OPERADS AND KNOT SPACES

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1. Introduction

Let E_m denote the space of embeddings of the interval $\mathbb{I} = [-1, 1]$ in the cube $\mathbb{I}^m$ with endpoints and tangent vectors at those endpoints fixed on opposite faces of the cube, equipped with an isotopy through immersions to the unknot – see Definition 5.1. By Proposition 5.17, E_m is homotopy equivalent to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \times \mathrm{Imm}(\mathbb{I}, \mathbb{I}^m)$. In [26], McClure and Smith define a cosimplicial object $\mathcal{O}^\bullet$ associated to any operad with multiplication $\mathcal{O}$, whose homotopy invariant totalization we denote $\widetilde{\mathrm{Tot}}(\mathcal{O}^\bullet)$ – see Definition 2.16 and Definition 2.5 below. Let $\mathcal{K}_m$ denote the mth Kontsevich operad, introduced in [21], whose entries are compactified configuration spaces and which is weakly equivalent to the little k-cubes operad [35] – see Definition 4.1 and Theorem 4.5 below.

Theorem 1.1. *The totalization of the Kontsevich operad $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$ is homotopy equivalent to the inverse limit of the Taylor tower approximations for E_m in the calculus of embeddings. Moreover, $\widetilde{\mathrm{Tot}}^{\mathrm{n}}(\mathcal{K}_{\mathrm{m}}^\bullet)$ is the nth degree approximation.*

Building on work of Goodwillie, Klein and Weiss [44, 15, 19, 18], and Volic [41, 42], we have the following.

Corollary 1.2. *For $m > 3$, E_m is weakly equivalent to $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$. For $m = 3$, all rational finite-type invariants of framed knots factor through a map from E_m to $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$.*

Applying the homology spectral sequence of a cosimplicial space, we have the following.

Corollary 1.3. *For $m > 3$, there is a spectral sequence with E^2 page given by the Hochschild cohomology of the degree $m-1$ Poisson operad which converges to the homology of E_m.*

These results resolve conjectures of Kontsevich from his address at the AMS Mathematical Challenges Conference in the summer of 2000 [22]. Kontsevich's insights had already motivated Tourtchine to give an algebraic description of the E^2-term of Vassiliev's spectral sequence along the lines of Corollary 1.3 in [38]. Our results are at the level of spaces and show that the disagreement which Tourtchine found between Hochschild cohomology of the Poisson operad and Vassiliev's E^2-term is accounted for by the difference between E_m and the classical knot space.

Our results bring together some recent developments in algebraic topology and its application to fields such as deformation theory and knot theory. In [34] we presented models for spaces of knots, including a cosimplical model which is analogous to the cosimplicial model for loop spaces. We build on these results in proving Theorem 1.1. In [26] McClure and Smith resolved the integral Deligne conjecture, showing that the totalization of an operad with multiplication has a two-cubes action both in the setting of chain complexes and that of spaces. We apply their results to establish the following.

Theorem 1.4. *For any m, there is a little two-cubes action on $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$. For $m > 3$, E_m is a two-fold loop space.*

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We conjecture that this two-cubes action on $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$ is compatible with a two-cubes action on the space of framed knots which has been recently defined by Budney [7], who goes on two show that long knots in dimension three are free over the two-cubes action. In future work we plan to investigate analogues of this freeness result in higher dimensions. A first step will be to construct operations compatible with this the two-cubes structure in the homology spectral sequence for an operad with multiplication, as McClure and Smith currently plan to do. On the E^1-term such operations will presumably coincide with Tourtchine's bracket, defined combinatorially in [38], but through their space-level construction would also be compatible with differentials and extend to further terms.

Some of the technical results developed in this paper may be of independent interest. We fully develop the operad structure on the simplicial compactification of configurations in Euclidean spaces. An operad structure on the canonical (Axelrod-Singer) compactification is known [13, 23], but does not yield an operad with multiplication and instead admits a map from Stasheff's A_∞ operad. Our approach to this operad structure blends geometry and combinatorics, revealing an operad structure on the standard simplicial model for the two-sphere.

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2. Background material

Our main results are stated in terms of operads with multiplication and their associated cosimplicial spaces. As a chance to set the choice of definitions and notation which will be most convenient, and as an opportunity to place all standard material together, we review this material here. For a more complete survey we highly recommend [28]. A reader familiar with these constructions may wish to skip this section and refer back for clarification as needed.

2.1. Cosimplicial spaces.

Definition 2.1. Let Δ denote the category with one object for every non-negative integer and where the morphisms from k to ℓ are the order-preserving maps from $[k] = \{0, \cdots, k\}$ to $[\ell] = \{0, \cdots, \ell\}$, ordered in the standard way. A cosimplicial object in a category $\mathcal{C}$ is a (covariant) functor from Δ to $\mathcal{C}$. A simplicial object is a contravariant functor from Δ to $\mathcal{C}$.

Cosimplicial objects are denoted $X^\bullet$, where X^k is the image of $[k]$ under the functor, also known as the kth entry. Simplicial objects are denoted $X_\bullet$. Central in the theory is the standard cosimplicial space $\Delta^\bullet$, whose kth entry is Δ^k, with vertices labelled by $[k]$, and which sends a morphism $[k] \to [\ell]$ to the linear map which extends this map on vertices.

Every order-preserving map $[k] \to [\ell]$ can be factored through elementary maps d^i, which are an isomorphism but for one element - namely i - not in their image, and elementary maps s^i, which are an isomorphism but for having i and $i+1$ in $[k]$ both mapping to $i \in [\ell]$. The corresponding maps between entries of simplicial and cosimplicial objects, called (co)face and (co)degeneracy maps, are often taken as a basis for their definition. The definitions are arranged so that the simplices of a simplicial complex form a simplicial set. Indeed, a simplicial set or simplicial space $X_\bullet$ may be realized, denoted $|X_\bullet|$, as the quotient space of the union of $X_i \times \Delta^i$ over all i by the relations $d_j x \times \beta \sim x \times d^j \beta$ and $s_j x \times \beta \sim x \times s^j \beta$ for all $x \in X^i$, $\beta \in \Delta^i$.

Cosimplicial spaces naturally arise when studying mapping spaces. The totalization of a cosimplicial space $\mathrm{Tot} X^\bullet$ is the space of natural transformations from $\Delta^\bullet$ to $X^\bullet$, which is first used tautologically to study mapping spaces as follows.

Definition 2.2. For any $X \in \mathcal{C}$, a symmetric monoidal category with product $\odot$, taking the product of X with itself gives rise to a functor $X^- : \mathrm{Set}^{op} \to \mathcal{C}$ which sends S to $\bigodot_{s \in S} X$. By composing a simplicial set $Y_\bullet : \Delta \to \mathrm{Set}^{op}$ with this functor, we obtain a cosimplicial object $X^{Y_\bullet}$.

Proposition 2.3. *If $Y_\bullet$ is a simplicial set and X is a space,* $\mathrm{Tot}(X^{Y_\bullet})$ *is homeomorphic to the space of maps from $|Y_\bullet|$ to X.*

For based X and $Y_\bullet$ we may replace X^{Y_n} by the subspace of $(x_1, \ldots, x_\alpha, \ldots)$, $\alpha \in Y_n$ where if b denotes the degenerate image of the basepoint of $Y_\bullet$ in Y_n we have that x_b is the basepoint of X. The resulting cosimplicial space, which we denote $X_\star^{Y_\bullet}$, has totalization homeomorphic to the space of based maps from $|Y_\bullet|$ to X. Another interesting example along these lines is the Hochschild cosimplicial vector space $A^{S^1_\bullet}$, whose associated chain complex computes Hochschild cohomology of a ring A.

Cosimplicial spaces are intimately connected with homotopy limits (in fact, homotopy limits are defined in terms of cosimplicial spaces in [5]). The nerve of a category C is the simplicial set $C_\bullet$, with C_i being the collection of i composable morphisms and structure maps defined through composing such maps or inserting identity maps (see for example Chapter 14 of [20]). Denote the realization of the nerve of C by BC, also called the classifying space. Recall that if c is an object of C, the category $C \downarrow c$ has objects which are maps with target c and morphisms given by morphisms in C which commute with these structure maps. The classifying space $B(C \downarrow c)$ is contractible because $C \downarrow c$ has a final object, namely c mapping to itself by the identity morphism. A morphism g from c to d induces a map from $C \downarrow c$ to $C \downarrow d$, so that $B(C \downarrow -)$ is itself a functor from C to spaces.

Definition 2.4. The homotopy limit of a functor E from a small category C to the category of spaces is $\mathrm{Nat}(B(C \downarrow -), E)$, the space of natural transformations from $B(C \downarrow -)$ to E.

For $X^\bullet$ fibrant in the standard model structure on cosimplicial spaces, that is those which satisfy the matching condition 10.4.6 of [5], Theorem 11.4.4 of [5] states $\mathrm{Tot}(X^\bullet) \simeq \underleftarrow{\mathrm{holim}}_\Delta X^\bullet$, an equivalence needed for many applications. For cosimplicial spaces which do not necessarily satisfy the matching condition, we use an alternate definition of totalization for which this equivalence is a tautology.

Definition 2.5.
- Let $\widetilde{\Delta^\bullet}$ be the cosimplicial space whose $[k]$th entry is $B(\Delta \downarrow [k])$ and whose structure maps are the standard induced maps.
- For a cosimplicial space $X^\bullet$ let $\widetilde{\mathrm{Tot}} X^\bullet$, called the homotopy invariant totalization, denote the space of natural transformations from $\widetilde{\Delta^\bullet}$ to $X^\bullet$.
- Let $\widetilde{\mathrm{Tot}}^k X^\bullet$ denote the space of natural transformations from the kth coskeleton of $\widetilde{\Delta^\bullet}$ to $X^\bullet$.
- Let Δ_k denote the full subcategory of Δ whose objects are $[i]$ for $i \leq k$. Let $i_k : \Delta_k \to \Delta$ be the inclusion functor.

In Section 15 of [27] the notation $\widetilde{\Delta^\bullet}$ and $\widetilde{\mathrm{Tot}}$ are used for any cofibrant replacement for $\Delta^\bullet$ and the corresponding totalization in the model structure on cosimplicial spaces where all objects are fibrant (in the usual model structure from [5], all objects are cofibrant). We choose particular models for definiteness, and so by definition $\widetilde{\mathrm{Tot}}(X^\bullet)$ is the homotopy limit of $X^\bullet$ and $\widetilde{\mathrm{Tot}}^k(X^\bullet) \cong \underleftarrow{\mathrm{holim}}\,(X^\bullet \circ i_k)$.

The cosimplicial category Δ is also intimately related to the category of subsets of a finite set.

Definition 2.6. Let $P(k)$ be the category of all subsets of $[k] = \{0, \cdots, k\}$ where morphisms are defined by inclusion. Let $P_0(k)$ be the full subcategory of non-empty subsets.

The connection of this category to the simplicial world is evident in the identification of $BP_0(k)$ with the barycentric subdivision of a k-simplex. More specifically, we define maps $\widetilde{\Delta}^k \to \Delta^k$, and thus $\mathrm{Tot}X^\bullet \to \widetilde{\mathrm{Tot}}X^\bullet$ for any $X^\bullet$, as the induced map on classifying spaces for the functor which sends some $[n] \xrightarrow{f} [k]$ in $(\Delta \downarrow [k])$ to the image of f, as a subset of $[k]$. There is also a translation between cosimplicial diagrams and those indexed by $P_0(k)$, which we will use in Section 6.

Definition 2.7. Let $c_k \colon P_0(k) \to \Delta$ be the functor which sends a subset S to the object in Δ with the same cardinality, and which sends and inclusion $S \subseteq S'$ to the composite $[i] \cong S \subset S' \cong [j]$, where $[i]$ and $[j]$ are isomorphic to S and S' respectively as ordered sets.

The following lemma is a consequence of Theorem 6.4 of [34], along with the observation that $\widetilde{\mathrm{Tot}}^k(X^\bullet) = \underleftarrow{\mathrm{holim}}\,(i_k \circ X^\bullet)$.

Lemma 2.8. *For $X^\bullet$ a cosimplicial space, $\underleftarrow{\mathrm{holim}}\,(X^\bullet \circ c_k)$ is weakly equivalent to $\widetilde{\mathrm{Tot}}^k X^\bullet$.*

For any cosimplicial space there are spectral sequences for the homotopy groups [5] and homology groups [6, 33] of its totalization, which we will apply in Section 7. The homotopy spectral sequence is straightforward, with convergence immediate from its definition through the tower of fibrations

$$\cdots \leftarrow \mathrm{Tot}^i X^\bullet \leftarrow \mathrm{Tot}^{i+1} X^\bullet \leftarrow \cdots,$$

whose homotopy inverse limit is $\mathrm{Tot}X^\bullet$. Unraveling the definitions we have the following.

Proposition 2.9. *For a cosimplicial space $X^\bullet$ there is a spectral sequence converging to $\pi_*(\widetilde{\mathrm{Tot}}X^\bullet)$ with $E^1_{-p,q} = \bigcap$ ker ${s^k}_* \subseteq \pi_q(X^p)$. The d_1 differential is the restriction to this kernel of the map $\Sigma_{i=0}^{p+1}(-1)^i {d^i}_* \colon \pi_q(X^{p-1}) \to \pi_q(X^p)$.*

The homology spectral sequence is more subtle, generalizing the Eilenberg-Moore spectral sequence. One of the precise statements as to the convergence of this spectral sequence arising from [6] goes as follows.

Theorem 2.10. *For a cosimplicial space $X^\bullet$ there is a spectral sequence with $E^1_{-p,q} = \bigcap$ ker ${s^k}_* \subseteq H_q(X^p; \mathbb{F}_p)$. The d_1 differential is the restriction to this kernel of the map*

$$\Sigma_{i=0}^{p+1}(-1)^i {d^i}_* \colon H_q(X^{p-1}; \mathbb{F}_p) \to H_q(X^p; \mathbb{F}_p).$$

This spectral sequence converges to $H_*(\mathrm{Tot}X^\bullet;\mathbb{F}_p)$ if X^k is simply connected for all k and $E^1_{-p,q}=0$ when $q \geq cp$ for some $c>1$.

Alternately, one may arrive at the same spectral sequence from E^2 forward with $E^1_{-p,q} = H_q(X^p)$ and d_1 defined as before, but not restricted to the kernel of the codegeneracies.

This theorem follows immediately from Theorem 3.2 of [6] as both of Bousfield's conditions, namely that $E^1_{-p,q}=0$ if $p>q$ and that only finitely many $E^1_{-p,q}$ with $q-p=n$ are non-zero for any given n, follow from the vanishing with $q \geq cp$ for some $c>1$.

These spectral sequences apply unchanged to the homotopy invariant totalization. If $X^\bullet$ is a cosimpicial space and $\underline{X^\bullet}$ is a fibrant replacement (as given by Proposition 8.1.3 and Theorem 15.3.4 in [20]) then

$$\widetilde{\mathrm{Tot}}(X^\bullet) = \mathrm{Maps}\left(\widetilde{\Delta^\bullet}, X^\bullet\right) \simeq \mathrm{Maps}\left(\widetilde{\Delta^\bullet}, \underline{X^\bullet}\right) \simeq \mathrm{Maps}\left(\Delta^\bullet, \underline{X^\bullet}\right) = \mathrm{Tot}(\underline{X^\bullet}).$$

Because homotopy and homology of the entries and structure maps of $\underline{X^\bullet}$ agree with those of $X^\bullet$, the identifications of the E^1-terms of the associated spectral sequences are unchanged.

2.2. **Operads.** We choose to define non-Σ operads in terms of a well-known [3, 24, 34, 35] category of rooted trees.

Definition 2.11.
- A rooted, planar tree (or rp-tree) is an isotopy class of finite connected acyclic graph with a distinguished vertex called the root, embedded in the upper half plane with the root at the origin. Univalent vertices of an rp-tree (not counting the root, if it is univalent) are called leaves.
- Given an rp-tree T and a set of edges E the contraction of T by E is the tree T' obtained by, for each edge $e \in E$, identifying its initial vertex with its terminal vertex (altering the embedding of the tree in a neighborhood of e) and removing e from the set of edges.
- Let Γ denote the category of rp-trees, where there is a unique morphism from T to T', denoted either $f_{T,T'}$ or c_E, if T' is the contraction of T along some set of non-leaf edges E.
- Each edge of the tree is oriented by the direction of the root path, which is the unique shortest path to the root. The vertex of an edge which is further from the root is called its inital vertex, and the vertex closer to the root is called its terminal vertex. We say that one vertex or edge lies over another if the latter is in the root path of the former. A non-root edge is called redundant if its initial vertex is bivalent.
- Both the collection of leaves in an rp-tree and the collections of edges with a given terminal vertex are ordered, using the clockwise orientation of the plane.
- A sub-tree of an rp-tree is a sub-graph consisting of a vertex v and some collection of vertices lying over v along with all edges for which these vertices are terminal, such that the resulting subgraph is connected. A sub-tree is an rp-tree through a linear isotopy which translates v to the origin.

See Figure 2.13 for some examples of objects and morphisms in Γ. Let Γ_n denote the full subcategory of trees with n leaves. Note that Γ_n differs from Φ_n of [34], which is naturally the full subcategory of rp-trees without bivalent vertices. Each Γ_n has a terminal object, namely the unique tree with one vertex, called the nth corolla γ_n as in [24]. We allow for the tree γ_0 which has no leaves, only a root vertex, and is the only element of Γ_0. For a vertex v let $|v|$ denote the number of edges for which v is terminal, usually called the arity of v.

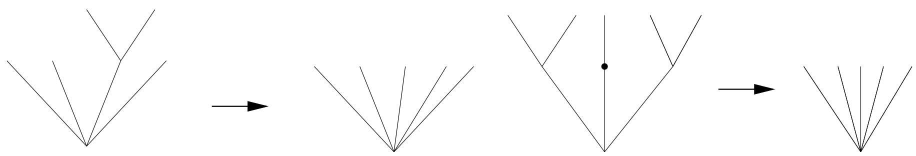
Definition 2.12. A non-Σ operad is a functor $\mathcal{O}$ from Γ to a symmetric monoidal category $(\mathcal{C},\odot)$ which satisfies the following axioms.

(1) $\mathcal{O}(T) = \odot_{v\in T}\mathcal{O}(\gamma_{|v|})$.
(2) $\mathcal{O}(\gamma_1) = \mathbf{1}_{\mathcal{C}} = \mathcal{O}(\gamma_0)$.
(3) If e is a redundant edge and v is its terminal vertex then $\mathcal{O}(c_{\{e\}})$ is the identity map on $\odot_{v'\neq v}F(\gamma_{v'})$ tensored with the isomorphism $(\mathbf{1}_{\mathcal{C}} \odot -)$ under the decomposition of axiom (1).

(4) If S is a subtree of T and if $f_{S,S'}$ and $f_{T,T'}$ contract the same set of edges, then under the decomposition of (1), $F(f_{T,T'}) = F(f_{S,S'}) \odot id$.

We sketch the equivalence of this definition with the standard ones. By axiom (1), the values of $\mathcal{O}$ are determined by its values on the corollas $\mathcal{O}(\gamma_n)$, which corresponds to $\mathcal{O}(n)$ in the usual operad terminology. Axioms (2) and (3) correspond to the unit condition. By axiom (4), the values of $\mathcal{O}$ on morphisms may be computed by composing morphisms on sub-trees, so we may identify some subset of basic morphisms through which all morphisms factor. We illustrate some basic morphisms in Γ which correspond to the $\circ_i$ operations and May's operad structure maps in Figure 2.13. Another basic class we consider is that of all morphisms $T \to \gamma_n$ where γ_n is a corolla and T is any tree; this class extends May's structure maps.

Figure 2.13.



Two morphisms in Γ which give rise to standard operad structure maps. The first corresponds to a $\circ_i$ operation, the second to one of May's structure maps.

Example 2.14.

- *The associative operad $\mathcal{A}$, defined in any symmetric monoidal category, has $\mathcal{A}(T) = \mathbf{1}_{\mathcal{C}}$, and $\mathcal{A}(T \to T') = id$ for all morphisms in Γ.*
- *Let Φ denote the full subcategory of rp-trees with no redundant edges (called the category of reduced trees in [24]) and let $P : \Gamma \to \Phi$ denote the functor which contracts all of the redundant edges of an rp-tree. The operad of planar trees, $Tree_n$ from Definition 1.41 of [24], is the operad in the category of sets which sends T to the set $(\Phi \downarrow P(T))$ of all $T' \in \Phi$ which map to $P(T)$. It sends a contraction of edges of T to the collection of contractions on the corresponding edges for trees over $P(T)$.*

Definition 2.15. A map between operads is a natural transformation which respects the decomposition of axiom (1) of Definition 2.12. An operad with multiplication is a non-Σ operad $\mathcal{O}$ equipped with a map from the associative operad $\mathcal{A}$.

The notion of operad with multiplication is due to Gerstenhaber and Voronov [12]. The canonical example is the endomorphism operad of an associative algebra $End(A)$. Algebras over an operad with multiplication are in particular associative algebras. An operad with multiplication in the category of spaces is an operad in the category of pointed spaces.

While we have taken a categorical approach to defining operads, we will take a more coordinatized approach to their associated cosimplicial objects. Recall the $\circ_i$ operations $\circ_i : \mathcal{O}(n) \odot \mathcal{O}(m) \to \mathcal{O}(n+m-1)$ which provide a basic set of morphisms for an operad, as illustrated in Figure 2.13. From section 3 of [26] we have the following.

Definition 2.16.

- Given an operad with multiplication $\mathcal{O}$, let μ denote the morphism $\mathcal{A}(2) = \mathbf{1}_{\mathcal{C}} \to \mathcal{O}(2)$.
- Define $d^i : \mathcal{O}(n) \to \mathcal{O}(n+1)$ by

$$d^i = \begin{cases} \mathbf{1}_{\mathcal{C}} \odot \mathcal{O}(n) \overset{\mu \odot id}{\to} \mathcal{O}(2) \odot \mathcal{O}(n) \overset{\circ_0}{\to} \mathcal{O}(n+1) & \text{if } i = 0 \\ \mathcal{O}(n) \odot \mathbf{1}_{\mathcal{C}} \overset{id \odot \mu}{\to} \mathcal{O}(n) \odot \mathcal{O}(2) \overset{\circ_i}{\to} \mathcal{O}(n+1) & \text{if } 0 < i < n+1 \\ \mathbf{1}_{\mathcal{C}} \odot \mathcal{O}(n) \overset{\mu \odot id}{\to} \mathcal{O}(2) \odot \mathcal{O}(n) \overset{\circ_{n+1}}{\to} \mathcal{O}(n+1) & \text{if } i = n+1. \end{cases}$$

- Define s^i as $\mathcal{O}(c_i)$ where $c_i : \gamma_n \to \gamma_{n-1}$ contracts the ith leaf of γ_n.
- Let $\mathcal{O}^\bullet$ be the cosimplicial object in $\mathcal{C}$ whose nth entry is $\mathcal{O}(n)$ and whose coface and codegeneracy maps are given by d^i and s^i above. If $\mathcal{C}$ is the category of vector spaces over a given field, let $HH^*(\mathcal{O})$ be the homology of the cochain complex defined by the cosimplicial abelian group $\mathcal{O}^\bullet$. If $\mathcal{C}$ is the category of spaces, we call $\widetilde{\mathrm{Tot}}(\mathcal{O}^\bullet)$ the totalization of $\mathcal{O}^\bullet$.

It is straightforward, and almost always left to the reader, to show that the maps d^i and s^i satisfy cosimplicial identities.

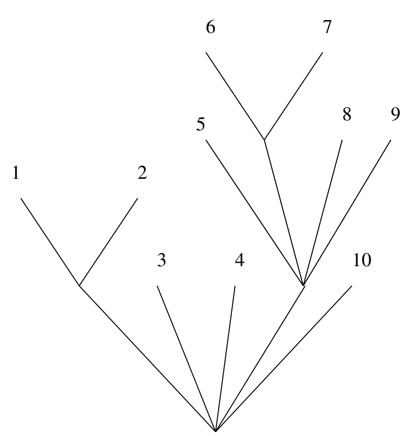
Remark. In the category of vector spaces, Tourtchine introduced the terminology $HH^*(\mathcal{O})$ because if A denotes an associative algebra and $\mathrm{End}(A)$ is its endomorphism operad then $HH^*(\mathrm{End}(A)) = HH^*(A)$, the usual Hochschild cohomology of A. We are not aware, however, of any sense in which Hochschild cohomology of operads is a cohomology theory for operads. Instead, Kontsevich conjectures that there is a suitable enriched homotopy structure on the category of operads of spaces such that $\widetilde{\mathrm{Tot}}(\mathcal{O}^\bullet)$ is the derived space of maps from the associative operad to $\mathcal{O}$.

3. The binomial operad

At the combinatorial heart of our work is the binomial operad, an intertwining of the sets $\binom{\mathbf{n}}{2}$, of distinct pairs of elements $(i,j) \in \mathbf{n} = \{1, \ldots, n\}$ with $i < j$ for definiteness, with rooted planar trees as follows. Recall that Set^{op}, the opposite category to the category of pointed sets, is symmetric monoidal with product is given by pointed union, denoted $\vee$, and unit given by the one-point set. Let S_+ denote the union of a set S with a disjoint base point.

Definition 3.1.
- The join of two leaves in an rp-tree is the first vertex (that is, the farthest from the root) at which their root paths coincide.
- Label both the leaves of an rp-tree and the edges which emanate from a given vertex v with elements of $\mathbf{n}$ and $\{1, \ldots, |v|\}$ respectively according to the order given by planar embedding. To an rp-tree T with n leaves and two distinct integers $i, j \in \mathbf{n}$ let v be the join of the leaves labelled i and j and define $J_v(i), J_v(j)$ to be the labels of the edges of v over which leaves i and j lie, as illustrated below.

Figure 3.2.



i, j	→	$J_v(i), J_v(j)$
6, 8	→	2, 3
3,4	→	2,3
3, 9	→	2,4
2, 10	→	1,5

- Let $\mathcal{B}$, the binomial operad, be the non-Σ operad in the category Set^{op} defined as follows:
  - $\mathcal{B}(T) = \bigvee_w \binom{|w|}{2}_+$, where w ranges over vertices of T.

– $\mathcal{B}(T \to \gamma_n)$, where γ_n is a corolla, is the function

$$(i,j) \in \binom{\mathbf{n}}{2} \mapsto (J_v(i), J_v(j)) \in \binom{|v|}{2} \subset \bigvee_{w \in T} \binom{|w|}{2}_+,$$

where v is the join of leaves i and j.

With our choice of definitions, it is straightforward to verify that $\mathcal{B}$ is an operad.

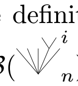
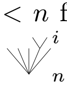
Theorem 3.3. *The cosimplicial object in Set^{op} associated to the binomial operad $\mathcal{B}^\bullet$ is naturally a simplicial set which is isomorphic to $S^2_\bullet$, the simplicial model for S^2.*

Proof. Recall that $S^2_\bullet = \Delta^2_\bullet/\partial\Delta^2_\bullet$, where $\Delta^2_\bullet$ is the standard simplicial model for Δ^2. The set n-simplices of Δ^2 is the set of $(x_0 \leq x_1 \leq \cdots \leq x_n) \in \{0,1,2\}^{n+1}$, so the cardinality of Δ^2_n is the $(n+1)$st triangular number. The ith face and degeneracy maps are defined by deleting and repeating x_i, respectively. To obtain $S^2_\bullet$ we identify all n-tuples in which one of $\{0,1,2\}$ does not appear to a single simplex in each degree, which is degenerate in positive degrees.

The nth entry of $S^2_\bullet$ is isomorphic to $\binom{\mathbf{n}}{2}_+$, the set of unordered pairs of points in $\mathbf{n}$, along with a disjoint point $+$ which is the image of $\partial\Delta^2_\bullet$ under the quotient map. The isomorphism records the indices j and k for which $x_{j-1} < x_j$ and $x_{k-1} < x_k$, when there are two such indices. When there are not two such indices, such a sequence is identified with the degenerate point $+$. Under this isomorphism d_i sends $+$ to $+$ and for $i \neq 0, n$ sends

$$(1) \qquad (j,k) \mapsto \begin{cases} (\delta_i(j), \delta_i(k)) & \text{if } \ \delta_i(j) \neq \delta_i(k) \\ + & \text{otherwise} \end{cases} \qquad \text{where} \quad \delta_i(j) = \begin{cases} j & \text{if } \ j \leq i \\ j-1 & \text{if } \ j > i. \end{cases}$$

For $i = 0$ and $i = n$ the basic formula is the same, but the (j,k) which get sent to $+$ are those with $j = 1$ or $k = n$, respectively. Similarly, s_i sends $+$ to $+$ and sends $(j,k) \mapsto (\sigma_i(j), \sigma_i(k))$ where $\sigma_i(j) = j$ if $j \leq i$ or $j+1$ otherwise.

By definition $\mathcal{B}^n = \binom{\mathbf{n}}{2}_+$, and it is straightforward to check that the structure maps of $S^2_\bullet$ and the associated cosimplicial object of $\mathcal{B}^\bullet$ coincide. To give an example, we unravel the definition of d^i with $0 < i < n$ for $\mathcal{B}^n$. These coface maps are given by composites $(\mathcal{B}(\gamma_n) \vee +) \xrightarrow{id \vee \mu} \mathcal{B}($$) \xrightarrow{\circ_i} \mathcal{B}(\gamma_{n+1})$, where  denotes the tree with n root edges and one trivalent internal vertex, which is terminal for the ith root edge. In Set^{op} the morphism $id \vee \mu$ corresponds to the collapse map in Set which sends $\binom{\mathbf{2}}{2}_+ \subset \binom{\mathbf{n}}{2}_+ \vee \binom{\mathbf{2}}{2}_+$ to the base point and is the identity on $\binom{\mathbf{n}}{2}_+$. The morphism $\circ_i$ sends $(i, i+1)$ to $(1,2) \in \binom{\mathbf{2}}{2}$ and sends all other (j,k) to $(\delta_i(j), \delta_i(k)) \in \binom{\mathbf{n}}{2}$. The composite of these two maps coincides with the definition of d_i for $S^2_\bullet$, as in Equation 1. □

The composite of $\mathcal{B}^\bullet : \Gamma \to \text{Set}^{op}$ with $X^- : \text{Set}^{op} \to \text{Top}$ gives rise to a functor which we call $X^{\mathcal{B}^\bullet}$, for which the axioms of an operad are immediate to verify. Theorem 3.3 implies the following.

Corollary 3.4. *For any X in a symmetric monoidal category $\mathcal{C}$, $X^{S^2_\bullet}$ canonically defines an operad through its isomorphism with $X^{\mathcal{B}^\bullet}$.*

We have yet to find any familiar interpretation for algebras over these operad in the categories of spaces and vector spaces. For spaces the operad structure on $X^{S^2_\bullet}$ does have consequences, as we explain in Example 7.4.

4. The Kontsevich operad

In this section we define an operad structure on the completion of configurations in Euclidean space up translation and scaling defined by Kontsevich [21]. The fact that one could define operads using the canonical completion of configuration spaces was noticed by Getzler and Jones [13] soon after this completion was introduced by Fulton-MacPherson [10] and Axelrod-Singer [1]. This operad structure was fully developed by Markl [23]. The variant with which we work was first proposed by Kontsevich [21], but Gaiffi [11] first pointed out the difference with the canonical completion. Indeed, while Kontsevich called the following the Fulton-MacPherson operad, we call it the Kontsevich operad to highlight the difference between the two constructions. Though this construction lacks some of the properties of the canoncial completion, in particular smoothness, is has diagonal and projection maps which satisfy simplicial identities exactly rather than up to homotopy. These properties led to this construction's independent discovery, its use, and its naming as the simplicial variant in [34].

We start by setting notation for products of spaces and maps, which we will use extensively.

Notation. If S is a finite set, X^S is the product $X^{\#S}$ where $\#S$ is the cardinality of S. For coordinates we use $(x_s)_{s\in S}$ or just (x_s) when S is understood. Similarly, a product of maps $\prod_{s\in S} f_s$ may be written$(f_s)_{s\in S}$ or just (f_s). Recall that $\mathbf{n} = \{1, \ldots, n\}$.

Definition 4.1.
- Let $C_n(\mathbb{R}^m)$ denote the space of $(x_i) \in (\mathbb{R}^m)^{\mathbf{n}}$ such that if $i \neq j$ then $x_i \neq x_j$. Let $\widetilde{C}_n(\mathbb{R}^m)$ be the quotient of $C_n(\mathbb{R}^m)$ by the equivalence relation generated by translating all of the x_i by some v or multiplying them all by the same positive scalar.
- For any $v \in \mathbb{R}^m - 0$, let $u(v) = \frac{v}{||v||}$, the unit vector in the direction of v.
- Let $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$ be the closure of the image of $\widetilde{C}_n(\mathbb{R}^m)$ under the map (π_{ij}) to $(S^{m-1})^{\binom{\mathbf{n}}{2}}$, where π_{ij} sends the equivalence class of (x_i) to $u(x_i - x_j)$.

Note that (π_{ij}) is not injective – it fails to be so on configurations in which all the x_i lie on some line – so $\widetilde{C}_n(\mathbb{R}^m)$ is not a subspace of $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$. But we do have the following theorem, a consequence of Corollary 4.5, Lemma 4.12 and Corollary 5.10 of [35].

Theorem 4.2. *The canonical map $\widetilde{C}_n(\mathbb{R}^m) \to \widetilde{C}_n\langle[\mathbb{R}^m]\rangle$ is a homotopy equivalence.*

What makes $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$ manageable is that we can characterize it as a subspace of $(S^{m-1})^{\binom{\mathbf{n}}{2}}$. It will be convenient to extend coordinates for $(u_{ij}) \in (S^{m-1})^{\binom{\mathbf{n}}{2}}$ by letting u_{ji} be $-u_{ij}$ when $j > i$.

Definition 4.3.
- A chain, or k-chain, in S is a collection $\{i_1i_2,\ i_2i_3,\ \ldots,\ i_{k-1}i_k\}$, with all $i_j \in S$ and $i_j \neq i_{j+1}$. Such indices label the edges of a path in the complete graph on S. A chain is a loop, or k-loop, if $i_k = i_1$. A chain is straight if it does not contain any loops. The reversal of a chain is the chain $i_ki_{k-1}, \ldots, i_2i_1$.
- A point $(u_{ij}) \in (S^{m-1})^{\binom{\mathbf{n}}{2}}$ is three-dependent if for any 3-loop L in $\mathbf{n}$ there exist $a_{ij} \geq 0$, with at least one non-zero, such that $\sum_{ij\in L} a_{ij}u_{ij} = 0$.
- If S has four elements and is ordered we may associate to a straight 3-chain $C = \{ij, jk, k\ell\}$ a permutation of S denoted $\sigma(C)$ which orders (i, j, k, ℓ). A complementary 3-chain C^* is a chain, unique up to reversal, which is comprised of the three pairs of indices not in C.
- A point in $(S^{m-1})^{\binom{\mathbf{n}}{2}}$ is four-consistent if for any $S \subset \mathbf{n}$ of cardinality four and any $v, w \in S^{m-1}$ we have that

$$\sum_{C\in\mathcal{C}^3(S)} (-1)^{|\sigma(C)|} \left(\prod_{ij\in C} u_{ij} \cdot v\right) \left(\prod_{ij\in C^*} u_{ij} \cdot w\right) = 0, \tag{2}$$

where $\mathcal{C}^3S$ is the set of straight 3-chains in S modulo reversal and $|\sigma(C)|$ is the sign of $\sigma(C)$.

Points in the image of $C_n(\mathbb{R}^m)$ under (π_{ij}) are three-dependent and four-consistent, and also satisfy $u_{ij} = -u_{ji}$, a condition we refer to as anti-symmetry. These properties also hold for $C_n\langle[\mathbb{R}^m]\rangle$, the closure, by continuity. Adding the converse, we have the following, which is Theorem 5.14 of [35].

Theorem 4.4. *$\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$ is the subspace of all three-dependent, four-consistent points in $(S^{m-1})^{\binom{n}{2}}$.*

We will define operad maps on the completions $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$ through coordinates of $(S^{m-1})^{\binom{n}{2}}$. Embed $(S^{m-1})^{\binom{n}{2}}$, and thus $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$, in $(S^{m-1})^{\binom{n}{2}+}$ as the subspace of $(u_{ij}) \times u_+$ with u_+ equal to the basepoint of S^{m-1}, which we choose to be the south pole $*_S = (0, \ldots, 0, -1)$.

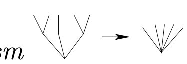
Theorem 4.5. *The operad structure on $(S^{m-1})^{\mathcal{B}^\bullet}$ restricts to the subspaces $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$.*

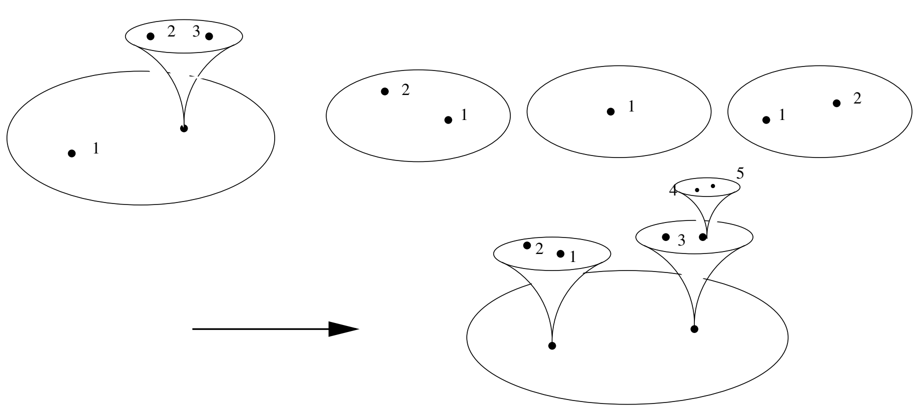
We call the resulting operad, whose nth entry is $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$, the Kontsevich operad $\mathcal{K}_m$.

Proof. Given a tree T, let $(u^v_{k\ell})$ be a point in $(S^{m-1})^{\mathcal{B}(T)}$, where v ranges over vertices of T and $k, \ell \in \binom{|\mathbf{v}|}{2}$. By Definition 3.1, the operad structure on $(S^{m-1})^{\mathcal{B}^\bullet}$ sends the morphism $T \to \gamma_n$ to the map given in coordinates by $(u^v_{k\ell}) \mapsto (w_{ij})_{i,j \in \binom{n}{2}}$, where $w_{ij} = u^v_{J_v(i),J_v(j)}$ and v is the join vertex of the leaves i and j.

We verify that if the $(u^v_{k\ell})$ satisfy three-dependence and four-consistency for each v, then so does (w_{ij}). For three-dependence, given some w_{ij}, w_{jk} and w_{ki}, there are two cases to consider. In the first case the join in T of leaves i and j lies over that of i and k, so that $w_{jk} = -w_{ki}$ or $0w_{ij} + 1w_{jk} + 1w_{ki} = 0$. In the second case the joins of i and j and k are all equal to the same v, so that the dependence of w_{ij}, w_{jk} and w_{ki} follows from that of $(u^v_{J_v(i)J_v(j)})$, $(u^v_{J_v(j)J_v(k)})$ and $(u^v_{J_v(k)J_v(i)})$. Four-consistency works similarly. Given indices i, j, k and ℓ the pairwise joins could all equal some v, in which case four consistency of these $\{w_{ij}\}$ follows from that of $\{u^v_{k\ell}\}$. Or, if for example the join of i and j lies over those of i, k and ℓ, then $w_{ik} = w_{jk}$ and $w_{i\ell} = w_{j\ell}$, so four-consistency will follow by the canceling of terms which agree but for opposite signs. □

In [35], we stratify $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$, and in particular the points added in closure. We will not need this stratification explicitly for our applications, but the related geometry is helpful in understanding the operad structure of $\mathcal{K}_m$. The stratification is indexed by rp-trees with no redundant edges, with the Tth stratum being the image of a map from $\left(\widetilde{C_{|v|}}(\mathbb{R}^m)\right)^{v \in V(T)}$ to $\widetilde{C}_n\langle[\mathbb{R}^m]\rangle$ sending $(x^v_i) \mapsto (u_{ij})$ with $u_{ij} = \pi_{J_v(i)J_v(j)}((x_i)_v)$. Studying this stratification helped lead us to the definition of the binomial operad. See section 3 and Theorem 5.14 of [35] for a full development of this geometry, which is illustrated in Figure 4.6.

Figure 4.6. *The effect of an operad structure map associated to the morphism* .



The standard completions $\widetilde{C_n}[\mathbb{R}^m]$ also constitute entries of an operad, which has been more intensively studied [13, 21, 23]. The reason we use $\mathcal{K}^m$ is the following.

Proposition 4.7. *The associative operad maps to the Kontsevich operad, for definiteness by choosing the basepoint* $(x_{ij}) \in \widetilde{C_n}\langle[\mathbb{R}^m]\rangle$ *with all* $x_{ij} = *_S$*, for all* n.

Finally, we give a comparison between the little disks operad, which we need to formalize, and the Kontsevich operad.

Definition 4.8.
- Recall that the space of n little disks in D^m, the unit disk, denoted $D^m(n)$ is the subspace of $C_n(D^m) \times (0,1]^n$ of $(x_i) \times (r_i)$ such that the balls $B(x_i, r_i)$ are contained in D^m and have disjoint interiors.
- Let T be a tree whose vertices consist of the root vertex v_0 and a terminal vertex v_e for each root edge e. Thus, $T \to \gamma_n$, where n is the number of leaves of T, gives rise to one of May's structure maps as in Figure 2.13. Given a label $i \in \mathbf{n}$ let $v(i)$ be the initial vertex for the ith leaf, let $o(i)$ be the label of leaf i within the ordering on edges of $v(i)$ and let $e(i)$ be the label of the root edge for which $v(i)$ is terminal.
- Define $\mathcal{D}^m(T \to \gamma_n)$ as follows

$$(x_i^v, r_i^v)_{1 \le i \le \# v}^{v \in V(T)} \mapsto (y_j, \rho_j)_{j \in \mathbf{n}} \quad \text{where} \quad y_j = x_{e(j)}^{v_0} + r_{e(j)}^{v_0} x_{o(j)}^{v(j)} \quad \text{and} \quad \rho_j = r_{e(j)}^{v_0} r_{o(j)}^{v(j)}.$$

Boardman and Vogt [4] and May [25] showed that algebras over $\mathcal{D}^m$ are m-fold loop spaces.

Theorem 4.9. *Let* T *be a tree with a vertex for each root edge as in Definition 4.8 above. The following diagram commutes up to homotopy,*

$$\begin{array}{ccc} \mathcal{D}^m(T) & \xrightarrow{\mathcal{D}^m(T \to \gamma_n)} & \mathcal{D}^m(n) \\ {\scriptstyle p_T}\downarrow & & {\scriptstyle p_n}\downarrow \\ \mathcal{K}^m(T) & \xrightarrow{\mathcal{K}^m(T \to \gamma_n)} & \mathcal{K}^m(n), \end{array}$$

where the vertical maps p_T *are the products of projections* $p_n : D^m(n) \to C_n(\mathbb{R}^m)$ *composed with the canonical maps* $C_n(\mathbb{R}^m) \to \widetilde{C}_n(\mathbb{R}^m) \to \widetilde{C}_n\langle[\mathbb{R}^m]\rangle$*. Moreover, the vertical maps are homotopy equivalences.*

Proof. We define the homotopy explicitly. Define $H : \mathcal{D}^m(T) \times (0,1] \to \widetilde{C}_n(\mathbb{R}^m)$ by sending (x_i^v, r_i^v) as in Definition 4.8 and $t \in (0,1]$ to the coset of $(y_j(t))$ with $y_j(t) = x_{e(j)}^{v_0} + t \cdot r_{e(j)}^{v_0} x_{o(j)}^{v(j)}$. We claim that H extends uniquely to $\overline{H} : \mathcal{D}^m(T) \times [0,1] \to \widetilde{C}_n\langle[\mathbb{R}^m]\rangle$, and that $\overline{H}$ coincides with $\mathcal{K}^m(T \to \gamma_n) \circ p_T$ when $t = 1$. Consider $u_{ij} = \pi_{ij}((y_k(t)))$. If the join of leaves i and j is one of the non-root vertices, so $v(i) = v(j)$, then u_{ij} will be equal to the unit vector in the direction of $x_{o(j)}^{v(j)} - x_{o(i)}^{v(j)}$, independent of t. If the join of leaves i and j is the root vertex, then as t approaches 0, u_{ij} approaches the unit vector in the direction of $x_{e(j)}^{v_0} - x_{e(i)}^{v_0}$. These limiting values coincide with the definition of $\mathcal{K}^m(T \to \gamma_n)(x_i^v)$.

That the projection $D^m(n) \to C_n(D^m)$ is a homotopy equivalence is standard, known since the definition of little cubes in [3], so by Theorem 4.2 the maps p_n are homotopy equivalences. □

In fact, [21] claims that these $\mathcal{D}^m$ and $\mathcal{K}^m$ are homotopy equivalent operads, which we assume to mean that there is a chain of equivalences of maps of operads, that is maps which commute with structure maps exactly. We will not need this stronger claim. Recall that the homology of an operad of spaces with field coefficients is an operad of vector spaces by the Künneth theorem. The homology of the little disks operad has a well-known description.

Definition 4.10. The kth entry of the degree n Poisson operad $\mathrm{Poiss}_n(k)$ is the submodule of the symmetric algebra on the free graded Lie algebra over k variables $x_1, \ldots, x_k$ spanned by monomials in which all variables appear exactly once. Monomials are graded by putting all x_i in degree zero and giving the bracket degree n. So for example $[x_1, x_3]\,[[x_4, x_2]x_5]$ and $x_1 x_2 \cdots x_5$ are elements of $\mathrm{Poiss}_3(5)$ of degree nine and zero respectively.

The map $\circ_i$: $\mathrm{Poiss}_n(j) \otimes \mathrm{Poiss}_n(k) \to \mathrm{Poiss}_n(j+k-1)$ sends $m_1 \otimes m_2$ to the monomial defined as follows.

- For each j, substitute x_{j+i-1} for x_j in m_2 to obtain $\overline{m_2}$.
- In m_1, substitute x_{j+i-1} for x_j if $j > i$ and $\overline{m_2}$ for x_i to obtain m.
- Reduce m according to the graded Leibniz rule
$$[a, bc] = [a, b]c + (-1)^{(|a|+n+1)|b|} b[a, c],$$
to obtain an element of $\mathrm{Poiss}_n(i + j - 1)$.

The following corollary is essentially a summary of Fred Cohen's famous calculation of the homology of $\mathcal{D}^m$ [9]. We also plan to give an exposition of this result in [36].

Theorem 4.11. *The homology of $\mathcal{K}^m$ is the degree $m-1$, Poisson operad* Poiss_{m-1}.

5. Models for spaces of knots and immersions arising from the calculus of embeddings

5.1. **A brief overview of the calculus of embeddings.** Our main theorems connect the theory of operads to Goodwillie calculus. We first informally introduce some terminology from the calculus of embeddings (see Weiss's [43] for an excellent introduction and [44] for a full treatment), and then precisely state the theorems we use. The main spaces with whom we are concerned are related to embeddings and immersions.

Definition 5.1. Let $\mathrm{Emb}(M, N)$ denote the space of embeddings of M in N, topologized as a subspace of the space of all maps, with the compact-open topology. Similarly, let $\mathrm{Imm}(M, N)$ be the space of immersions of M in N. If M and N have boundary, we usually specify some boundary conditions. In particular, if $M = \mathbb{I}$ and $N = \mathbb{I}^m$, we let $*_+ = (0, \ldots, 0, 1) \in \mathbb{I}^m$, $*_- = (0, \ldots, 0, -1)$ and demand that the endpoints of $\mathbb{I}$ map to $*_+$ and $*_-$ with tangent vectors $*_S$.

By results of Palais [30], these spaces are dominated by simplicial complexes and thus homotopy equivalent to CW-complexes [29].

In the calculus of embeddings, one views spaces of embeddings, immersions and other moduli in differential topology as functors from the poset of open subsets of M to topological spaces, a philosophy originally due to Gromov. Ultimately interested in the value of the functor at the open set which is all of M, one tries to use homotopy limits to interpolate that value from values of the functor at simple open sets, namely those which are diffeomorphic to a union of open balls. Functors for which interpolation using a finite number of balls works perfectly are called polynomial, and those for which interpolation works in the limit as the number of balls tends to infinity are called analytic. Weiss shows in [44] that polynomial functors are those which satisfy higher-order Mayer-Vietoris conditions, and Goodwillie-Weiss show in [15] that analyticity follows from satisfying those conditions through an increasing range of connectivity. More formally we have the following.

Definition 5.2.
- For any manifold W of dimension m let $\mathcal{U}(M)$ be the category of open subsets of M under inclusion, and let $\mathcal{U}_k(M)$ be the full sub-category of $\mathcal{U}(M)$ of open sets diffeomorphic to $\sqcup_i \mathbb{R}^m$, where $i \leq k$.
- For any contravariant functor F from $\mathcal{U}(M)$ to spaces let T_kF be the functor which sends W to $\underleftarrow{\mathrm{holim}}_{U \in \mathcal{U}_k(W)} F(U)$.
- Let $b_k(F) : T_kF \to T_{k-1}F$ be the canonical natural transformation defined by restricting $\mathcal{U}_k(M)$ to $\mathcal{U}_{k-1}(M)$, and let $T_\infty F$ be the homotopy inverse limit of the T_kF over these restrictions.
- Let $\eta_k(F) : F \to T_kF$ be the canonical natural transformation arising from the maps $F(M) \to F(U)$ for $U \in \mathcal{U}_k(M)$. If by context F is understood, we will use the simpler notation η_k.
- The natural transformations η_k commute with the b_k, so let $\eta_\infty : F \to T_\infty F$ be the limiting natural transformation.

The sequence $T_0F \overset{b_1}{\leftarrow} T_1F \overset{b_2}{\leftarrow} T_2F \leftarrow \cdots$ is called the Taylor tower for F. Analyticity means that the homotopy inverse limit of this tower is weakly equivalent to F. The motivating example for this circle of ideas is that of immersions.

Theorem 5.3. *If* $\dim(M) < \dim(N)$ *then for* $k \geq 1$, $\eta_k : \mathrm{Imm}(U, N) \to T_k\mathrm{Imm}(U, N)$ *is a weak equivalence for any* $U \subseteq M$.

This theorem follows from Example 2.3 of [44], which says that immersions are a linear functor, and the commentary after Theorem 5.1 of [44]. The embedding functor is not polynomial but by theorems of Goodwillie, Klein and Weiss it is analytic. The following Theorem is essentially Corollary 2.5 of [15].

Theorem 5.4. *If* $\dim(M) < \dim(N) - 2$ *then* $\eta_\infty(\mathrm{Emb})$ *is a weak equivalence.*

For $\dim(M) < \dim(N) - 2$ as stated, this theorem requires deep disjunction results of Goodwillie, and surgery results of Goodwillie-Klein [19, 18]. If $\dim(M)$ is less than roughly $\frac{\dim(N)}{2}$ there are much easier methods, using only the Blakers-Massey theorem and dimension counting, for proving the needed higher-order Mayer-Vietoris conditions.

5.2. **Knot space models through homotopy limits of configuration spaces.** Definition 5.2 of $T_k\mathrm{Emb}(M, N)$ is ephemeral, but the building blocks, namely spaces of embeddings of balls, are essentially configuration spaces. Goodwillie, Klein and Weiss have given more concrete models for the spaces in this Taylor tower (or for the homotopy fibers of $T_k \to T_{k-1}$, which are called layers), either as spaces of sections, as in section nine of [44], or as mapping spaces with strongly defined equivariance properties, as in [16]. In the case of knots we have developed three closely-related models for these polynomial approximations [34] and used them for both computational and geometric applications [32, 8]. These models all utilize completions of configuration spaces constructed similarly to the Kontsevich operad.

Definition 5.5.
- Let $A_n\langle[\mathbb{I}^m]\rangle$ be the product $(\mathbb{I}^m)^{\mathbf{n}} \times (S^{m-1})^{\binom{\mathbf{n}}{2}}$, with coordinates $(x_i) \times (u_{ij})$.
- Let $C_n\langle[\mathbb{I}^m]\rangle$ be the closure of the image of $C_n(\mathbb{I}^m)$ under $\iota \times (\pi_{ij})$, where ι is the inclusion of $C_n(\mathbb{I}^m)$ in $(\mathbb{I}^m)^{\mathbf{n}}$.
- Let $C_n\langle[\mathbb{I}^m, \partial]\rangle$ be the closure in $C_{n+2}\langle[\mathbb{I}^m]\rangle$ of the subspace of $C_{n+2}(\mathbb{I}^m)$ with $x_1 = *_+ = (0, \ldots, 0, 1)$ and $x_{n+2} = *_-$.

In [35] we study $C_n\langle[\mathbb{I}^m]\rangle$ by relating it to the canonical compactification $C_n[\mathbb{I}^m]$, which is a manifold with corners. We characterize $C_n\langle[\mathbb{I}^m]\rangle$ as a subspace of its defining ambient space, as stated for $\widetilde{C}_n\langle[\mathbb{R}^k]\rangle$ in Theorem 4.4. The following is essentially Theorem 5.14 of [35].

Theorem 5.6. $C_n\langle[I^k]\rangle$ *is the subspace of* $(x_i) \times (u_{ij})$ *such that* $(u_{ij}) \in \widetilde{C}_n\langle[\mathbb{R}^k]\rangle$ *and if* $x_i \neq x_j$ *then* u_{ij} *is* $u(x_j - x_i)$.

In our models, we need diagonal maps between configuration spaces. The idea is to add a point "infinitesimally far" from one point in a configuration, but to do so entails choosing a unit tangent vector at that point.

Definition 5.7. Let $C'_n\langle[\mathbb{I}^m]\rangle = C_n\langle[\mathbb{I}^m]\rangle \times (S^{m-1})^{\mathbf{n}}$. Let $A'_n\langle[\mathbb{I}^m]\rangle = (\mathbb{I}^m \times S^{m-1})^{\mathbf{n}} \times (S^{m-1})^{\binom{\mathbf{n}}{2}}$, which is canonically diffeomorphic to $(\mathbb{I}^m)^{\mathbf{n}} \times (S^{m-1})^{\mathbf{n}^2}$. We use coordinates for this latter presentation of the form $(x_i) \times (u_{ij})$ with i and j possibly equal.

As in Theorem 4.5 we define maps between the $C'_n\langle[\mathbb{I}^m]\rangle$ at the level of the ambient spaces $A'_n\langle[\mathbb{I}^m]\rangle$, using Theorem 5.6 to check that they restrict appropriately. We are aided by the following combinatorial shorthand.

Definition 5.8.
- Given a map of sets $\sigma : R \to S$ let p^X_σ or just p_σ denote the map from X^S to X^R which sends $(x_i)_{i \in S}$ to $(x_{\sigma(j)})_{j \in R}$.

- Given $\sigma : \mathbf{m} \to \mathbf{n}$, define $A_\sigma : A'_n\langle[\mathbb{I}^m]\rangle \to A'_m\langle[\mathbb{I}^m]\rangle$ as $p_\sigma^{\mathbb{I}^m} \times p_{\sigma^2}^{S^{k-1}}$.

Proposition 5.9 (Proposition 6.6 of [35]). *The restriciton A_σ to $C'_n\langle[\mathbb{I}^m]\rangle$ maps to $C'_m\langle[\mathbb{I}^m]\rangle$. If σ sends $1 \to 1$ and $n \to m$ then A_σ also restricts to a map, which we call F_σ, from $C'_{n-2}\langle[\mathbb{I}, \partial]\rangle$ to $C'_{m-2}\langle[\mathbb{I}, \partial]\rangle$.*

We may now define diagonal maps on compactified configuration spaces with tangential data.

Definition 5.10. Let $\delta^i : C'_n\langle[\mathbb{I}^m, \partial]\rangle \to C'_{n+1}\langle[\mathbb{I}^m, \partial]\rangle$ be F_{σ_i} where $\sigma_i : \underline{n+3} \to \underline{n+2}$ sends j to itself if $j \leq i$ or $j-1$ if $j > i$.

A final key property of this compactification is that it is functorial for embeddings. The proof of the following theorem is identical to that of Corollary 4.8 of [35], using Theorem 5.8 of [35] and the fact that $C_n\langle[\mathbb{I}]\rangle = \Delta^n$. Recall that for a nonzero vector $v \in \mathbb{R}^m$, $u(v) = \frac{v}{||v||}$.

Theorem 5.11. *For an embedding $f : \mathbb{I} \to \mathbb{I}^k$ there is an evaluation map $ev_n(f) : \Delta^n \to C_n\langle[\mathbb{I}^k]\rangle$ which extends the map from the interior of Δ^n to $C'_n(\mathbb{I}^k)$ sending (t_i) to $(f(t_i)) \times (u(f'(t_i)))$.*

One of the main themes of [34] is connecting this evaluation map with the calculus of embeddings. Applying this calculus to embeddings of the unit interval is simpler than to embeddings of higher-dimensional manifolds because the category $\mathcal{U}_k(\mathbb{I})$ may be replaced by the category of subsets of a finite set (see Definition 2.6).

Definition 5.12. Let $\mathcal{D}_k^m$ be the functor from $P_0(k)$ to spaces which sends $S \subseteq [k]$ to $C'_{\#S-1}\langle[\mathbb{I}^m, \partial]\rangle$ and which sends the inclusion $S \subset S \cup j$ to the map δ^i where i is the number of elements of S less than j. Let $D_k^m = \underset{\longleftarrow}{\text{holim}}\, \mathcal{D}_k^m$.

In the notation of [34], $\mathcal{D}_k^m$ would be $\mathcal{D}_k\langle[\mathbb{I}^m]\rangle$. Because the realization of $P_0(k)$ is Δ^k and all of the maps δ^i are inclusions of subspaces, D_k^m is a subspace of $\text{Maps}(\Delta^k, C'_k\langle[\mathbb{I}^m, \partial]\rangle)$. If $f \in \text{Emb}(\mathbb{I}, \mathbb{I}^m)$ is a knot, $ev_k(f)$ defines an element of D_k^m, as we may simply check that if $t_j = t_{j+1}$ for some point $(t_i) \in \Delta^k$ then the image of $ev_k(f)\,((t_i))$ is in the image of δ^j. By abuse, let ev_k denote the adjoint map from $\text{Emb}(\mathbb{I}, \mathbb{I}^m)$ to D_k^m. Building on the simpler "cutting method" definition of $T_k\text{Emb}(\mathbb{I}, \mathbb{I}^m)$, as described in Section 3 of [34], Lemma 5.18 and the proof of Theorem 5.3 of [34] establish the following.

Theorem 5.13. *D_k^m is homotopy equivalent to $T_k\text{Emb}(\mathbb{I}, \mathbb{I}^m)$, and ev_k agrees with η_k in the homotopy category.*

Though not historically expressed in these terms, there is a similar theorem for immersions of an interval.

Definition 5.14.
- Let $d^i : (S^{m-1})^j \to (S^{m-1})^{j+1}$ be the ith diagonal inclusion, which by convention for $i = 0$ and $i = j+1$ are insertion of the basepoint $*_S$ as the first, respectively last, coordinate.
- Let $\mathcal{G}_k^m$ be the functor from $P_0(k)$ to spaces which sends $S \subseteq [k]$ to $(S^{m-1})^{\#S-1}$ and which sends the inclusion $S \subset S \cup j$ to the diagonal map d^i where i is the number of elements of S less than j.
- Let $G_k^m = \underset{\longleftarrow}{\text{holim}}\, \mathcal{G}_k^m$.

As was true for D_k^m, G_k^m is a subspace of the space of maps from Δ^k to the terminal space of $\mathcal{G}_k^m$, namely $(S^{m-1})^k$. The evaluation map for immersions is the unit derivative map. By abuse, let $ev_k : \text{Imm}(\mathbb{I}, \mathbb{I}^m) \to G_k^m$ send f to the map which sends $t_1, \ldots, t^k$ to $uf'(t_1), \ldots, uf'(t_k)$.

Theorem 5.15. *If $k \geq 1$, G_k^m is homotopy equivalent to $T_k\text{Imm}(\mathbb{I}, \mathbb{I}^m)$, and thus to $\text{Imm}(\mathbb{I}, \mathbb{I}^m)$. Moreover, ev_k agrees with η_k in the homotopy category.*

Sketch of proof. There are many ways to establish this theorem. By the Hirsh-Smale theorem [37], $\text{Imm}(\mathbb{I}, \mathbb{I}^m)$ is homotopy equivalent to ΩS^{m-1}, through the unit derivative map. But $ev_1 : \text{Imm}(\mathbb{I}, \mathbb{I}^m) \to \underset{\longleftarrow}{\text{holim}}\,(* \to S^{m-1} \leftarrow *)$ is also the unit derivative map, which establishes the theorem for $k = 1$. For the other k, we may use Lemma 2.8, since $\mathcal{G}_k^m$ is the pull-back of the standard cosimplicial model for ΩS^{m-1} through the functor c_k of Definition 2.7. The kth totalization of this cosimplicial model, which is fibrant, is homeomorphic to ΩS^{m-1} if $k \geq 1$, from which it follows that $\mathcal{G}_k^m$ is homotopy equivalent to ΩS^{m-1}. The map from ΩS^{m-1} to the kth totalization, and thus G_k^m, is through evaluation of the unit derivative. □

Let $\tau : \mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \to \mathrm{Imm}(\mathbb{I}, \mathbb{I}^m)$ denote the inclusion. Let $\rho_k^m : \mathcal{D}_k^m \to \mathcal{G}_k^m$ denote the map of diagrams defined on each entry by projection from $C'_n\langle[\mathbb{I}^m, \partial]\rangle = C_n\langle[\mathbb{I}^m, \partial]\rangle \times (S^{m-1})^n$ onto $(S^{m-1})^n$, and let p_k^m also denote the induced map on homotopy limits.

Proposition 5.16. *The square*

$$\begin{array}{ccc} \mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) & \xrightarrow{\tau} & \mathrm{Imm}(\mathbb{I}, \mathbb{I}^m) \\ {\scriptstyle ev_k}\downarrow & & {\scriptstyle ev_k}\downarrow \\ D_k^m & \xrightarrow{p_k^m} & G_k^m. \end{array}$$

commutes. Moreover, p_k^m agrees with $T_k(\tau)$ in the homotopy category.

Sketch of proof. The commutativity of the diagram is immediate from the definitions. That p_k^m agrees with $T_k(\tau)$ in the homotopy category ultimately follows from the fact that for U a disjoint union of $k+2$ open intervals, two of which contain endpoints of $\mathbb{I}$ and thus are fixed at one end, we have $\mathrm{Emb}(U, \mathbb{I}^m) \simeq C'_k\langle[\mathbb{I}^m, \partial]\rangle$, $\mathrm{Imm}(U, \mathbb{I}^m) \simeq (S^{m-1})^k$ and the inclusions from embeddings to immersions coincides with projection, as in the definition of p_k^m. □

5.3. **A closer look at E_m.**

Proposition 5.17. *The inclusion $\tau : \mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \to \mathrm{Imm}(\mathbb{I}, \mathbb{I}^m)$ is null-homotopic, so*

$$E_m \simeq \mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega\mathrm{Imm}(\mathbb{I}, \mathbb{I}^m) \simeq \mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega^2 S^{m-1}.$$

Proof. Given $f \in \mathrm{Emb}(\mathbb{I}, \mathbb{I}^m)$ consider the map $\rho(f) : \Delta^2 \to S^{m-1}$ which sends t_1, t_2 to either $u(f(t_2) - f(t_1))$ if $t_1 \neq t_2$ or $u(f'(t))$ if $t_1 = t_2 = t$. We may view $\rho(f)$ as a homotopy between $ev_1(f)$, which is the restriction to the $t_1 = t_2$ edge, and the restriction to the $t_1 = 0$ and $t_2 = 1$ edges. But the restriction to these latter two edges is canonically null-homotopic, since their images lie in the southern hemisphere of S^{m-1}. Thus, ev_1 restricted to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m)$ is null-homotopic. Since ev_1 is an equivalence on $\mathrm{Imm}(\mathbb{I}, \mathbb{I}^m)$ this implies that the inclusion of $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m)$ is null-homotopic.

That $E_m \simeq \mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega\mathrm{Imm}(\mathbb{I}, \mathbb{I}^m)$ is immediate from its definition as the homotopy fiber of this inclusion, and that this is in turn weakly equivalent to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega^2 S^{m-1}$ follows from the Hirsch-Smale theorem [37]. □

The embedding space E_m is related to the space of framed knots when $m = 3$. Because the normal bundle of an embedded interval in $\mathbb{I}^m$ is essentially an oriented bundle over S^1, it is trivial. By fixing one framing of the normal bundle, all others are related to it by a map from $\mathbb{I}$ to $SO(m-1)$ fixed at the endpoints of $\mathbb{I}$. The space of framed knots is thus homotopy equivalent to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega SO(m-1)$. For $m = 3$ we get $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^3) \times \Omega S^1$, which is homotopy equivalent to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^3) \times \mathbb{Z}$. There is thus a suspension map η on the second factor to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^3) \times \Omega^2 S^2 \simeq E_3$.

Proposition 5.18. *The suspension map η from the space of framed knots in $\mathbb{I}^3$ to E_3 is a bijection on components.*

Framings naturally arise in defining knot invariants through our operad model because of this bijection. We also see $\Omega^2 S^3$ as the homotopy fiber of η, which might give a new viewpoint on the results of [2].

6. The main result

We assemble our work to this point to prove the main result. As needed for the calculus of functors, extend E_m to be a functor on the open sets of $\mathbb{I}$ by sending U to the homotopy fiber of the inclusion $\mathrm{Emb}(U, \mathbb{I}^m) \to \mathrm{Imm}(U, \mathbb{I}^m)$. We will recover models for E_m from those for embeddings and immersion spaces.

Lemma 6.1. *If A and B are two functors from $\mathcal{U}(M)$ to spaces with a natural transformation τ between them, and F is defined by $F(U) = \mathrm{hofib}\ (\tau : A(U) \to B(U))$, then $T_k(F) = \mathrm{hofib}\ (T_k(A) \to T_k(B))$.*

Proof. The equality is immediate from the definition of T_k, since taking homotopy fibers commutes with taking homotopy limits. □

In defining a fiber to ρ_k^m we are led to the following.

Definition 6.2. Let $e_i : A_n\langle[\mathbb{I}^m]\rangle \to A_{n+1}\langle[\mathbb{I}^m]\rangle$ send $(u_{j\ell})$ to $(v_{j\ell})$ where $v_{i,i+1} = *_S$, the basepoint of S^{m-1} and other $v_{j\ell}$ are equal to $u_{\sigma_i(j)\sigma_i(\ell)}$, where as before $\sigma_i(j) = j$ or $j-1$ if $j < i$ or $j > i$ respectively. By abuse, also use e_i to denote its restriction to $C_n\langle[\mathbb{I}^m, \partial]\rangle$ mapping to $C_{n+1}\langle[\mathbb{I}^m, \partial]\rangle$ as one can check using Theorem 5.6.

Alternately, $e_i : C_n\langle[\mathbb{I}^m, \partial]\rangle \to C_{n+1}\langle[\mathbb{I}^m, \partial]\rangle$ is the restriction of δ^i to $C_n\langle[\mathbb{I}^m, \partial]\rangle \times (*_S)^n \subset C_n'\langle[\mathbb{I}^m, \partial]\rangle$.

Definition 6.3. Let $\mathcal{F}_k^m$ be the functor from $P_0(k)$ to spaces which sends $S \subseteq [k]$ to $C_{\#S-1}\langle[\mathbb{I}^m, \partial]\rangle$ and which sends the inclusion $S \subset S \cup j$ to the map e^i where i is the number of elements of S less than j. Let $F_k^m = \underleftarrow{\mathrm{holim}}\, \mathcal{F}_k^m$.

Theorem 6.4. *F_k^m is homotopy equivalent to $T_k E_m$. For $m > 3$, $\eta_\infty : E_m \to \underleftarrow{\mathrm{holim}}\, T_k E_m$ is a weak equivalence.*

Proof. We use the models D_k^m and G_k^m for $T_k\mathrm{Emb}$ and $T_k\mathrm{Imm}$ as given in Theorems 5.13 and 5.15 respectively. By Proposition 5.16, $p_k^m : D_k^m \to G_k^m$ agrees with T_k of the inclusion from embeddings to immersions. Applying Lemma 6.1 with $A = \mathrm{Emb}(-, \mathbb{I}^m)$, $B = \mathrm{Imm}(-, \mathbb{I}^m)$, and the natural transformation between them be the standard inclusion, we have that $T_k E_m = \mathrm{hofib}\ p_k^m$.

If a map diagrams indexed by $P_0(k)$ is a fibration object-wise, then the induced map on homotopy limits is a fibration and the fiber is given by the homotopy limit of the fibers object-wise (see for example Lemma 3.5 of [8]). Because ρ_k^m is a fibration object-wise, we thus identify $\mathrm{hofib}\ p_k^m = \mathrm{hofib}\ (\underleftarrow{\mathrm{holim}}\, \rho_k^m)$ with such a homotopy limit of object-wise fibers. By our definition, the diagram of fibers is $\mathcal{F}_k^m$, whose homotopy limit is F_k^m, establishing the first half of the theorem.

The second half of the theorem is immediate from Theorems 5.3 and 5.4. □

Note that because of Proposition 5.17, we could extend E_m to a functor on $\mathcal{U}(\mathbb{I})$ by setting $E_m^!(U) = \mathrm{Emb}(U, \mathbb{I}^m) \times \Omega\mathrm{Imm}(U, \mathbb{I}^m)$. The extension $E_m^!$ would lead to a set of approximations to E_m different from the F_k^m. Also, while E_m decomposes as a product in Proposition 5.17, at the level of entries of diagrams it is in the approximation to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m)$ that we see a product.

Recall Proposition 4.7 that $\mathcal{K}_m$ is an operad with multiplication, which using Definition 2.16 has an associated cosimplicial object. We translate from F_k^m to $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$, essentially through the standard projection $C_n\langle[\mathbb{I}^m, \partial]\rangle \to \widetilde{C_n}\langle[\mathbb{R}^m]\rangle$. We modify both $C_n\langle[\mathbb{I}^m, \partial]\rangle$ and this projection to define a natural transformation.

Definition 6.5.
- Let $\varepsilon <= \frac{1}{6}$. For $x \in \mathbb{R}^m$, let $d_+(x)$ be the distance in $\mathbb{R}^m$ from x to $*_+ = (0, \ldots, 0, 1)$ and $d_-(x)$ be the distance to $*_-$. Let $\gamma_j : \mathbb{R}^m \to \mathbb{R}$ be projection onto the jth coordinate.

- Let $C_n\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ be the subspace of $(x_i) \times (u_{ij}) \in C_n\langle[\mathbb{I}^m, \partial]\rangle$ where if $d_+(x_i)$ and $d_+(x_j)$ are less than ε and $i < j$ then $\gamma_k(x_i) = \gamma_k(x_j)$ for $k < m$ and $\gamma_m(x_i) \geq \gamma_m(x_j)$. Moreover, if $x_i = x_j$ and $i < j$ then $u_{ij} = *_S$.
- Let $\mathcal{F}^m_{k,\varepsilon}$ be the functor from $P_0(k)$ to spaces which sends $S \subseteq [k]$ to $C_{\#S-1}\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ and which sends the inclusion $S \subset S \cup j$ to the map e^i where i is the number of elements of S less than j.
- Let $F^m_{k,\varepsilon} = \underset{\longleftarrow}{\text{holim}}\, \mathcal{F}^m_{k,\varepsilon}$.

Proposition 6.6. *The map $F^m_{k,\varepsilon} \to F^m_k$, induced by the natural transformation $\iota : \mathcal{F}^m_{k,\varepsilon} \to \mathcal{F}^m_k$ which at each entry is the canonical inclusion, is a homotopy equivalence.*

Proof. It suffices to check ι is a homotopy equivalence object-wise, for which we adapt the machinery developed in [35] for compactified configuration spaces. Both $C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ and $C_k\langle[\mathbb{I}^m, \partial]\rangle$ are quotients of the canonical Axelrod-Singer compactifications which we call $C_k[\mathbb{I}^m, \partial_\varepsilon]$ and $C_k[\mathbb{I}^m, \partial]$ respectively; see Definitions 1.3 and 4.18 of [35] for the definition of $C_k[\mathbb{I}^m, \partial]$, which can be modified as in Definition 6.5 for $C_k[\mathbb{I}^m, \partial_\varepsilon]$. These quotient maps are homotopy equivalences, by the proof of Theorem 5.10 of [35], which applies verbatim in these cases.

$C_k[\mathbb{I}^m, \partial_\varepsilon]$ retracts to its subspace $C_k[\mathbb{I}^m - N^\varepsilon_\pm]$, where $N^\varepsilon_\pm$ is the union of the ε neighborhoods of $*_+$ and $*_-$ by scaling the x_i by $1-\varepsilon$. Both $C_k[\mathbb{I}^m, \partial]$ and $C_k[\mathbb{I}^m - N^\varepsilon_\pm]$ are manifolds with corners (see Theorem 4.4 of [35]), and thus are homotopy equivalent to their interiors, $C_k(Int(\mathbb{I}^m))$ and $C_k(Int(\mathbb{I}^m - N^\varepsilon_\pm))$ respectively. But these interior configuration spaces are diffeomorphic, since $Int(\mathbb{I}^m)$ and $Int(\mathbb{I}^m - N^\varepsilon_\pm)$ are. Composing this diffeomorphism with the previous homotopy equivalences establishes the equivalence of $C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ and $C_k\langle[\mathbb{I}^m, \partial]\rangle$ and thus establishes the result.

□

We use $C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ because they readily project to $\widetilde{C_k}\langle[\mathbb{R}^m]\rangle$.

Definition 6.7.
- Let $(a_i)^m_{i=1}$, $a_i \in \mathbb{R}$ denote a point in $\mathbb{R}^m$. Define $\lambda_+ : (\mathbb{R}^m - *_+) \to \mathbb{R}^m$ by sending (a_i) to (b_i) where if $i \neq m$ then $b_i = a_i$ and
$$b_m = \begin{cases} \frac{\varepsilon a_m}{d_+(a_i)} & d_+(a_i) < \varepsilon \\ a_m & d_+(a_i) \geq \varepsilon. \end{cases}$$
Define $\lambda_- : (\mathbb{R}^m - *_-) \to \mathbb{R}^m$ similarly, and let $\lambda = \lambda_+ \circ \lambda_-$.
- Define $\pi_k : C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle \to \widetilde{C_k}\langle[\mathbb{R}^m]\rangle \subset (S^{m-1})^{\binom{k}{2}}$ by sending $(x_i) \times (u_{ij})$ to (v_{ij}) where v_{ij} is:
  - $u(\lambda(x_i) - \lambda(x_j))$ if $x_i \neq x_j$ and neither equals $*_+$ or $*_-$.
  - The Jacobian on λ applied to u_{ij} if $x_i = x_j$.
  - $*_S$, if $x_i \neq x_j$ and either $x_i = *_+$ or $x_j = *_-$.

Proposition 6.8. *π_k is continuous.*

Proof. We first identify π_k on the subspace t $C_k\langle[\mathbb{I}^m - (*_+ \cup *_-)]\rangle$ with the composite of $C_k\langle[\lambda]\rangle$, the map on configuration spaces induced by the embedding λ (see Corollary 4.8 of [35]), and the canonical projection $C_k\langle[\mathbb{R}^m]\rangle$ to $\widetilde{C_k}\langle[\mathbb{R}^m]\rangle$. What remains is to check continuity on the subspace in which some $x_n = *_+$. Consider a sequence $\{(x^\ell_i), (u^\ell_{ij})\}^\infty_{\ell=1}$ with limit point $(x^\infty_i) \times (u^\infty_{ij})$, so that $x^\infty_n = *_+$. We show that its image under π_k has v_{nj} which approaches $*_S$ if $x^\infty_j \neq *_+$ or $n < j$ or which approaches $-*_S$ otherwise. For each j, either $x^\infty_j \in N^\varepsilon_+$, which is also true for ℓ sufficiently large, in which case u_{nj} must be $*_S$ if $n < j$ or $-*_S$ if $n > j$, so that the sequence v_{nj} would be eventually constant at $*_S$ or $-*_S$. Or if $x^\infty_j \notin N^\varepsilon$ then as $x^\ell_n \mapsto *_+$ the last coordinate of $\lambda_+({x_i}^\ell)$ becomes arbitrarily large. Because ${x_j}^\ell \mapsto x_j$ stays in $\mathbb{I}^m$ we have $u(\lambda(x^\ell_n) - \lambda(x^\ell_j)) \mapsto *_S$. Continuity when some $x_n = *_-$ works similarly. □

We now may assemble our main result, Theorem 1.1, which casts the embedding calculus tower for E_m in the language of operads. For convenience, we restate the theorem here.

Theorem 6.9. *The kth approximation to E_m in the embedding calculus, namely $T_k E_m$, is weakly equivalent to* $\widetilde{\mathrm{Tot}}^k \mathcal{K}_m^\bullet$.

Proof. We will check that the maps π_k assemble to a natural transformation of functors from $\mathcal{F}_k^m$ to $\mathcal{K}_m^\bullet \circ c_k$, with c_k as in Definition 2.7, which gives rise to a weak equivalence on homotopy limits. Theorem 6.4 then says that the homotopy limit of $\mathcal{F}_k^m$ is weakly equivalent to $T_k E_m$, and Lemma 2.8 implies that the homotopy limit of $\mathcal{K}_m^\bullet \circ c_k$ is weakly equivalent to $\widetilde{\mathrm{Tot}}^k \mathcal{K}_m^\bullet$, establishing the theorem.

For the assembled π_k to be a natural transformation, we must have $\pi_k \circ e^i = d^i \circ \pi_k$. For most i this is immediate to check, as repeating coordinates and passing to the quotient $\widetilde{C}_k\langle[\mathbb{R}^m]\rangle$ are processes which clearly commute. The $i = 0$ and $i = k + 1$ cases require the modifications we made in Definition 6.7. For $\mathcal{K}_m^\bullet \circ c_k$ we trace through Definitions 2.16 and 3.1, Theorem 4.5 and Proposition 4.7 see that d^{k+1} takes a point $(u_{ij}) \in \widetilde{C}_k\langle[\mathbb{R}^m]\rangle$ leaves all these u_{ij} unchanged and adds $u_{i,k+1} = *_S$ for all i to obtain a point in $\widetilde{C}_{k+1}\langle[\mathbb{R}^m]\rangle$. On the other hand, e^{k+1} adds the $k + 1$st point to the configuration at $*_-$, which under π_k will also lead to all $u_{i,k+1} = *_S$. The $i = 0$ case works similarly.

The fact that the assembled π_k induce a weak equivalence on homotopy limits follows from it being a homotopy equivalence object-wise. We already know from the proof of Proposition 6.6 that $C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ is homotopy equivalent to the subspace $C_k(Int(\mathbb{I}^m - N_\pm^\varepsilon))$, which is diffeomorphic to $C_k(\mathbb{R}^m)$. Composed with this diffeomorphism on this subspace, π_k is the standard projection $C_k(\mathbb{R}^m) \to \widetilde{C_k}(\mathbb{R}^m)$ followed by the canonical map to $\widetilde{C_k}\langle[\mathbb{R}^m]\rangle$ which is a homotopy equivalence by Corollaries 4.5 and 5.9 of [35]. □

7. Observations and consequences

7.1. **Spectral sequences.** The results in this section parallel those of section 7 of [34]. Applying the homotopy spectral sequence of Proposition 2.9 for $\mathcal{K}_m^\bullet$ we immediately have the following.

Theorem 7.1. *There is a spectral sequence converging to $\pi_*(\widetilde{\mathrm{Tot}}\mathcal{K}_m^\bullet)$ with*

$$E_1^{-p,q} = \bigcap \, ker \, {s^k}_* \subseteq \pi_q(C_p(\mathbb{R}^m)).$$

The d_1 differential is the restriction to this kernel of the map

$$\Sigma_{i=0}^{p+1}(-1)^i {d^i}_* \colon \pi_q(C_{p-1}(\mathbb{R}^m)) \to \pi_q(C_p(\mathbb{R}^m)).$$

Theorem 6.4 implies that this spectral sequence computes homotopy groups of E_m when $m \geq 4$. Except for in the $p = 1$ column, this spectral sequence coincides exactly with that studied with rational coefficients in [32], so we do not give a more explicit description here. The rows of this spectral sequence have also been examined by Kontsevich [22].

For $m = 3$, the case of classical knots, we conjecture that $\eta_k : E_m \to T_k E_m$ is a universal type-$(k-1)$ framed knot invariant over the integers. For $k \leq 3$, we may deduce this from the main results of [8]. We conjecture that the entries $E^\infty_{-k,k}$ of this spectral sequence are isomorphic to the module of primitive weight systems of degree $k - 1$ over the integers, which would be a first step to this conjecture. We have checked that for small k, the group $E^2_{-k,k}$ as described purely algebraically in Theorem 7.1 is isomorphic to this module of primitives, but have not resolved this algebraic question in general.

In light of Theorem 4.11, the homology spectral sequence from Theorem 2.10 has a pleasant description. Recall Definition 2.16, which for operads of vector spaces introduces the notation of $HH^*(\mathcal{O})$ for the total cohomology of the associated cosimplicial object.

Theorem 7.2. *There is a spectral sequence with $E^2_{-p,q} = HH^{p,q}(\mathrm{Poiss}_m)$ which for $m \geq 4$ converges to the homology of $\widetilde{\mathrm{Tot}}\mathcal{K}_m^\bullet$, and thus of E_m.*

Proof. If we use the second description of the homology spectral sequence from Theorem 2.10, then $E^1_{-*,*}$ will be $H_*(\mathcal{K}_m^\bullet)$, which is the Poisson operad by Theorem 4.11. The induced operad with multiplication structure on the Poisson operad is the standard one. Thus, the d^1 differential will coincide with the differential for total cohomology of the Poisson operad, and the E^2 term will be the total (or Hochschild) cohomology of the Poisson operad as stated.

It remains to check the convergence conditions of Theorem 2.10. In the case of the Kontsevich operad, the entries $\mathcal{K}_m^k = \widetilde{C}_k\langle[\mathbb{R}^m]\rangle$ are homotopy equivalent to $C_k(\mathbb{R}^m)$, which are simply connected if $m \geq 3$. Using the first definition of Theorem 2.10, we start with $H_*(C_p(\mathbb{R}^m))$ and explicitly understand the kernels of the maps s^i_*. We use Theorem 4.11 and Definition 4.10 to identify $H_*(C_p(\mathbb{R}^m))$ in terms of products of brackets in variables $x_1, \ldots, x_p$. Tracing through the definitions of the associated cosimplicial object, s^i sends a product of brackets in the x_j to either zero, if the variable x_i appears in a bracket, or the monomial obtained by removing x_i and relabeling x_j to x_{j-1} for $j > 1$, if x_i does not appear in a bracket. Therefore to be in the kernel of all of the s^i, all of the variables x_i must appear in a bracket, so there must be at least $\frac{k}{2}$ brackets, leading to a total degree of at least $\frac{k(m-1)}{2}$. For $m > 3$, this is greater than k and thus gives the estimate needed for application of Theorem 2.10. □

This spectral sequence in rational cohomology can also be viewed as arising for the homotopy groups of the Taylor tower for the functor to spectra $U \mapsto \mathbb{Q} \wedge E_m(U)$, the rational Eilenberg-MaClane spectrum smashed with $E_m(U)$. For $m = 3$, Volic's results [41, 42] imply that the map from the knot space to this Taylor tower serves as a universal framed finite-type invariant over the rational numbers.

7.2. **A little two-cubes action from the McClure-Smith framework.** Theorem 1.1 fits perfectly into the framework created by McClure and Smith in their solution of the Deligne conjecture [26]. One of their central results is the following.

Theorem 7.3. *The totalization of the associated cosimplicial object of an operad with multiplication admits an action of an operad equivalent to the little 2-cubes operad, as does its homotopy invariant totalization.*

Proof. We are simply collecting results from [26] and [27]. For the standard totalization, we are simply quoting Theorem 3.3 in [26]. For the homotopy invariant totalization, Theorem 15.3 of [27] says that $\widetilde{\mathrm{Tot}}$ of any cosimplicial space with what they call a Ξ^2-structure has an action of an operad equivalent to the little 2-cubes. Proposition 10.3 of [27] identifies an operad with multiplication structure on a sequence of spaces with a Ξ^2 structure. □

Example 7.4. *Consider the cosimplicial model for the space of maps from S^2 to X, namely $X^{S^2_\bullet}$. By Theorem 3.3 $S^2_\bullet \cong \mathcal{B}^\bullet$, so there is an operad structure on this collection of spaces. In order to get an operad with multiplication, we restrict each $X^{\mathcal{B}^n} = (x_\alpha)$ to the subspace in which $x_+ = *$, where $+$ is the basepoint of $\mathcal{B}^n$ and $*$ is the base point of X. The operad structure maps restrict appropriately, and we obtain $X_\star^{S^2_\bullet}$, to which the associative operad maps at each level to the point with all $x_\alpha = *$.*

Applying Theorem 7.3, the totalization of $X_\star^{S^2_\bullet}$ is a little 2-cubes space, and we know that its totalization is $\Omega^2 X$. McClure and Smith fully develop this example (in fact for all cases, not just $n = 2$) in Section 11 of [27]. *They show that the little 2-cubes action which arises in this example coincides with the standard one.*

We can immediately establish Theorem 1.4, one of our main results, which for convenience we restate here.

Theorem 7.5. *For any m, there is a little two-cubes action on $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$. For $m > 3$, E_m is a two-fold loop space.*

Proof. Applying Theorem 7.3 for the Kontsevich operad with its given multiplication establishes the two-cubes action.

By Theorem 1.1, if $m \geq 4$, E_m is homotopy equivalent to $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$, so it has a 2-cubes action as well. But E_m is connected for $m \geq 4$, since both $\Omega(\mathrm{Imm}(\mathbb{I}, \mathbb{I}^m)) \simeq \Omega^2 S^{m-1}$ and $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m)$ are connected (that the latter space is connected is because any path through maps from an embedding to the standard one becomes an isotopy once put in general position). So by the recognition theorem of [3, 4], E_m is a 2-fold loop space. □

We expect this two-cubes action to be important for closer examination of the homotopy type, in particular the homology, of E_m. In [7], Budney constructs a little two-cubes action directly on a different space closely related to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m)$, namely the space of framed knots. He goes on to show that the two-cubes action is free when $m = 3$, generated by the components of prime knots. He identifies the homotopy types of a large class of prime components. It would be interesting to see if Budney's two-cubes action is related to ours on E_m, and if there is a two-cubes action to be found on $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m)$ itself. Perhaps this two-cubes action respects the product decomposition of E_m. There is a possible first step towards such a result as follows.

We defined $\mathcal{K}_m$, following Theorem 4.5, with entries as subspaces of those in $(S^{m-1}_\star)^{\mathcal{B}^\bullet}$, which is $(S^{m-1}_\star)^{S^2_\bullet}$ by Theorem 3.3. Let $\iota^\bullet$ denote the corresponding map of operads with multiplication and thus of associated cosimplicial spaces. We conjecture that the following diagram, in which the bottom arrow is the standard projection along with the identification given by the Hirsch-Smale Theorem, commutes

$$\begin{array}{ccc} \widetilde{\mathrm{Tot}}(\mathcal{K}^\bullet_m) & \xrightarrow{\widetilde{\mathrm{Tot}}(\iota^\bullet)} & \widetilde{\mathrm{Tot}}((S^{m-1}_\star)^{S^2_\bullet}) \\ \simeq\uparrow & & \cong\downarrow \\ \mathrm{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega\mathrm{Imm}(\mathbb{I}, \mathbb{I}^m) & \longrightarrow & \Omega^2 S^{m-1}. \end{array}$$

If it does, then we may be able to use the McClure-Smith machinery to define a two-cubes action on the homotopy fiber of $\iota^\bullet$, whose totalization would be $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^m)$.

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