MONOIDAL UNIQUENESS OF STABLE HOMOTOPY THEORY

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ABSTRACT. We show that the monoidal product on the stable homotopy category of spectra is essentially unique. This strengthens work of this author with Schwede on the uniqueness of models of the stable homotopy theory of spectra. Also, the equivalences constructed here give a unified construction of the known equivalences of the various symmetric monoidal categories of spectra (S-modules, $\mathscr{W}$-spaces, orthogonal spectra, simplicial functors) with symmetric spectra. As an application we show that with an added assumption about underlying model structures Margolis' axioms uniquely determine the stable homotopy category of spectra up to monoidal equivalence.

1. INTRODUCTION

The homotopy category of spectra, obtained by inverting the weak equivalences of spectra, has long been known to have a symmetric monoidal product induced by the smash product [1, 23]. Recently, several categories of spectra have been constructed which have symmetric monoidal smash products on the point set level [5, 9, 12]. In this paper we consider the uniqueness properties of the monoidal product on the models for spectra and on the homotopy category of spectra.

On the homotopy category, Corollary 3.3 shows that under weak hypotheses if there is a natural transformation $A \wedge B \longrightarrow A \wedge' B$ between two monoidal products which both have the sphere spectrum, $\mathbb{S}$, as the unit, then this transformation is an isomorphism on all objects. Thus, the main obstruction to showing that two monoidal products are equivalent is constructing a natural transformation between them.

To construct such natural transformations we consider the models of spectra, rather than the homotopy category. Instead of restricting to the known models of spectra we consider any stable Quillen model category $\mathcal{C}$ with a compatible symmetric monoidal product, that is, a *stable, monoidal model category*; see Definitions 2.1, 2.4. The compatibility conditions ensure that the product on $\mathcal{C}$ has a derived functor on the homotopy category, $\mathrm{Ho}(\mathcal{C})$, which is exact with respect to the triangulated structure arising from the stability of $\mathcal{C}$ [7, Proposition 6.4.1]. Under the weak additional hypothesis that $\mathcal{C}$ has a good desuspension of the unit, see Definition 6.1, we construct a natural isomorphism, see Definition 2.2, between the derived product on $\mathrm{Ho}(\mathcal{C})$ and the smash product on the homotopy category of spectra, $\mathrm{Ho}(\mathrm{Sp})$.

Theorem 1.1. *Let $\mathcal{C}$ be a stable, monoidal model category with a good desuspension of the unit such that the monoidal product commutes with colimits. Then there is a strong monoidal functor from* $\mathrm{Ho}(\mathrm{Sp})$ *to* $\mathrm{Ho}(\mathcal{C})$.

This statement is proved as Theorem 6.2. This functor provides a highly structured monoidal equivalence between $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathrm{Sp})$ under hypotheses first developed in [19].

Corollary 1.2. *Assume $\mathcal{C}$ satisfies the hypotheses of Theorem 1.1. If the unit, $\mathbb{I}$, is a small weak generator for which $[\mathbb{I}, \mathbb{I}]_*^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by the identity map of $\mathbb{I}$,*

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then there is a π_*^s-linear, triangulated, monoidal equivalence between the homotopy category of $\mathcal{C}$ and the homotopy category of spectra.

This shows that the only monoidal product on Ho(Sp) which has an underlying model satisfying these weak hypotheses is the usual smash product. This statement, its converse, and several other equivalent conditions are proved as Corollary 6.3. More formally, we prove in Section 3 that with an added underlying model structure (see Definition 3.5) the axioms introduced by Margolis in [13] uniquely determine the stable homotopy category of spectra.

Theorem 1.3. *Suppose that $\mathscr{S}$ is a stable homotopy category in the sense of* [13, Chapter 2 §1] *which has an underlying model category. Then $\mathscr{S}$ is monoidally equivalent to the stable homotopy category of spectra.*

As mentioned above, we also consider the uniqueness of the symmetric monoidal models of spectra. We show that the strong monoidal functor on the homotopy category level lifts to a Quillen functor, see Definition 2.3, on the model category level under the additional hypothesis that $\mathcal{C}$ is simplicial. This strengthens the uniqueness of the homotopy category to the whole homotopy theory. The reference model for spectra here is the positive stable model structure on the category of symmetric spectra. Again, we require that $\mathcal{C}$ has a simplicial desuspension of the unit, see Definition 4.4.

Theorem 1.4. *Let $\mathcal{C}$ be a simplicial, stable, monoidal model category with a simplicial desuspension of the unit such that the monoidal product commutes with the simplicial action. Then there is a strong monoidal functor from Sp^Σ to $\mathcal{C}$. Moreover, this functor is a left Quillen adjoint.*

This statement is proved as Theorem 4.6. Under additional hypotheses on the unit of the product in $\mathcal{C}$ from [19], this functor provides a monoidal Quillen equivalence between $\mathcal{C}$ and Sp^Σ.

Corollary 1.5. *Assume $\mathcal{C}$ satisfies the hypotheses of Theorem 1.4. The unit, $\mathbb{I}$, is a small weak generator for which $[\mathbb{I},\mathbb{I}]_*^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by the identity map of $\mathbb{I}$ if and only if there is a simplicial monoidal Quillen equivalence between $\mathcal{C}$ and the positive stable model structure on Sp^Σ.*

This shows that up to monoidal Quillen equivalence there is a unique symmetric monoidal model of spectra which satisfies the above hypotheses. Other equivalent conditions are stated in Corollary 4.7. In Section 5 this uniqueness is extended to modules and algebras.

As shown in Remark 4.5, all of the known symmetric monoidal model categories of spectra, (S-modules [5], symmetric spectra [9], simplicial functors [10], orthogonal spectra [12], topological symmetric spectra [12], and $\mathscr{W}$-spaces [12]) satisfy the hypotheses of Corollary 4.7. In fact, as shown in Remarks 4.8 and 6.5 a simplicial or good desuspension of the unit is a necessary condition for having a functor from symmetric spectra to $\mathcal{C}$ with the properties listed in Theorems 4.6 and 6.2. Remark 4.8 shows that one can choose the desuspension of the unit so that the monoidal Quillen equivalences constructed in Theorem 4.6 recover those constructed in [12] and [17]. This gives a unified approach to the constructions developed in [12, 17].

The initial impetus for this paper was an observation of Hovey which appeared in an early version of [9]. Corollary 3.3 is a modification and generalization of that observation. This work continues the study begun in [19] on uniqueness properties of models of spectra where the monoidal product was ignored. The construction of the functors in the simplicial case, see Section 4, builds on the special cases developed in [12] and [17]. The construction of functors in the non-simplicial case builds on the treatment of cosimplicial frames in [19]. The new ingredient is a monoidal product on cosimplicial frames that has not been considered before.

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2. Model category preliminaries

In this section we recall the relevant definitions. A monoidal model category is a model category with a compatible symmetric monoidal product. Note that we do require the product to be symmetric even though that term is suppressed in the name 'monoidal model category'. The compatibility is expressed by the pushout product axiom below. This compatibility is analogous to the simplicial axiom of [14, Chapter II §2]. In particular, the product on a monoidal model category induces a derived product on the homotopy category which is symmetric monoidal. Monoidal model categories have been studied in [18] and [7]. Here, instead of requiring a closed monoidal structure, we use the weaker hypotheses that the product commutes with colimits.

Definition 2.1. A model category $\mathcal{C}$ is a *monoidal model category* if it is endowed with a symmetric monoidal structure and satisfies the following pushout product axiom and unit axiom. We denote the symmetric monoidal product by $\wedge$ and the unit by $\mathbb{I}$.

Pushout product axiom. Let $i\colon A \longrightarrow B$ and $j\colon X \longrightarrow Y$ be cofibrations in $\mathcal{C}$. Then the map

$$i \,\Box\, j\colon\ A \wedge L \cup_{A\wedge K} B \wedge K \longrightarrow B \wedge L$$

is a cofibration which is a weak equivalence if either i or j is a weak equivalence.

Unit axiom. If the unit is not cofibrant then fix a cofibrant replacement $u\colon Q\mathbb{I} \longrightarrow \mathbb{I}$ which is a trivial fibration from a cofibrant object $Q\mathbb{I}$. Then for any cofibrant object X the map $u \wedge X\colon Q\mathbb{I} \wedge X \longrightarrow \mathbb{I} \wedge X$ is a weak equivalence.

Definition 2.2. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ between symmetric monoidal categories is *lax monoidal* if there is a map $\eta\colon \mathbb{I}_{\mathcal{D}} \longrightarrow F(\mathbb{I}_{\mathcal{C}})$ and a transformation $\phi\colon FA \wedge_{\mathcal{D}} FB \longrightarrow F(A \wedge_{\mathcal{C}} B)$, natural in both variables, such that the coherence diagrams for commutativity, associativity, and unital properties commute. The functor F is *strong monoidal* if η and ϕ are isomorphisms.

Similarly, a *simplicial model category* is a model category with a compatible action of simplicial sets. A simplicial functor is a functor that is compatible with this structure. See [14, Chapter II §1, 2].

Next we define the appropriate equivalences of model categories and monoidal model categories.

Definition 2.3. A pair of adjoint functors between model categories is a *Quillen adjoint pair* if the right adjoint preserves trivial fibrations and fibrations between fibrant objects. This is equivalent to the usual definition [7, Definition 1.3.1] by [2, Corollary A.2]. A Quillen adjoint pair induces adjoint total derived functors between the homotopy categories [14, Chapter I §4 Theorem 3]. A Quillen functor pair is a *Quillen equivalence* if the total derived functors are adjoint equivalences of the homotopy categories. A *monoidal Quillen equivalence* is a Quillen equivalence between monoidal model categories with a strong monoidal left adjoint functor L such that $L(Q\mathbb{I}) \longrightarrow L(\mathbb{I})$ is a weak equivalence. If one functor in an adjoint equivalence is strong monoidal then so is the other, so both the left and right total derived functors of a monoidal Quillen equivalence are strong monoidal. Similarly, a *monoidal equivalence* is an equivalence via strong monoidal functors.

In this paper we actually consider only stable model categories. Recall from [14, Chapter I §2] or [7, Definition 6.1.1] that the homotopy category of a pointed model category supports a suspension functor Σ with a right adjoint loop functor Ω.

Definition 2.4. A *stable model category* is a pointed, complete and cocomplete category with a model category structure for which the functors Ω and Σ on the homotopy category are inverse equivalences.

Certain extra structures on the homotopy category of a stable model category are key here. The homotopy category is naturally a triangulated category [22]. The suspension functor defines the shift functor and the cofiber sequences of [14, Chapter I §3] define the distinguished triangles (the fiber sequences agree up to sign [7, Theorem 7.1.11]); see [7, Proposition 7.1.6] for more details. We have required a stable model category to have all limits and colimits so that its homotopy category has infinite sums and products. The homotopy category of a stable model category also has a natural action of the ring π_*^s of stable homotopy groups of spheres [19, Construction 2.3]. If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is the left adjoint of a Quillen adjoint pair between stable model categories, then the total left derived functor $LF : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ of F is π_*^s-linear and an exact functor [19, Lemma 6.1], [7, Proposition 6.4.1].

For objects A and X of a triangulated category $\mathcal{T}$ we denote by $[A, X]_*^{\mathcal{T}}$ the graded abelian homomorphism group defined by $[A, X]_m^{\mathcal{T}} = [A[m], X]^{\mathcal{T}}$ for $m \in \mathbb{Z}$, where $A[m]$ is the m-fold shift of A. If $\mathcal{T}$ is a π_*^s-triangulated category, then the groups $[A, X]_*^{\mathcal{T}}$ form a graded π_*^s-module.

Definition 2.5. An object G of a triangulated category $\mathcal{T}$ is called a *weak generator* if it detects isomorphisms; i.e., a map $f : X \longrightarrow Y$ is an isomorphism if and only if it induces an isomorphism between the graded abelian homomorphism groups $\mathcal{T}(G, X)_*$ and $\mathcal{T}(G, Y)_*$. An object G of $\mathcal{T}$ is *small* if for any family of objects $\{A_i\}_{i \in I}$ whose coproduct exists the canonical map

$$\bigoplus_{i \in I} \mathcal{T}(G, A_i) \longrightarrow \mathcal{T}(G, \coprod_{i \in I} A_i)$$

is an isomorphism.

3. Margolis' uniqueness conjecture

In this section we apply our monoidal uniqueness results to Margolis' conjecture about the uniqueness of the stable homotopy category. Margolis introduced axioms for a stable homotopy category in [13]. He then conjectured that these axioms uniquely specify the stable homotopy category of spectra up to a monoidal, triangulated equivalence of categories. In [19], any stable homotopy category satisfying Margolis' axioms and having an underlying model category was shown to be triangulated equivalent to the stable homotopy category of spectra. Here we strengthen that result to a monoidal, triangulated equivalence.

First we consider a more general setting than Margolis' stable homotopy categories. The following proposition shows that under weak hypotheses a lax monoidal functor between two monogenic, monoidal, triangulated categories is strong monoidal.

Definition 3.1. A *monogenic, monoidal, triangulated category* is a triangulated category $\mathcal{T}$ with arbitrary coproducts and with a symmetric monoidal, bi-exact smash product $\wedge$ which commutes with suspensions and coproducts such that the unit $\mathbb{I}$ is a small, weak generator.

Proposition 3.2. *Assume $(\mathcal{T}, \wedge, \mathbb{I})$ and $(\mathcal{T}', \wedge', \mathbb{I}')$ are two monogenic, monoidal, triangulated categories. Suppose that $F\colon \mathcal{T} \longrightarrow \mathcal{T}'$ is a lax monoidal, exact functor with unit map $\eta\colon \mathbb{I}' \longrightarrow F(\mathbb{I})$ and natural transformation $\phi\colon FA \wedge' FB \longrightarrow F(A \wedge B)$. If F commutes with coproducts, η is an isomorphism and $\phi\colon F\mathbb{I} \wedge' F\mathbb{I} \longrightarrow F(\mathbb{I} \wedge \mathbb{I})$ is an isomorphism, then F is a strong monoidal.*

Proof. Consider the subcategory of objects A in $\mathcal{T}$ such that $\phi\colon FA \wedge' F\mathbb{I} \longrightarrow F(A \wedge \mathbb{I})$ is an isomorphism. By the assumptions on F, $\wedge$ and $\wedge'$, both source and target commute with triangles and coproducts. So this subcategory is a localizing subcategory which contains $\mathbb{I}$. Since $\mathbb{I}$ is a small, weak generator it follows that this subcategory is the whole category. This follows from [8, Theorem 2.3.2]; see also [20, Lemma 2.2.1]. Now fix any A and consider the subcategory of objects B in $\mathcal{T}$ such that $\phi\colon FA \wedge' FB \longrightarrow F(A \wedge B)$ is an isomorphism. Again this is a localizing subcategory which contains $\mathbb{I}$, and hence it is the whole category. Thus, ϕ is an isomorphism for any A and B. □

Since the stable homotopy category of spectra is a monogenic, monoidal, triangulated category, this gives the following corollary.

Corollary 3.3. *Assume that $\wedge$ and $\wedge'$ are two monogenic, monoidal, triangulated structures on the homotopy category of spectra, both with unit the sphere spectrum, $\mathbb{S}$. If the identity functor is lax monoidal and the unit map η and the natural transformation ϕ evaluated on the unit are isomorphisms, then the identity functor gives a monoidal equivalence between these two structures.*

So the only obstruction to showing that the smash product of spectra is unique up to monoidal equivalence on the homotopy category is constructing a natural transformation between any two monoidal products. Our solution is to assume there is an underlying model category satisfying the hypotheses of Theorem 6.2. We state this result for Margolis' stable homotopy categories.

Definition 3.4. A *stable homotopy category* in the sense of [13, Chapter 2 §1] is a monogenic, monoidal, triangulated category $\mathscr{S}$ with an exact and strong symmetric monoidal equivalence $R : \mathcal{SW}_{\mathrm{f}} \longrightarrow \mathscr{S}^{\mathrm{small}}$ between the Spanier-Whitehead category of finite CW-complexes ([21], [13, Chapter 1, §2]) and the full subcategory of small objects in $\mathscr{S}$.

As shown in [19, Section 3], such an equivalence induces a π_*^s-linear structure on the triangulated category $\mathscr{S}$. In fact, we could weaken the definition above to only require that there is such an equivalence R with the full subcategory of $\mathcal{SW}$ on the spheres S^n for n an integer.

Definition 3.5. A stable homotopy category $\mathscr{S}$ *has an underlying model category* if there is a monoidal, π_*^s-linear, exact equivalence $\Phi\colon \mathscr{S} \longrightarrow \mathrm{Ho}(\mathcal{C})$ with $\mathcal{C}$ a stable, monoidal model category with a good desuspension of the unit (see Definition 6.1) such that $\wedge$ commutes with coproducts.

Proof of Theorem 1.3. Since $\mathscr{S}$ has an underlying model category, there is an equivalence $\Phi\colon \mathscr{S} \longrightarrow \mathrm{Ho}(\mathcal{C})$ with all of the properties mentioned in Definition 3.5. Since the properties of a small, weak generator are determined on the homotopy category level, the image $X \in \mathrm{Ho}(\mathcal{C})$ under Φ of the unit object in $\mathscr{S}$ is a small weak generator of the homotopy category of $\mathcal{C}$. Because the equivalence Φ is monoidal and π_*^s-linear, X is isomorphic to the unit and satisfies the hypotheses on the unit in Corollary 1.2. Thus, the homotopy category of $\mathcal{C}$, and hence also $\mathscr{S}$, is monoidally equivalent to the ordinary stable homotopy category of spectra. □

4. Simplicial monoidal model categories

Throughout this section we assume that $\mathcal{C}$ is a simplicial, stable, monoidal model category. Here, under one more technical hypothesis, we construct a Quillen adjoint pair between $\mathcal{C}$ and the positive stable model category on Sp^Σ. Then under hypotheses from [19] we conclude that this produces a monoidal Quillen equivalence between $\mathcal{C}$ and Sp^Σ. First we recall the positive model structure from [12].

Definition 4.1. In the positive stable model structure on Sp^Σ a map f is a weak equivalence if it is a stable equivalence, [9, 12]. The map f is a positive trivial fibration if f_n is a trivial fibration for $n > 0$. The positive cofibrations and positive fibrations are then determined by the respective right and left lifting properties with respect to the trivial fibrations and the trivial cofibrations.

In [12] only symmetric spectra over topological spaces are considered, but the arguments can be easily modified to apply to symmetric spectra over simplicial sets. The identity functor from the usual to the positive structure is a right Quillen functor since (trivial) fibrations are in particular positive (trivial) fibrations.

Theorem 4.2. [12, Theorem 14.2, Proposition 14.6] *The positive stable model structure on Sp^{Σ} forms a stable, monoidal model category. The identity functor induces a monoidal Quillen equivalence between the positive stable model structure and the usual stable model structure on Sp^{Σ}.*

Denote the unit in Sp^{Σ} by $\mathbb{S}$. Note $\mathbb{S}$ is not cofibrant in the positive stable model category. To fix its cofibrant replacement for the unit axiom of the monoidal model category structure, first consider the nth evaluation functor Ev_n on symmetric spectra which lands in Σ_n-equivariant spaces. For X a Σ_n-space, the left adjoint F'_n is defined by $(F'_n X)_k \cong \Sigma_{n+k} \wedge_{\Sigma_n \times \Sigma_k} (X \wedge S^k)$. This is a slight variant of the free functor F_n studied in [9]. Note that $F'_1 \cong F_1$ and $F'_0 \cong F_0$. Then define the cofibrant replacement of $\mathbb{S}$ as the weak equivalence $Q\mathbb{S} = F'_1 S^1 \longrightarrow F_0 S^0 = \mathbb{S}$ induced by the identity map in level one.

Proposition 4.3. *The fibrant objects in the positive stable model structure are the positive Ω-spectra. That is, X is fibrant if X_n is fibrant for $n > 0$ and $X_n \longrightarrow \Omega X_{n+1}$ is a weak equivalence for $n > 0$. A map f between positive Ω-spectra is a fibration if each f_n is a fibration for $n > 0$.*

Proof. The description of the fibrant objects follows from [12, Theorem 14.2]. The description of the fibrations follows from the fact that the positive stable model structure is a localization of the positive level model structure [12, Theorem 14.1]. In a localized model structure the fibrations between fibrant objects are the fibrations in the original model structure. So here they are the positive level fibrations. This statement also follows from the positive variants of [9, Lemma 3.4.12] or [12, Proposition 9.5]. □

The construction of a monoidal adjoint pair of functors requires one other technical hypothesis.

Definition 4.4. A *simplicial desuspension of the unit* is a cofibrant object $\mathbb{I}_c^{-1}$ with a weak equivalence $\eta\colon\ \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$.

Remark 4.5. If the unit $\mathbb{I}$ in $\mathcal{C}$ is fibrant, then a simplicial desuspension exists. Since $\mathcal{C}$ is stable, there is a cofibrant object X whose suspension in the homotopy category is isomorphic to $\mathbb{I}$. Since $\mathcal{C}$ is simplicial and X is cofibrant its suspension is modeled by $X \otimes S^1$. Since $X \otimes S^1$ is cofibrant and $\mathbb{I}$ is fibrant the isomorphism in the homotopy category is realized by some weak equivalence in $\mathcal{C}$.

A simplicial desuspension of the unit exists in every known symmetric monoidal model category of spectra. In the diagram categories of spectra investigated in [12] and their simplicial analogues [9, 10], (orthogonal spectra, symmetric spectra, and simplicial functors or $\mathscr{W}$-spaces) the simplicial desuspension can be chosen as the object denoted $F_1 S^0$, with the weak equivalence $\eta\colon\ F_1 S^1 \longrightarrow F_0 S^0$, see [12, Definition 1.3, Remark 4.7]. The S-modules of [5] are all fibrant, so the previous paragraph applies.

Recall that a monoidal model category is a model category $\mathcal{C}$ with a symmetric monoidal product that is compatible with the model structure; see Definition 2.1. For the next statement we require that the simplicial action commutes with the monoidal product. That is, for X, Y in $\mathcal{C}$ and K in $\mathcal{S}_*$ there are natural coherent isomorphisms $(X \wedge Y) \otimes K \cong X \wedge (Y \otimes K)$.

Theorem 4.6. *Let $\mathcal{C}$ be a stable, simplicial monoidal model category with a simplicial desuspension of the unit such that the monoidal product commutes with the simplicial action. Then there exists a Quillen adjoint functor pair between $\mathcal{C}$ and the positive stable model structure on Sp^{Σ},*

$\mathbb{I} \wedge -\colon Sp^\Sigma \longrightarrow \mathcal{C}$ and $\mathrm{Hom}(\mathbb{I}, -)\colon \mathcal{C} \longrightarrow Sp^\Sigma$. These functors are simplicial, the left adjoint $\mathbb{I} \wedge -$ is strong monoidal, and $\mathbb{I} \wedge Q\mathbb{S} \longrightarrow \mathbb{I} \wedge \mathbb{S}$ is a weak equivalence.

Remark 4.8 below shows that the existence of such a Quillen adjoint pair implies the existence of a simplicial desuspension. Combining Theorem 4.6 with results from [19] gives the following corollary.

Corollary 4.7. *Let $\mathcal{C}$ be a simplicial, stable, monoidal model category with a simplicial desuspension of the unit such that the monoidal product commutes with the simplicial action. The following conditions are equivalent:*

1. *There is a π_*^s-linear triangulated equivalence between the homotopy category of $\mathcal{C}$ and the homotopy category of Sp^Σ which takes the unit $\mathbb{I}$ of the monoidal product in $\mathcal{C}$ to the unit $\mathbb{S}$ of Sp^Σ.*
2. *The unit, $\mathbb{I}$, is a small weak generator for which $[\mathbb{I}, \mathbb{I}]_*^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by the identity map of $\mathbb{I}$.*
3. *There is a simplicial, monoidal Quillen equivalence between $\mathcal{C}$ and the positive stable model structure on Sp^Σ.*
4. *There is a zig-zag of monoidal Quillen equivalences between $\mathcal{C}$ and the usual stable model structure on Sp^Σ.*

Proof. Condition (1) implies condition (2) since the properties of $\mathbb{I}$ mentioned in (2) hold for $\mathbb{S}$ and are determined by the π_*^s-linear triangulated homotopy category. Condition (3) implies condition (4) because the positive stable model structure is monoidally Quillen equivalent to the usual stable model structure on Sp^Σ by Theorem 4.2. Since Quillen functors induce π_*^s-linear triangulated functors on the homotopy categories by [7, Proposition 6.4.1] and [19, Lemma 6.1] and monoidal functors preserve the unit, condition (4) implies condition (1).

Next we show that given condition (2) the simplicial Quillen adjoint pair constructed in Theorem 4.6 is a Quillen equivalence. Since $\mathbb{I} \wedge -$ is strong monoidal, $\mathbb{I} \wedge \mathbb{S} \cong \mathbb{I}$. Also, $\mathbb{I} \wedge Q\mathbb{S} \longrightarrow \mathbb{I} \wedge \mathbb{S}$ is a weak equivalence, so $\mathbb{I} \wedge^L \mathbb{S} \cong \mathbb{I}$. The total derived functor $\mathbb{I} \wedge^L -$ is exact by [7, Proposition 6.4.1]. So $\mathbb{I} \wedge^L \mathbb{S}[n] \cong \mathbb{I}[n]$ where $X[n]$ denotes the nth shift of X for any integer n. This isomorphism and the derived adjunction for $\mathbb{I} \wedge^L -$ and $\mathrm{RHom}(\mathbb{I}, -)$ produce the following natural isomorphisms

$$\pi_* \mathrm{RHom}(\mathbb{I}, Y) \cong [\mathbb{S}[*], \mathrm{RHom}(\mathbb{I}, Y)] \cong [\mathbb{I}, Y]_*^{\mathrm{Ho}(\mathcal{C})}.$$

Since $\mathbb{I}$ is a weak generator, $\mathrm{RHom}(\mathbb{I}, -)$ detects isomorphisms. So to show that this pair is a Quillen equivalence we need to show that for any symmetric spectrum A the unit of the adjunction $A \longrightarrow \mathrm{RHom}(\mathbb{I}, \mathbb{I} \wedge^L A)$ is an isomorphism. Consider the full subcategory $\mathcal{T}$ of such objects. Note that for $A = \mathbb{S}$ in homotopy this map is the map $[\mathbb{S}, \mathbb{S}]_* \longrightarrow [\mathbb{I}, \mathbb{I}]_*$ induced by $\mathbb{I} \wedge^L -$. This map of free π_*^s-modules takes the identity map of $\mathbb{S}$ to the identity map of $\mathbb{I}$. Hence it is also an isomorphism by condition (2). So $\mathbb{S}$ is contained in $\mathcal{T}$. Since $\mathbb{I}$ is small, $\mathrm{RHom}(\mathbb{I}, -)$ commutes with coproducts by the display above. Hence, since left adjoints commute with coproducts and total derived functors between stable model categories are exact, the composite $\mathrm{RHom}(\mathbb{I}, \mathbb{I} \wedge^L -)$ is an exact functor which commutes with coproducts. So $\mathcal{T}$ is a localizing subcategory which contains the generator $\mathbb{S}$ of symmetric spectra. Hence $\mathcal{T}$ is the whole category. Thus, these derived functors induce an equivalence of homotopy categories. □

Proof of Theorem 4.6. We first construct the functor $\mathrm{Hom}(\mathbb{I}, -)\colon \mathcal{C} \longrightarrow Sp^\Sigma$. Let $\mathbb{I}_c^{-1}$ be a simplicial desuspension of the unit in $\mathcal{C}$ with a weak equivalence $\eta\colon \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$. Let $\mathbb{I}_c^{-n} = (\mathbb{I}_c^{-1})^{\wedge n}$ be the n-fold smash product of $\mathbb{I}$ where $X^0 = \mathbb{I}$. Notice that in general $\mathbb{I}_c^0$ is not cofibrant. For Y in $\mathcal{C}$, define the nth level of $\mathrm{Hom}(\mathbb{I}, Y)_n$ to be the simplicial mapping space $\mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-n}, Y)$.

The symmetric group on n letters acts on $\mathbb{I}_c^{-n}$ by permuting the factors and hence also acts on $\mathrm{Hom}(\mathbb{I}, Y)_n$. The structure map

$$\mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-n}, Y) \longrightarrow \Omega^m \,\mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-(n+m)}, Y) \cong \mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-(n+m)} \otimes S^m, Y)$$

is induced by $\mathrm{map}_{\mathcal{C}}(\sigma, Y)$ with σ defined as

$$\sigma_{n,m}\colon\ \mathbb{I}_c^{-(n+m)} \otimes S^m \cong \mathbb{I}_c^{-n} \wedge (\mathbb{I}_c^{-1} \otimes S^1)^m \overset{\mathrm{id}\wedge(\eta)^m}{\longrightarrow} \mathbb{I}_c^{-n} \wedge (\mathbb{I})^m \cong \mathbb{I}_c^{-n}.$$

Since the adjoint of $\mathrm{map}_{\mathcal{C}}(\sigma, Y)$ is $\Sigma_n \times \Sigma_m$ equivariant, this makes $\mathrm{Hom}(\mathbb{I}, Y)$ into a symmetric spectrum. Here we have used the fact that the simplicial action and the monoidal product commute.

Since $\mathcal{C}$ is a simplicial model category and $\mathbb{I}_c^{-n}$ is cofibrant for $n > 0$, $\mathrm{Hom}(\mathbb{I}, -)$ of a (trivial) fibration is a (trivial) fibration in levels $n > 0$. Since $\mathbb{I}_c^{-1} \otimes S^1$ is cofibrant η factors as $\mathbb{I}_c^{-1} \otimes S^1 \longrightarrow Q\mathbb{I} \longrightarrow \mathbb{I}$ where $Q\mathbb{I} \longrightarrow \mathbb{I}$ is the fixed cofibrant replacement of $\mathbb{I}$ given in the monoidal model structure on $\mathcal{C}$. Since $\mathcal{C}$ is monoidal and η is a weak equivalence, $\sigma_{n,1}$ is a weak equivalence between cofibrant objects for $n > 0$. Hence $\mathrm{Hom}(\mathbb{I}, -)$ takes a fibrant object to a positive Ω-spectrum, which is a fibrant object in the positive stable model structure. Thus $\mathrm{Hom}(\mathbb{I}, -)$ takes trivial fibrations to positive trivial fibrations and fibrations to positive fibrations between positive fibrant objects by Proposition 4.3. So $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen adjoint.

Next we consider the left adjoint $\mathbb{I} \wedge -\colon Sp^{\Sigma} \longrightarrow \mathcal{C}$. Using the definition of $F'_n X$, $\mathbb{I} \wedge F'_n X$ is isomorphic to $\mathbb{I}_c^{-n} \otimes_{\Sigma_n} X$ since both corepresent the functor which takes Y in $\mathcal{C}$ to the space of Σ_n-equivariant maps from X to $\mathrm{Hom}(\mathbb{I}, Y)_n$. So $\mathbb{I} \wedge Q\mathbb{S} \longrightarrow \mathbb{I} \wedge \mathbb{S}$ is isomorphic to the weak equivalence $\eta\colon\ \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$.

To evaluate $\mathbb{I} \wedge -$ on an arbitrary symmetric spectrum A, note that A can be built as the coequalizer of the following diagram:

$$\bigvee_n F'_{n+1}(\Sigma_{n+1} \wedge_{\Sigma_n} (A_n \wedge S^1)) \implies \bigvee_n F'_n A_n$$

Here one map is induced by the map $A_n \wedge S^1 \longrightarrow A_{n+1}$ and the other is induced by smashing $F'_n A_n$ with the map $F'_1 S^1 \longrightarrow F'_0 S^0$ which is the adjoint of the identity map on S^1 in level one. Since $\mathbb{I} \wedge -$ must commute with colimits, $\mathbb{I} \wedge A$ is defined as the coequalizer of the diagram:

$$\bigvee_n \mathbb{I}_c^{-(n+1)} \otimes_{\Sigma_n} (A_n \wedge S^1) \implies \bigvee_n \mathbb{I}_c^{-n} \otimes_{\Sigma_n} A_n$$

Again the first map is induced by $A_n \wedge S^1 \longrightarrow A_{n+1}$ and the second map uses the fact that the simplicial action and monoidal product in $\mathcal{C}$ commute to give the isomorphism

$$\mathbb{I}_c^{-(n+1)} \otimes_{\Sigma_n} (A_n \wedge S^1) \cong (\mathbb{I}_c^{-n} \otimes_{\Sigma_n} A_n) \wedge (\mathbb{I}_c^{-1} \otimes S^1)$$

along with the map $\eta\colon\ \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$.

Next we consider the monoidal properties of these adjoint functors. First $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal; since the simplicial action and monoidal product commute the product of maps induces $\mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-n}, A) \wedge \mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-m}, B) \longrightarrow \mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-(n+m)}, A \wedge B)$. These fit together to give a natural map $\mathrm{Hom}(\mathbb{I}, A) \wedge \mathrm{Hom}(\mathbb{I}, B) \longrightarrow \mathrm{Hom}(\mathbb{I}, A \wedge B)$. The unit map $F'_0 S^0 = \mathbb{S} \longrightarrow \mathrm{Hom}(\mathbb{I}, \mathbb{I})$ is given by sending the non-base point of S^0 to the identity map of $\mathbb{I}$ in simplicial degree zero of $\mathrm{Hom}(\mathbb{I}, \mathbb{I})_0 = \mathrm{map}_{\mathcal{C}}(\mathbb{I}, \mathbb{I})$.

The left adjoint of a lax monoidal functor is automatically lax comonoidal. The adjoint of the unit map is an isomorphism $\mathbb{I} \wedge \mathbb{S} \longrightarrow \mathbb{I}$. Denote the adjoint pair by L and R. Then the counit and unit of the adjunction and the lax monoidal structure of R give

$$L(A \wedge B) \longrightarrow L(RLA \wedge RLB) \longrightarrow LR(LA \wedge LB) \longrightarrow LA \wedge LB.$$

Here in fact $L = \mathbb{I} \wedge -$ is strong monoidal because this map is an isomorphism. To show this we only need to consider the special case where $A = F'_n X$ and $B = F'_m Y$ for X a Σ_n-space and Y a Σ_m-space since the general case follows by using the coequalizer diagrams above. Then

$$L(A \wedge B) = LF'_{n+m}(\Sigma_{n+m} \wedge_{\Sigma_n \times \Sigma_m} X \wedge Y) \cong \mathbb{I}_c^{-(n+m)} \otimes_{\Sigma_n \times \Sigma_m} X \wedge Y.$$

Again commuting the simplicial action and the monoidal product shows this last term is isomorphic via the transformation displayed above to $(\mathbb{I}_c^{-n} \otimes_{\Sigma_n} X) \wedge (\mathbb{I}_c^{-m} \otimes_{\Sigma_m} Y) = LA \wedge LB$.

Finally, these adjoint functors $\mathrm{Hom}(\mathbb{I}, -)$ and $\mathbb{I} \wedge -$ are simplicial functors. This follows by various adjunctions from the isomorphism $\mathrm{Hom}(\mathbb{I}, Y^K) \cong \mathrm{Hom}(\mathbb{I}, Y)^K$ given by the simplicial structure on $\mathcal{C}$. □

Remark 4.8. If there is a Quillen adjoint pair between $\mathcal{C}$ and the positive stable model structure on Sp^Σ with a strong monoidal, simplicial left adjoint L which takes $Q\mathbb{S} \longrightarrow \mathbb{S}$ to a weak equivalence, then a simplicial desuspension of the unit exists. Set $\mathbb{I}_c^{-1} = L(F'_1 S^0)$. The map $\eta\colon \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$ is then given by $L(F'_1 S^0) \otimes S^1 \longrightarrow L(F'_1 S^1) \longrightarrow L(F_0 S^0)$. Since $F'_1 S^1 \cong Q\mathbb{S}$, the second map is a weak equivalence. The first map is a weak equivalence because it is the cofiber of the weak equivalence $L(F_1 S^0) \otimes \Delta[1]_+ \longrightarrow L(F_1 S^0 \otimes \Delta[1]_+)$ by the isomorphism $L(F_1 S^0) \otimes (S^0 \vee S^0) \longrightarrow L(F_1 S^0 \otimes (S^0 \vee S^0))$.

This also gives a procedure for recovering the known equivalences between symmetric monoidal model categories of spectra as $\mathbb{I} \wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$ for some choice of a simplicial desuspension of the unit. For the monoidal functors constructed in [12] ($\mathbb{P}$ and $\mathbb{U}$ between orthogonal spectra and symmetric spectra and between $\mathscr{W}$-spaces and symmetric spectra), the chosen desuspension of the unit is $\mathbb{P}(F_1 S^0) \cong F_1 S^0$ [12, Definition 1.3, Remark 4.7]. The monoidal functors (Λ and Φ) between S-modules and symmetric spectra as defined in [17] are isomorphic to $\mathbb{I} \wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$ since $\Lambda(F_1 S^0) \cong S_c^{-1}$.

Remark 4.9. If $\mathcal{C}$ is a cofibrantly generated, proper, stable model category then [16, Proposition 4.4] shows that $\mathcal{C}$ is Quillen equivalent to a simplicial model category structure on the category of simplicial objects, $\mathcal{C}^{\Delta^{\mathrm{op}}}$. If the product on $\mathcal{C}$ commutes with coproducts then the level prolongation of the product commutes with the simplicial action. Using [6, Proposition 16.11.1, Theorem 16.4.2], one can show that if $\mathcal{C}$ is a monoidal model category then the simplicial model category from [16] is also monoidal. Hence, under these conditions, one can apply the constructions in this section to the simplicial, stable, monoidal model category on $\mathcal{C}^{\Delta^{\mathrm{op}}}$. This remark can also be applied if $\mathcal{C}$ is simplicial and the product does not commute with the simplicial action but does commute with coproducts. We treat the non-simplicial case in even more generality in Section 6.

5. Modules and Algebras

In this section, we show that the functors constructed in Theorem 4.6 induce Quillen adjoint pairs on modules and algebras. Since $\mathbb{I} \wedge -$ is strong monoidal and $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal, these functors restrict to adjoint functors on subcategories of modules and algebras. Since we want the restriction of $\mathrm{Hom}(\mathbb{I}, -)$ to be a right Quillen adjoint, we assume that in the model structures on categories of modules or algebras over $\mathcal{C}$ a morphism is a weak equivalence or fibration if it is one in the underlying model structure on $\mathcal{C}$. The next proposition states sufficient conditions for this assumption to hold.

Proposition 5.1. [18, Theorem 4.1] *Assume $\mathcal{C}$ is a cofibrantly generated, monoidal model category that satisfies the monoid axiom* [18, Definition 3.3]*. If the objects in $\mathcal{C}$ satisfy certain smallness conditions* [18, Lemma 2.3]*, then the category of left R-modules (for a fixed monoid*

R) and the category of R-algebras (for a fixed commutative monoid R) are model categories with fibrations and weak equivalences determined in $\mathcal{C}$.

Theorem 5.2. *Assume $\mathcal{C}$ satisfies the hypotheses and condition (2) of Corollary 4.7 and that the conclusion of the previous proposition holds. If $\mathbb{I} \wedge -$ preserves weak equivalences between stably cofibrant symmetric spectra, then $\mathbb{I} \wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$ induce a Quillen equivalence between*

1. *$(\mathbb{I} \wedge R)$-modules and the positive stable model category of R-modules for R a cofibrant symmetric ring spectrum, and*
2. *$(\mathbb{I} \wedge R)$-algebras and the positive stable model category of R-algebras for R a commutative symmetric ring spectrum which is cofibrant as a symmetric spectrum.*

These statements also hold with the usual stable model category replacing the positive one if $\mathbb{I}$ is cofibrant.

Note that since $\mathbb{S}$ is cofibrant as a symmetric spectrum the last statement implies that the category of symmetric ring spectra, $\mathbb{S}$-algebras, and the category of monoids in $\mathcal{C}$, $\mathbb{I}$-algebras, are Quillen equivalent. We have not considered categories of commutative algebras here because their analysis is more technical to generalize and is treated for all known cases in [12, §16] and [17, Theorem 5.1].

Remark 5.3. The hypothesis that $\mathbb{I} \wedge -$ preserves weak equivalences between stably cofibrant symmetric spectra is satisfied when $\mathcal{C}$ is any one of the symmetric monoidal model categories of orthogonal spectra, $\mathscr{W}$-spaces [12], simplicial functors [10], or S-modules [5]. Since the unit is cofibrant in the first three cases, this follows from the next proposition. This holds in the case of S-modules by [17, Theorem 3.1] and the fact that $\mathrm{Hom}(\mathbb{I}, -)$ detects and preserves weak equivalences.

Proposition 5.4. *If $\mathbb{I}$ is cofibrant and $\mathcal{C}$ satisfies the hypotheses and condition (2) of Corollary 4.7, then $\mathbb{I} \wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$ form a Quillen equivalence between $\mathcal{C}$ and the usual stable model category of symmetric spectra. Hence $\mathbb{I} \wedge -$ preserves weak equivalences between cofibrant symmetric spectra.*

Proof. If $\mathbb{I}$ is cofibrant, then $\mathrm{Hom}(\mathbb{I}, -)_0 = \mathrm{map}(\mathbb{I}, -)$ also preserves (trivial) fibrations. Hence $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen adjoint functor from $\mathcal{C}$ to the usual stable model category of symmetric spectra. The statements follow from the same proof as given in Corollary 4.7. ☐

Proof of Theorem 5.2. Since the (trivial) fibrations in the categories of $(\mathbb{I} \wedge R)$-modules and $(\mathbb{I} \wedge R)$-algebras are determined on the underlying category, the restriction of $\mathrm{Hom}(\mathbb{I}, -)$ in both cases is still a right Quillen adjoint functor to the positive model structure. Since $\mathbb{I}$ is assumed to be a weak generator by condition (2) of Corollary 4.7, $\mathrm{Hom}(\mathbb{I}, -)$ preserves and detects weak equivalences. So by [9, Lemma 4.1.7] we only need to show that $\psi_A\colon A \longrightarrow \mathrm{Hom}(\mathbb{I}, (\mathbb{I} \wedge A)^f)$ is a weak equivalence for A a positive cofibrant object in R-modules or R-algebras where $(\mathbb{I} \wedge A)^f$ is a fibrant replacement. Note that since fibrations are determined on the underlying category a fibrant replacement as a module or algebra restricts to a fibrant replacement in $\mathcal{C}$.

Under the given conditions on R, if A is cofibrant in the positive model category of R-modules or R-algebras then A is cofibrant as a symmetric spectrum. By [12, Proposition 14.6] the identity functor from the positive stable model structure on R-modules to the usual stable model structure on R-modules is a Quillen left adjoint. So if A is a positive cofibrant R-module then it is a cofibrant R-module. Since R is assumed to be cofibrant as a symmetric ring spectrum it is cofibrant as a symmetric spectrum by [12, Theorem 12.1(v)]. Hence, by [12, Theorem 12.1(ii)], A is cofibrant as a symmetric spectrum. Again by [12, Proposition 14.6], if A is a positive cofibrant R-algebra, then it is a cofibrant R-algebra. Then by [12, Theorem 12.1 (ii), (v)] it follows that A is cofibrant as a symmetric spectrum.

We now show that ψ_B is a weak equivalence for B any cofibrant symmetric spectrum. It then follows that $\mathbb{I} \wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$ restrict to Quillen equivalences on the positive stable model categories of R-modules and R-algebras. The proof of Corollary 4.7 shows that ψ_A is a weak equivalence for A any positive cofibrant symmetric spectrum. Given a cofibrant symmetric spectrum B, choose a positive cofibrant replacement $\phi\colon cB \longrightarrow B$. Since $\mathbb{I} \wedge -$ preserves weak equivalences between cofibrant objects and positive cofibrant objects are cofibrant, $\mathbb{I} \wedge cB \longrightarrow \mathbb{I} \wedge B$ is a weak equivalence. Then one can choose fibrant replacements and a lift $(\mathbb{I} \wedge \phi)^f$ so that $\psi_B \circ \phi = \mathrm{Hom}(\mathbb{I}, (\mathbb{I} \wedge \phi)^f) \circ \psi_{cB}$. Thus, ψ_B is a weak equivalence, since $\mathrm{Hom}(\mathbb{I}, -)$ preserves weak equivalences between fibrant objects and ϕ and ψ_{cB} are weak equivalences.

Note that if $\mathbb{I}$ is cofibrant then $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen adjoint functor to the usual stable model structures. So the last statement follows similarly. □

6. Non-simplicial case

In this section we consider the case when the given stable, monoidal model category $\mathcal{C}$ is not simplicial. Under similar conditions as Corollary 4.7, we can still produce a monoidal equivalence of the homotopy category of $\mathcal{C}$ and the homotopy category of symmetric spectra. Since $\mathcal{C}$ is not simplicial, we need a new definition for a desuspension of the unit.

Definition 6.1. Let X be a cofibrant object in $\mathcal{C}$. A *cylinder object* for X is an object $X \times I$ with a factorization of the fold map $X \amalg X \xrightarrow{i} X \times I \xrightarrow{p} X$ such that i is a cofibration and p is a weak equivalence. A model for the *suspension*, ΣX, is the cofiber of $X \amalg X \xrightarrow{i} X \times I$ for some cylinder $X \times I$ [14, Chapter I §1, 2]. A *good desuspension* of the unit is a cofibrant object $\mathbb{I}_c^{-1}$ with a weak equivalence $\eta\colon \Sigma\mathbb{I}_c^{-1} \longrightarrow \mathbb{I}$ for some model of the suspension.

Theorem 6.2. *Let $\mathcal{C}$ be a stable, monoidal model category with a good desuspension of the unit such that $\wedge$ commutes with colimits. Then there is a Quillen adjoint pair between $\mathcal{C}$ and the positive stable model structure on Sp^Σ, again denoted by $\mathbb{I} \wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$, such that the total left derived functor $\mathbb{I} \wedge^L -$ is strong monoidal. Moreover, $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal, $\mathbb{I} \wedge \mathbb{S} \cong \mathbb{I}$, and $\mathbb{I} \wedge Q\mathbb{S} \longrightarrow \mathbb{I} \wedge \mathbb{S}$ is a weak equivalence.*

As with the simplicial desuspension, the existence of a functor with the properties listed for $\mathbb{I} \wedge -$ implies the existence of a good desuspension; see Remark 6.5. Again by combining this result with results from [19] we obtain the following corollary.

Corollary 6.3. *Let $\mathcal{C}$ be a stable, monoidal model category with a good desuspension of the unit such that $\wedge$ commutes with colimits. The following conditions are equivalent:*

1. *There is a π_*^s-linear triangulated equivalence between the homotopy category of $\mathcal{C}$ and the homotopy category of Sp^Σ which takes the unit $\mathbb{I}$ of the monoidal product in $\mathcal{C}$ to the unit $\mathbb{S}$ of Sp^Σ.*
2. *The unit, $\mathbb{I}$, is a small weak generator for which $[\mathbb{I}, \mathbb{I}]_*^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by the identity map of $\mathbb{I}$.*
3. *$\mathcal{C}$ and Sp^Σ are Quillen equivalent via functors whose derived functors are strong monoidal.*
4. *There is a π_*^s-linear, triangulated, monoidal equivalence between the homotopy category of $\mathcal{C}$ and the homotopy category of Sp^Σ.*

The proof of this corollary is similar to the proof of Corollary 4.7. The proof of the only non-trivial step, that condition (2) implies that the functors produced by Theorem 6.2 form a Quillen equivalence, is in fact the same as for Corollary 4.7.

To construct the right adjoint $\mathrm{Hom}(\mathbb{I}, -)$ we use *cosimplicial frames* since $\mathcal{C}$ is not simplicial. These *cosimplicial resolutions* were first used in [4] to construct function complexes on homotopy categories, but in [3] this theory has been extended to provide function complexes on model categories. Our main reference here is [7, Chapter 5], see also [19].

Given a cosimplicial object $X^\cdot$ in $\mathcal{C}^\Delta$ and a pointed simplicial set K denote the coend [11, Chapter IX §6] in $\mathcal{C}$ by $X^\cdot \otimes_\Delta K$ see also [7, Chapter 5 §7]. Define $X^\cdot \otimes K$ by $(X^\cdot \otimes K)^n = X^\cdot \otimes_\Delta (K \wedge \Delta[n]_+)$. Notice $X^\cdot \otimes K$ and $X^\cdot \otimes_\Delta K$ are objects in different categories ($\mathcal{C}^\Delta$ and $\mathcal{C}$ respectively.) Set $S^m = (S^1)^m$ and denote $X^\cdot \otimes S^m$ by $\Sigma^m(X^\cdot)$. If $X^\cdot$ is a cosimplicial object and Y is an object of $\mathcal{C}$ then $\mathcal{C}(X^\cdot, Y)$ is a simplicial set with degree n the set of $\mathcal{C}$-morphisms $\mathcal{C}(X^n, Y)$. There is an adjunction isomorphism $\mathcal{C}(X^\cdot \otimes K, Y) \cong \mathrm{map}(K, \mathcal{C}(X^\cdot, Y))$. This shows that $X^\cdot \otimes (K \wedge L) \cong (X^\cdot \otimes K) \otimes L$ since they both represent the same functor. In particular, $\Sigma^m(X^\cdot)$ is the mth iterated suspension of $X^\cdot$.

We consider the Reedy model category on $\mathcal{C}^\Delta$, the cosimplicial objects on $\mathcal{C}$ [15], [7, Theorem 5.2.5]. An object $A^\cdot$ is *Reedy cofibrant* if the map $A^\cdot \otimes \partial\Delta[k]_+ \longrightarrow A^\cdot \otimes \Delta[k]_+ \cong A^k$ is a cofibration for each k. A *cosimplicial frame* is then a Reedy cofibrant object of $\mathcal{C}^\Delta$ such that each of the codegeneracy and coface maps are weak equivalences. That is, cosimplicial frames are the Reedy cofibrant, homotopically constant objects. A cosimplicial frame A is called a cosimplicial frame of the cofibrant object A^0 in [7, Chapter 5].

The category $\mathcal{C}^\Delta$ has a symmetric monoidal product, defined on each level by the symmetric monoidal product on $\mathcal{C}$. That is, $(A^\cdot \wedge B^\cdot)^n \cong A^n \wedge B^n$. The following proposition collects several useful properties of the proceeding constructions.

Proposition 6.4. *Let $A^\cdot$ and $B^\cdot$ be cosimplicial frames in $\mathcal{C}^\Delta$ where $\mathcal{C}$ is a monoidal model category such that $\wedge$ commutes with colimits.*

1. *$A^\cdot \wedge B^\cdot$ is a cosimplicial frame.*
2. *$\Sigma A^\cdot$ is a cosimplicial frame.*
3. *There is a natural level equivalence $\Sigma(A^\cdot \wedge B^\cdot) \longrightarrow (\Sigma A^\cdot) \wedge B^\cdot$.*

Proof. For part 1, note that $A^\cdot \wedge B^\cdot$ is homotopically constant since the smash product of two maps which are each weak equivalences between cofibrant objects is a weak equivalence. The monoidal product also preserves Reedy cofibrant objects. By [6, Proposition 16.11.1], since $\wedge\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ preserves cofibrations and $\wedge$ commutes with colimits, the prolongation $\wedge\colon \mathcal{C}^\Delta \times \mathcal{C}^\Delta \longrightarrow \mathcal{C}^\Delta$ also preserves cofibrations. This uses [6, Theorem 16.4.2] to recognize that the Reedy model category on $\mathcal{C}^\Delta \times \mathcal{C}^\Delta$ agrees with the Reedy model category on $(\mathcal{C} \times \mathcal{C})^\Delta$.

For part 2, the map $\Sigma A^\cdot \otimes (\partial\Delta[k]_+ \longrightarrow \Delta[k]_+)$ is isomorphic to the map $A^\cdot \otimes (S^1 \wedge \partial\Delta[k]_+ \longrightarrow S^1 \wedge \Delta[k]_+)$. By [7, Proposition 5.7.1], if $A^\cdot$ is Reedy cofibrant then this map is a cofibration. So $\Sigma A^\cdot$ is Reedy cofibrant. Since each map $S^1 \wedge \Delta[n]_+ \longrightarrow S^1 \wedge \Delta[n+1]_+$ is a trivial cofibration, the coface maps of $\Sigma A^\cdot$ are trivial cofibrations [7, Proposition 5.7.2]. Since $s^i d^i = \mathrm{id}$ the codegeneracy maps are also weak equivalences.

For part 3, the coend defining $\Sigma(A^\cdot \wedge B^\cdot)^m$ is a colimit of copies of $(A^k \wedge B^k)$ indexed by the non-base point k-simplices of $S^1 \wedge \Delta[m]_+$. Use the map $k \longrightarrow m$ in Δ determined by the k-simplex of $\Delta[m]_+$ to induce a map $B^k \longrightarrow B^m$. These maps are all compatible and define a map $\Sigma(A^\cdot \wedge B^\cdot)^m \longrightarrow \Sigma(A^\cdot \wedge B^m)^m \cong \Sigma(A^\cdot)^m \wedge B^m \cong (\Sigma(A^\cdot) \wedge B^\cdot)^m$. Here $(A^\cdot \wedge B^m)^k \cong A^k \wedge B^m$ and we have used the fact that $\wedge$ commutes with colimits. Since parts 1 and 2 show that this is a map between cosimplicial frames, it is a level equivalence if degree zero is a weak equivalence. The map $A^\cdot \wedge B^\cdot \longrightarrow A^\cdot \wedge B^0$ is a level equivalence between Reedy cofibrant objects by part 1 and the monoidal model structure on $\mathcal{C}$. Hence $(A^\cdot \wedge B^\cdot) \otimes_\Delta S^1 \longrightarrow (A^\cdot \wedge B^0) \otimes_\Delta S^1 \cong (A^\cdot \otimes_\Delta S^1) \wedge B^0$ is a weak equivalence by [7, Proposition 5.7.1]. □

Proof of Theorem 6.2. To define the right adjoint $\mathrm{Hom}(\mathbb{I}, -)$ we consider cosimplicial frames related to $\mathbb{I}$. First, let $\omega^0\mathbb{I}$ be the constant cosimplicial object on $\mathbb{I}$. Since $\mathbb{I}$ is not necessarily cofibrant $\omega^0\mathbb{I}$ is not necessarily a cosimplicial frame. Since $\mathcal{C}$ has a good desuspension of the unit, one can build a cosimplicial frame $\omega^1\mathbb{I}$ of the cofibrant object $\mathbb{I}_c^{-1}$. The first two cosimplicial degrees of $\omega^1\mathbb{I}$ are determined by the cylinder object on $\mathbb{I}_c^{-1}$; define $(\omega^1\mathbb{I})^0 = \mathbb{I}_c^{-1}$ and $(\omega^1\mathbb{I})^1 = \mathbb{I}_c^{-1} \times I$. Define the coface maps as the two inclusions $X \longrightarrow X \amalg X \xrightarrow{i} X \times I$ and define the codegeneracy map as the map $X \times I \xrightarrow{p} X$. Using the factorization properties in $\mathcal{C}$ one can inductively define the higher levels of $\omega^1\mathbb{I}$, see the proof of [7, Theorem 5.1.3]. Since $\Sigma(X^{\cdot})$ is the cofiber of $X^{\cdot} \otimes (S^0 \vee S^0) \longrightarrow X^{\cdot} \otimes \Delta[1]_+$, $\Sigma(\omega^1\mathbb{I})^0$ is the cofiber of i, that is, a model for the suspension of $\mathbb{I}_c^{-1}$. Then the weak equivalence $\eta\colon\ \Sigma\mathbb{I}_c^{-1} \longrightarrow \mathbb{I}$ extends to a level equivalence $\eta^{\cdot}\colon\ \Sigma\omega^1\mathbb{I} \longrightarrow \omega^0\mathbb{I}$.

Define $\omega^n\mathbb{I} = (\omega^1\mathbb{I})^{\wedge n}$ for $n > 0$. Using these desuspensions of the unit we define the right adjoint $\mathrm{Hom}(\mathbb{I}, -)$ in level n to be $\mathcal{C}(\omega^n\mathbb{I}, -)$. The symmetric group on n letters acts here by permuting the factors of $\omega^n\mathbb{I}$. The structure maps are induced by the map η. Proposition 6.4 part 3 provides a level equivalence $\phi\colon\ \Sigma^m(A^{\wedge m} \wedge B) \longrightarrow (\Sigma A)^{\wedge m} \wedge B$. The isomorphism of $\Sigma^m(X^{\cdot})$ with the m-fold iterated suspension of $X^{\cdot}$ induces a $\Sigma_m \times \Sigma_n$-equivariant level equivalence where Σ_m acts trivially on the target:

$$\Sigma^m(\omega^{m+n}\mathbb{I}) \xrightarrow{\phi} (\Sigma\omega^1\mathbb{I})^{\wedge m} \wedge \omega^n\mathbb{I} \xrightarrow{(\eta^{\wedge m})\wedge \mathrm{id}} (\omega^0\mathbb{I})^{\wedge m} \wedge \omega^n\mathbb{I} \cong \omega^n\mathbb{I}.$$

Applying $\mathcal{C}(-, Z)$ to the displayed composition and taking adjoints gives the $\Sigma_m \times \Sigma_n$-equivariant structure map

$$S^m \wedge \mathcal{C}(\omega^n\mathbb{I}, Z) \longrightarrow \mathcal{C}(\omega^{m+n}\mathbb{I}, Z).$$

Let $Q\omega^0\mathbb{I}$ denote the constant cosimplicial object on $Q\mathbb{I}$, the chosen cofibrant replacement of $\mathbb{I}$. Then since degree zero of η factors through $Q\mathbb{I}$, η factors as two level equivalences $\Sigma\omega^1\mathbb{I} \longrightarrow Q\omega^0\mathbb{I} \longrightarrow \omega^0\mathbb{I}$. Hence $(\eta)^{\wedge m} \wedge \mathrm{id}_{A^{\cdot}}$ for any cosimplicial frame $A^{\cdot}$ is a level equivalence by the monoidal model structure on $\mathcal{C}$. Since $\omega^n\mathbb{I}$ for $n > 0$ is a cosimplicial frame by Proposition 6.4 part 1, each map $\Sigma^m(\omega^{m+n}\mathbb{I}) \longrightarrow \omega^n\mathbb{I})$ with $n > 0$ is a weak equivalence. By the pointed version of [7, Corollary 5.4.4], $\mathcal{C}(A^{\cdot}, -)$ preserves fibrations and trivial fibrations when $A^{\cdot}$ is a cosimplicial frame and for Z fibrant $\mathcal{C}(-, Z)$ takes level equivalences between cosimplicial frames to weak equivalences. Hence, $\mathrm{Hom}(\mathbb{I}, -)$ takes fibrant objects to positive Ω-spectra and (trivial) fibrations to positive level (trivial) fibrations. Thus $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen functor by [2, Corollary A.2] since positive stable fibrations between positive Ω-spectra are positive level fibrations by Proposition 4.3.

To show that the total left derived functor $\mathbb{I} \wedge^L -$ is strong monoidal we first show that $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal. For the unit map take the non-base point of S^0 to the identity map in simplicial degree zero of $\mathrm{Hom}(\mathbb{I}, \mathbb{I})^0$. The monoidal product on $\mathcal{C}$ induces a natural map $\mathcal{C}(\omega^m\mathbb{I}, A) \wedge \mathcal{C}(\omega^n\mathbb{I}, B) \longrightarrow \mathcal{C}(\omega^{m+n}\mathbb{I}, A \wedge B)$. Assembling these levels produces a natural map $\mathrm{Hom}(\mathbb{I}, A) \wedge \mathrm{Hom}(\mathbb{I}, B) \longrightarrow \mathrm{Hom}(\mathbb{I}, A \wedge B)$. Hence, $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal.

As in the simplicial case, since $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal since its left adjoint $\mathbb{I} \wedge -$ is lax comonoidal. Also, $\mathbb{I} \wedge F_n'X \cong (\omega^n\mathbb{I} \otimes_{\Sigma_n} X)^0$ because they represent the same functor in $\mathcal{C}$. So $\mathbb{I} \wedge -$ takes the cofibrant replacement $Q\mathbb{S} \cong F_1'S^1 \longrightarrow F_0S^0 \cong \mathbb{S}$ to the weak equivalence $\eta\colon\ (\omega^1\mathbb{I} \otimes S^1)^0 \longrightarrow \mathbb{I}$.

The comonoidal structure on $\mathbb{I} \wedge -$ induces a natural transformation $\mathbb{I} \wedge^L (A \wedge^L B) \longrightarrow (\mathbb{I} \wedge^L A) \wedge^L (\mathbb{I} \wedge^L B)$. The coherence of the unit diagram and the fact that $\mathbb{I} \wedge^L \mathbb{S} \cong \mathbb{I}$ shows that this map is an isomorphism for $A = \mathbb{S}$ and any B. For any fixed B both the source and target are exact functors in A which commute with coproducts. So for any fixed B the subcategory of objects A where this transformation is an isomorphism is a localizing subcategory which contains the generator $\mathbb{S}$. Hence this transformation is an isomorphism for all A and B. So $\mathbb{I} \wedge^L -$ is strong monoidal. □

Remark 6.5. Let $\mathcal{C}$ be a monoidal model category with a Quillen adjoint pair between $\mathcal{C}$ and the positive stable model category on Sp^Σ with left adjoint $L\colon\ Sp^\Sigma \longrightarrow \mathcal{C}$. If $L(Q\mathbb{S}) \longrightarrow L(\mathbb{S})$ is a weak equivalence and $L(\mathbb{S}) \cong \mathbb{I}$ then $\mathcal{C}$ has a good desuspension of the unit. Define $\mathbb{I}_c^{-1} = L(F_1S^0)$, with cylinder $L(F_1\Delta[1]_+)$ and model of the suspension $L(F_1S^1)$. These definitions have all the necessary properties since L preserves positive cofibrations and weak equivalences between positive cofibrant objects. Define $\eta\colon\ L(F_1S^1) \longrightarrow L(F_0S^0)$ as the adjoint of the identity map on level one. In fact, the cosimplicial frame $\omega^1\mathbb{I}$ can be defined by $(\omega^1\mathbb{I})^n = L(F_1\Delta[n]_+)$.

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