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The Osgood-Schoenflies Theorem Revisited

— *in honor of Ludmila Vsevolodovna Keldysh and her students on the centenary of her birth* —

by Laurent Siebenmann(*)

The very first unknotting theorem of a purely topological character established that every compact subset of the euclidean plane that is homeomorphic to a circle can be moved onto a round circle by a globally defined self-homeomorphism of the plane. This difficult hundred year old theorem is here celebrated with a partly new elementary proof, and a first but tentative account of its history. Some quite fundamental corollaries of the proof are sketched, and some generalizations are mentioned.

1 Introduction

This retrospective article presents an elementary, and hopefully direct and clear, geometric proof of what is usually called the (classical planar) Schoenflies Theorem; it is stated as (ST) in §4 below — with mention of its early history, including W.F. Osgood's rarely cited contributions. This (ST) is essentially the fact — surprising in view of known fractal curves — that every compact subset of the cartesian plane $\mathbb{R}^2$ that is homeomorphic to the circle $\mathbb{S}^1$, is necessarily the frontier in $\mathbb{R}^2$ of a set homeomorphic to the 2-disk. Beware that the 'Generalized Schoenflies theorem' of B. Mazur [Maz] and M. Brown [Brow1] — proved five decades later and valid in all dimensions — does *not* imply (ST) since it assumes a condition of flatness (or local flatness [Brow2]).

The Schoenflies Theorem (ST) is, in three respects, more awkward than other equally crucial and famous theorems of plane topology, notably the Jordan Curve Theorem and Invariance of Domain, which are stated as (JCT) and (IOD) in §3. Indeed, most extant proofs of (ST) lack three features of some well-known proofs for (JCT) and (IOD):

(i) to be essentially homological.

(ii) to apply in all dimensions to prove an important result.

(iii) to be easily motivated, remembered, and explained.

I have encountered exceptions to the above dismissive judgements. R.H. Bing's exposition [Bing5] 1983 can claim feature (iii); it is similar to an earlier one in [Newm], but clearer. Moreover, A. Chernavsky's highly original proof [Cher] of the natural extensions of (ST) to dimensions > 4 could perhaps be adapted to claim both features (ii) and (iii); however, I find his argument unnecessarily mysterious when adapted to dimension 2. Also worth special attention is the proof of (ST) by S. Cairns [Cair] (with contributions from John Nash, then a graduate student). Likewise, the proofs based on the theory of conformal mappings [Osg3][OsgT][Cara1–3][Koeb1–3][Stud][Keré][Cour].(†) As for non-conformal proofs, those in textbooks include [Newm][HallS][Why2][Mois2][WhyD][MohT] while those in research monographs include [Keré][Moor2][Wild][Why1][Kura][Keld][Bing5].

The proof of (ST) to follow achieves neither (i) nor (ii); but it will, perhaps, be most fully credited with feature (iii). It is exceptional in using a striking combinatorial design (Figure 6-a) well known in plane hyperbolic geometry; this simple design lays bare the crucial proof certifying the homeomorphism establishing (ST). Additionally, our policy is to classify all PL (= piecewise linear) surfaces encountered; hopefully this will add insight at slight extra cost. On the other hand, at a low level, our techniques are fairly typical of textbook proofs — in using a mixture of some general topology and some PL topology

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(†) Not to mention conformal proofs of (ST) in treatises on complex analysis, for example the mid-century ones by A.I. Markushevich or E. Hille. Furthermore, some solutions of the famous Plateau problem (of finding an area minimizing and well behaved disk map spanning a given embedded continuous closed curve in R^3) are conformal in character, and imply (ST) when applied to planar curves; see the books by R. Courant (1950), M. Struwe (1988), and J. Jost (1994) — which were kindly pointed out by Jürgen Jost.

of Euclidean space; such methods have more than didactic merit, since they dominate the study of topological embeddings of manifolds in dimensions > 2, cf. [Keld][Daver2].

Interestingly, in dimension 2 itself, conformal methods dominated for a good part of the last century, thanks to early initiatives of Osgood [Osg3] 1903, and Carathéodory [Cara3] 1913. The latter established a theory of "prime ends", cf. [Stud][Koeb3][Keré], which generalizes (ST) to an analysis of the frontier of any connected open subset U with connected complement in a closed 2-manifold M — a theory having since had important applications to dynamics, cf. [Mat1][BarK][Epst].

In 1982, J. Mather [Mat2] provided a bootstrapping purely topological treatment of the topological aspects of "prime ends" complementing earlier (but hard to locate) ones of P. Urysohn [Ury4] (cf. [Ury1–3]) of the 1920s, and of M.H.A. Newman [Newm] 1939. Mather writes: "It does not seem possible to give a brief account of Carathéodory's theory which does not rely on some [other] deep theory." This should be a warning to some readers, but a challenge to others. Mather includes a remarkable consequence, cf. [Newm][BreB], generalizing (ST):— any contractible open subset U in M, has a natural "prime end" compactification $\widehat{U}$ that is a 2-disk, and, if the frontier δU in M is locally connected, then the closure $\bar{U}$ of U in M is naturally the quotient of $\widehat{U}$ by a continuous map of the circle boundary $\partial\widehat{U}$ onto the frontier δU. In high dimensions, there is related current research concerning 'mapping cylinder neighborhoods', see [Quinn] and [Daver2, §47].

Purely topological methods have led to other deep results related to (ST), notably several characterizations of 2-manifolds, cf. [Moor2][Wild][Bing1], for example, L. Zippin's characterization [Zipp][HallS] of the 2-sphere, which states roughly this: A Peano continuum B, in which there is at least one embedded circle and in which the statement of the Jordan Curve Theorem (JCT) holds true, is necessarily homeomorphic to the 2-sphere. Compare R.D. Edwards' characterization of manifolds of dimension > 4 [Lat][Daver2].

Leaving aside such difficult extensions of (ST), the last section §9 gives, mostly as exercises, a few important corollaries of (ST) and its easy PL (= piecewise linear) analog. One is the approximability by PL homeomorphisms of merely continuous homeomorphisms between PL manifolds.

I have collected many references to the early literature concerning (ST) with the help of the recently constituted electronic version of the 1868–1942 review journal [JFM]. This literature is curiously disconnected — for example: the *earliest* explicit reference I have thus far encountered to Schoeflies' original (but partial) proof of (ST) in 1906 [Scho3] is in L.V. Keldysh's 1966 monograph [Keld]!

I am indebted to Alexey Chernavsky and Jean Cerf for encouragements and criticism that helped to improve my exposition at several points. Lucien Guillou alerted me to the fascinating and relevant histories of (JCT) in [DosT][Gugg] and their bibliographies. In translating this article for its Russian edition, Prof. Chernavsky provided, not just welcome collections of errata, but several interesting references, notably P.S. Alexandroff's reminiscences [Alxf], and Urysohn's cited work on prime ends.

2 Notions and notations

All spaces are by assumption metrizable topological spaces unless the contrary is stated. A space is **connected** if it cannot be expressed as the disjoint union of two non-empty subsets that are both open and closed. A **component** of X, will, in this article, be understood to mean a connected subset that is both open and closed. A subset A of X is **bounded** if its closure in X is compact. If X itself is compact, then a subset A of X is closed if and only if it is compact.

The following easy "Frontier Crossing Lemma" of general topology (dating from Brouwer [Brou1] if not earlier) will often be used without mention: (FCL) *Let Y be a subspace of space X. If a connected set C in X contains points of both Y and $X - Y$, then C also contains points of the frontier δY of Y in X.* Beware that δY depends on X although our notation does not indicate this; the frontier of Y in Y itself is always empty.

Given (possibly non-continuous) map $f : X \to Y$ and also a finite collection X_i of closed subsets of X, with $X = \bigcup_i X_i$, the map f is continuous if and only if all the restrictions $f|X_i : X_i \to Y$ are continuous.

A map $f : X \to Y$ is said to be **surjective** or **onto** (respectively **injective**) if, for every point y in Y, there exists at least one (respectively, at most one) point x in X such that $f(x) = y$. This f is said to be **bijective** or **one-to-one** if it is both injective and surjective.

An **embedding** $f : X \to Y$ is an injective and continuous map of topological spaces that gives a homeomorphism onto its image $f(X)$.

A map $f : X \to Y$ of topological spaces is said to be an **open map** if, for every open subset A of X, the image $f(A)$ is open in Y. Replacing 'open' by 'closed' in this sentence yields the definition of a **closed map**.

The image of a compact set under a continuous map is necessarily compact. Hence a continuous map $f : X \to Y$ of compact spaces is closed. And if this f is bijective, then f is also open and a homeomorphism.

If there is no contrary indication, a map of spaces will normally be supposed continuous.

Concerning PL objects and maps, see for example the first few pages of [RourS]. Every (metrizable) PL object is PL homeomorphic to a locally finite simplicial complex. The symbol $\cong$ will denote PL homeomorphism, whilst $\approx$ denotes ordinary homeomorphism. The boundary of a manifold M is denoted ∂M and its (manifold) interior is $M - \partial M$, denoted $\mathrm{Int} M$. An n-simplex Δ will always be identified with the n-dimensional convex hull of $(n+1)$ points in a real linear space. Those points and/or the dimension may be specified as arguments, as for the 1-simplex $\Delta^1(p,q)$. The formal boundary $\partial\Delta$ is the union of all its faces of dimension $< n$, and coincides with the boundary $\partial\Delta$ of Δ as a PL manifold.

For dimensions $n \geq 1$, the standard PL (= piecewise linear) n-disk B^n is $[-1,1]^n \subset \mathbb{R}^n$, and its frontier and boundary is the PL $(n-1)$-sphere S^{n-1}. The interior of B^n is $\mathrm{Int}\, B^n = (-1,1)^n = B^n - S^n$.

We shall also encounter the smooth n-ball $\mathbb{B}^n$ (respectively the smooth n-sphere $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$) consisting of all points in $\mathbb{R}^n$ at Euclidean distance ≤ 1 (respectively exactly 1) from the origin of $\mathbb{R}^n$. These do not have a natural PL structure. However, there is a homeomorphism $\rho : B^n \to \mathbb{B}^n$ sending S^{n-1} to $\mathbb{S}^{n-1}$; and with some effort, it can be chosen to be PL on $B^n - S^{n-1}$.

It would be quite possible to use $\mathcal{C}^\infty$ smooth manifolds and maps instead of PL objects and maps; see [Miln]; the required techniques are slightly less elementary, but perhaps more important to undergraduates.

We are chiefly interested in embeddings into S^2 and $\mathbb{R}^2$. Since $\mathbb{R}^2$ is PL homeomorphic to the complement of any point in S^2, we can study any compact set X in S^2 that omits at least one point of S^2, by regarding it as a compact set in $\mathbb{R}^2$, and conversely.

A **Jordan curve** (respectively **arc**) C is a compact subset of S^2 or $\mathbb{R}^2$ that is homeomorphic to S^1 (respectively to B^1). These notions also make good sense in any space homeomorphic to S^2 or $\mathbb{R}^2$.

If d_n, $n = 1, 2, 3, \ldots$ is a sequence of real numbers, the unorthodox phrase "d_n converges $\mathcal{O}(n)$" will mean that d_n converges to 0 as n converges to infinity.

3 Homologically provable results that we exploit

Proofs of the following basic results using chiefly homology theory, are widely understood by students (even undergraduates) who have studied topology for a year or two. Our proof of (ST) and its complements will freely use (JCT).

JORDAN CURVE THEOREM (JCT) — [Jord2], 1887.
In $\mathbb{R}^2$ or $\mathbb{S}^2$, the complement of a Jordan curve C has exactly two components, say D_- and D_+. Furthermore, C is the frontier of both D_- and D_+.

As 'universal' homological proof of the first clause of (JCT), we cite Alexander duality, as expounded for example in [Dold1], of which an immediate corollary is this invariance principle: (JA) *The number of components of the complement $S^n - X$ of any compact* *subset X of S^n is a an intrinsic topological invariant of X itself — indeed of its 'Borsuk shape'.* The note [Dold2] offers a simple and elegant proof of (JA) – leaving aside its last clause concerning shape; it relies on a homology suspension isomorphism and this rudimentary unknotting lemma: *The embedding of any compact set in $\mathbb{R}^n$ is unique up to* (i) *the 'stabilization' by inclusion $\mathbb{R}^n \to \mathbb{R}^{2n}$, and* (ii) *homeomorphism of $\mathbb{R}^{2n}$ that is the identity outside a compact set.* The proof of this lemma uses Tietze's well known extension theorem [Tiet3] 1914. The partial proof of (JA) in [Dold2] can be generalized to fully establish (JA) — including the shape clause — using Chapman's stable correspondence [Chap] between shape and complement in $\mathbb{R}^n$. For a full proof of (JA) based on a study maps to S^{n-1} viewed up to homotopy, see [tomD] (or [Dugu] for $n = 2$).

As for the second 'frontier' clause of (JCT), we now recall a classic two part argument that adapts to any dimension. (i) The complement of any Jordan arc J in $\mathbb{S}^2$ is connected by (JA). (ii) Given a point p on C and any Jordan arc $J \subset C$ not containing p, the connected open set $\mathbb{S}^2 - J$ contains the disjoint Jordan domains D_- and D_+. Hence, by (FCL) of §2, it meets the frontier δD_- of D_- in $\mathbb{S}^2$, indeed, necessarily in $C - J$. Since $C - J$ can lie in any prescribed neighborhood of p, it follows that p lies in the frontier of D_- in δD_-. Similarly $p \in \delta D_+$. □

There is a strong school of thought, see [Vebl][Alex1][Scho4][DosT], that (JCT) was not fully proved by Jordan [Jord2]; however, I am unaware of specific objections, other than those of Schoenflies [Scho5] 1924, which Schoenflies himself considered non-fatal. Jordan's arguments do seem to involve less than any proof that I fully understand; compare Keldysh [Keld, Chap II, Lemme 4.1].

The first complete proof of (JCT) seems to be O. Veblen's [Vebl] 1905; this intricate proof developed Schoenflies' notion [Scho3] 1902 of PL path access in $\mathbb{R}^2 - C$ from any given point of $\mathbb{R}^2 - C$ to (at least) a dense set of points of C. Similar proofs appear in most textbooks featuring (ST). Incidentally, some access path technology seems essential to most proofs of (ST) itself; see (SAL) in next section.

Jordan's 1887 exposition *assumes* (JCT) for the case of PL Jordan curves; this case was discussed by Schoenflies in 1896 [Scho1], see [Jord1]. A pleasant inductive proof for PL Jordan curves largely due to N. Lennes 1903, 1911 (see [DosT]) and to M. Dehn (see [Gugg]) is sketched in the first remark of §7.

By a **Jordan domain**, we will mean a bounded and connected open set D in a space homeomorphic to $\mathbb{R}^2$ or $\mathbb{S}^2$, whose frontier is a Jordan curve C. Its compact closure $B = D \cup C$ is called a **sealed Jordan domain**. With this language, (JCT) shows that, for every Jordan curve C in S^2, the complement $S^2 - C$ consists of exactly two Jordan domains. The main result (ST) to be expounded reveals that D and B are homeomorphic to $\mathbb{R}^2$ and $\mathbb{B}^2$ respectively.

Here are two easy but useful corollaries of (JCT) and (FCL). Let C and C' be two Jordan curves in $\mathbb{R}^2$. Let D be the (unique) Jordan domain with frontier C and let B be its compact closure. Define D' and B' similarly for C'.

JORDAN SUBDOMAIN LEMMA *If $C' \subset B$, then $B' \subset B$.* □

JORDAN DOMAIN DISJUNCTION LEMMA *Suppose C' does not intersect D, and C does not intersect D' (equivalently, suppose C stays outside or on C', and reciprocally). Then either $B = B'$, or else D and D' do not intersect (although B and B' may intersect).* □
In many proofs of the Schoenflies theorem, these lemmas are unmentioned but silently applied; we will endeavor to cite them explicitly wherever they are needed.

Almost as famous and useful as (JCT) is:

INVARIANCE OF DOMAIN (IOD) — see [Brou2]. *Every embedding $h : U \to \mathbb{R}^2$ of an open subset U of $\mathbb{R}^2$ into $\mathbb{R}^2$ is an open mapping.*

L. Bieberbach 1913 (see [DosT]), attributes the first proof of (IOD) to one E. Jürgens in a 1879 habilitation thesis in Halle, which F. Hausdorff [Haus, p. 468] 1914 seems to identify as published in Leipzig 1879. For our proof of (ST), one needs (IOD) only in the special case when h is PL. That case is easy to prove for any dimension by induction on dimension, since h is then (affine) linear on the simplices of a linear triangulation of U. Here is a classic proof of (IOD) based on (JA) above, and adaptable to all dimensions: For any 2-ball B in U, the complement $\mathbb{R}^2 - h(B)$ is connected, by (JA). Also, $\mathbb{R}^2 - h(\partial B)$ has two (open) components, again by (JA); clearly one is $\mathbb{R}^2 - h(B)$ and the other is $h(B - \partial B)$. Consequently $h(B - \partial B)$ is open, which implies that h is an open mapping. $\square$

4 Statement of the main result

What is called the Schoenflies Theorem goes beyond (JCT) and better describes the embedding of C as follows:

SCHOENFLIES THEOREM (ST) *Let B be a sealed Jordan domain in S^2 or $\mathbb{R}^2$ with frontier the Jordan curve C. There exists a homeomorphism $H : B^2 \to B$ sending S^1 onto C.*

HISTORICAL NOTES. Jordan's reputedly inconclusive arguments for (JCT) in [Jord1] 1887 paradoxically made measurable progress towards (ST); they essentially prove that every complementary component D of $S^2 - C$ is homeomorphic to $\mathbb{R}^2$. In more detail, D is a nested union of sealed Jordan domains $B_i \subset D$ whose frontiers are a PL Jordan curves C_i; each B_i is $\approx B^2$ by [Jord1], while all successive differences are $\approx$ to annuli, again by [Jord1]. As the 1866 date of [Jord1] might suggest, the proofs seem obscure to modern eyes, cf. [Hirs, Chap 9]. But see the exercise under (PLCT) of §7, which indicates how to prove that $D \cong \mathbb{R}^2$ as directly as possible.

In 1887, the fractal curves of Klein would have made any mathematician hesitate to conjecture that $B \approx \mathbb{B}^2$. The first clear assertion that $B \approx \mathbb{B}^2$ of which I am aware is by Wm. F. Osgood in 1903 [Osg3]; he had already proved in [Osg2] 1900 (cf. [Scho2]) that D is conformally homeomorphic to $\mathrm{Int}\mathbb{B}^2 \approx \mathbb{R}^2$ in spite of examples in [Osg1] where C has positive Lebesgue measure. Full proofs of (ST) by Osgood and several other mathematicians came a decade later [Cara1–3][Koeb1–3][OsgT][Stud]; all these first generation proofs used complex variable theory, i.e. conformal mappings. (The proof in [OsgT] was accepted by Koebe in his [JFM] review and also by Courant in [Cour].) Schoenflies [Scho4,§13] 1906 gave the second clear statement of (ST). He also ventured the first proof; it begins by correctly establishing the easy PL version of (ST) (cf. §7), and concludes well by using an infinite tessellation; but he seems to make a significant blunder in between — claiming to prove something impossible. Namely that, for any nested sequence of PL Jordan curves $C_i \subset D$ (as asserted by Jordan), a subsequence can be parametrized, say by $c_i : S^1 \to C_i \subset \mathbb{R}^2$, so that the c_i converge to a topological parametrization $c : S^1 \to C$ of C.(*) In 1902, Schoenflies [Scho3] had discovered an interesting characterization of Jordan curves in terms of access paths, one that may well have motivated Carathéodory's theory of prime ends [Cara3] 1913. The first complete proof of (ST) not based on conformal mappings may be H. Tietze's long argument [Tiet1, (b)][Tiet2, SatzIII] 1914. Or it may be that in L. Antoine's thesis [Ant1, Chap.I] 1921, an argument of reasonable length that is detailed at the mentioned point where [Scho4,§13] seems to blunder. See also [Moor1] and [Keld; pp. 63–81]). The name "Schoenflies Theorem" to designate (ST) seems to originate with R.L. Wilder [Wild, I.6 and III.5.9].

Natural generalizations of (ST) to all dimensions $n > 2$ have now been proved by surprisingly diverse and difficult methods — for $n = 3$ by [Bing3] and [Bing4] 1961; for $n \geq 5$ by Chernavsky [Cher] 1973, and independently by Daverman-Price-Seebeck [PriS] [Daver1] 1973; and finally for $n = 4$ by Freedman-Quinn [FreeQ] in the 1980s. All of these require a local fundamental group condition usually called 1-ULC, as the Antoine-Alexander horned 2-sphere in S^3 first revealed in 1924 [Ant2][Alex2][Alex3].

Our proof of (ST) in §6 will also prove

COMPLEMENT (ST+) *The homeomorphism $H : B^2 \to B$ can be chosen to extend any given homeomorphism $S^1 \to C$, and to be PL (= piecewise linear) on $B^2 - S^1$.*

(*) Does this blunder explain why Schoenflies' pioneering proof long went unmentioned in the literature? I think not. Some 18 years later, Kerékjártó in [Keré, p.72] 1924, after a very condensed proof of (ST), seems to assert as a corollary of that proof, that Schoenflies' falacy above is true — indeed even without the above-mentioned subsequencing and reparametrization!

5 Two tools for our proof of (ST) and (ST+)

The first tool collects simplest special cases of (ST) enhancing them with extra piecewise linearity.

ALMOST PL SCHOENFLIES THEOREM (APLST) *Statement* (ST) *holds true in case* C *is* PL *(= piecewise linear) except at a (possibly empty) finite set* X *of points in* C. *Furthermore, in that case, a homeomorphism* H *can be built, from the standard* PL *disk* B^2, *onto the sealed Jordan domain* B, *such that* H *is* PL *except possibly at* $H^{-1}(X)$.

This result follows quickly from the classification of noncompact PL surfaces, cf. Kerékjártó [Keré]. For completeness, it will be proved in §7, along with the following complement.

COMPLEMENT (APLST+) *If* $G : \partial B^2 \to C$ *is a homeomorphism that is* PL *except at* $G^{-1}(X)$, *then* H *offered by* (APLST) *can be made to extend* G.

The second tool (SAL) is a 2-dimensional analog of a key 3-dimensional result of R.H. Bing [Bing2], which was essential to his proof of the analog of (ST) in dimension 3 [Bing4]. (SAL) is not new, but a (partly new?) geometric proof will be provided in §8.

SIDE APPROXIMATION LEMMA (SAL) *Let* J *be any Jordan arc in the Jordan curve* $C \subset \mathbb{R}^2$ *and let* $B \subset \mathbb{R}^2$ *be the sealed Jordan domain with frontier* C. *There exists a Jordan arc* $J' \subset B$ *with the same end points* $\{P_0, P_1\}$ *as* J, *such that* $J' \cap C = \{P_0, P_1\}$ *and* $J' - C$ *is* PL. *Furthermore, one can choose* J' *to lie in any prescribed neighborhood of* J.

REMARK. (SAL) can clearly be deduced from (ST) and its complement (ST+). □

(SAL) can be regarded as a mostly homological result. Indeed, it can be proved homologically that every open Jordan domain is 0-lc in $\mathbb{R}^2$ — using a local form of Alexander duality valid in all dimensions and codimensions, see [Wild][Dold1], or [Dold2]. Then, from this 0-lc property, one can derive (SAL).

6 The core of the proof of (ST) and (ST+)

This is the section that hopefully holds a bit of novelty for topologists. We prove (ST) and (ST+) assuming (APLST), (APLST+) and (SAL), plus the homologically accessible result (JCT) and the trivial PL case of (IOD).

Without loss, we can assume that the sealed Jordan domain B that is given for (ST), lies in $\mathbb{R}^2$. Indeed, if B is originally given in S^2, let P be any point in the component of $S^2 - C$ distinct from $D = B - C$ which is provided by by (JCT). Then we identify $\mathbb{R}^2$ by a PL homeomorphism to $S^2 - P \supset B$.

The use of $\mathbb{R}^2$ as ambient space rather than S^2 is helpful because the (Euclidean) metric of R^2 has the special feature:

6.1 EUCLIDEAN METRIC PROPERTY *The diameter of any subset* X *of* B *is realized as the Euclidean distance between two points of its frontier* δX *in* $\mathbb{R}^2$. □

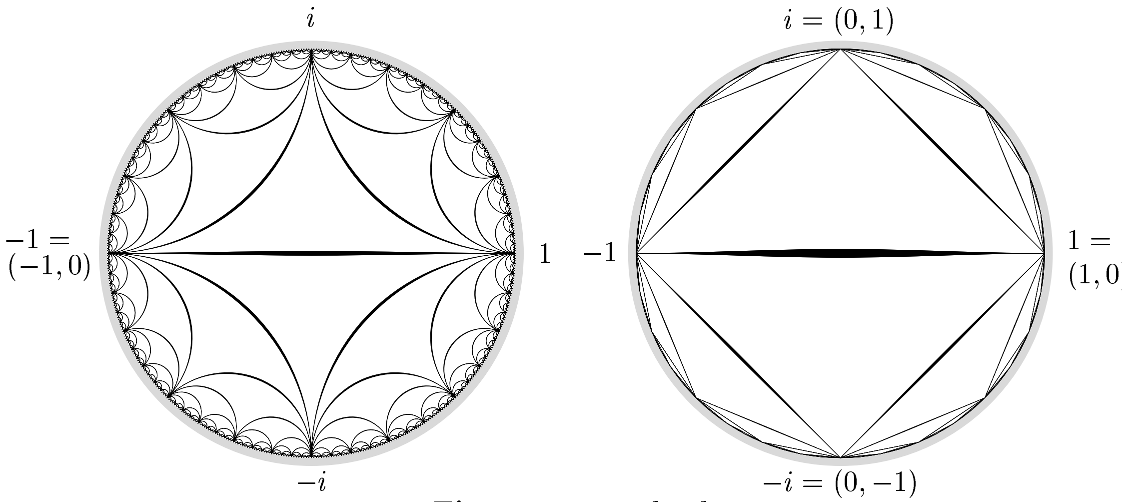
The continuity of the wanted homeomorphism $B^2 \to B$ will be checked using such diameters.

Fix any embedding $c : \mathbb{S}^1 \to \mathbb{R}^2$ of the unit circle $\mathbb{S}^1$ onto the Jordan curve C. We shall extend c to a homeomorphism $h : \mathbb{B}^2 \to B$.

As observed in §2, there is a homeomorphism ρ of the square B^2 onto the smooth unit disk $\mathbb{B}^2$ that is PL on the interior of the square. Then the composition $H = h \circ \rho$ will establish (ST) and (ST+). Thus, it will suffice now to construct h to be PL on IntB^2 and then verify that it is a homeomorphism.

6.2 Constructing a tessellation $\mathcal{T}$ of $\mathbb{B}^2$ and a map $h : \mathbb{B}^2 \to B$

The rough idea is to gradually build up h using the tiles of an infinite tessellation (=tiling) $h(\mathcal{T})$ of B that is combinatorially isomorphic by h to the tessellation of the naturally compactified hyperbolic plane generated by the three reflections in the sides of a triangle with all three of its vertices on the limit circle $\mathbb{S}^1 \subset \mathbb{R}^2 = \mathbb{C}$; see Figure 6-a.



Figures 6-a and 6-b

More precisely, we first define a standard infinite tessellation $\mathcal{T}$ by linear simplices of the smooth unit disk $\mathbb{B}^2$ in $\mathbb{R}^2$, as illustrated in Figure 6-b. Then, using $\mathcal{T}$, we will build a homeomorphism $h : \mathbb{B}^2 \to B$ that is PL on the open disk $\mathrm{Int}\,\mathbb{B}^2$.

Here, a **tessellation** of a space X means a triangulation of X as a (unordered) simplicial complex, also called $\mathcal{T}$, except for one difference: the given bijective identification, say $\tau : \mathcal{T} \to X$, of the simplicial complex $\mathcal{T}$ to the space X is assumed continuous but *not necessarily* a homeomorphism. However, τ necessarily induces a topological embedding into X of any (compact!) finite subcomplex T of $\mathcal{T}$ — since the induced continuous bijection $\tau| : T \to \tau T$ is a homeomorphism (§2). In particular, the infinite tessellation $\mathcal{T}$ of $\mathbb{B}^2$ in Figure 6-b gets a compact topology from $\mathbb{B}^2$ whereas the standard (weak or metric) simplicial topologies of $\mathcal{T}$ are noncompact.

$\Pi(k)$ with $k \geq 0$ denotes the convex hull of the k complex k-th roots of unity in $\mathbb{C} = \mathbb{R}^2$. It is the standard (solid) k-gon. Note the degenerate cases $\Pi(1) = 1 = (1,0)$ and $\Pi(2) = \Delta^1(-1,1)$. All other $\Pi(k)$ are 2-dimensional.

To be quite specific (see Figure 6-b), we define the 0-simplices (=vertices) of the tessellation $\mathcal{T}$ to be all the continuously many points of the boundary circle $\mathbb{S}^1$ of $\mathbb{B}^2$; we define the 1-simplices (= edges) to be the edges of the regular convex 2^n-gons $\Pi(2^n)$, $n \geq 1$. Finally, we define the *open* 2-simplices (= faces) of $\mathcal{T}$ to be the connected components of $(\mathrm{Int}\,\Pi(2^n)) - \Pi(2^{n-1})$, $n \geq 2$, the *closed* 2-simplices being their respective closures in $\mathbb{R}^2$.

Let T_n be $\Pi(2^n)$ with the *finite* triangulation inherited from $\mathcal{T}$.

The construction of h will mention circular arcs $A(p,q)$ for points p and q in $\mathbb{S}^1$. Such an arc $A(p,q)$ is always the shorter arc between p and q. Except that, when p and q are antipodal, $A(p,q)$ denotes the counterclockwise arc p to q. Thus $A(p,q) = A(q,p)$, except that, when $q = -p$, one has $A(p,q) = -A(q,p)$ in $\mathbb{C}$.

STEP (0) *Define the restriction $h|\mathbb{S}^1$ to be a given Jordan curve parametrization $c : \mathbb{S}^1 \to C \subset B$.* □

Recall that the points of S^1 are exactly the vertices of the tessellation $\mathcal{T}$.

A **chord** of B is a Jordan arc $J \subset B$ such that $J \cap C = \partial J$. We call it an **nice chord** if $J - \partial J$ is PL.

The next step of the construction uses an arbitrary sequence ε_n, $n \geq 1$, of positive numbers that tends to 0, i.e. it converges $\mathcal{O}(n)$.

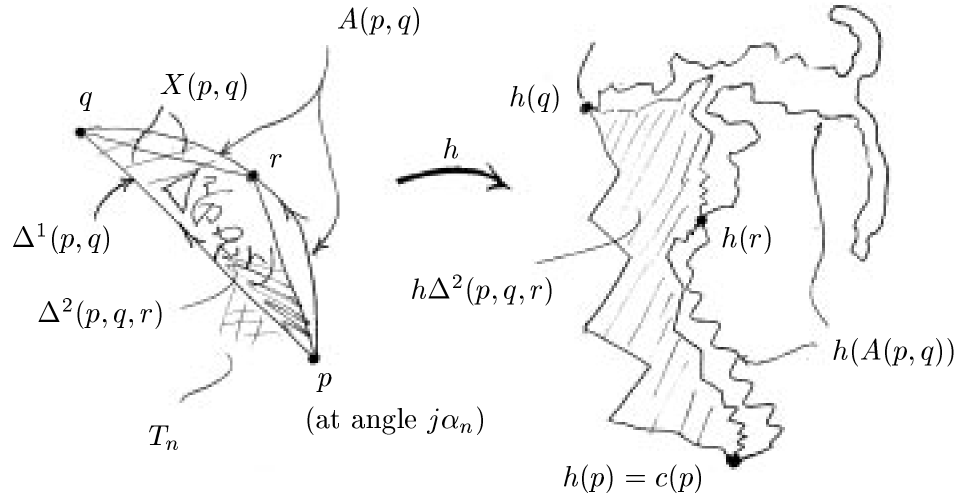
STEP (1) *For each 1-simplex $\Delta^1(p,q)$ of $\mathcal{T}$ construct a nice chord $\delta(p,q)$ in B joining $c(p)$ to $c(q)$ in such a way that:*

(i) *No two of these chords of B intersect except (possibly) on C.*

(ii) *If the 1-simplex $\Delta^1(p,q)$ of $\mathcal{T}$ lies in the boundary of $T_n = \Pi(2^n)$, then the nice chord $\delta(p,q)$ lies in the ε_n-neighborhood in B of the Jordan arc $cA(p,q) \subset C$.*

Since simplices, chords etc. are unoriented, $\delta(p,q) = \delta(q,p)$.

Figure 6-c



Construction for (1) We initialize by applying (SAL) to obtain a nice chord $\delta(1,-1) = \delta(-1,1)$ of the closure B of D in $\mathbb{R}^2$, one that joins $c(1)$ to $c(-1)$.

Suppose inductively that $\delta(p,q)$ has been defined for every 1-simplex $\Delta^1(p,q)$ in T_n, with $n \geq 1$. There are 2^n distinct 2-simplices S of $\mathcal{T}$ in T_{n+1}, that are not in T_n. Each such S is of the form $S = \Delta^2(p,q,r)$ where $S \cap T_n$ is $\Delta^1(p,q)$, an edge of $\Pi(2^n)$, and r is the midpoint of the circular arc $A(p,q)$ of length $2\pi/2^n$ from p to q in $\mathbb{S}^1$.

Use (SAL) to obtain a nice chord $\delta(q,r)$ joining $c(q)$ to $c(r)$ in the sealed Jordan domain $B(p,q) \subset B$ whose frontier is the Jordan curve $\delta(p,q) \cup cA(q,p)$, requiring that $\delta(q,r)$ lie in the ε_{n+1} neighborhood of $cA(q,r)$.

Once again, use (SAL) to obtain a nice chord $\delta(r,p)$ joining $c(r)$ to $c(p)$ in the sealed Jordan domain whose frontier is the Jordan curve $\delta(p,q) \cup \delta(q,r) \cup cA(r,p)$, requiring that $\delta(r,p)$ lie in the ε_{n+1} neighborhood of $cA(r,p)$.

Then $\delta(p,q)$, $\delta(q,r)$ and $\delta(r,p)$ lie in $B(p,q)$ and meet only on C.

When this has been done for each such S, the nice chord $\delta(p,q)$ is defined for every 1-simplex $\Delta^1(p,q)$ in T_{n+1}. Then induction on n completes Step (1). □

Step (1+) *For each (unoriented) edge* $\Delta^1(p,q)$ *of* $\mathcal{T}$ *define a homeomorphism* $h| : \Delta^1(p,q) \to \delta(p,q)$ *that maps* p *to* $c(p)$, q *to* $c(q)$, *and is* PL *on* $\Delta^1(p,q) - \{p,q\}$. □

Here and elsewhere, $h|$ informally denotes a map that is going be a restriction of h. Observe that, after Step (1+), the map h is well defined and continuous on the boundary ∂S of every 2-simplex $S = \Delta^2(p,q,r)$ of $\mathcal{T}$, and PL except (possibly) at the 3 vertices; it maps onto the frontier of the sealed Jordan region $B(p,q,r)$ in B with boundary $\delta(p,q) \cup \delta(q,r) \cup \delta(r,p)$.

Step (2) *For each (unoriented) 2-simplex* $S = \Delta^2(p,q,r)$ *of* $\mathcal{T}$ *define a homeomorphism* $h| : \Delta^2(p,q,r) \to B(p,q,r)$, *so that:*

(i) *On each 1-simplex face of* S, *this* $h|S$ *is the 1-simplex mapping defined in* Step (1+).

(ii) $h|S$ *is* PL *except at the vertices* p, q, r *— which are the points of* $(h|S)^{-1}(C)$.

Construction for (2) The required extension $h|S$ is provided by the Almost PL Schoenflies Theorem (APLST) with its complement (APLST+), see §5. □

At this point $h : \mathbb{B}^2 \to B$ is well defined as a map of sets and it is clearly PL on $\mathrm{Int}\,\mathbb{B}^2$.

6.3 Proof that $h : \mathbb{B}^2 \to B$ is a homeomorphism

ASSERTION (A) *The map $h : \mathbb{B}^2 \to B$ is injective.*

PROOF OF (A). By Steps (0), (1), and (1+) of its construction, this h is injective on the union of the simplices of $\mathcal{T}$ of dimensions 0 and 1. By Step (2) it is injective on each individual 2-simplex. Then (A) follows from the Jordan Domain Disjunction Lemma of §3. □

ASSERTION (B) $h|\mathrm{Int}\,\mathbb{B}^2$ *is continuous.*

PROOF OF (B). For this, it suffices to verify continuity of $h|U$ on a collection of open sets U of $\mathbb{B}^2$ that cover $\mathrm{Int}\,\mathbb{B}^2$. Now $h|T_k$ is continuous since it is continuous on each simplex, and T_k is a finite complex. A fortiori, $h|\mathrm{Int}\,T_k$ is continuous. But the open sets $\mathrm{Int}\,T_k$ form an open cover of $\mathrm{Int}\,\mathbb{B}^2$ because $\Pi(2^n)$ contains the scaled 2-disk $\cos(2\pi/2^{n+1})\mathbb{B}^2$. □

ASSERTION (C) *There exists a sequence $\eta_1, \eta_2, \eta_3, \ldots$ converging $\mathcal{O}(n)$ such that, for every chordal sector X of $\mathbb{B}^2$ that is one of the 2^n sealed components of $\mathbb{B}^2 - \Pi(n)$, the diameter $\mathrm{Diam}\,h(X)$ of $h(X)$ is $\leq \eta_n$.*

PROOF OF (C). By definition, X is the convex hull in $\mathbb{R}^2$ of a unique circle arc $A = A(j\alpha_n, (j+1)\alpha_n)$ where $\alpha_n = 2\pi i/2^n$ and $0 \leq j < 2^n$. Also $X \cap \Pi(n)$ is $\Delta = \Delta^1(j\alpha_n, (j+1)\alpha_n)$. By Step (1) of the construction of h, this linear chord Δ of $\mathbb{B}^2$, which is the frontier of X in $\mathbb{B}^2$, is mapped by h to a nice chord lying in the ε_n-neighborhood in B of $h(A)$.

CLAIM
$$\mathrm{Diam}\,h(X) = \mathrm{Diam}\,h(\partial X) \leq \mathrm{Diam}\,h(A) + 2\varepsilon_n \qquad (*)$$

PROOF OF CLAIM. In $(*)$, the relation $\leq$ follows from the metric triangle inequality, but the equality $=$ is not obvious. However, $\geq$ in place of $=$ is obvious from inclusion. So the remaining task is to establish $\leq$ in place of $=$, which we do as follows.

The Euclidean Metric Property of §6.1 tells us that $\mathrm{Diam}\,h(\partial X) = \mathrm{Diam}\,(Y)$, where Y denotes the sealed Jordan domain with frontier $h(\partial X)$. It is not (yet) clear that Y is $h(X)$. But Y contains $h(A)$ by its definition, and it contains $h(S_i)$ for every triangular tile S_i of $\mathcal{T}$ lying in the chordal sector X; this is a consequence of the Jordan Subdomain Lemma of §3. Hence Y contains all of $h(X)$ and so $\mathrm{Diam}\,h(X) \leq \mathrm{Diam}\,Y$, as required to complete the proof of the claim. □

Continuing the proof of (C), note that, since the restriction $h|\mathbb{S}^1$ is continuous and $\mathbb{S}^1$ is compact, $h|\mathbb{S}^1$ is uniformly continuous, in the sense that, for any $\zeta > 0$, there exists a $\xi > 0$ such that if $A' \subset \mathbb{S}^1$ is of diameter $< \xi$ then the diameter $\mathrm{Diam}\,h(A')$ is $< \eta$. It follows that, if $\zeta_n = \mathrm{Max}\,\mathrm{Diam}\,(h(A))$, where A ranges over the 2^n arcs of $\mathbb{S}^1$ into which the 2^n-th roots of 1 cut $\mathbb{S}^1$, then ζ_n converges $\mathcal{O}(n)$.

From $(*)$ we conclude, setting $\eta_n = \zeta_n + 2\varepsilon_n$, that $\mathrm{Diam}\,h(X) \leq \eta_n$, where η_n converges $\mathcal{O}(n)$. This proves Assertion (C). □

ASSERTION (D) $h : \mathbb{B}^2 \to B \subset \mathbb{R}^2$ *is continuous.*

PROOF OF (D). In view of (B), it suffices to prove continuity of h at an arbitrary point p in $\mathbb{S}^1$. We distinguish two cases:

CASE (I) *The angle of p is not a dyadic rational multiple of 2π.*

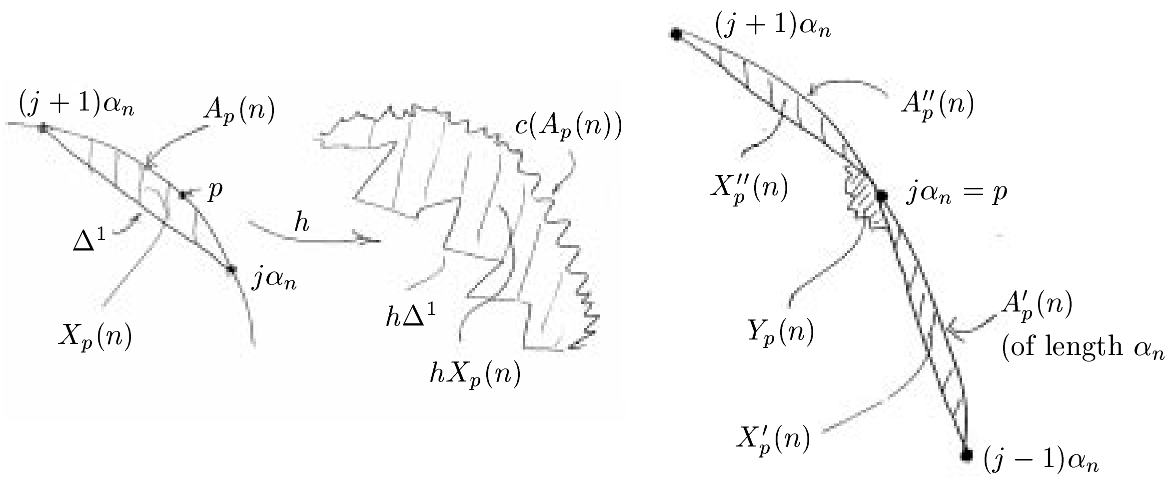
PROOF OF CASE (I) — see Figure 6-d.

For all $n \geq 1$, the point p of $\mathbb{S}^1$ lies in the interior in $\mathbb{B}^2$ of exactly one chordal sector $X = X_p(n)$ cut out of $\mathbb{B}^2$ by $\Pi(2^n)$ as described in Assertion (C). By Assertion (C), $\mathrm{Diam}\,h(X_p(n))$ converges $\mathcal{O}(n)$, which proves continuity of h at p, thus completing the proof of Case (I). □

CASE (II) *The angle of p is a dyadic rational multiple of 2π.*

PROOF OF CASE (II) — see Figure 6-e.

Suppose p has angle $2\pi\ell/2^m$ where ℓ is odd and $m \geq 1$. Then p is a vertex of the regular polygon $\Pi(2^n)$ precisely if $n \geq m$. The new difficulty here is that the 2^n chordal



Figures 6-d and 6-e

sectors X of $\mathbb{B}^2$ cut out by $\Pi(2^n)$ do not include *any* neighborhood of p in $\mathbb{B}^2$, whilst we want a basis of neighborhoods as n varies to be able to test continuity of h at p.

However, for all $n \geq m$, the point p in $\mathbb{S}^1$ lies at the intersection of exactly *two* adjacent chordal sectors $X'_p(n)$ and $X''_p(n)$ of $\mathbb{B}^2$, having respective extremities $(j-1)\alpha_n$, $j\alpha_n$ and $j\alpha_n$, $(j+1)\alpha_n$, where $\alpha_n = 2\pi i/2^n$, and $0 \leq j < 2^n$. Their union is not quite, but almost, a neighborhood of p in $\mathbb{B}^2$. We define the neighborhood $X_p(n)$ of p in $\mathbb{B}^2$ to be (see Figure 6-e illustrating $X_p(n)$ as a glider in flight):

$$X_p(n) = X'_p(n) \cup X''_p(n) \cup Y_p(n)$$

where $Y_p(n)$ is a small neighborhood of p in the convex full regular 2^n-gon $\Pi(2^n)$, so chosen that:

$$\operatorname{Diam} h(Y_p(n)) < \eta_n \qquad (**)$$

This is possible because $h|T_n$ is continuous by construction.

We conclude that the image under h of the neighborhood $X_p(n)$ of p in $\mathbb{B}^2$ has diameter bounded by the sum of the diameters of its three parts:

$$\operatorname{Diam} h(X_p(n)) \leq \operatorname{Diam} h(X'_p(n)) + \operatorname{Diam} h(X''_p(n)) + \operatorname{Diam} Y_p(n) \leq 3\eta_n$$

where the last $\leq$ uses $(*)$ of Assertion (C) and $(**)$ above. It follows that $\operatorname{Diam} h(X_p(n))$ converges $\mathcal{O}(n)$, which proves continuity of h at any dyadic point p.

Assertion (D) is now proved in both possible cases. □

Assertion (E) *The restriction* $h| : \operatorname{Int}\mathbb{B}^2 \to D = B - C$ *is an open map.*

Proof of (E). Apply the invariance of domain theorem (IOD) in the easy case of PL maps. □

Lemma *Let* $f : X \to Y$ *be a closed and continuous map, and let* U *be open in* Y. *Then the restriction* $f| : f^{-1}(U) \to U$ *is a closed map.* □

Assertion (F) $h : \mathbb{B}^2 \to B$ *is bijective.*

Proof of Assertion (F). By Assertion (A), h is injective. Since h maps $\mathbb{S}^1$ bijectively to C, it suffices to show that h maps $\operatorname{Int}\mathbb{B}^2 = \mathbb{B}^2 - \mathbb{S}^1$ surjectively to $D = B - C$.

Since h is continuous, and $\mathbb{B}^2$ is compact, h is closed. It follows, from the above lemma, that $h| : \operatorname{Int}\mathbb{B}^2 \to D = B - C$ is a closed map. As it is also open by Assertion (E), and both $\operatorname{Int}\mathbb{B}^2$ and D are connected, we conclude that the open and closed subset $h(\operatorname{Int}\mathbb{B}^2)$ of D is all of D; thus h is surjective as well as injective. □

Since h is, by Assertion (F), a bijective continuous map of compact spaces, it is a homeomorphism (see §2). This completes the proof of the Schoenflies Theorem (ST) and its complement (ST+). □

COMMENT. The pace of the above proof is extremely leisurely, indeed prudish — like that in [Keld] — in comparison with many, say those in [Bing5][Newm][Keré]. This is perhaps *advisable* given the history in §4; and it is *possible* without straining the reader's stamina, thanks to our simplified outline.

7 Proof of some PL Schoenflies Theorems — see §5

This is the first of two sections establishing tools for the proof of (ST). We begin with

CLASSICAL PL SCHOENFLIES THEOREM (PLST) — see [Scho4, §13] 1906.
Every PL *Jordan curve* C *in* $\mathbb{R}^2$ *bounds a* PL 2*-disk.*

We will need some easy lemmas that are left as exercises, cf. [RourS].

LEMMA (1) A compact and convex or star-shaped sealed Jordan domain in $\mathbb{R}^2$ with PL frontier is necessarily PL homeomorphic to B^2. In particular, B^2 is PL homeomorphic to the PL cone $\mathrm{Cone}(B^1)$ on B^1 and also to the PL cone $\mathrm{Cone}(S^1)$ on S^1. □

LEMMA (2) Any PL self-homeomorphism of S^1 extends (by coning) to a PL self-homeomorphism of $B^2 \cong \mathrm{Cone}(S^1)$. Likewise any PL self-homeomorphism of a closed interval in S^1 extends (by coning) to a PL self-homeomorphism of $B^2 \cong \mathrm{Cone}(B^1)$. □

LEMMA (3) Any naturally cyclicly or anti-cyclicly ordered finite sequence of N points in S^1 is equivalent to any other by a PL self homeomorphism of S^1. □

LEMMA (4) If X is a finite simplicial 2-complex expressed as a union $X = X_1 \cup X_2$ of two subsomplexes, each $\cong B^2$, so that $X_1 \cap X_2 \cong B^1$, then X is PL homeomorphic to B^2. □

TOPOLOGICAL VERSIONS Four lemmas parallel to the above four, but without the epithets PL, and with ordinary homeomorphism ($\approx$) in place of PL homeomorphism ($\cong$), are also true and have similar proofs. □

PROOF OF (PLST). We proceed by induction on the number $N(C)$ of 2-simplices, linear in $\mathbb{R}^2$, needed to triangulate the (compact) sealed Jordan domain B with frontier C. Since, by Lemma (1), (PLST) is true for $N(C) = 1$, we can assume $N(C) > 1$.

There is always a 2-simplex S in the triangulation of B with at least one edge in C. By a short case analysis, one or two edges of the boundary of this S must always split the triangulation X of B as envisioned in Lemma (4). Since X_1 and X_2 are $\cong B^2$ by inductive assumption, Lemma (4) shows that $B \cong B^2$. □

REMARKS ON PROOFS OF THE PL CASE OF (JCT) The above easy proof of (PLST) requires the Jordan curve theorem for PL Jordan curves — call this (PL JCT). Indeed, (PL JCT) provides us with the sealed Jordan domain B on which the above induction turns.

(I) A GEOMETRICAL PROOF OF (PL JCT) One proceeds by induction on the number of corners of $C \subset \mathbb{R}^2$. We merely give some hints: Since the case when C is convex is trivial to prove directly, one can assume there is a linear segment Δ^1 in the frontier of the convex hull of C that intersects C in its end points $\partial\Delta^1$ only. This provides two Jordan curves C' and C'' with intersection Δ^1 and union $C \cup \Delta^1$. Since each has fewer corners than C, one can argue inductively ... □

(II) A COMBINATORIAL HOMOLOGY PROOF OF (PL JCT). The homology in question has coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and one calculates it for a finite simplicial 2-complex X using the chain complex $\mathrm{C}_*(X;\mathbb{Z}_2) : 0 \leftarrow \mathrm{C}_0 \leftarrow \mathrm{C}_1 \leftarrow \mathrm{C}_2 \leftarrow 0$ in which $\mathrm{C}_i = \mathrm{C}_i(X;\mathbb{Z}_2)$ can be identified to the set of finite subsets of the set of (unoriented) i-simplices of X. Now let X be a linear triangulation of the convex hull of C in $\mathbb{R}^2$, making the given PL Jordan curve C a subcomplex; then note that the set of 1-simplexes of X in C is naturally a cycle in $\mathrm{C}_1(X;\mathbb{Z}_2)$. Since X is contractible, $H_*(X;\mathbb{Z}_2) = 0$; in particular, the 1-cycle C is the boundary of a 2-chain B; this B is a set of 2-simplices whose union is a compact connected 2-manifold with boundary C. Using these facts as a lemma, one shows that any PL Jordan curve C in S^2 is the common boundary and frontier of compact connected submanifolds B_1 and B_2, so that $C = B_1 \cap B_2$. Then $B_1 \cup B_2$ is a closed PL sumanifold of S^2 and hence all of S^2. □

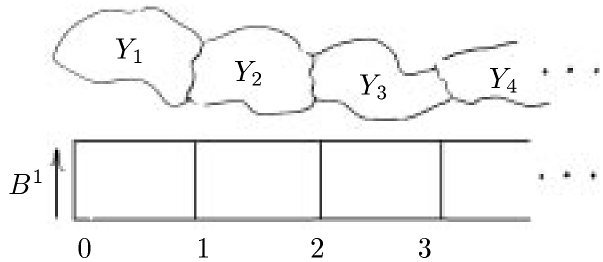
The proof of (APLST) uses three more lemmas.

LEMMA (5) — see Figure 7-a. *If a* PL *2-manifold Y is expressed as an infinite union*

$$Y = Y_1 \cup Y_2 \cup Y_3 \cup \cdots$$

where $Y_j \cong B^2$, and $Y_j \cap Y_{j+1} \cong B^1$ for all $j \geq 1$, while $Y_j \cap Y_{j+k} = \emptyset$ for $k > 1$, then $Y \cong [0,\infty) \times B^1 \cong B^1 \times [0,1) \cong (B^1 * i) - \{i\} \subset \mathbb{C}$, where $*$ denotes join and $i = \sqrt{-1}$. □

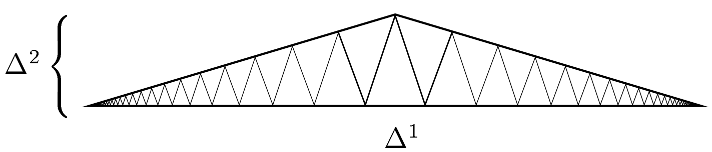
Figure 7-a

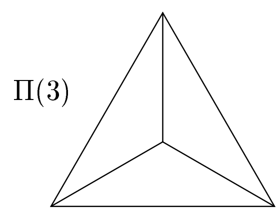


SUBLEMMA (6) *Let Δ^2 in $\mathbb{R}^2$ be a linear 2-simplex. Consider a self-homeomorphism φ of a 1-simplex Δ^1 that is a face of Δ^2. Suppose that φ fixes the end points $\partial\Delta^1$ and is* PL *on* $\mathrm{Int}\,\Delta^1 = \Delta^1 - \partial\Delta^1$. *Then φ can be extended to a self-homeomorphism Φ of Δ^2 that is* PL *on $\Delta^2 - \partial\Delta^1$ and fixes pointwise $\partial\Delta^2 - \mathrm{Int}\,\Delta^1$.*

PROOF OF (6). Form an infinite but locally finite linear triangulation of $\Delta^2 - \partial\Delta^1$ such that φ is linear on simplices — see Figure 7-b. Then, on $\Delta^2 - \partial\Delta^1$, the extension Φ is the unique symplexwise linear map that fixes all vertices outside of Δ^1; then one further extends by the identity on the two points $\partial\Delta^1$ to define Φ on all of Δ^2. □

Figures 7-b and 7-c





LEMMA (7) *Consider a self-homeomorphism φ of ∂B^2 that respects a finite set Y of N points in ∂B^2 and is* PL *on $\partial B^2 - Y$. Such a φ can be extended to an self-homeomorphism Φ of B^2 that is* PL *on $B^2 - Y$.*

PROOF OF (7). By Lemmas (2) and (3), it suffices to prove this when ϕ fixes X pointwise. We initially assume that $N \geq 3$. Identify B^2 by a PL homeomorphism to the standard solid regular N-gon $\Pi(N)$, sending Y to the vertices. Triangulate $\Pi(N)$ by coning the natural triangulation of the N-gon frontier to the origin of $\mathbb{R}^2$; see Figure 7-c. Applying Lemma (6) to the N resulting 2-simplexes simultaneously, but independently, we get the wanted extension Φ.

If $N = 0$, then Lemma (2) establishes Lemma (7).

If $N = 1$, (respectively $N = 2$), then the proof for $N = 4$ applied with data symmetric under rotation of angle $\pi/2$ (respectively angle π) establishes Lemma (7) by passage to the PL orbit space, which is a PL 2-disk. □

In §5, we mentioned that (APLST) follows from the known PL classification of surfaces. Here is the relevant part of the classification; it is formulated to additionally help to prove the Side Approximation Lemma (SAL) in the next section.

DEFINITION A PL surface M is **irreducible** if every PL circle embedded in M is the boundary of a PL 2-disk in M.

NONCOMPACT IRREDUCIBLE PL SURFACE CLASSIFICATION THEOREM (PLCT)

Let M be a connected noncompact irreducible PL *surface.*

(i) *If $\partial M = \emptyset$, then $M \cong R^2$.*

(ii) *If M has $N \geq 1$ boundary components, then $M \cong \mathbb{B}^2 - \widehat{X}$ where $\widehat{X}$ is any set of N points of $\partial B^2 = S^1$.*

For the proof, we fix any PL triangulation T of M.

PROOF OF (i). Arbitrarily large connected compact subcomplexes K of T are easily constructed. For $\varepsilon > 0$ small, the closed ε-neighborhood $L = N_\varepsilon(K)$ of K in the simplicial metric of T is a compact PL submanifold of M. By irreducibility of M, any boundary component C of L, is the boundary of a PL 2-disk B_C in M. The connected set $\mathrm{Int} L \supset K$ lies entirely in one of (a) B_C or (b) $M - B_C$. Case (b) cannot hold for all the finitely many components C of ∂L, for then M would be compact. Hence case (a) occurs.

We now know that M is an ascending union of open sets $\mathrm{Int} B_C \cong \mathrm{Int} B^2 \cong R^2$. But it is well known that any PL manifold that is an ascending union of open sets $\cong \mathbb{R}^n$ is itself $\cong \mathbb{R}^n$. The proof is an interesting exercise. See [RourS]. □

EXERCISE Combining (i) above with (PL JCT) and (PLST) prove the following: *If X is a compact connected subset of S^2 and U is any component of $S^2 - X$, then $U \cong \mathbb{R}^2$.* □

PROOF OF (ii) FOR $N = 1$. Build L an arbitrarily large compact connected PL submanifold as for case (i), but assure that that meets ∂M. Consider its frontier δL in M. It is a compact PL 1-manifold and each component of δL is a circle C in $\mathrm{Int} M$ or an arc J with $\partial J = J \cap \partial M$. A disk B_C with boundary a circle component C of δL cannot contain L since it lies in $\mathrm{Int} M$. For each arc component J of δL, there is a unique arc $J' \subset \partial M$ with the same boundary $\partial J = \partial J'$. Together they form a circle C_J, which, by irreducibility of M, bounds a PL disk B_J in M. Arguing as for (i), we see that exactly one such B_J contains L.

We now know that M is an ascending union of open sets $\mathrm{Int} B_J \cong R^2_+ \cong R^1 \times [0, \infty) \cong B^2 - \{i\}$. But any PL manifold that is an ascending union of open sets $\cong \mathbb{R}^n_+$ is itself $\cong \mathbb{R}^n_+$. Alternatively, one can apply Lemma (5). □

PROOF OF (ii) FOR $N > 1$. We induct on N. Since M is connected, there exists a PL path $g : [0,1] \to M$ joining two distinct components of ∂M and meeting ∂M only at its endpoints. By a canonical shortcutting procedure, one can make g an embedding onto a PL arc $\gamma \subset g([0,1])$; for more on this shortcutting, see the proof of (PLCL) in the next section; it was already used by Jordan [Jord2] in 1887. Splitting M at γ yields two components M_1 and M_2 with intersection γ. Indeed, if the splitting yielded just one component, M would not be irreducible. Each M_i is a connected noncompact irreducible PL surface with $N_i \geq 1$ boundary components. Further, $N = N_1 + N_2$; whence N_1 and N_2 are $< N$. Applying induction on N and Lemma (7), part (ii) easily follows. In more detail: choose a PL Jordan arc J in $B^2 - \widehat{X}$ so that $J \cap S^1 = \partial J$, and J splits B^2 into two PL disks B_i containing respectively subsets $\widehat{X}_i$ of N_i points of $\widehat{X}$. Then, inductive hypothesis offers PL homeomorphisms $h_i : B_i - \widehat{X}_i \to M_i$, and Lemma (7) allows us to adjust them so that the restrictions $h_1|J$ and $h_2|J$ induce one and the same PL homeomorphism onto γ. Now h_1 and h_2 together define the required PL homeomorphism $B^2 - \widehat{X} \cong M$. □

The following was used directly (but for $N = 3$ only) in proving (ST) in §6.

ALMOST PL SCHOENFLIES THEOREM (APLST) *Let C be a Jordan curve in $\mathbb{R}^2$ that is PL (= piecewise linear) except at a (possibly empty) finite set X of $N = N(X)$ of points in C. A homeomorphism H can be built from the standard PL disk B^2 onto the sealed Jordan domain B with frontier C; furthermore, H can be PL except possibly at $h^{-1}X \subset \partial B^2$.*

PROOF OF (APLST) FOR $N = 0$. This is exactly (PLST). □

PROOF OF (APLST) FOR $N > 0$. By (PLCT) there is a PL homeomorphism $H_0 : B^2 - \widehat{X} \longrightarrow B - X$ where $\widehat{X}$ and X are sets of exactly N points in S^1 and C respectively. The following Compactification Lemma (8) from general topology assures us that there is an extension of H_0 to a continuous surjection $H : B^2 \to B$ with $H(\widehat{X}) = X$. Since a surjection of sets of N elements is bijective, H is bijective. Then, by compactness, H is a homeomorphism (see §2). □

Compactification Lemma (8) *Let X and Y be compact Hausdorff spaces; let A and B be finite subsets of X and Y respectively such that $X-A$ is dense in X and $Y-B$ is dense in Y. Consider a continuous map $g : X-A \longrightarrow Y-B$ that is proper in the sense that, for every compact set K in $Y-B$, the preimage $g^{-1}(K)$ is compact in $X-A$. Suppose that each point $a \in A$ has arbitrarily small compact neighborhoods X_a in X such that $X_a - \{a\}$ is connected. Then g extends uniquely to a continuous map $G : X \to Y$. Moreover, G is surjective if g is surjective.*

Remarks.

(i) The connectivity assumption is essential, as the homeomorphism $(\mathbb{S}^1 - (1,0)) \approx \mathrm{Int}\mathbb{B}^1$ reveals.

(ii) This lemma can be regarded as a fragment of the 'end compactification' theory of Kérékjartó and Freudenthal.

(iii) The assumption that the X_a in (8) are compact is inessential because the closure of a connected set is always connected.

Proof of (8) It suffices to establish the existence of a continuous extension G of g. Indeed, the asserted uniqueness of G follows from the density of $X-A$ in X. The implication "g surjective implies G surjective", follows from the density of $Y-B$ in Y.

As for existence, we begin by disposing of the case when B consists of a single point. Then Y is the well known Alexandroff or one-point compactification of the space $(Y-B)$ by the 'infinity' point B. The extension G is obtained by sending all of A to B. Its continuity is an easily proved and standard fact about Alexandroff compactification, cf. [Dugu]. Note that, in this case, the assumptions mentioning connectivity are superfluous; also A need only be compact, rather than finite.

It remains to prove existence of G in the case when B is ≥ 2 points.

Assertion $(*)$ With the data of (8), for any point a in A, the following exist: a point b in B; a compact neighborhood Y_b of b in Y with $Y_b \cap B = \{b\}$; and a compact neighborhood X_a of a in X such that $g(X_a - \{a\}) \subset Y_b - \{b\}$.

Proof that G exists if $(*)$ is true. With the data provided by $(*)$, consider, for each a in A the (proper!) restriction $g_a : X_a - \{a\} \longrightarrow Y_b - \{b\}$ of g. The proved case of (8) where B is a single point assures us a continuous extension of this g_a to a continuous map $G_a : X_a \to Y_b \hookrightarrow Y$. These G_a together yield an extension G of g. It is continuous since g and these G_a are continuous and agree on the open cover of X formed by $X-A$ and the interiors in X of these X_a. □

Proof of $(*)$ — using connectivity. Choose, for the points b in B, pairwise disjoint compact neighborhoods Y_b in Y; we denote by Y_B their union. Since $Y_B - B$ is a neighborhood of (Alexandroff) infinity of $Y-B$, (i.e. $Y - Y_B$ has compact closure in $Y-B$), while g is proper, it follows that $g^{-1}(Y_B - B)$ is a neighborhood of infinity in $X-A$, whence $A \cup g^{-1}(Y_B - B)$ is a neighborhood of A in X. For any given a in A, choose a neighborhood X_a of a in X, with $(X_a - \{a\})$ connected, and so small that $(X_a - \{a\}) \subset g^{-1}(Y_B - B)$. As the preimages $g^{-1}(Y_b - \{b\})$ form a closed partition of $g^{-1}(Y_B - B)$, this connectedness implies that $(X_a - \{a\})$ lies entirely in some one $g^{-1}(Y_b - \{b\})$. □

It remains to prove (APLST+), which supported (APLST) in §6.

Complement (APLST+) to (APLST) — see §5.
If f: $\partial B^2 \to C$ is a homeomorphism that is PL *except at $f^{-1}(X)$, then $H : B^2 \to B$ offered by* (APLST) *can be (re)chosen to extend f.*

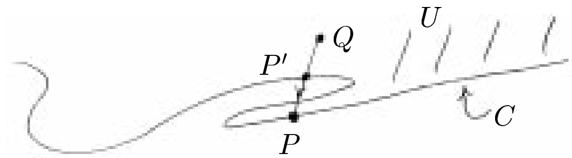
Proof of (APLST+). This follows immediately from (APLST) and Lemma (7). □

8 Proof of the Side Approximation Lemma (SAL) — see §5

This is the second and last section devoted to tools for the proof of the Schoenflies Theorem (ST). We begin with a couple of lemmas valid in all dimensions.

Linear Access Lemma (LAL), cf. [Vebl][Brou1]. *Consider any open subset D of $\mathbb{R}^n$ with its closure B and its frontier C. In C consider any point P and an $\varepsilon > 0$. There exists a compact linear arc L in B such that $L \cap C$ is a single point ε-near to P. If C is* PL *near P then $L \cap C$ can be P itself.*

Figure 8-a



Proof of (LAL) — see Figure 8-a. The case when C is PL near P is immediate from the locally cone-like behavior of a PL object [RourS]. Otherwise, since C is the frontier of $D = B - C$, there is a point Q of D that is ε-near to P (for Euclidean distance). This assures that the oriented linear segment $\Delta = \Delta^1(Q,P)$ in $\mathbb{R}^n$ running from Q to P is also ε-near to P. Letting P' be the first point on Δ that lies on C, the linear segment $L = \Delta^1(Q,P')$ establishes the lemma. □

PL Chord Lemma (PLCL) *Given again the data of* (LAL), *suppose that D is connected, and that $P' \neq P$ is a second given point in C. Then there exists a compact* PL *arc L in B such that $L \cap C = \partial L$ and L joins a point ε-near P to a point ε-near P'. Furthermore, if C is* PL *near P and near P', then one can have $\partial L = \{P, P'\}$.*

Proof of (PLCL). Applying (LAL) twice, and then the connectivity of $B - C$ assured by (JCT), we get a PL path $\lambda : [0,1] \to B$ so that $\Lambda = \lambda([0,1])$ is as required for L, except for one fault: λ may not be injective. Taking, in order, all possible shortcuts to λ offered by the self-intersections of this path, we (canonically!) derive an injective PL path whose image $L \subset \Lambda$ is a PL Jordan arc with the required properties. □

Data We now restrict to dimension 2 and consider a Jordan curve $C \subset \mathbb{R}^2$ that by (JCT) is the frontier of a unique Jordan domain D. In the sealed Jordan domain $B = C \cup D$ we consider what has been called an **nice chord J of B**, which is, by definition, a Jordan arc J in B such that $\partial J = J \cap C$ and $J \cap D$ is PL. A **nice sector X of B** is defined to be a sealed Jordan domain in $\mathbb{R}^2$ with frontier $J \cup J_0$ where J is a nice chord of B and J_0 is a Jordan arc in C with $\partial J = \partial J_0$.

Side Compression Lemma (SCL) *With the above data, let J be an nice chord of B and let X be the nice sector of B whose frontier in $\mathbb{R}^2$ is the Jordan curve $J \cup J_0$. Then for any open neighborhood U of J_0 in B there exists another nice sector X' of B such that* (a) *$X' \subset U \cap X$,* (b) *$X' \cap C = J_0$, and* (c) *the frontier J' of X' in B is an nice chord of B that coincides with J near C.*

Proof of (SCL). The PL manifold $X - J_0$ with boundary J is irreducible by (PLST) of the last section; hence, by the PL classification theorem of the last section, there is a PL homeomorphism $H : \mathbb{R}^1 \times [0,\infty) \to X - J_0$. Since $X - U$ is a compact set in $X - J_0$, it is contained in the image by H of any square $[-r,r] \times [0,2r]$ with $r > 0$ sufficiently large. Then the required nice sector of B can be $X' = X - H((-r,r) \times [0,2r])$. □

Remark Nothing has been done to assure nice convergence of H in B; it is quite possible, for example, that the image by H of some infinite linear ray in $\mathbb{R}^2_+ = \mathbb{R}^1 \times [0,\infty)$ (but not in $\mathbb{R}^1 \times \{0\}$), has all of J_0 as limit set.

Peripheral Niceness Proposition (PNP) *With the same data, let p be any point of the Jordan curve C. There exist arbitrarily small compact neighborhoods of p in B that are nice sectors of B.*

Proof of (PNP). Let V be any open neighborhood of p in B. By the PL Chord Lemma (PLCL), there exists a nice sector X of B such that $X \cap C$ is a Jordan arc J_0 and $p \in (J_0 - \partial J_0)$. Now (SCL) provides a nice sector $X' \subset V$ of B with $X' \cap C = J_0$. It is the required neighborhood of p in B. □

Piecewise Linear Access Theorem (PLAT) *Let $C \subset \mathbb{R}^2$ be a Jordan curve and B its sealed Jordan domain.*

(i) *For any point* p *in* C, *there exists a Jordan arc* J *in the sealed Jordan region* B *with frontier* C *such that* $J \cap C = p$ *and* $J - p$ *is* PL.

(ii) *For any two distinct points* p *and* q *in* C, *there exists a Jordan arc* J *in* B *such that* $J \cap C = \{p, q\}$ *and* $J - \{p, q\}$ *is* PL.

PROOF OF (i). By the Peripheral Niceness Proposition (PNP) there exists an infinite descending sequence $X_1, X_2, X_3, \ldots$ of compact neighborhoods of p in B such that their intersection $\bigcap_i X_i$ is p and their respective frontiers in B are pairwise disjoint and nice chords J_i of B. Choose a point p_i in $J_i - \partial J_i$. Let Y_i be the compact closure of $X_i - X_{i+1}$. Its frontier δY_i in $\mathbb{R}^2$ is a Jordan curve containing the disjoint Jordan arcs J_i and J_{i+1} By (PLCL) there is a PL Jordan arc K_i in Y_i such that $\partial K_i = \{p_i, p_{i+1}\}$. The union $\{p\} \cup \bigcup_i K_i$ is the wanted Jordan arc J. □

PROOF OF (ii). One proof is similar. Alternatively, one can piece together two arcs offered by (i) using the method of the proof of (PLCL). □

The result needed in the proof of (ST) is now within easy reach.

SIDE APPROXIMATION LEMMA (SAL) *Let* J *be any Jordan arc in the Jordan curve* $C \subset \mathbb{R}^2$ *and let* $B \subset \mathbb{R}^2$ *be the sealed Jordan domain with frontier* C. *There exists a Jordan arc* $J' \subset B$ *with the same end points* $\{P_0, P_1\}$ *as* J, *such that* $J' \cap C = \{P_0, P_1\}$ *and* $J' - C$ *is* PL. *Further, one can choose* J' *in any prescribed neighborhood of* J.

PROOF OF (SAL). This is an immediate consequence of part (ii) of (PLAT) and the Side Compression Lemma (SCL). □

9 Some consequences of (ST) and (PLST)

Many proofs will be left partly or wholly as exercises — hopefully all pleasant!

UNKNOTTING THEOREM [Scho4, §13]. *If* C *and* C' *are Jordan curves in* $\mathbb{R}^2$ *there exists a self-homeomorphism* $H : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $H(C) = C'$. *Furthermore, if a homeomorphism* $h : C \to C'$ *is given, then* H *can coincide with* h *on* C.

HINTS FOR PROOF. Given h, (ST) of §4 provides a homeomorphism $H_0 : B \to B'$ extending h where B and B' are the sealed Jordan domains with frontiers C and C' respectively. The same argument extends H_0 to a self-homeomorphism H_1 of the 1-point compactification $\mathbb{R}^2 \cup \infty = \mathbb{S}^2$. One can adjust H_1 on $\mathbb{R}^2 - B$ to fix ∞. Then the required H is the restriction of H_1 to $\mathbb{R}^2$. □

GRAPH DATA A **graph** is a topological space homeomorphic to a locally finite simplicial 1-complex. Consider a graph Γ embedded as a closed subset of a topological 2-manifold M without boundary.

LOCAL GRAPH TAMING THEOREM (LGTT)
With above data, let P *be any point in* Γ. *There exists an open neighborhood* U *of* P *in* M *and an embedding* $f : U \to \mathbb{R}^2$ *such that the image* $h(U \cap G)$ *is a* PL *subset of* $\mathbb{R}^2$.

Here are three exercises leading to a proof of (LGTT) from (ST).

(1) *Without loss of generality,* Γ *is homeomorphic to the cone on* $N \geq 1$ *points and* M *is* $\mathbb{S}^2$. (This is assumed for the rest of the proof.) □

(2) *The case* $N \leq 2$.

HINTS FOR (2). Suppose then that N is 1 or 2. The graph Γ is then a Jordan arc in $\mathbb{R}^2$. Identify $\mathbb{S}^2 \approx \mathbb{R}^2 \cup \infty$ by homeomorphism, and form the 2-fold branched cyclic covering of $\mathbb{S}^2$ branched at the two end points ∂G. In it apply (ST) to the Jordan curve covering Γ and examine the quotient map to $\mathbb{S}^2 \supset \mathbb{R}^2$. □

(3) *The case* $N \geq 3$.

HINTS FOR (3). Proceed by induction on N. Choose a Jordan arc J in Γ joining any pair among the N distinct extremal (i.e. non-separating) points. Then form the 2-fold branched cyclic covering of $\mathbb{S}^2$ branched at the two end points ∂J, and complete the induction by proceeding much as for (2). □

Next come basic uniqueness theorems for PL structures on 2-manifolds. See [Mois2] for alternative proofs also accessible to students.

SURFACE HAUPTVERMUTUNG (SH) *Consider any topological homeomorphism* $g : M \to W$ *of* PL *2-manifolds. Let* $A \subset M$ *be closed and* $B \subset M$ *be compact. Suppose that* g *is* PL *near* A. *Then there exists another homeomorphism* $h : M \to W$ *that is* PL *near* $A \cup B$, *and coincides with* g *near* A *and outside a compact subset of* M.

SURFACE HAUPTVERMUTUNG WITH APPROXIMATION (SH+) *Consider any topological homeomorphism* $g : M \to W$ *of metric* PL *2-manifolds. Let* A *be a closed subset of* M *near which* g *is* PL, *and let* $\varepsilon : M \to \mathbb{R}$ *be any strictly positive continuous function. Then there exists a self-homeomorphism* $\gamma : M \to M$ *fixing all points near* A, *such that* $g\gamma : M \to W$ *is a* PL *homeomorphism, and such that, for all points* x *in* M, *the distance in* M *from* x *to* $\gamma(x)$ *is* $< \varepsilon(x)$.

These statements make sense in all dimensions (i.e. with any integer $n > 0$ in place of 2) and are collectively known as the "Hauptvermutung for manifolds" (first formulated in print by Steinitz in 1907). However, they are only true for $n \leq 3$ (see [Mois1–2]), or with extra hypotheses (see [KirS1] and [FreeQ]). See [Sieb] for more historical notes and references.

The proofs will be rather formal consequences of the following three 'handle lemmas': $\mathfrak{H}(2,0)$, $\mathfrak{H}(2,1)$, $\mathfrak{H}(2,2)$:

$\mathfrak{H}(2,k)$ HANDLE LEMMA FOR DIMENSION 2 AND INDEX $k \leq 2$, — cf. [KirS1].
The Surface Hauptvermutung (SH) *holds true under the following set of extra conditions:*
(i) $M = \mathbb{R}^2$ *and* W *is an open subset of* $\mathbb{R}^2$.
(ii) *The closed set* A *is empty if* $k = 0$ *and is* $(\mathbb{R}^k - \mathrm{Int}\, B^k) \times \mathbb{R}^{2-k}$ *in general.*
(iii) *The compact set* B *(which is is called the k-handle core) is the origin if* $k = 0$ *and is in general the* k*-disk* $B^k \times 0$.

Condition (i) permits us to use in M and W our results on Jordan domains.

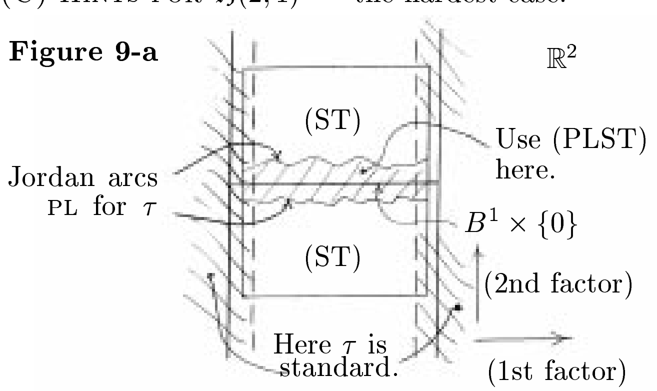
The following exercises outline proofs of these lemmas needing little more than (ST) and (PLST).

(A) HINTS FOR $\mathfrak{H}(2,0)$. Apply (ST+) in W to the Jordan curve $g(\partial\, 2B^2)$. □

(B) HINTS FOR $\mathfrak{H}(2,2)$. Choose $\lambda < 1$ so near to 1 that h is PL on $(\mathbb{R}^2 - \partial\lambda B^2)$. Then apply (PLST) just once to the PL circle $h(\mathbb{R}^2 - \partial\lambda B^2)$ in $W \subset \mathbb{R}^2$. □

(C) HINTS FOR $\mathfrak{H}(2,1)$ — the hardest case.

Figure 9-a



In W, apply (PLST) once, and (ST) twice, as indicated in Figure 9-a, to three adjacent sealed Jordan domains. For convenience we have identified W to M_τ where τ is the unique PL manifold structure on M making $h : M_\tau \to W$ a PL homeomorphism. The two shared PL Jordan arcs in the frontier of the PL sealed Jordan domain are provided by the easy PL Chord Lemma (PLCL) of §8. The coning lemmas of §7, particularly the PL and topological versions of Lemma (2), can serve to fit together the embeddings that (PLST) and (ST) provide. □

REMARKS

(1) The proof for index 0, in case (A) above, could imitate case (C) using, in place of (ST+), the simpler statements (ST) and (PLST).

(2) The use of (ST) in the above proof is very mild since the Jordan curves met obviously can be (bi-)collared, cf. [Brow2][KirS2]. Thus, for example, the proof of (ST) by M. Brown [Brow1] under this stronger hypothesis (it is valid for all dimensions) could here replace our proof of (ST).

(3) Some classical proofs of (SH) use no more than (PLST), but they seem to share some of the complexities of our proof of (ST).

(4) The Alexander isotopy theorem of [Alex4] shows that the homeomorphism h promised by $\mathfrak{H}(2,k)$ is (topologically) isotopic to g fixing points in A and all points outside a compact set in $\mathbb{R}^2$.

We delay the indications for the proof of (SH) because the proof of (SH+) seems more interesting.

Definition For any (closed) simplex σ of a simplicial complex $\mathcal{T}$, the **star** $\mathrm{St}(\sigma)$ of σ in $\mathcal{T}$ is the union of all (closed) simplices of $\mathcal{T}$ that contain σ.

Proof of (SH+). Let U be an open neighborhood of A in M on which g is PL. Choose a PL triangulation $\mathcal{T}$ of M so fine that: (a) if a simplex σ of $\mathcal{T}$ intersects A then it lies in U, (b) for any vertex v of $\mathcal{T}$ the diameter of its star neighborhood $\mathrm{St}(v,\mathcal{T})$ in $\mathcal{T}$ is less than the minimum of ε on the compact set $\mathrm{St}(v)$, and (c) some open neighborhood of $g(\mathrm{St}(v))$ in W is a PL chart of W in the sense that it admits an open PL embedding into $\mathbb{R}^2$.

For each vertex v of $\mathcal{T}$, PL identify to $\mathbb{R}^2$ a small open neighborhood $N(v) \subset \mathrm{St}(v)$ while assuring that the $N(v)$ are disjoint for distinct vertices v. Apply $\mathfrak{H}(2,0)$ in $N(v)$ to obtain a self-homeomorphism g_v of $N(v)$ with compact support in $N(v)$ such that the composition $g\gamma_v$ is PL near v. Doing this independently for all such $N(v)$, we obtain a self-homeomorphism g_0 of M so that the composition $h_0 = g\gamma_0 : M \to W$ is PL near all vertices of $\mathcal{T}$.

Next, for each edge e of $\mathcal{T}$ disjoint from A, PL identify to $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$ a small open neighborhood $N(e) \subset \mathrm{St}(e)$ of $e - \partial e$ in M, in such a way that: (a) $\mathbb{R}^1 \times 0$ corresponds to $e - \partial e$, (b) the homeomorphism h_0 is PL near $(\mathbb{R}^1 - \mathrm{Int} B^1) \times \mathbb{R}^1$, and (c) the sets $N(e)$ are disjoint for distinct edges. Apply $\mathfrak{H}(2,1)$ in $N(e)$ to obtain a self-homeomorphism g_e of $N(e)$ with compact support in $N(e)$ such that the composition $h_e = h_0\gamma_e$ is PL near e. Doing this independently for all such $N(e)$, we obtain $\gamma_1 : M \to M$ so that $h_1 = h_0\gamma_1 = h\gamma_0\gamma_1$ is PL near all vertices and edges of $\mathcal{T}$.

Finally, for each face (2-simplex) f of $\mathcal{T}$ disjoint from A, let $N(f)$ be $f - \partial f$ PL identify it to $\mathbb{R}^2$ in such a way that h_1 is PL near $\mathbb{R}^2 - \mathrm{Int} B^2$. The sets $N(f)$ are necessarily disjoint for distinct faces f. Apply $\mathfrak{H}(2,2)$ in $N(f)$ to obtain a self-homeomorphism g_f of $N(f)$ with compact support in $N(f)$ such that $g\gamma_f$ is PL near f. Doing this independently for all such $N(f)$, we obtain $\gamma_2 : M \to M$ such that that $h_2 = h_1\gamma_2$ is PL on all simplices of $\mathcal{T}$, hence on all of M.

Observe that, by construction, each of γ_1, γ_2, γ_3 respects the vertex star $\mathrm{St}(v)$ for each vertex v of $\mathcal{T}$. Hence the composition $\gamma = \gamma_0\gamma_1\gamma_2$ does likewise and it follows, for the metric d on M, that $d(\gamma(x),x) < \varepsilon(x)$ for all x. □

Hints for proof of (SH). Carry through the proof of (SH+) above with the following modifications:

(a) Eliminate the approximation conditions — say by choosing ε identically 1 and replacing metric d by $d/1+d$.

(b) Replace $\mathcal{T}$ by the finite subcomplex of all (closed) simplices that meet the compact set B.

The resulting self-homeomorphism γ of M then has compact support since it is a finite composition of homeomorphisms with compact support. The required h is the composition $h = g\gamma$. □

Exercise Deduce (SH) from (SH+). □

Exercise Using Remark (4) following $\mathfrak{H}(2,k)$, establish versions with isotopy of (SH) and (SH+). □

Surface Triangulation Theorem (STT) — see [Radó] 1925.
Consider any topological 2-manifold M without boundary. There always exists a PL *structure τ on M. Furthermore, if σ is a given* PL *structure on an open subset U of M, then τ can coincide with σ on U.*

PROOF OF (STT). This proof is valid for non-metrizable manifolds. Zorn's Lemma clearly applies to the partially ordered set of all pairs (V,α) consisting of an open subset V of M and a PL structure α on V — the ordering being by PL inclusion. Thus, there is a maximal pair (V,τ) containing (U,σ).

Seeking a contradiction, suppose V is not all of M; select a point x in $M-V$ and an open neighborhood W of x in M that is homeomorphic to $\mathbb{R}^2$; use the homeomorphism to endow W with a PL structure β. Let $\varepsilon : W \to \mathbb{R}$ be a continuous function that is positive on $W\cap V$ and zero elsewhere on W. (SH+) then provides a PL homeomorphism $f:(W\cap V)_\beta \to (W\cap V)_\tau$ that is ε-near the identity map of $W\cap V$, and hence extends by the identity to a PL isomorphism $g: W_\beta \to W_{g(\beta)}$. Now $g(\beta)$ and τ agree on $(W\cap V)$ and hence define a PL structure on $(W\cup V)$ showing that (V,τ) is *not* maximal, the contradiction proving (SH+). □

REMARKS ON BOUNDARIES AND EMBEDDED GRAPHS

(a) In this section we have paid little or no attention to 2-manifolds with non-empty boundary. However, similar results can easily be *deduced* for them with the help of M. Brown's collaring theorems [Brow1], [KirS2, App A of Essay I,].

(b) In a similar manner, one can deduce versions of (STT) and (SH) for pairs (M,Γ) with Γ a graph as introduced for (LGTT). Indeed, using (LGTT), one can adapt M. Brown's collaring theorems to deal with a slightly generalized notion of collaring for Γ in M. This extension of (STT) and (SH) is greatly facilitated if one takes the time to show first that there is a natural construction of a 2-manifold with boundary M_Γ such that $\mathrm{Int}\, M_\Gamma = M-\Gamma$ and M_Γ maps naturally onto M sending ∂M_Γ onto Γ, — indeed immersively wherever Γ is a 1-manifold. For PL pairs (M,Γ), this M_Γ is well known and called **the splitting of M at Γ**; its construction in terms of a PL triangulation of (M,Γ) is obvious. Given M_Γ, the relevant (generalized) collarings of Γ in M are in natural one-to-one correspondence with the usual sort of collarings for ∂M_Γ in M_Γ.

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