COMPUTATIONS IN COMPLEX EQUIVARIANT BORDISM THEORY

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1. Introduction

Bordism theory is fundamental in algebraic topology and its applications. In the early sixties Conner and Floyd introduced equivariant bordism as a powerful tool in the study of transformation groups. In the late sixties, tom Dieck introduced homotopical bordism in order to refine understanding of the localization techniques employed by Atiyah, Segal and Singer in index theory. Though equivariant bordism theories are fundamentally important they have been mysterious from a computational point of view, even for cyclic groups of order p where only a result by Kosniowski for geometric bordism [11] gives any explicit information. Recently, interest in equivariant bordism has grown in the wake of work on Morava K-theory of classifying spaces by Hopkins, Kuhn and Ravenel [10], as a theorem of Greenlees and May [9] shows that the homotopical equivariant bordism ring MU_*^G determines the structure of E-homology and cohomology of classifying spaces for any complex-oriented homology theory E. Moreover, Kriz [12] has recent computations of $MU_*^{\mathbb{Z}/p}$, which have quite a different form from ours.

In this paper we present the first computations of the ring structure of the coefficients of equivariant bordism. We have a result which establishes an algebraic framework in which to understand equivariant bordism for any group such that any proper subgroup is contained in a proper normal subgroup. This class of groups includes abelain groups and p-groups. Our general result is computationally satisfying when one can find a suitable representation of MU_*^G. For abelian groups, the map to completion at the augmentation ideal seems to be such a representation. We state our results in the abelian case as follows.

Let ρ denote the standard one-dimensional representation of S^1. Let n denote the representation where S^1 acts trivially on $\mathbb{C}^n$. Choose as a model of the three-sphere the set of all pairs of complex numbers (z_1, z_2) such that $|z_1|^2 + |z_2|^2 = 1$.

Definition 1.1. *Let $\mathbb{P}(V)$ denote the $(|V| - 1)$-dimensional complex projective space of lines in V. In particular, $\mathbb{P}(n \oplus \rho)$ is the projective space with an S^1 action where in homogeneous coordinates $\zeta \in S^1$ acts by multiplication on the last coordinate.*

Definition 1.2. *Define $\Gamma(X)$ for any stably complex S^1-manifold to be the stably complex manifold*

$$\Gamma(X) = X \times_{S^1} S^3 \sqcup (-\overline{X}) \times \mathbb{P}(1 \oplus \rho),$$

where S^3 has the standard Hopf S^1-action, $-\overline{X}$ is the manifold obtained from X by imposing a trivial action on X and taking the opposite orientation, and the S^1-action on $X \times_{S^1} S^3$ is given by

$$\zeta \cdot [m, z_1, z_2] = [\zeta \cdot m, z_1, \zeta z_2].$$

Inductively define $\Gamma^i(X)$ *to be* $\Gamma(\Gamma^{i-1}(X))$, *where* $\Gamma^0(X) = X$.

The importance of the construction Γ is that it gives an explicit model by which we can compute the image of a geometric class $[X]$ (that is, a class in the image of the Pontrjagin-Thom map) in the completion of $MU_*^{S^1}$ at its augmentation ideal.

Theorem 1.1. *Let* $[M]$ *be class in* $MU_*^{S^1}$ *which is the image under the Pontrjagin-Thom map of the class in geometric bordism represented by the complex* S^1*-manifold* M. *The image of* $[M]$ *under the map from* $MU_*^{S^1}$ *to its completion at its augmentation ideal, which is isomorphic to* $MU_*[[x]]$, *is the power series*

$$[\alpha(M)] + [\alpha(\Gamma(M))]x + [\alpha(\Gamma^2(M))]x^2 + \cdots ,$$

where $\alpha(\Gamma^i(M))$ *is the manifold obtained from* $\Gamma^i(M)$ *simply by forgetting the* G*-action.*

Definition 1.3. *Let* $Y_i(x) \in MU_*[[x]]$ *be the image of the class* $[\mathbb{P}(i \oplus \rho)]$ *under the completion map.*

We are now ready to state our main theorems. Let $T = (S^1)^k$. Though we will prove the following results from our techniques, we cite them now for convenience. Recall (from [14] for example) that for abelian groups G the completion of the equivariant bordism ring MU_*^G at its augmentation ideal (which is the kernel of the forgetful map to ordinary bordism) is isomorphic to $MU^*(B_G)$. For the torus, $MU^*(B_T)$ is isomorphic to $MU_*[[x_1, \cdots x_k]]$. Moreover, Landweber computed that more generally $MU^*(B_G)$ is the quotient of $MU^*(B_T)$ by the ideal $([d_i]_F x_i)$, where $[d]_F x$ denotes the d-series of the universal formal group law over MU_* and the d_i are the orders of the cyclic factors of G when it is put in canonical form, with $d_i = 0$ for a factor of S^1.

Theorem 1.2. *et* G *be an abelian group. If we embed* G *in a torus, the restriction map* $MU_*^T \to MU_*^G$ *is surjective.*

Theorem 1.3. *The image of* MU_*^T *in its completion at the augmentation ideal is contained in the minimal sub-ring* S *of* $MU_*[[x_1, \cdots , x_k]]$ *which satisfies the following two properties:*

- S *contains the series* $Y_i(f)$ *where* f *ranges over* $[m_1]_F x_1 +_F \cdots +_F [m_k]_F x_k$, $Y_0(f) = f$ *and* $Y_i(f)$ *for* $i > 0$ *are defined above.*
- *If* $\alpha f \in S$ *then* $\alpha \in S$, *for* $f = [m_1]_F x_1 +_F \cdots +_F [m_k]_F x_k$.

We can recover the image of MU_*^G in its completion at the augmentation ideal for general G by embedding G in a torus T and reducing the image of MU_*^T in $MU_*[[x_i]]$ modulo the ideal $([d_i]_F x_i)$ as above.

The structure of this paper is as follows. In the next two sections we review basic definitions. In section 4 we establish the well-known connection between taking fixed sets and localization in equivariant homotopy theory, in our setting of equivariant bordism. Section 5 is brief, introducing the algebraic language which has proven to be most useful in equivariant bordism theory. We prove our main theorems in section 6. We give geometric applications in section 7, in particular answering a question of Milgram about the geometry of Lens spaces and one of Bott about S^1 actions on complex surfaces.

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2. Prelimanaries

Until otherwise noted, the group G is a compact Lie group.

All G actions are assumed to be continuous, and G-actions on manifolds are assumed to be smooth. For any G-space X, we let X^G denote the subspace of X fixed under the action of G. The space of maps between two G-spaces, which we denote $\mathrm{Maps}(X,Y)$ has a G-action by conjugation. We denote its subspace of G-fixed maps by $\mathrm{Maps}^G(X,Y)$. Throughout, E_G will be a contractible space on which G is acting freely. And B_G, the classifying space of G, is the quotient of E_G by the action of G.

We will always let V and W be finite-dimensional complex representations of G. All representations, and in fact all G-vector bundles, are assumed to carry a G-invariant inner product. Our G-vector bundles will always have paracompact base spaces. We will use the same notation for a G-bundle over a point as for the corresponding representation. Let $|V|$ denote the dimension of V as a complex vector space. The sphere S^V is the one-point compactification of V, based at 0 if a base point is needed. And the sphere $S(V)$ is the unit sphere in V with inherited G-action. For a G-vector bundle E, let $T(E)$ denote its Thom space, which is the cofiber of the unit sphere bundle of E included in the unit disk bundle of E. Thus for V a representation $T(V) = S^V$.

Let $R^+(G)$ denote the monoid (under direct sum) of isomorphism classes of complex representations of G, and let $R(G)$ denote the associated Grothendieck ring (where multiplication is given by tensor product). We let $Irr(G)$ denote the set of isomorphism classes of irreducible complex representations of G, and let $Irr^*(G)$ be the subset of non-trivial irreducible representations. Recall from the introduction that ρ is the standard representation of S^1. We will by abuse use ρ to denote the standard representation restricted to any subgroup of S^1. We use n or $\mathbb{C}^n$ to denote the trivial n-dimensional complex representation of a group. We will sometimes think of representations as group homomorphisms, and talk of their kernels, images, and so forth.

For a ring R we let $R\left[\frac{1}{e_i}\right]$ denote the localization of R in which $e_i \in R$ are inverted.

3. Definition of MU^G and Basic Properties Assumed

There are two basic definitions of bordism, geometric and homotopy theoretic. Equivariantly, these two theories are not equivalent, and we will comment on this difference later in this section.

Our main concern is the homotopy theoretic version of complex equivariant bordism, as first defined by tom Dieck [7]. Fix $\mathcal{U}$, a complex representation of G which contains all irreducible representations of G infinitely often. If there is ambiguity possible we specify the group by writing $\mathcal{U}(G)$. Let $BU^G(n)$ be the Grassmanian of complex n-dimensional linear subspaces of $\mathcal{U}$. Let ξ_n^G denote the tautological complex n-plane bundle over $BU^G(n)$. As in the non-equivariant setting, the bundle ξ_n^G over $BU^G(n)$ serves as a model for the universal complex n-plane bundle.

Definition 3.1. *We let TU^G be the pre-spectrum, indexed on all complex subrepresentations of $\mathcal{U}$, defined by taking the Vth entry to be $T(\xi^G_{|V|})$ (it suffices to define entries of a prespectrum only for complex representations). Define the bonding maps by noting that for $V \subseteq W$ in $\mathcal{U}$, letting $V^\perp$ denote the complement of V in W, we have*

$$S^{V^\perp} \wedge T(\xi^G_{|V|}) \cong T(V^\perp \times \xi^G_{|V|}).$$

Then use the classifying map

$$V^\perp \times \xi^G_{|V|} \to \xi^G_{|W|}$$

to define the corresponding map of Thom spaces. Pass to a spectrum in the usual way, so that the Vth de-looping is given by

$$\lim_{W \supseteq V} \Omega^{V^\perp}(T(\xi^G_{|W|})),$$

to obtain the homotopical equivariant bordism spectrum MU^G.

We list some standard properties of MU^G. For a thorough treatment of foundations, see the volume [14].

By construction, the homology theory which is represented by MU^G has suspension isomorphisms by any real representation of G. Such suspension isomorphisms are necessary for instance when constructing equivariant versions of transfer. The classifying map of the Whitney sum

$$\xi^G_{|V|} \times \xi^G_{|W|} \to \xi^G_{|V \oplus W|}$$

gives rise to a map

$$T(\xi^G_{|V|}) \wedge T(\xi^G_{|W|}) \to T(\xi^G_{|V \oplus W|}),$$

which defines a multiplication on MU^G. The unit element is represented by the maps $S^V \to T(\xi^G_{|V|})$ induced by passing to Thom spaces the classifying map of V viewed as a G-bundle over a point. Thus in the usual way the coefficients MU^G_* form a ring and $MU^G_*(X)$ is a module over MU^G_*. Note that the coefficient rings of spectra which are fully equivariant (that is, indexed over $\mathcal{U}$) are in general not commutative even in the graded sense. But MU^G exhibits a periodicity which simplifies matters.

Definition 3.2. *Let $V \subset \mathcal{U}$ be of dimension n. Then the classifying map $V \to \xi_n^G$ induces a map of Thom spaces $S^V \to T(\xi_n^G)$, which represents an element $t_V \in MU^G_{V-2n}$ known as a* Thom class.

Proposition 3.1. *The Thom class t_V is invertible.*

Proof. It is straightforward to show that the class represented by the map $S^{2n} \to T(\xi^G_{|V|}) \in MU^G_{2n-V}$ induced by the classifying map $\mathbb{C}^n \to \xi^G_{|V|}$ is the multiplicative inverse of t_V. □

Corollary 3.2. *The group $MU^G_V(X)$ is naturally isomorphic to $MU^G_{2|V|}(X)$.*

The isomorphism is given by multiplication by t_V. And this isomorphism is easily extended via suspension isomorphisms to real representations of G. Thus the grading of MU^G_*, which we have so far treated using subspaces of $\mathcal{U}$, reduces to a $\mathbb{Z}$-grading. And we have chosen specific compatible isomorphisms to reduce to this grading, namely multiplication by Thom classes.

Convention. *We grade all MU^G_*-modules over the integers.*

The most pleasant way to produce classes in MU^G_* is from equivariant stably almost complex manifolds. Recall that there is an real analog of $BU^G(n)$, which we call $BO^G(n)$, and which is the classifying space for all G-vector bundles.

Definition 3.3. *A tangentially complex G-manifold is a pair (M, τ) where M is a smooth G-manifold and τ is a lift to $BU^G(n)$ of the map to $BO^G(2n)$ which classifies $TM \times \mathbb{R}^k$ for some k.*

We can define bordism equivalence in the usual way to get a geometric version of equivariant bordism.

Definition 3.4. *Let $\Omega^{U,G}_*$ denote the ring of tangentially complex G-manifolds up to bordism equivalence.*

Classes in geometric bordism give rise to classes in homotopical bordism through the Pontrjagin-Thom construction.

Definition 3.5. *Define a map $PT\colon \Omega^{U,G}_* \to MU^G_*$ as follows. Choose a representative M of a bordism class. Embed M in some sphere S^V, avoiding the basepoint and so that the normal bundle ν has a complex structure. Identify the normal bundle with a tubular neighborhood of M in S^V. Define $PT([M])$ as the composite*

$$S^V \xrightarrow{c} T(\nu) \xrightarrow{T(f)} T(\xi_{|\nu|}),$$

where c is the collapse map which is the identity on ν and sends everything outside ν to the basepoint in $T(\nu)$, and $T(f)$ is the map on Thom spaces given rise to by the classifying map $\nu \to \xi_{|\nu|}$.

The proof of the following theorem translates almost word-for-word from Thom's original proof.

Theorem 3.3. *The map PT is a well-defined graded ring homomorphism.*

The Pontrjagin-Thom homomorphism is not an isomorphism equivariantly as it is in the ordinary setting, which is illustrated by the existence of the following classes in MU^G_*.

Definition 3.6. *Compose the map $S^V \to T(\xi^G_n)$, in Definition 3.2 of the Thom class with the evident inclusion $S^0 \to S^V$ to get an element $e_V \in MU^G_{-2n}$ which is called the* Euler class *associated to V.*

We will see that Euler classes e_V associated to representations V such that $V^G = \{0\}$ are non-trivial. Thus MU^G_* is not connective, a feature which already distinguishes it from $\Omega^{U,G}_*$ as well as MU_* itself. The key difference between the equivariant and ordinary settings is the lack of transversality equivariantly. For example, if $V^G = \{0\}$ the inclusion of S^0 into S^V cannot be deformed equivariantly to be transverse regular to $0 \in S^V$.

Finally, we introduce maps relating bordism rings for different groups. Recall that ordinary homotopical bordism MU can be defined using Thom spaces as in our definition of MU^G but without any group action present.

Definition 3.7. *Define the augmentation map $\alpha\colon MU^G \to MU$ by forgetting the G-action on MU^G. When G is abelian and H is a subgroup of G define res_H to be the map from $MU^G_* \to MU^H_*$ by restricting the G-action to an H-action.*

The map res_H is well-defined because any representation of H extends to a representation of G so that the Thom space $T(\xi^G_n)$ coincides with $T(\xi^H_n)$ when its G-action is restricted to an H-action.

Definition 3.8. *Define the inclusion map $\iota\colon MU \to MU^G$ by composing a map $S^n \to T(\xi_n)$ with the inclusion $T(\xi_n) \to T(\xi^G_n)$. More generally, we may define an inclusion from MU^G to any $MU^{G\times H}_*$ by imposing a trivial H-action on a G-map from a sphere to Thom space and including $T(\xi^G_n)$ into the $T(\xi^{G\times H}_n)$ by including $\mathcal{U}(G)$ as the H-fixed set of $\mathcal{U}(G \times H)$ and passing to Thom spaces.*

On coefficients, ι defines an MU_*-algebra structure on MU^G_*. The kernel of α on coefficients is called the *augmentation ideal*. For example, the Euler class e_V is in the augmentation ideal as the map $S^0 \to S^V$ in its definition is null-homotopic when forgetting the G-action.

Definition 3.9. *Let $\bar{x}$ denote the image of $x \in MU^G_*$ under $\iota \circ \alpha$.*

Then the augmentation ideal contains, and is clearly generated by, elements of the form $x - \bar{x}$. On the other hand, ι is injective, which follows from the following proposition which is proved for example in [14].

Proposition 3.4. *The composite $\alpha \circ \iota\colon MU \to MU$ is homotopic to the identity map.*

4. The Connection Between Taking Fixed Sets and Localization

Localization plays a central role in our computations. The connection between localization, in the commutative algebraic sense, and "taking fixed sets" has been a fruitful theme in equivariant topology. Computational work in equivariant bordism in the seventies, the last time such work was seriously taken up, focused on construction of spectral sequences based on taking fixed sets. Our progress has arisen from taking the idea of localization more seriously.

The following lemma provides translation between localization and topology.

Lemma 4.1. *As rings, $\widetilde{MU^G_*}(S^{\oplus_\infty V}) \cong MU^G_*[\frac{1}{e_V}]$.*

Proof. Apply $\widetilde{MU^G_*}$ to the identification $S^{\oplus_\infty V} = \varinjlim S^{\oplus_n V}$. After applying the suspension isomorphisms $\widetilde{MU^G_*}(S^{\oplus_k V}) \cong \widetilde{MU^G_{*+|V|}}(S^{\oplus_{k+1} V})$, the maps in the resulting directed system are multiplication by the e_V. $\square$

After inverting suitable Euler classes, the resulting localization is computable.

Definition 4.1. *A* full *set of representations of G is a set of representations $\{W_i\}$ such that ${W_i}^G = \{0\}$ for all i but for any $H \subset G$ there is an i such that ${W_i}^H \neq \{0\}$.*

Full sets of representations are useful because $Z = S^{\oplus_i(\oplus_\infty W_i)}$ has fixed sets $Z^G = S^0$ while Z^H is contractible for any $H \subset G$. There are full sets of representations for any group such that any proper subgroup is contained in a proper normal subgroup. In particular, abelian groups and p-groups have full sets of representations.

Theorem 4.2. *Let $\{W_j\}$ be a full set of representations of G. Then the localization of complex G-equivariant bordism obtained by inverting the Euler classes e_{W_j} is a Laurent algebra tensored with a polynomial algebra, given abstractly as follows:*

$$MU_*^G\left[\left\{\frac{1}{e_{W_j}}\right\}\right] \cong MU_*\left[e_V, e_V^{-1}, Y_{i,V}\right],$$

where V ranges over irreducible representations of G, i ranges over the positive integers, and where as indicated by notation e_V is the image of the Euler class $e_V \in MU_*^G$ under the canonical map to the localization.

This theorem follows from a series of elementary, fairly standard lemmas. Our first lemma, taken with Lemma 4.1 and the existence of full sets of representations, establishes the strong link between localization and taking fixed sets.

Lemma 4.3. *Let X be a finite G-complex and let Z be a G-space such that $Z^G \simeq S^0$ and Z^H is contractible for any proper subgroup of G. Then the restriction map*

$$Maps^G(X, Y \wedge Z) \to Maps(X^G, Y^G)$$

is a homotopy equivalence.

Proof. The fiber of this restriction map can be identified with the space of G-maps which are trivial on X^G, which is homotopy equivalent to the space of G-maps $\mathrm{Maps}^G(X/X^G, Y \wedge Z)$. Using the skeletal filtration of X/X^G, we can then filter this mapping space by spaces

$$\mathcal{M}_i = \Omega^k \mathrm{Maps}^G(G/H_+, Y \wedge Z),$$

where H is a proper subgroup of G. A standard change-of-groups lemma says that $\mathcal{M}_i$ is homeomorphic to $\mathrm{Maps}(S^0, (Y \wedge Z)^H)$. But $(Y \wedge Z)^H$ is contractible, and thus so are the $\mathcal{M}_i$, and thus so is the fiber of the restriction map. □

We now translate this lemma to the stable realm. For simplicity, let us suppose that our G-spectra are indexed over the real representation ring. We can do so by choosing specific representatives of isomorphism classes of representations. Let $K_n \subset K_{n+1}$ denote a sequence of representations which eventually contain all irreducible representations infinitely often and such that ${K_n}^\perp \subset K_{n+1}$ contains precisely one copy of the trivial representation. If G is finite, we can let K_n be the direct sum of n copies of the regular representation.

Definition 4.2. *Let X be a G-prespectrum. We define the geometric fixed sets spectrum $\Phi^G X$ by passing from a prespectrum $\phi^G X$ defined as follows. We let the entry $\{\phi^G X\}_n$ be $(X_{K_n})^G$, the G-fixed set of the K_n-entry of X. The bonding maps are composites*

$$(X_{K_n})^G \longrightarrow (\Omega^{{K_n}^\perp} X_{(n+1)K})^G \longrightarrow \Omega^{({K_n}^\perp)^G}(X_{K_{n+1}})^G = \Omega(X_{K_{n+1}})^G,$$

where the first map is a restriction of a bonding map of X, and the second map is restriction to fixed sets of the loop space.

Lemma 4.4. *Let Z be as in Lemma 4.3. Then for any G-prespectrum X, the prespectra $(X \wedge Z)^G$ and $\Phi^G X$ are homotopy equivalent.*

Proof. Looking at the definition of $(X \wedge Z)^G$, consider

$$(\Omega^W (X_{W \oplus V} \wedge Z))^G.$$

Lemma 4.3 implies that restriction to fixed sets from this mapping space to $\Omega^{W^G}(X_{W \oplus V})^G$ is a homotopy equivalence. Choosing $V = K_n$, we see that $\Omega^{W^G}(X_{W \oplus K_n})^G$ is an entry of $\phi^G X$. The bonding maps clearly commute with these restriction to fixed sets maps, so we have an equivalence of spectra. □

Thus, the computation of MU_*^G localized at a full set of Euler classes reduces to the computation of $(\Phi^G MU^G)_*$. But for this latter computation we can use the geometry of Thom spaces. Because passing from prespectra to spectra commutes with taking smash products (a consequence of uniqueness of adjoints and the analogous statement for function spectra) we have

$$\Phi^G MU^G \cong Z \wedge MU^G \cong Z \wedge TU^G \cong \Phi^G TU^G,$$

where we recall from Definition 3.1 that TU^G denotes the equivariant Thom prespectrum. We thus proceed with analysis of fixed-sets of Thom spaces.

We first need the following basic fact about equivariant vector bundles.

Proposition 4.5. *And let E be a G-vector bundle over a base space with trivial G-action X. Then E decomposes as a direct sum*

$$E \cong \bigoplus_{V \in Irr(G)} E_V,$$

where $E_V \cong \widetilde{E} \otimes V$ for some vector bundle $\widetilde{E}$.

Lemma 4.6 (tom Dieck)**.** *The G-fixed set of the Thom space of ξ_n^G is homotopy equivalent to*

$$\bigvee_{W \in R^+(G)_n} T(\xi_{|W^G|}) \wedge \left(\prod_{V \in Irr^*(G)} BU(\langle V|W \rangle) \right)_+ ,$$

where $R^+(G)_n$ is the subset of dimension n representations in $R^+(G)$ and $\langle V|\alpha\rangle$ is the coefficient of V in α.

Proof. The universality of ξ_n^G implies that $(BU^G(n))^G$ is a classifying space for n-dimensional complex G-vector bundles over trivial G-spaces. Using Proposition 4.5 we see that this classifying space is

$$\coprod_{\beta \in R^+(G)} \left(\prod_{V \in Irr(G)} BU(\langle V|\beta \rangle) \right),$$

where $\langle V|\beta\rangle$ is the coefficient of V in β. Over each component of this union, the universal bundle decomposes as $\xi_1 \times \xi_2$, where ξ_1 is the universal vector bundle over the factor of $BU(n)$ corresponding to the trivial representation. The fixed set of ${\xi_2}^G$ associated to the trivial representation is the entire bundle while the fixed set ${\xi_2}^G$ is the zero section. The result now follows by passing to Thom spaces. $\square$

Theorem 4.2 now follows from the computation $MU_*(BU) \cong MU_*[Y_i]$, which is standard as in [1], where the Pontrjagin product using the multiplication on BU defines the ring structure on $MU_*(BU)$. Note that multiplication by Euler classes and their inverses serves to change components.

We will also need the following geometric point of view, which dates back to Conner and Floyd. The following precise formulation is due to Costenoble.

Proposition 4.7. *Let M be a tangentially complex G-manifold Then the normal bundle ν of a connected component M_0^G of M^G in M is a complex vector bundle.*

Proof. Let η be a complex G-bundle over M whose underlying real bundle is $TM \times \mathbb{R}^k$, as given by the tangential unitary G-structure of M. Then by Proposition 4.5, $\eta|_{M_0^G}$ decomposes as a complex G-bundle

$$\eta|_{M_0^G} \cong \eta_1 \oplus \bigoplus_{\rho\in Irr^0(G)} \eta_\rho,$$

where η_1 has trivial G-action. But we can identify η_1 as having underlying real bundle equal to $TM^G \times \mathbb{R}^k$. So the normal bundle ν underlies $\bigoplus_{\rho\in Irr^0(G)} \eta_\rho$, which gives ν the desired complex structure. $\square$

This proposition following would not be true if in the defintion of complex G-manifold we chose a complex structure on either the stable normal bundle or on $TM \times V$ for an arbitrary V as opposed to $\mathbb{R}^k$. In these cases we could only guarantee that normal bundles to fixed sets would be stably complex.

Definition 4.3. *Let*

$$F_* = \bigoplus_{\alpha\in R^+(G)} MU_{*-|\alpha|}\left(\prod_{V\in Irr(G)} BU(\langle V|\alpha\rangle)\right)$$

For a tangentially complex G-manifold M enumerate the connected components of M^G, and label each component with the corresponding number as in M_i^G. Define the homomorphism $\varphi\colon \Omega_*^{U,G} \to F_*$ as sending a class $[M] \in \Omega_n^{U,G}$ to the sum of the M_i^G with reference map to the product of classifying spaces which classifies its normal bundle.

This geometric picture of taking fixed sets of G-actions on manifolds fits nicely with the homotopy theoretic picture we have been developing so far.

Proposition 4.8 (tom Dieck)**.** *The following diagram commutes*

$$\begin{tikzcd} \Omega_*^{U,G} \arrow[r, "\varphi"] \arrow[d, "PT"] & F_* \arrow[d, "i"] \\ MU_*^G \arrow[r, "\lambda"] & (\Phi^G MU^G)_*. \end{tikzcd}$$

We can explicitly compute φ. A key fact which follows from standard knowledge of $MU_*(BU)$ is the following.

Proposition 4.9. *We can choose the class $Y_{i,V}$ of Theorem 4.2 to be represented by the image under the map i of Proposition 4.8 of the complex projective space $\mathbb{P}^i$ with reference map to the Vth factor in $\prod_{Irr^*(G)} BU$ which classifies the dual to the tautological line bundle.*

Proposition 4.10. *Let V be an irreducible representation of G. The image of $[\mathbb{P}(n \oplus V)]$ in the localization $MU_*^G\left[\left\{\frac{1}{e_{W_j}}\right\}\right]$ is $Y_{n,V} + X$, where X is $(e_{V^*})^{-n}$ for one-dimensional V and is zero otherwise.*

Proof. We use homogeneous coordinates. There are two possible components of the fixed sets. The points whose coordinates "in V" are zero, constitute a fixed $\mathbb{P}^{n-1}$. Alternately, when all other coordinates are zero the resulting submanifold is the space of lines in V, which is an isolated fixed point when V is one-dimensional and is a projective space with no fixed points, as V has no non-trivial invariant subspaces, when V has higher dimension. $\square$

Definition 4.4. *Let R_0 be the sub-ring of MU_*^G generated by Euler classes and by the classes $[\mathbb{P}(n \oplus V)]$.*

Corollary 4.11. *The inclusion of R_0 into MU_*^G induces an isomorphism after inverting the Euler classes.*

5. A Little Algebra

Definition 5.1. *Let R_0 be a sub-ring of a ring R. The* saturation *of R_0 at a subset $\{e_i\}$ of R is the smallest sub-ring S of R which contains R_0 and such that for any $\alpha \in R$, if $\alpha e_i \in S$ then $\alpha \in S$.*

We may think of the saturation of R_0 at $\{e_i\}$ as the ring of iterated fractions $\dfrac{\frac{a_0}{e_{i_0}} - a_1}{e_{i_1}} \cdots$, where $a_i \in R_0$. The saturation is canonically filtered as $R_0 \subseteq R_1 \subseteq \cdots S$, where R_j is the minimal sub-ring containing R_{j-1} and quotients of elements of R_{j-1} by some e_i.

Proposition 5.1. *Let R_0 be a sub-ring of a ring R such that the inclusion of R_0 into R becomes an isomorphism after inverting elements $\{e_i\}$ of R_0, and such that R_0 contains the kernel of each of the canonical homomorphisms $\lambda_i\colon R \to R\left[\frac{1}{e_i}\right]$. Then the saturation of R_0 at the $\{e_i\}$ is in fact all of R.*

Proof. We first show that the saturation of R_0 at the $\{e_i\}$ contains the kernel of the canonical homomorphism $\lambda\colon R \to R\left[\{\frac{1}{e_i}\}\right]$. Suppose x is in the kernel of λ so that $x e_{i_1}^{p_1} e_{i_2}^{p_2} \cdots e_{i_k}^{p_k} = 0$.

Then $xe_{i_2}^{p_2}\cdots e_{i_k}^{p_k}$ annihilates $e_{i_1}^{p_1}$ and so is in the kernel of λ_{i_1} and thus R_0. But then x is in the saturation of R_0.

Now for any $r \in R$, $\lambda(r) = b/m$ where m is a monomial in the e_i and by assumption $b \in R_0$. Then $r \cdot m - b$ is in the kernel of λ and thus in R_0 by the previous paragraph, so that $m \cdot r$ is in R_0. Hence r is in the saturation of R_0 at $\{e_i\}$. □

6. Computations of MU_*^G

Theorem 6.1. *Let G be a group such that any proper subgroup is contained in a proper normal subgroup. Let R_0 be the sub-ring of MU_*^G described above. The saturation at the Euler classes of any sub-ring which contains R_0 and the annihilators of Euler classes is in fact all of MU_*^G.*

Proof. Take Corollary 4.11 and Proposition 5.1 together. □

Describing MU_*^G as the saturation of one of its sub-rings seems circular in reasoning. But we will prove facts about MU_*^G which make this description systematic. And as seen through the map from MU_*^G to its completion at its augmentation ideal, this description as a saturation works well and seems to be the simplest description possible.

We now focus on the case of abelian groups. We start by finding the annihilators of Euler classes.

Definition 6.1. *Given a representation V let $K(V)$ denote its kernel, which is the subgroup of G which acts trivially on V.*

Definition 6.2. *For an abelian group G we let $s_{G/H} \in MU_*^G$ denote the image of G/H, considered as a (framed, thus complex) G-manifold, under the Pontrjagin-Thom map.*

Theorem 6.2. *Let V be a non-trivial irreducible representation of an abelian group G. The kernel of the localization map $\lambda_V\colon MU_*^G \to MU_*^G\left[\frac{1}{e_V}\right]$ is principal, generated by $s_{G/K(V)}$ which is zero if $G/K(V)$ is isomorphic to S^1.*

Proof. Apply $\widetilde{MU_*^G}$ to the cofiber sequence $S(\oplus_\infty V)_+ \to S^0 \to S^{\oplus_\infty V}$. By Lemma 4.1, the second map of the sequence becomes the localization map λ_V. Thus the kernel of λ_V is the image of $MU_*^G(S(\oplus_\infty V)) \to MU_*^G$. We can begin to compute $MU_*^G(S(\oplus_\infty V))$. Because $S(\oplus_\infty V)$ is fixed by $K(V)$ but then is a free $G/K(V)$ space we can use an Adams transfer to deduce that

$$MU_*^G(S(\oplus_\infty V)) \cong MU_{*-|G/K(V)|}^{K(V)}(B_{G/K(V)}).$$

As $B_{G/K(V)}$ here has a trivial $K(V)$ action there is an Atiyah-Hirzebruch spectral sequence with

$$E_{p,q}^2 \cong H_p(B_{G/K(V)}, MU_q^{K(V)}),$$

whose $\lim^1$ term vanishes as $H_p(B_{G/K(V)})$ is finite, converging to $MU_*^{K(V)}(B_{G/K(V)})$. Because $G/K(V)$ is cyclic, its homology is generated by the unit class and images under the classifying map of fundamental classes of $S(W)/G$ for any W which has a free action away from zero. We deduce that $MU_*^G(S(\oplus_\infty V))$ is generated as an MU_*^G-module by the image of the unit class and the spheres $S(\oplus_n V)$. But the spheres $S(\oplus_n V)$ bound the disks $D(\oplus_n V)$, so the image is generated by the image of the unit class which is $s_{G/K(V)}$. And if $s_{G/K(V)}$ is isomorphic to S^1 then as a G-space it is $S(V)$ and thus is zero in MU^G_*. $\square$

Corollary 6.3. *For a torus T the map from MU^T_* to its localization by inverting Euler classes is injective.*

This corollary points to why localization methods have been so successful in studying S^1-manifolds.

Remark. *The analog of Theorem 6.2 is not true for geometric bordism for finite groups. If $G = \mathbb{Z}/4$ then s_G is in the image of $\widetilde{\Omega^{U,G}_*}(S(\oplus_\infty \rho^{\otimes 2})_+) \to \Omega^{U,G}_*$. Let $H \cong \mathbb{Z}/2 \subset G$. In MU^G_*, we have $s_G = s_{G/H} \cdot q$ where q is a quotient of $e_{\rho^{\otimes 2}}$ by e_ρ. But this class q is not in the image of the Pontrjagin-Thom map, and in fact s_G is not divisible by $s_{G/H}$ in $\Omega^{U,G}_*$.*

The following corollary is originally due to Comezaña [14].

Corollary 6.4. *For G abelian, MU^G_* is concentrated in even degrees.*

Proof. Our original R_0 and the kernels of these localizations are concentrated in even degrees. When we divide by Euler classes to construct the saturation we continue to be concentrated in even degrees. $\square$

For abelian groups, the restriction maps to subgroups are useful for organization.

Theorem 6.5. *Let G be abelian, and let V be a representation of G. The kernel of the restriction map $MU^G_* \to MU^{K(V)}_*$ is principal, generated by e_V.*

Proof. Apply ${MU_G}^*$ to the cofiber sequence

$$S(V)_+ \xrightarrow{i} S^0 \xrightarrow{j} S^V.$$

The image of j is by definition the set of multiples of e_V. To analyze the first map we apply $\widetilde{MU}^*_G$ to the composite of maps

$$G/K(V)_+ \overset{k}{\hookrightarrow} S(V)_+ \xrightarrow{l} S^0.$$

Since $G/K(V)$ and $S(V)$ are spaces which are fixed by $K(V)$ but are free $G/K(V)$ spaces, the fact that the fixed-point spectra $(MU_G)^H$ and $(MU_H)^H$ are equivalent spectra implies that $\widetilde{MU}^*_G(G/K(V)_+) \cong \widetilde{MU}^*_{K(V)}(S^0)$ and $\widetilde{MU}^*_G(S(V)_+) \cong \widetilde{MU}^*_{K(V)}(S(V)/G_+)$. Moreover, the composite

$$l^* \circ k^* \colon \widetilde{MU}^*_G(S^0) \to \widetilde{MU}^*_{G/K(V)}(S^0)$$

coincides with the restriction map. But by Corollary 6.4,

$$k^* \colon \widetilde{MU}^*_{K(V)}(S(V)/G_+) = MU^*_{K(V)}(S^1) \to MU^*_{K(V)}$$

is an isomorphism in even degrees. Hence, i^* coincides with the augmentation map in even degrees, which by Corollary 6.4 constitute all of MU^G_*. $\square$

Remark. *There is a pleasing geometric construction which reflects the divisibility of classes in the augmentation ideal by Euler classes. Let $f\colon X \to Y$ be an S^1-equivariant map of based spaces which is null-homotopic upon forgetting the S^1 action. Let $F\colon X \times I \to Y$ be the null-homotopy. Construct an S^1-equivariant map $f_{\Sigma(F)}\colon X \times I \times S^1 \to Y$ by sending*

$$(x, t, \zeta) \mapsto \zeta \cdot F(\zeta^{-1} \cdot x, t).$$

This map passes to the quotient

$$X \times I \times S^1 / \left(\{X \times 0 \times S^1\} \cup \{X \times 1 \times S^1\} \cup \{* \times I \times S^1\}\right),$$

which is $S^\rho \wedge X$. When restricted to $S^0 \wedge X \subset S^\rho \wedge X$ this map coincides with the orginal f, and thus gives a "quotient" of f by the class $S^0 \hookrightarrow S^\rho$.

Corollary 6.6. *The augmentation ideal of MU_*^G is generated by classes $\{e_{V_i}\}$ such that $G \overset{\times V_i}{\to} (S^1)^k$ is injective.*

Proof. We may assume the inclusions of the kernels of V_i into G are split. Express x in the augmentation ideal as

$$x = [x - r_1(x)] + [r_1(x) - r_1 \circ r_2(x))] + r_1 \circ r_2(x) - \cdots - r_1 \circ \cdots \circ r_k(x),$$

where r_i is the map from MU_*^G to itself defined by composing the restriction map to $MU_*^{K(V_i)}$ with the inclusion map of Definition 3.8. By assumption on the V_i, the composite $r_1 \circ \cdots \circ r_k(x)$ will be $\overline{x}$ which is zero. And each difference $y - r_i(y)$ is in the kernel of the restriction map to $K(V_i)$ and thus by Theorem 6.5 is divisible by e_{V_i}, yielding x as a linear combination of the e_{V_i}. □

We now reduce our study to the case in which G is a torus.

Theorem 6.7. *Let H be a subgroup of an abelian group G. Then the restriction map $MU_*^G \to MU_*^H$ is surjective.*

Proof. We use the characterization of MU_*^H as a saturation. To do so we must take care in choosing lifts of H-actions to G-actions. Factor the inclusion of H into G as $H \to \widetilde{H} \to G$ where $\widetilde{H}$ is a subgroup of G which contains H, which is of the same rank as H, and whose inclusion into G is split. We fix a splitting $G \cong \widetilde{H} \times F$. Then to lift an H-representation to a G-representation, we first lift it to a $\widetilde{H}$-representation and then to a G-representation by having F act trivially.

Because representations lift, the H actions on projective spaces $\mathbb{P}(n \oplus V)$ lift to G actions, so the corresponding classes lift to MU_*^G. The classes $s_{H/K(V)}$ when zero-dimensional lift as the quotient of $e_{\widetilde{V}}$ by any e_W where $\widetilde{V}$ is a lift of V to G as above and W is any representation with minimal kernel contained in the kernel of $\widetilde{V}$. Specifically, compose an equivariant map from S^W to $S^{\widetilde{V}}$ with the unit map $S^{\widetilde{V}} \to T(\xi^G_{|V|})$ to get a class which restricts to $s_{H/K(V)}$.

As MU_*^H is the saturation of the classes above at Euler classes, to prove the theorem it suffices to show that if a is divisible by an Euler class and a lifts to MU_*^G then its quotient by that Euler class lifts to MU_*^G. Let us denote the lift of a to MU_*^G by $\widetilde{a}$. Inductively, we may assume that $\widetilde{a}$ is "acted on trivially" by F, by which we mean that the restriction of $\widetilde{a}$ to MU_*^F is in the image of the inclusion map from MU_* to MU_*^F, as is the case for liftings of the classes above. That a is divisible by some e_V is equivalent to, by Theorem 6.5, the restriction of a to $MU_*^{K(V)}$ being zero. And since $\widetilde{a}$ is a lift of a its restriction to $K(V)$ is zero. Moreover, as $\widetilde{a}$ is acted on trivially by F and the restriction of $\widetilde{a}$ to $MU_* \subset MU_*^{K(V)}$ is zero, $\widetilde{a}$ restricted to MU_*^F is zero. If we lift V to a G-representation $\widetilde{V}$ as above then $K(\widetilde{V}) = K(V) \times F$ so that $\widetilde{a}$ restricts to zero in $MU_*^{K(\widetilde{V})}$. Hence $\widetilde{a}$ is divisible by $e_{K(\widetilde{V})}$, and the resulting quotient is a lift of the quotient of a by e_V modulo some annihilator of e_V, which we already have shown lifts. Finally, we note that the quotient of $\widetilde{a}$ by $e_{\widetilde{V}}$ will itself be acted on trivially by F. ☐

Definition 6.3. *Fix a splitting s_H of res_H as a map of sets. And let $\tau_H = s_H \circ res_H$.*

Definition 6.4. *Impose an ordering $\leq$ on representations of T. Let Σ be a sequence of representations $\Sigma = \{V_1, V_2, \cdots V_m\}$ such that $V_i \leq V_{i+1}$ and let $\Sigma' = \{V_1, \cdots V_{m-1}\}$. For any $x \in MU_*^T$ inductively define x_Σ as the unique class such that $e_{V_m} \cdot x_\Sigma = x_{\Sigma'} - \tau_{K(V_m)}(x_{\Sigma'})$.*

Theorem 6.8. *As an MU_*-algebra, MU_*^T is generated by classes $\mathbb{P}(n \oplus V)_\Sigma$ and $(e_V)_\Sigma$, as n ranges over positive integers, V ranges over irreducible representations of T and Σ ranges over sequences of ordered representations as above.*

Writing down relations among these generators is not enlightening because we do not have explicit understanding of the maps τ_H. So we focus instead on the map from MU_*^T to its completion at its augmentation ideal.

Theorem 6.9. *Let $T = (S^1)^k$. And given this decomposition let V_i be the representation of T which restricts to ρ, the standard representation, on the ith factor and which restricts to the trivial representation on the other factors. Let I be the augmentation ideal of MU_*^T. Then $(MU_*^T)_{\hat{I}}$ is isomorphic to $MU_*[[x_i]]$, the power series ring on i indeterminates, where $e_{V_i} \mapsto x_i$.*

The proof of this theorem is straightforward, as we have that the inclusion of MU_* into MU_*^G is split by the augmentation map and that for a torus T the annihilators of Euler classes are zero. This theorem also follows from a completion theorem originally due to Löffler and proved by Comezaña and May in [14]. In fact, for abelian G completion of MU_*^G at its augmentation ideal coincides with the natural transformation of complex-oriented equivariant cohomology theories

$$MU_G^*(X) \mapsto MU_G^*(X \wedge E_{G+}) \cong MU^*(X \wedge_G E_{G+}).$$

We are now positioned to understand the image of MU_*^G in its completion, which has been our goal.

Proposition 6.10. *Choose the isomorphism $(MU_*^{S^1})_{\hat{I}} \cong MU_*[[x]]$ where $e_\rho \mapsto x$ as in Theorem 6.9. The image of the Euler class $e_{\rho^{\otimes n}}$ in this completion is $[n]_F x$, the n-series in the formal group law over MU_*.*

Proof. As the map from MU_*^G to its completion is a map of complex-oriented equivariant cohomology theories, the Euler class of the bundle V over a point gets mapped to the Euler class of $V \times_G E_G$ over B_G. For $G = S^1$, $V = \rho^{\otimes n}$ the resulting bundle is the nth-tensor power of the tautological bundle over B_{S^1}, whose Euler class is by definition the n-series. ☐

Remarkably, we can understand the image of geometric classes under completion as well. Recall Definition 1.2 of $\Gamma(X)$ for an S^1 manifold X. We give an alternate description now.

Definition 6.5. *Let X be an S^1 manifold. We let $\eta(X)$ be the S^1-equivariant bundle over $\mathbb{P}(1 \oplus \rho)$ defined by taking the union of $X \times D(\rho)$ with $\overline{X} \times D(\rho^*)$ over their boundaries, where the clutching function $X \times S^1 \to X$ is given by the S^1 action on X and the stably complex structure on the quotient is defined by using that on X and the standard complex structure on $\mathbb{P}(1 \oplus \rho)$.*

Proposition 6.11. *For any S^1 manifold, $\Gamma(X) = \eta(X) - \overline{X} \times \mathbb{P}(1 \oplus \rho)$.*

Note that if in the definition of $\eta(X)$ we use not the standard complex structure on $\mathbb{P}^1$ but the null-bordant complex structure, we have that $\Gamma(X) = \eta(X)$ without any correcting term. This η-construction applied when X is a point gives rise to bounded flag manifolds, as studied extensively by Ray and his collaborators [15].

Theorem 6.12. *For any complex S^1-manifold X,*

$$e_\rho \cdot [\Gamma(X)] = [X] - [\overline{X}]$$

in $MU_*^{S^1}$.

Proof. By Corollary 6.3 it suffices to check the equality in a localization of $MU_*^{S^1}$ by inverting Euler classes. By Proposition 4.8 we can compute the image of $[X]$, $[\overline{X}]$ and $[\Gamma(X)]$ in the localization at a full set of Euler classes by computing fixed sets with normal bundle data. The result follows easily as the fixed sets of $\Gamma(X)$ are clearly those of X crossed with ρ along with an $\overline{X}$ crossed with ρ. In the localization, crossing with ρ coincides with dividing by e_ρ. □

Theorem 1.1, which states that the image of a geometric class $[M]$ in $MU_*^{S^1} \cong MU_*[[x]]$ is

$$\alpha([M]) + \alpha([\Gamma(M)])x + \alpha([\Gamma^2(M)])x^2 + \cdots,$$

follows as a corollary of Theorem 6.12.

And we can now also prove the our main theorem in the case of abelian groups which we can state now as follows. Recall that $Y_i(x)$ is defined to be the image of $[\mathbb{P}(i \oplus \rho)]$ under the completion map to $MU_*[[x]]$, and $Y_0(x) = x$.

Theorem 6.13 (Restatement of Theorem 1.3)**.** *The image of MU_*^T in its completion at the augmentation ideal is contained in the saturation of $MU_*[Y_i(f)]$ at the series $f \in MU_*[[x]]$, where f ranges over $[m_1]_F x_1 +_F \cdots +_F [m_k]_F x_k$.*

Proof. First note that in general the image of the saturation of a sub-ring $R_0 \subset R$ at $\{e_i\}$ under a homomorphism f is clearly contained in the saturation of $f(R_0)$ at $\{f(e_i)\}$. Given Theorem 6.1 and Theorem 6.2 which together say that MU_*^T is the saturation of Euler classes and classes $[\mathbb{P}(i \oplus V)]$ at the Euler classes, along with the computations of these classes under completion in Proposition 6.10 and Theorem 1.1 we are almost done. It suffices to check that the image of $[\mathbb{P}(i \oplus \rho^{\otimes n}]$ is $Y_i([n]_F x)$, which follows from the fact that the S^1 action on $[\mathbb{P}(i \oplus \rho^{\otimes n}]$ is pulled back from the S^1 action on $[\mathbb{P}(i \oplus \rho)]$ by the degree n homomorphism from S^1 to itself. □

To recover the theorem for all abelian groups we cite Theorem 6.7 and the fact that after completion, restricting to a subgroup $K(V)$ coincides with taking the quotient of the power series ring modulo the image under completion of e_V.

Remark. *The completion at the augmentation ideal is a natural place in which to view MU_*^G as a saturation. Suppose $f = a_0 + a_1x + a_2x^2 + \cdots$ is in the image of $MU_*^{S^1}$ under completion. Then because R_0 contains MU_* we have that $f - a_0 = a_1x + \cdots$ is in the image of the saturation. But the saturation property at $x = \hat{I}(e_\rho)$ implies that $a_1 + a_2x + a_3x^2 + \cdots$ is in the image. More generally, any $a_i + a_{i+1}x + \cdots$ is in the image. So the property of a series being in the image of the completion map depends only on the tail of the series. Perhaps there is an analytic way to define this image.*

A similar statement to this theorem holds for K-theory, again for abelian groups. For example, $K^*_{S^1} \cong \mathbb{Z}[\rho, \rho^{-1}]$. If we complete at the augmentation ideal, a generator for which we choose the Euler class $\rho - 1$, we see that the image is generated over the integers by x, which is the 1-series and by $-x + x^2 - x^3 + \cdots$ which is the -1-series in the multiplicative formal group law. In fact all the n-series are in this image, but they are already in the subring generated by the 1 and -1 series as the multiplicative formal group law is a polynomial. Moreover, the image is saturated, but once again one does not produce new classes in this way.

Because the completion at the augmention ideal is a natural place in which to understand equivariant bordism, we hope that the map from MU_*^G to this completion is injective. We also hope to understand whether this image is all of the saturation in the completion or whether the containment is proper. These questions are linked.

Definition 6.6. *We say that a homomorphism $f\colon R \to B$ is* clean *(with respect to $\{e_i\}$) if the image of the saturation of R_0 at $\{e_i\}$ is the saturation of the image of R_0 at the image of $\{e_i\}$*

Theorem 6.14. *For $G = \mathbb{Z}/p$ the homomorphism from MU_*^G to its completion at its augmentation ideal $MU_*[[x]]/([p]_F x)$ is clean with respect to Euler classes.*

Proof. The following diagram commutes

$$\begin{array}{ccc} MU_*^G & \xrightarrow{\hat{I}} & MU_*[[x]]/([p]_F x) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle f \mapsto f(0)} \\ MU_* & \xrightarrow{=} & MU_*. \end{array}$$

If $\hat{I}(a)$ is divisible by some $[n]_F x$ in the completion for some $n < p$ then it maps to 0 in MU_*. So $\alpha(a) = 0$. Corollary 6.6 implies that a is divisible by $e_{\rho^{\otimes n}}$, which is what was to be shown. ∎

For general abelian groups such a proof does not work because for we do not know that the map $MU_*^{K(V)} \to (MU_*^{K(V)})_{\hat{I}}$ is injective except for when $K(V)$ is the trivial group. In fact, the completion map $MU_*^G \to (MU_*^G)_{\hat{I}}$ is clean with respect to Euler classes if and only if this completion map is injective for all subgroups of G.

There are alternate representations of MU_*^G for G abelian which are clean and injective.

Definition 6.7. *Let $\lambda_H\colon MU_*^G \to MU_*^H\left[\frac{1}{e_{W_i}}\right]$ be the composition of the restriction map from MU_*^G to MU_*^H and the canonical map from MU_*^H to its localization by inverting a full set of Euler classes.*

Theorem 6.15. *The map*

$$\prod_{H\subseteq G} \lambda_H\colon MU_*^G \to \prod_{H\subseteq G} MU_*^H\left[\frac{1}{e_{W_i}}\right]$$

is injective for G abelian.

Proof. We introduce the localization filtration. Consider

$$MU_*^G \to MU_*^G\left[\frac{1}{e_{V_1}}\right] \to MU_*^G\left[\frac{1}{e_{V_1}}, \frac{1}{e_{V_2}}\right] \to \cdots,$$

where the V_i are non-trivial irreducible representations. As in Theorem 6.2, we can fit the map

$$MU_*^G\left[\frac{1}{e_{V_i}}\right] \to MU_*^G\left[\frac{1}{e_{V_i}}, \frac{1}{e_W}\right]$$

into an exact sequence whose third term is $MU_{*-1}^{K(W)}\left[\frac{1}{e_{V_i}}\right](B_{G/K(W)})$ and compute the kernels of the localization maps as generated by $s_{G/K(W)}$. But then these kernels map injectively to $MU_{*-1}^{K(W)}\left[\frac{1}{e_{V_i}}\right]$, from which by induction in the filtration we can prove that MU_*^G maps injectively to their product. $\square$

By Proposition 4.8, this theorem says that we have a good understanding of equivariant bordism theory by restricting to subgroups and then looking at fixed-sets with normal bundle data. In fact this point of view was our starting point. But it is difficult to discern for any given fixed-set data whether is the fixed-set data of a G manifold, or from the localization point of view whether fixed-set data is integral. The saturation has proven to be a way to generate all integral fixed-set data given a relatively small amount of such data with which to begin. Unfortunately, we do not know how to compute the image of the saturation of R_0 at the Euler classes under $\prod \lambda_H$ to get a faithful description of MU_*^G even though we know how to compute the images of the classes in R_0 themselves.

In the context of looking at fixed sets, there is a geometric interpretation of the canonical filtration of MU_*^G arising from its construction as a saturation. For geometric classes $[M]$ in MU_*^G define the depth of isotropy to be the maximum length of a chain of proper containments $M^{H_1} \subset M^{H_2} \subset \cdots \subset M$, taking the minimum over bordism representatives to get a well-defined number. The R_{i-1} in which a geometric class in MU_*^G first appears corresponds roughly to the depth of its isotropy. All $[\mathbb{P}(i \oplus V)]$ have depth 1 as there are no subgroups H such that $\mathbb{P}(i\oplus V)^G \subset \mathbb{P}(i\oplus V)^H \subset \mathbb{P}(i \oplus V)$. But for example, the quotient $[V]$ of $[\mathbb{P}(1\oplus\rho)] - [\mathbb{P}(1\oplus\rho^{\otimes 3})]$ by $e_{\rho^{\otimes 2}}$ in $MU_*^{S^1}$ will have $V^{S^1} \subset V^{\mathbb{Z}/2} \subset V$.

7. Applications and Further Remarks

In this section we give an assortment of applications and indicate directions for further inquiry.

Our first application answers a question about bordism of free $\mathbb{Z}/n$-manifolds posed to us by Milgram. It is well-known that the spheres $S(\oplus_k \rho^{\otimes m})$ for any m relatively prime to n generate $MU_*(B_{\mathbb{Z}/n})$ as an MU_*-module. How are these bases related?

Theorem 7.1. *Let m and n be relatively prime. Let $Q(x)$ be a quotient of x by $[m]_F x$ modulo $[n]_F x$ in $MU_*[[x]]$. Define $a_i \in MU_*$ by $(Q(x))^k = a_0 + a_1 x + a_2 x^2 + \cdots$. Then*

$$[S(\oplus_k \rho^{\otimes m})] = a_0[S(\oplus_k \rho)] + a_1[S(\oplus_{k-1}\rho)] + \cdots + a_{k-1}[S(\rho)]$$

in $MU_*(B_{\mathbb{Z}/n})$.

Proof. We use an analog of the simple fact that if M is a G-manfold and $M \smallsetminus M^G$ has a free G-action then $[\partial\nu(M^G)] = 0$ in $MU_*(B_G)$, where $\partial\nu(M^G)$ is the boundary of a tubular neighborhood around the fixed set M^G. The null-bordism is defined by $M \smallsetminus \nu(M^G)$. If the fixed points of M are isolated, this will give rise to a relation among spheres with free G-actions.

Let $\alpha_0 = q^k$ where q is a quotient of e_ρ by $e_{\rho^{\otimes m}}$ in $MU_*^{\mathbb{Z}/n}$. Inductively, let α_i be a quotient of $\alpha_{i-1} - \overline{\alpha_{i-1}}$ by e_ρ. Then the "fixed sets" of α_k are given by

$$\lambda(\alpha_k) = {e_{\rho^{\otimes m}}}^{-k} - \overline{\alpha_0}{e_\rho}^{-k} - \overline{\alpha_1}{e_\rho}^{-k+1} - \cdots - \overline{\alpha_{k-1}}{e_\rho}^{-1}.$$

As ${e_V}^{-1}$ corresponds to a tubular neighborhood of an isolated fixed point in geometric bordism, we can deduce via transversality arguments for free G-actions that

$$[S(\oplus_k \rho^{\otimes m})] - \overline{\alpha_0}[S(\oplus_k \rho)] - \overline{\alpha_1}[S(\oplus_{k-1}\rho)] - \cdots - \overline{\alpha_{k-1}}[S(\rho)] = 0$$

in $MU_*(B_{\mathbb{Z}/n})$. But the image of α_0 in $(MU_*^{\mathbb{Z}/n})_{\hat{I}} \cong MU_*[[x]]/[n]_F x$ is $(Q(x))^k$ from which we can read off that $\overline{\alpha_i} = a_i$.

Note that our expressions in $MU_*(B_{\mathbb{Z}/n})$ are independent of the indeterminacy in choosing q and the α_i. $\square$

This old idea of using G-manifolds to bound and thus give insight into free G-manifolds has been codified by Greenlees's introduction of local cohomology to equivariant stable homotopy theory [8]. Essentially, $MU_*(B_G)$ is isomorphic to the cohomology of the Koszul complex

$$\bigotimes_{V \in B} \left(MU_*^G \to MU_*^G \left[\frac{1}{e_V} \right] \right)$$

where S is a set of representations of G such that G acts freely on $\prod_{V \in B} S(V)$. We have reproduced known Tor classes in $MU_*(B_{(\mathbb{Z}/2)^k})$ using this method. Results along these lines will appear in [17]

Our next application is in answer to a question posed to us by Bott. For simplicity, define the fp-signature of an isolated fixed point in an S^1 manifold whose normal bundle is isomorphic to the representation $\bigoplus_{i=1}^k \rho^{\otimes m_i}$ to be the unordered k-tuple $\{m_1, \cdots, m_k\}$. The fp-signature of an S^1 action with isolated fixed points is the unordered set of fp-signatures of the fixed points. For example, the fp-signature of $\mathbb{P}[1 \oplus \rho^{\otimes m} \oplus \rho^{\otimes n}]$ is $\{\{m, n\}, \{-m, n -$

$m\}, \{m-n, -n\}\}$. Bott's question is as to whether there are any other fp-signatures which arise from S^1 actions on $\mathbb{CP}^2$ with three isolated fixed points. In fact, a surface admitting an S^1 action with three isolated fixed points is a surprisingly restrictive condition.

Theorem 7.2. *Let M be a complex four (real)-dimensional S^1-manifold with three isolated fixed points. Then the fp-signature of M is $\{\{m,n\}, \{-m, n-m\}, \{m-n, -n\}\}$ for some integers m and n. Moreover, forgetting the S^1 action M is bordant to $\mathbb{CP}^2$.*

Proof. For convenience, let refer to the Euler class $e_{\rho^{\otimes n}} \in MU_*^{S^1}$ by e_n. A complex S^1 manifold M with three isolated fixed points defines a class in $MU_*^{S^1}$ whose image under λ is

$$\lambda([M]) = e_a^{-1}e_b^{-1} + e_c^{-1}e_d^{-1} + e_f^{-1}e_g^{-1}$$

for some integers $a, \cdots, g$. We let T denote

$$e_ce_de_fe_g + e_ae_be_fe_g + e_ae_be_ce_d \in MU_*^{S^1}.$$

And we say that a class in a localization is integral when it is in the image of the canonical map to that localization.

Without loss of generality, assume a is greatest of the integers $a, \cdots, g$ in absolute value. As T must be divisible by e_a, Theorem 6.5 implies that T restricted to $MU_*^{\mathbb{Z}/a}$ must be zero. The Euler class e_n restrict non-trivially to $MU_*^{\mathbb{Z}/a}$ unless $a|n$. Therefore one of c, d, f, g, say c must be equal to $\pm a$. We first claim that this number must be $-a$. Look in the localization of $MU_*^{S^1}\left[\{\widetilde{\tfrac{1}{e_a}}\}\right]$ defined by inverting all Euler classes except for e_a This localization is itself a saturation, in this case only at e_a. Suppose that $|b|, |d|, |f|, |g| < |a|$ and that

$$e_a^{-1}e_b^{-1} + e_a^{-1}e_d^{-1} + e_f^{-1}e_g^{-1}$$

is integral, by which we mean in the image of this partial localization map. Then we must have that $e_b^{-1} + e_d^{-1}$ divisible by e_a and thus is zero in $MU_*^{\mathbb{Z}/a}\left[\frac{1}{e_V}\right]$, which is impossible. Cases where some of $|b|, |d|, |f|, |g|$ are equal to $|a|$ can be treated similarly.

As $c = -a$ and again assuming that $|b|, |d|, |f|, |g| < |a|$, consider the class $\lambda_{\widetilde{a}}([M]) - \lambda_{\widetilde{a}}([\mathbb{P}(1+\rho^{\otimes a})])e_d^{-1}$ in $MU_*^{S^1}\left[\{\widetilde{\tfrac{1}{e_a}}\}\right]$. Its image under full localization

$$e_a^{-1}e_b^{-1} - e_a^{-1}e_d^{-1} + e_f^{-1}e_g^{-1},$$

which implies that $e_b^{-1} - e_d^{-1}$ is divisible by e_a in $MU_*^{S^1}\left[\{\widetilde{\tfrac{1}{e_a}}\}\right]$ or that $b \equiv d \pmod a$. But because $|b|, |d| < |a|$ we have that $d = a \mp b$ depending on whether b is positive or negative. We can once again eliminate cases where some $|n| = |a|$.

Finally, as $c = -a$ and $d = b - a$ consider $\lambda([M] - [\mathbb{P}(1 \oplus \rho^{\otimes a} \oplus \rho^{\otimes b})])$, which will be equal $e_f^{-1}e_g^{-1} - e_{a-b}^{-1}e_{-b}^{-1}$. Case analysis of necessary divisibilities as we have been doing implies that this difference must be zero.

Finally, by Theorem 6.2 this fixed-set data determines $[M]$ as in S^1-equivariant homotopical bordism uniquely, so that $[M]$ must equal $[\mathbb{P}[1 \oplus \rho^{\otimes m} \oplus \rho^{\otimes n}]]$ in $MU_4^{S^1}$. Forgetting the S^1 action, this equality gives rise to a bordism between $[M]$ and $[\mathbb{CP}^2]$.

∎

In fact, we have show that any complex four-dimensional S^1-manifold M with three fixed points is "stably cobordant" to one of the standard actions on $\mathbb{P}^2$. The question of whether this equivalence gives rise to an actual equivariant bordism between M and such a $\mathbb{P}^2$ reduces to a transversality question. In fact, transversality questions which arise in real equivariant bordism are universal in a sense [6]. Hence, our techniques could be useful in studies which require control of the failure of equivariant transversality.

Equivariant bordism also underlies the study of equivariant genera. Our work illuminates known results and could lead to further developments. Ochanine proved that any (strongly) multiplicative genus, which is a genus whose value on the total space of a fiber bundle of manifolds is the product of the genus of the base and fiber, is rigid in the sense that the equivariant genus obtained using the Borel construction and "integration along the fiber" will not depend on the G-action on the manifold (see for example [16]). But such an equivariant genus factors through completion of equivariant bordism at its augmentation ideal, and Theorem 1.1 show that coefficients of terms other than the constant term in the image under completion will be differences of twisted bundles over $\mathbb{P}^1$ and trivial bunldes over $\mathbb{P}^1$. Any (strongly) multiplicative genus will thus vanish on these terms.

Finally, we reiterate that the theories we have been studying provide a unified framework in which to study the characteristic classes $E^*(BG)$ for any complex-oriented theory E. We hope that our understanding of relevant commutative algebra can lead to new insights into these characteristic classes.

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