(c) $d_5(v_2^{3^{n+2}t+8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n+1}) = \begin{cases} 0 & n=0,1 \\ \pm v_2^{3^{n+2}t+5\cdot 3^n+3(3^{n-1}-1)/2}\xi b_{10}^2/v_1 & n>1 \end{cases}$

(d) $d_5(v_2^{3^n(9t+5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}) = \pm v_2^{3^{n+1}(3t+1\pm 1)+3(3^n-1)/2-1}\psi_1 b_{10}^2/v_1$
$(n>0)$

(e) $d_5(v_2^{9t+8}\xi/v_1^4) = 0$

(e') $d_5(v_2^{9t+2}\xi/v_1^4) = -v_2^{9t-1}\psi_1 b_{10}^2/v_1$

PROOF. For the cases (d) and (e'), $d_5(\delta_2(v_2^{b(n)}\xi/v_1^{4\cdot 3^n})) = d_5(v_2^{b(n)-3^n-1}\psi_1) = \pm v_2^{b(n)-3^n-2}\xi b_{10}^3 = \pm\delta_7(v_2^{b(n)-3^n-2}\psi_1 b_{10}^2/v_1)$, since $b(n)-3^n-1 \equiv 0$ or 3 mod 9, where $b(0) = 9t+2$ and $b(n) = 3^n(9t+5\pm 3)+(3^n-1)/2$ for $n>0$. Therefore Lemma 8.4 (2) implies (d) and (e'), since $(J)^{s,u} = 0$ for $(7, 16(b(n)-3^n-2)+44)$ and $(7, 16(9t-1)+44)$.

Take $x/v_1^a \in F_n$. For the cases (a), (a'), (b), (c) and (e), $d_5(\delta_*(x/v_1^a)) = 0$ by [11, Prop.s 8.4, 9.13]. Therefore, $d_5(x/v_1^a) \in J$ by Lemma 8.4 (2). Comparing degrees, we have (a) for $n>0$ and (a'), and (c) for $n=0$. Besides,

$$\begin{aligned} d_5(v_2^{9t+3}/v_1^3) &= k_1 v_2^{9t+1}h_{10}b_{10}^2/v_1 + k_2 v_2^{9t}\xi b_{11}\zeta_2/v_1 \\ d_5(v_2^{9t-3}/v_1^3) &= k_3 v_2^{9t-5}h_{10}b_{10}^2/v_1 + k_4 v_2^{9t-6}\xi b_{11}\zeta_2/v_1 \end{aligned}$$

$$\begin{aligned} d_5(v_2^{3^{n+2}t+3^{n+1}}h_{10}/v_1^{2\cdot 3^{n+1}+1}) &= k_5 v_2^{3^{n+2}t+3(3^n-1)/2}\xi b_{10}^2/v_1 \quad (n\ge 0)\\ d_5(v_2^{3^{n+2}t+8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n+1}) &= k_6 v_2^{3^{n+2}t+5\cdot 3^n+3(3^{n-1}-1)/2}\xi b_{10}^2/v_1 \quad (n\ge 1)\\ d_5(v_2^{9t+8}\xi/v_1^4) &= k_7 v_2^{9t+6}\xi b_{10}^2\zeta_2/v_1 + k_8 v_2^{9t+5}h_{11}b_{10}^3/v_1 \end{aligned}$$

for some $k_i\in \boldsymbol{Z}/3$ $(1\le i\le 8)$. Since $v_2^{9t+5}h_{11}b_{10}^3/v_1$ is hit by $d_5$ of $v_2^{9t+7}b_{10}/v_1$ by Proposition 8.5, we take $k_8=0$ by replacing $v_2^{9t+8}\xi/v_1^4$ by $v_2^{9t+8}\xi/v_1^4 - (\pm k_8)v_2^{9t+7}b_{10}/v_1$, that is,

$$d_5(v_2^{9t+8}\xi/v_1^4) = k_7 v_2^{9t+6}\xi b_{10}^2\zeta_2/v_1.$$

Now we determine the numbers $k_i$ for $1\le i\le 7$. Consider the following charts (up to signs)

$$\begin{array}{lclcl} k_2v_2^{-1}\psi_1 b_{10}^3\zeta_2/v_1 & & & & \\ \quad\nwarrow d_5 & & & & \\ k_1v_2h_{10}b_{10}^2/v_1^2 + k_2b_{11}\xi\zeta_2/v_1^2 & \xrightarrow{v_1} & k_1v_2h_{10}b_{10}^2/v_1 + k_2b_{11}\xi\zeta_2/v_1 & & \\ & & \quad\nwarrow d_5 & & \\ & & v_2^3/v_1^3 & \xrightarrow{\delta_0} & v_2^2h_{10} \end{array}$$

and

$$\begin{array}{lclcl} k_3v_2^{-7}b_{10}^5/v_1 + k_4v_2^{-7}\psi_1 b_{10}^3\zeta_2/v_1 & & & & \\ \quad\nwarrow d_5 & & & & \\ k_3v_2^{-5}h_{10}b_{10}^2/v_1^2 + k_4v_2^{-6}b_{11}\xi\zeta_2/v_1^2 & \xrightarrow{v_1} & k_3v_2^{-5}h_{10}b_{10}^2/v_1 + k_4v_2^{-6}b_{11}\xi\zeta_2/v_1 & & \\ & & \quad\nwarrow d_5 & & \\ & & v_2^{-3}/v_1^3 & \xrightarrow{\delta_0} & v_2^{-4}h_{10} \end{array}$$

obtained from Propositions 3.4 and 8.5. Then $k_2=0$ since $v_2^2h_{10}$ is a permanent cycle by [11]. Besides Corollary 8.7 shows that $k_1=1$. The numbers $k_3$ and $k_4$ are seen to be zero by Lemma 8.6, since $v_2^{-4}h_{10}$ is also a permanent cycle by [11, Cor. 10.7]. By similar charts, the third and the fourth equations imply

$$\begin{aligned} i_*(v_2^{3^{n+2}t+(3^{n+1}-1)/2}\xi) &= k_5 v_2^{3^{n+2}t+3(3^n-1)/2}\xi b_{10}^2/v_1^2 \quad (n>0)\\ i_*(v_2^{3^{n+2}t+5\cdot 3^n+(3^n-1)/2}\xi) &= k_6 v_2^{3^{n+2}t+5\cdot 3^n+3(3^{n-1}-1)/2}\xi b_{10}^2/v_1^2 \quad (n\ge 1) \end{aligned}$$

up to sign using Propositions 3.4 and 8.5. In fact, for $n>0$, $v_2^{3^{n+2}t+(3^{n+1}-1)/2}\xi$ and $v_2^{3^{n+2}t+5\cdot 3^n+(3^n-1)/2}\xi$ are permanent cycles by [11, Cor. 10.7]. Now compare with Corollary 8.7, and we obtain

$$k_5 = 1 \text{ if } n > 0, \text{ and } k_6 = \begin{cases} 0 & n = 1 \\ 1 & n > 1. \end{cases}$$

If $n = 0$, then $k_5 = 1$ by applying Lemma 8.6 (3) to the following chart (up to sign):

$$\begin{array}{ccccc} & & v_2^{-2}\psi_0 b_{10}^4 & \xrightarrow{i_*} & v_2^{-2}\psi_0 b_{10}^4/v_1 \\ & & \nwarrow d_9 & & \nwarrow d_5 \\ v_2^3 h_{10}/v_1^7 & \xrightarrow{\delta_1} & v_2\xi & & \xi b_{10}^2/v_1^2. \end{array}$$

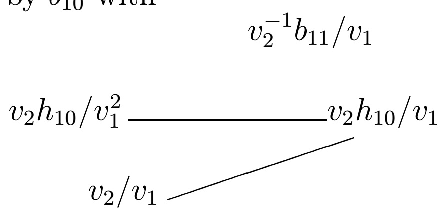
Consider again the chart (up to sign)

$$\begin{array}{ccccc} k_7 v_2^4 \psi_0 b_{10}^5 \zeta_2/v_1 & & & & \\ \nwarrow d_5 & & & & \\ k_7 v_2^6 \xi b_{10}^3 \zeta_2/v_1^2 & \xrightarrow{v_1} & k_7 v_2^6 \xi b_{10}^6 \zeta_2/v_1 & & \\ & & \nwarrow d_5 & & \\ & & v_2^8 \xi/v_1^4 & \xrightarrow{\delta_2} & v_2^6\psi_1 \pm v_2^7 \xi\zeta_2 \end{array}$$

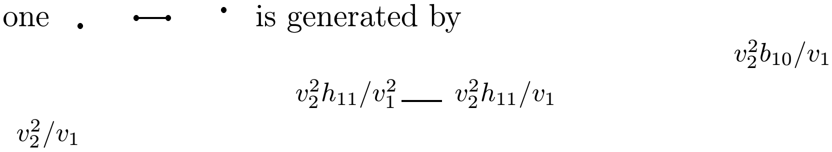
obtained from Propositions 3.4 and 8.5. Since $v_2^6\psi_1 \pm v_2^7\xi\zeta_2$ is a permanent cycle by [11, Cor. 10.7], we obtain $k_8 = 0$. q.e.d.

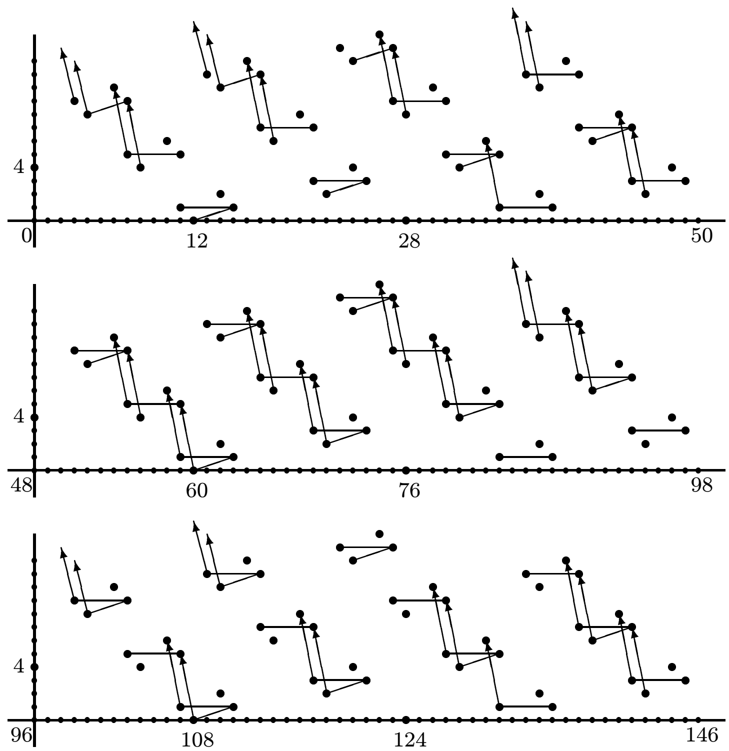
These propositions give us the following charts of $E_2$-term with $d_5$, in which holizontal lines of length 4 denote multiplication by $v_1$, lines of slope 1/3, multiplication by $h_{10}$ and lines of slope 1/11, multiplication by $h_{11}$. The differential $d_5$ is expressed by arrows of slope $-5$. Besides, the same pattern of period $(10, 2)$ denotes multiplication by $b_{10}$.

The following is the one on $F\otimes(\boldsymbol{Z}/3)[b_{10}]$, where starting from dimension 12 is multiples by $b_{10}$ with

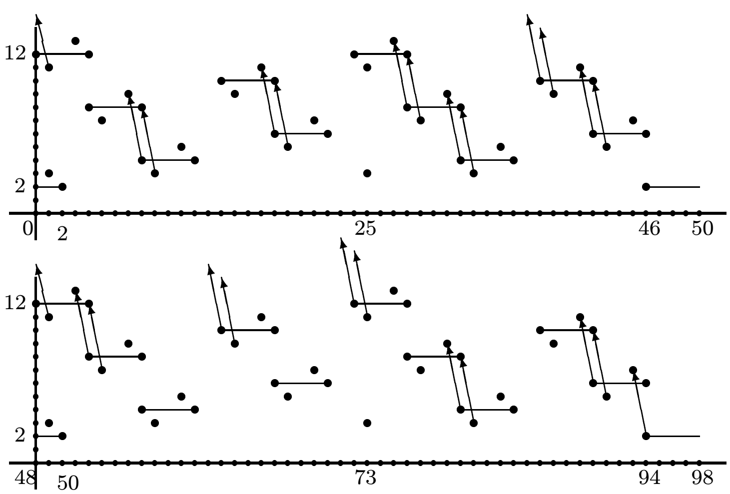


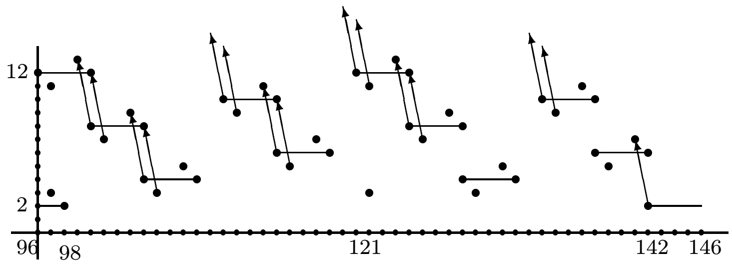

The other one is generated by






The next one is the $E_2$-term with $d_5$ on $F^*\otimes(\boldsymbol{Z}/3)[b_{10}]$. Each dot can be read off from the degree of the generator.






For the part with $\zeta_2$, just shift the dimension. With these charts, the relation $\beta_1^6 = 0 \in \pi_*(S^0)$ implies the following

PROPOSITION 8.9. *On the elements of* $E_6^*(W)$ *originating* $(F \bigoplus F^*) \bigotimes (\boldsymbol{Z}/3)[b_{10}] \bigotimes \Lambda(\zeta_2)$, $d_9 : E_9^s(W) \to E_9^{s+9}(W)$ *is obtained by the followings*:

(a) $d_9(v_2^{9t+2} b_{11}/v_1) = v_2^{9t+1} h_{10} b_{10}^5/v_1^2$

(b) $d_9(v_2^{9t+4} h_{10}/v_1) = v_2^{9t+1} b_{10}^5/v_1$

(c) $d_9(v_2^{9t+8} h_{11}/v_1) = v_2^{9t+4} b_{11} b_{10}^4/v_1$

(d) $d_9(v_2^{9t+8} b_{10}/v_1) = v_2^{9t+5} h_{11} b_{10}^5/v_1^2$

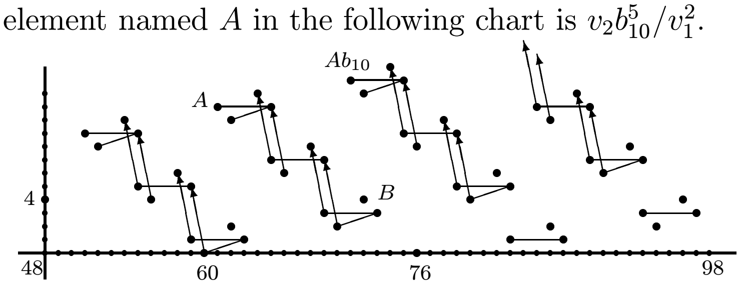
(a)* $d_9(v_2^{9t+5} \psi_1/v_1) = v_2^{9t+3} \xi b_{10}^5/v_1^2$

(b)* $d_9(v_2^{9t+6} \xi/v_1) = v_2^{9t+3} \psi_0 b_{10}^4/v_1$

(c)* $d_9(v_2^{9t} \xi/v_1) = v_2^{9t-2} \psi_1 b_{10}^5/v_1$

(d)* $d_9(v_2^{9t+1} \psi_0/v_1) = v_2^{9t-3} b_{11} \xi b_{10}^4/v_1^2$

PROOF. Since the proof of each equation is similar, we only prove (a) here. The element named $A$ in the following chart is $v_2 b_{10}^5/v_1^2$.



Let $\overline{E}_n$ denote the $n$-th term of the Adams-Novikov resolution such that $\operatorname{holim}_{\overleftarrow{n}} \overline{E}_n = L_2 S^0$. Then $E_2^s(\overline{E}_{12} \wedge W) \cong E_2^s(L_2 W)$ if $s < 12$, and $= 0$ if $s > 12$. By observing the above chart, we see that $v_2 h_{10}/v_1^2$ is a permanent cycle in the Adams-Novikov spectral sequence and yields an element of $\pi_*(\overline{E_{12}} \wedge W)$. Since $\beta_1^6 = 0 \in \pi_*(S^0)$ and $\pi_*(S^0)$ acts on any homotopy groups, the element $Ab_{10} = v_2 h_{10} b_{10}^6 / v_1^2$ must be killed by some element. Note here that $\beta_1$ is represented by $b_{10}$. The chart shows us the only candidate for it is $B$ in the chart which is $v_2^2 b_{11} b_{10}/v_1$. Since we have a map $L_2 W \to \overline{E_{12}} \wedge W$, the relation $d_9(v_2^2 b_{11} b_{10}/v_1) = Ab_{10}$ in $E_r^*(\overline{E_{12}} \wedge W)$ also holds in $E_r^*(L_2 W)$. Thus, dividing by $b_{10}$ gives (a). q.e.d.

The following completes the computation of the differentials.

PROPOSITION 8.10. *On the elements of* $E_6^*(W)$ *originating* $F_n \otimes \Lambda(\zeta_2)$, $d_9$ *is given by the followings*:

(a) $d_9(v_2^{3^{n+1}(3t+1)}/v_1^{4\cdot 3^n - 1}) = 0$

(a') $d_9(v_2^{3^{n+1}(3t-1)}/v_1^{4\cdot 3^n - 1}) = 0$

(c) $d_9(v_2^{9t+8} h_{10}/v_1^{11}) = v_2^{9t+3} \xi b_{10}^4 / v_1^2$

(e) $d_9(v_2^{9t+8} \xi / v_1^4) = 0$

PROOF. Consider the total degree

$$|v_2^{3^{n+1}(3t+1)}/v_1^{4\cdot 3^n - 1}| \equiv \begin{cases} 100 \mod 144 & n = 1 \\ 4 \mod 144 & n \geq 2, \end{cases}$$

$$|v_2^{3^{n+1}(3t-1)}/v_1^{4\cdot 3^n - 1}| \equiv \begin{cases} 84 \mod 144 \\ 100 \mod 144 \\ 4 \mod 144 \end{cases}.$$

Then the chart shows that nothing can be hit by $d_9$ of these elements. Thus (a) and (a') follows. In the same way, $|v_2^{9t+8}\xi/v_1^4| \equiv 118 \mod 144$, and nothing can be the target. This is (e). For (c), we compute

$$\delta_{11}(d_9(v_2^8 h_{10}/v_1^{11})) = d_9(v_2^5 \xi) = v_2^2 \psi_0 b_{10}^4 = \delta_{11}(v_2^3 \xi b_{10}^4 / v_1^2).$$

q.e.d.

THEOREM 8.11. *The $E_{10}$-term $E_{10}(W)$ is isomorphic to the direct sum of $k(1)_*$-modules $(K(1)_*/k(1)_*)\otimes\Lambda(h_{10},\zeta_2)$, $\sum_{n\geq 0}\widetilde{F}_n\otimes\Lambda(\zeta_2)$ and $(\widetilde{F}\oplus\widetilde{F}^*)\otimes\Lambda(\zeta_2)$ for $k(1)_*$-modules in* (2.7).

PROOF OF THEOREM 2.8. Since $E_{10}^*(W)$ has a holizontal vanishing line by Theorem 8.11, we have $E_{10}^{*,*}(W) = E_\infty^{*,*}(W)$. Furthermore, there arises no extension problem in the spectral sequence, since $\pi_*(L_2W)$ is a $\pi_*(V(0))$-module and so $(\boldsymbol{Z}/3)$-vector space. Therefore we obtain the homotopy groups $\pi_*(L_2W) = E_{10}^*(W)$. q.e.d.

## §9. $\beta$-elements

The $\beta$-elements in the $E_2$-term for $\pi_*(S^0)$ are defined in [6]. Here we modifies it in the $E_2$-term $H^*E(2)$ for $\pi_*(L_2S^0)$ as follows: Let $0\to E(2)_*\xrightarrow{3} E(2)_*\to E(2)_*/(3)\to 0$ and $0\to E(2)_*/(3)\to v_1^{-1}E(2)_*/(3)\to M_1^1\to 0$ be short exact sequences, and $\delta: H^*E(2)_*/(3)\to H^{*+1}E(2)_*$ and $\delta': H^*M_1^1\to H^{*+1}E(2)_*/(3)$ the connecting homomorphisms associated to the short exact sequences, respectively. Then for an element of the form $v_2^a/v_1^b$ in $H^0M_1^1$, we define

$$\beta_{a/b} = \delta\delta'(v_2^a/v_1^b)\in H^*E(2)_*$$

and $\beta_a = \beta_{a/1}$, which is essential in the $E_2$-term $H^*E(2)_*$ for $\pi_*(L_2S^0)$.

Consider the cofiber sequences defining the spectra $V(0)$ and $W$: $S^0\xrightarrow{3} S^0\xrightarrow{i} V(0)\xrightarrow{j}\Sigma S^0$ and $V(0)\xrightarrow{\lambda} L_1V(0)\xrightarrow{\iota} W\xrightarrow{\pi}\Sigma V(0)$, respectively. If an element $v_2^a/v_1^b$ is permanent cycle, then so is $\beta_{a/b}$, by Geometric Boundary theorem (*cf.* [9]).

PROOF OF THEOREM 2.12. By Theorem 2.8, we see that $v_2^j/v_1$ for $j\equiv 0,1,2,3,5,6 \bmod 9$ are permanent cycles. Thus 'if' part is shown. 'Only if' part is shown in [11]. q.e.d.

PROOF OF THEOREM 2.13. The element $v_2^a/v_1^b$ with $9|a$ is in $\widetilde{F}_1$ or $\widetilde{F}_n$ of (2.7), and so it is permanent by Theorem 2.8. For the case $9\not| a$, the part (a) follows from Theorem 2.12. $v_2^{9t\pm 3}/v_1^b$ comes from $\widetilde{F}_0$ of (2.7), and we obtain the part (b). q.e.d.

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