BORDISM OF SEMI-FREE S^1-ACTIONS

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Abstract. We calculate the geometric and homotopical (or stable) bordism rings associated to semi-free S^1 actions on complex manifolds, giving explicit generators for the geometric theory. To calculate the geometric theory, we prove a case of the geometric realization conjecture, which in general would determine the geometric theory in terms of the homotopical. The determination of semi-free actions with isolated fixed points up to cobordism complements similar results from symplectic geometry.

1. Introduction

In this paper we describe both the geometric and homotopical bordism rings associated to S^1-actions in which only the two simplest orbit types, namely fixed points and free orbits, are allowed. Our work is of further interest in two different ways. To make the computation of geometric semi-free bordism, in Theorem 3.11 we prove the semi-free case of what we call the geometric realization conjecture (Conjecture 2.7), which if true in general would determine the ring structure of geometric S^1-bordism from the ring structure of homotopical S^1-bordism given in [11]. Additionally, an application of our results to semi-free actions with isolated fixed points which we state now gives results parallel to results from symplectic geometry [10]. Let $\mathbb{P}(\mathbb{C} \oplus \rho)$ denote the space of complex lines in $\mathbb{C} \oplus \rho$ where ρ is the standard complex representation of S^1 (in other words, the Riemann sphere with S^1 action given by the action of the unit complex numbers.)

Theorem 1.1. *Let S^1 act semi-freely with isolated fixed points on M, compatible with a stable complex structure on M. Then M is equivariantly cobordant to a disjoint union of products of $\mathbb{P}(\mathbb{C} \oplus \rho)$.*

This result should be compared with the second main result of [10], which states that when M is connected a semi-free Hamiltonian S^1 action on M implies that M has the same Borel equivariant cohomology and equivariant Chern classes as a product of such $\mathbb{P}^1$'s. Based on their results and ours, we make the following.

Conjecture 1.2. *A semi-free Hamiltonian S^1 action with isolated fixed points on a connected manifold is equivariantly diffeomorphic to a product of $\mathbb{P}(\mathbb{C} \oplus \rho)$'s.*

As Theorem 1.1 lead us to the more general computation of Theorem 3.12, it would also be interesting to see if there is an analog of Theorem 3.12 for Hamiltonian S^1-actions. In general, the symplectic and cobordism approaches to transformation groups have remarkable overlaps in language (for example, localization by inverting Euler classes of representations plays a key role in each theory), though the same words sometimes have different precise meanings. A synthesis of these techniques could perhaps address Conjecture 1.2 or other interesting questions within transformation groups.

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2. Review of complex equivariant bordism

The foundational results of this section are taken from [7], and the computational results are taken from [11]. If V is a representation of G equipped with a G-invariant inner product, let S^V denote its one-point compactification, let $D(V)$ be the unit disk in V, and let $S(V)$ be the boundary of $D(V)$, namely the unit sphere in V.

Let $\Omega_*^{U,G}$ denote geometric complex equivariant bordism (see for example chapter 28 of [7]), the bordism ring whose representatives are stably complex G-manifolds up to cobordisms in which the bounding manifolds have stably complex G-actions which extend the actions on their boundaries. By equipping these manifolds with maps to a space X we define an equivariant homology theory $\Omega_*^{U,G}(X)$, which extends to a homology theory on pairs by using manifolds with boundary. Note that this equivariant homology theory has suspension isomorphisms only when G acts trivially on the suspension coordinate.

Let MU_*^G denote stable complex equivariant bordism, represented by a direct limit of homotopy classes of maps from spheres S^V to Thom spaces (see for example section 3 of [11]). Alternately, by a result of Bröcker and Hook [2], $MU_*^G(X)$ is isomorphic to a direct limit which has non-trivial G-suspensions built in, namely $\lim_V \Omega_*^{U,G}(X \times D(V), X \times S(V))$, where V ranges over the real representations of G. Thus, MU_*^G is also represented by stably complex G-manifolds M with boundary equipped with a map to $D(V)$ for some representation V such that the boundary of M maps to $S(V)$. Through either the Pontryagin-Thom construction or through the Bröcker-Hook result, there is a canonical a map $\phi\colon \Omega_*^{U,G} \to MU_*^G$. The following result allows us to use homotopy theory to effectively study G-manifolds.

Theorem 2.1 (Comezaña, Theorem 28.5.4 of [7]). *If A is abelian, the map $\phi\colon \Omega_*^{U,A} \to MU_*^A$ is injective.*

What makes MU_*^A at all manageable is that it in turn maps injectively to a localization which is computable. To set this stage, we recall some of the main results from [11].

Euler classes play a central role in equivariant bordism. Vector spaces with G-action, viewed as vector bundles over a point, may have interesting characteristic classes in the equivariant cohomology of a point. For a complex representation V let $e_V \in MU_G^n(pt.) = MU_{-n}^G$, where n is the dimension of V over the reals, denote the Euler class of V. Under the Bröcker-Hook isomorphism, e_V is represented by a point (as a zero-manifold) mapping to zero in $D(V)$. It is intuitively clear, and shown in chapter 15 of [7], that e_V is non-trivial if $V^G = 0$. Thus, MU_*^G contains many classes in arbitrarily negative degrees. Euler classes multiply by the rule $e_V e_W = e_{V\oplus W}$. Let S be the multiplicative set of non-trivial Euler classes. Let $Z_{n,V}$ be the class in MU_*^G represented by $\mathbb{P}(\mathbb{C}^n \oplus V)$, the space of lines in $\mathbb{C}^n \oplus V$ with induced G-action.

Theorem 2.2 (Theorem 1.2.5 of [11]). *There are inclusions of MU_*^T-algebras.*

$$MU_*[e_V, Z_{n,V}] \to MU_*^T \xrightarrow{\lambda} S^{-1}MU_*^T \cong MU_*[e_V^{\pm 1}, Z_{n,V}],$$

where V ranges over non-trivial irreducible representations of T, $n \geq 2$, and λ is the canonical localization map.

Thus, understanding of MU_*^T follows from understanding of divisibility by Euler classes. Crucial insight is provided by the following theorem, which is proved by applying MU_T^* to the cofiber sequence $S(V) \to * \to S^V$. Let $K(V)$ denote the kernel of the representation V.

Theorem 2.3 (Theorem 1.2 of [11]). *There is an isomorphism of $MU_*^T/(e_V)$ with $MU_*^{K(V)}$, where the quotient map from MU_*^T to $MU_*^T/(e_V)$ coincides with the restriction map r_V from MU_*^T to $MU_*^{K(V)}$.*

By carefully splitting the restriction map r_V and composing that splitting with r_V itself, in [11] we define idempotent operations $\beta_V\colon MU_*^T \to MU_*^T$ such that $x - \beta_V(x)$ is divisible by e_V. By Theorem 2.2, the quotient of $x - \beta_V(x)$ by e_V is unique, and we call that quotient $\Gamma_V(x)$. We say Γ_V is a Conner-Floyd operation. If $I = V_1, \cdots, V_k$ is a finite sequence of non-trivial irreducible representations let $\Gamma_I(x) = \Gamma_{V_k}\Gamma_{V_{k-1}}\cdots\Gamma_{V_1}x$. Fix an ordering on the non-trivial irreducible representations of T which includes the relation $V < W$ if $K(V) \subset K(W)$. Call a finite sequence of representations admissible if it respects this ordering.

The following is the main theorem of [11].

Theorem 2.4. *MU_*^T is generated as an MU_*-algebra by the classes $\Gamma_I(e_V)$ and $\Gamma_I(Z_{n,V})$, where V ranges over non-trivial irreducible representations, I ranges over all admissisible sequences of non-trivial irreducible representations, and n ranges over natural numbers.*

Relations include the following:

(1) $e_V\Gamma_V(x) = x - \beta_V(x)$
(2) $\Gamma_V(\beta_V(x)) = 0$
(3) $\Gamma_V(e_V) = 1$
(4) $\Gamma_V(x)y = (x - \beta_V(x))\Gamma_V(y) + \Gamma_V(x)\beta_V(y)$
(5) $\Gamma_{VW}x = \Gamma_{WV}x - \Gamma_{WV}\beta_W(x) - \Gamma_{WV}(e_W\beta_V(\Gamma_W x))$

where V and W range over non-trivial irreducible representations of T and x and y are any classes in MU_*^T. For $T = S^1$, these relations are complete.

Note that while $MU_*^{S^1}$ is known as a ring, Ω_*^{U,S^1} is not known because it is not known which classes $\Gamma_I(x)$, and in particular which $\Gamma_I(e_V)$ can be realized by classes in Ω_*^{U,S^1}. As with much of the theory, this question is best understood by applying the localization map λ. By Propositions 4.13 and 4.5 of [11], one may compute the image of a class in $\Omega_*^{U,S^1} \subset MU_*^{S^1}$ under λ by investigating fixed sets and normal bundle data of that manifold. Indeed, this is one of the main results of the paper [4].

Definition 2.5. *Let P_* denote the sub-algebra $MU_*[e_V^{-1}, Z_{n,V}]$ of $MU_*[e_V^{\pm 1}, Z_{n,V}]$.*

In other words P_* is the sub-algebra of $S^{-1}MU_*^T$ generated by those generators which are in positive degrees. Note that this algebra is called F_* in [11].

Proposition 2.6 (See Proposition 4.13 of [11])**.** *The image of Ω_*^{U,S^1} under λ lies in P_*.*

We make the following conjecture, a converse to Proposition 2.6, which would determine the ring structure of Ω_*^{U,S^1} from that of $MU_*^{S^1}$ and its image under λ.

Conjecture 2.7 (The realization conjecture)**.** *If $\lambda(x) \in P_*$ then $x \in \Omega_*^{U,S^1}$.*

As evidence for this conjecture, we will prove it in the semi-free setting.

3. Semi-free bordism

We now focus on the case where $T = S^1$ and the S^1 action on manifolds in question is semi-free, so that points are either fixed or freely acted upon by S^1.

Definition 3.1. *Let Ω_*^{SF} denote the subring of Ω_*^{U,S^1} of bordism classes represented by semi-free actions.*

For a semi-free action, the normal bundle of a fixed set will have representation type on the fiber of ρ or ρ^*, where ρ is the standard one-dimensional representation of S^1 and ρ^* is its conjugate.

Note here that bordisms between our semi-free manifolds are allowed to have general S^1-action, so we are looking at the image in Ω^{U,S^1} of the theory which we may call $\Omega_*^{SF!}$ in which all manifolds in question have semi-free action. But in fact, by looking at the families exact sequence for the family consisting of all of S^1 and the identity subgroup (see chapter 15 of [7], or [12]) both of these theories fit in long exact sequences

$$\cdots \to MU_{*-1}(BS^1) \xrightarrow{i} \Omega_*^{SF} \xrightarrow{\lambda} MU_*\left(\left(\coprod_{n>0} BU(n)\right)^2\right) \xrightarrow{\partial} MU_{*-2}(BS^1) \to \cdots,$$

which map to each other and thus can be shown to be isomorphisms by the five-lemma. Here, if we have M mapping to BS^1 we may pull back the canonical S^1-bundle to get a free S^1 manifold, so that i is inclusion of free S^1-manifolds into the semi-free theory. And λ in this setting sends a semi-free bordism class to the bordism class of the normal bundle (split according to appearance of ρ and ρ^* in the fiber). Finally, ∂ sends a manifold with a direct sum two bundles over it (classified by a map some $BU(i) \times BU(j)$) to the free S^1 manifold given by the sphere bundle of that bundle, where S^1 acts as ρ on the first factor and ρ^* on the second.

In fact, by identifiying $MU_*\left(\left(\bigsqcup_{n>0} BU(n)\right)^2\right)$ as a sub-ring of $P_* \subset S^{-1}MU_*^{S^1}$ (again, see Proposition 4.13 of [11]) we have the following.

Lemma 3.2. *The image under λ of Ω_*^{SF} lies in the sub-ring $P_*^{SF} = MU_*[e_\rho^{-1}, e_{\rho^*}^{-1}, Z_{n,\rho}, Z_{n,\rho^*}]$ of P_*.*

We are lead to the following.

Definition 3.3. *Let MU_*^{SF} be the subring of $MU_*^{S^1}$ which maps under λ to $\mathbb{Z}[e_\rho^{\pm 1}, e_{\rho^*}^{\pm 1}, Z_{n,\rho}, Z_{n,\rho^*}]$.*

Our main results are computations of MU_*^{SF} and then, remarkably, Ω_*^{SF} as MU_*-algebras. First, we pause to consider semi-free manifolds with isolated fixed points. In fact, at first we analyzed this case because of its independent interest [10] and then realized it could be used as a base case in a filtration to compute Ω_*^{SF}. Later, we found that we could compute Ω_*^{Sf} more directly from Theorem 2.4 and deduce Theorem 1.1 from that computation. Now we choose to present the isolated fixed point case independently from the more general semi-free case (Theorem 3.12, as the two approaches are complementary.

Under the identification of Lemma 3.2, semi-free actions with isolated fixed points have image under λ which sit in the subring $\mathbb{Z}[e_\rho^{-1}, e_{\rho^*}^{-1}]$. In particular $\lambda(\mathbb{P}(\mathbb{C} \oplus \rho) = e_\rho^{-1} + e_{\rho^*}^{-1}$.

Theorem 3.4. *The intersection of $\lambda(MU_*^{S^1})$ with the subring $\mathbb{Z}[e_\rho^{-1}, e_{\rho^*}^{-1}]$ is the subring $\mathbb{Z}[e_\rho^{-1} + e_{\rho^*}^{-1}]$.*

This theorem, along with Theorem 2.1 and Theorem 2.2, implies Theorem 1.1 and thus characterizes semi-free actions with isolated fixed points up to cobordism. Our main tool in this direct proof is application of Theorem 2.3, which for $V = \rho$ or ρ^* says that reduction modulo e_ρ or e_{ρ^*} coincides with the augmentation map from $MU_*^{S^1}$ to MU_*.

Proof of Theorem 3.4. Let R denote the subring $\mathbb{Z}[e_\rho^{-1}, e_{\rho^*}^{-1}]$ of $S^{-1}MU_*^{S^1}$, and let Q denote the subring $\mathbb{Z}[e_\rho^{-1} + e_{\rho^*}^{-1}]$ of R, so $Q = \lambda(\mathbb{Z}[\mathbb{P}(1 \oplus \rho)])$.

Since R is graded and lies in positive degrees, we may induct by degree, focusing on homogeneous elements. Suppose that $a_0 e_\rho^{-n} + a_1 e_\rho^{-(n-1)} e_{\rho^*} + \cdots + a_n e_{\rho^*}^{-n}$ is equal to $\lambda(x)$. Consider $y = e_{\rho^*}(x - a_0[\mathbb{P}(\mathbb{C} \oplus \rho)]^n)$. The image $\lambda(y)$ is in R and is in degree $2(n-1)$, thus by induction hypothesis we may deduce that y is in $\mathbb{Z}[\mathbb{P}(\mathbb{C} \oplus \rho)]$, and thus must be equal to an integral multiple of $\mathbb{P}(\mathbb{C} \oplus \rho)^{n-1}$. But this is not possible since by Theorem 2.3 the image of e_{ρ^*} under augmentation is zero, thus so is the image of y, whereas it is well-known that $(\mathbb{P}^1)^{n-1}$ is non-zero in MU_* for any $n > 0$.

Finally, we must estabish the base case, which is for the degree two part of R. Here we want to establish that if $a e_\rho^{-1} + b e_{\rho^*}^{-1}$ is $\lambda(x)$ for some x, then $a = b$. By subtracting $b\mathbb{P}(\mathbb{C} \oplus \rho)$ from x, it suffices to show that no non-zero integral multiple of e_ρ^{-1} is in the image of λ. But if $\lambda(z) = c e_\rho^{-1}$, then $\lambda(e_\rho z) = c$, so that $e_\rho z = c$ by Theorem 2.2, which implies that $0 = c$ once we apply the augmenation map to the equality. □

Now we proceed with the computation of MU_*^{SF}, which follows from Theorem 2.4 by noting that any class in $MU_*^{S^1}$ is in MU_*^{SF} if and only if the only representations which appear in its definition are ρ and ρ^*. For these representations, we have $K(\rho)$ and $K(\rho^*)$ are the trivial subgroup of S^1 and thus the idempotents β_ρ and β_{ρ^*} project onto the split image of MU_* in MU_*^T. In the "stable manifolds" interpretation of Bröcker and Hook, β_ρ and β_{ρ^*} take a class $M \to D(V)$ and impose a trivial S^1-action on both M and $D(V)$. Let $\overline{x}$ denote $\beta_\rho(x)$.

Definition 3.5. *Let B be the set of MU_*^T elements $\{e_\rho, e_{\rho^*}, Z_{n,\rho},$ and $Z_{n,\rho^*}\}$ where $n \geq 2$. Order B by the degree of the classes, with the additional needed convention that $Z_{n,\rho} < Z_{n,\rho^*}$ and $e_\rho < e_{\rho^*}$.*

Theorem 3.6. *MU_*^{SF} is generated as a ring by classes $\Gamma_\rho^i \Gamma_{\rho^*}^j(x)$ where $x \in B$ and if $x = e_{\rho^*}$, $j = 0$. Relations are*

(1) $e_\rho \Gamma_\rho(x) = x - \bar{x} = e_{\rho^*}\Gamma_{\rho^*}(x)$,
(2) $\Gamma_{\rho^*\rho}(x) = \Gamma_{\rho\rho^*}(x) + \overline{\Gamma_\rho(x)}\Gamma_{\rho\rho^*}(e_\rho)$,
(3) $\Gamma_V(x)(y - \bar{y}) = (x - \bar{x})\Gamma_V(y)$, *where V is ρ or ρ^*,*
(4) $\Gamma_{\rho\rho^*}(e_\rho) = \Gamma_{\rho^*\rho}(e_{\rho^*})$.

Additionally, we require the calculuations $\bar{e_V} = 0$ and $\Gamma_V(e_V) = 1$, where V is ρ or ρ^*.

An additive basis over MU_* is given by monomials $\Gamma_\rho^i \Gamma_{\rho^*}^j(x)m$ where $x \in B$ and m is a monomial in the $y \geq x$ in B.

Proof. The computation of ring structure is an immediate application of Theorem 2.4, using the fact that any class in MU_*^T is in MU_*^{SF} if and only if the only representations which appear in its definition are ρ and ρ^*.

The identification of the additive basis follows from the fact that one may take any product $\prod \Gamma_\rho^{i_k}\Gamma_{\rho^*}^{j_k}(x_k)$ where $x_k \in B$ and use relation 3 repeatedly to reduce to a sum of monomials such that only the minimal element of B appearing in each monomial is operated on by any Γ_ρ or Γ_{ρ^*}. Then, one uses relation 2 to reorder these operations so that the Γ_{ρ^*} are applied before the Γ_ρ. □

We now turn to a computation of Ω_*^{SF} by proving the version of Conjecture 2.7 for semi-free actions. We start by making geometric constructions of Γ_ρ and Γ_{ρ^*} on classes represented by honest G- manifolds. These constructions follow ones made by Conner and Floyd (hence the name given to the general opeations Γ_V).

Lemma 3.7. $\lambda(\Gamma_\rho(x)) = e_\rho^{-1}(\lambda(x) - \bar{x})$ *and similarly* $\lambda(\Gamma_{\rho^*}(x)) = e_{\rho^*}^{-1}(\lambda(x) - \bar{x})$.

Definition 3.8. *Define $\gamma(M)$ for any stably complex S^1-manifold to be the stably complex S^1-manifold*

$$\gamma(M) = M \times_{S^1} S^3 \sqcup (-\overline{M}) \times \mathbb{P}(\mathbb{C} \oplus \rho),$$

where S^3 has the standard Hopf S^1-action and the S^1-action on $M \times_{S^1} S^3$ is given by

$$\zeta \cdot [m, z_1, z_2] = [\zeta \cdot m, z_1, \zeta z_2]\,. \tag{1}$$

Define $\bar{\gamma}(M)$ similarly with the quotient of $M \times S^3$ now being by the S^1 action in which τ sends $m, (z_1, z_2)$ to $\tau m, (\tau z_1, \tau^{-1} z_2)$ and with induced S^1 action on the quotient given by

$$\zeta \cdot [m, z_1, z_2] = \left[\zeta \cdot m, z_1, \zeta^{-1} z_2\right]. \tag{2}$$

Proposition 3.9. *Let M be a stably complex S^1-manifold. Then $\Gamma_\rho[M] = [\gamma(M)]$ and $\Gamma_{\rho^*}[M] = [\bar{\gamma}(M)]$ in $MU_*^{S^1}$.*

Proof. By Lemma 3.7 and the injectivity of λ, it suffices to check the fixed sets of $\gamma(M)$ and $\bar{\gamma}(M)$. One set of fixed points of $\gamma(M)$ are points $[m, z_1, z_2]$ such that m is fixed in M and $z_2 = 0$. This fixed set is diffeomorphic to M^G, and its normal bundle is the normal bundle of M^G in M crossed with the representation ρ. In the localization, crossing with ρ coincides with multiplying by e_ρ^{-1}. The second set of fixed points are $[m, z_1, z_2]$ such that $z_1 = 0$. This set of fixed points is diffeomorphic to M, and its normal bundle is the trivial bundle ρ^{-1}.

Hence, if $x = \lambda([M])$, then the image of $[\gamma(M)]$ is $xe_\rho^{-1} + \overline{M}e_{\rho^{-1}}^{-1}$. By subtracting the image of $\overline{M} \times \mathbb{P}(\mathbb{C}\oplus\rho)$ we are left with $xe_\rho^{-1} - \overline{M}e_\rho^{-1}$, which by Lemma 3.7 is $\lambda(\Gamma_\rho([M]))$.

The analysis is similar for $\bar{\gamma}(M)$. □

Thus, the classes $\Gamma_I(\mathbb{P}(\mathbb{C}^n \oplus \rho))$ and $\Gamma_I(\mathbb{P}(\mathbb{C}^n \oplus \rho^*))$ can be realized geometrically. Along similar lines we have the following.

Lemma 3.10. $\Gamma_{\rho\rho^*}(e_\rho) = \mathbb{P}(\mathbb{C} \oplus \rho)$.

Proof. The equality of these classes follows from computation of their image under λ. Following the methods of tom Dieck [4] as applied in Proposition 4.14 of [11], the isolated fixed point of $\mathbb{P}(\mathbb{C}\oplus\rho)$with normal bundle ρ contributes a term of e_ρ^{-1} to its image under λ, and similarly the other fixed point contributes an $e_{\rho^*}^{-1}$. Thus, $\lambda(\mathbb{P}(\mathbb{C}\oplus\rho)) = e_\rho^{-1} + e_{\rho^*}^{-1}$. To show that this is also $\lambda(\Gamma_\rho\Gamma_{\rho^*}(e_\rho))$, by applying Lemma 3.7 twice it suffices to compute that $\overline{\Gamma_{\rho^*}(e_\rho)} = -1$. This equality in turn follows from giving a Bröcker-Hook model for $\Gamma_{\rho^*}(e_\rho)$ as $D(\rho^*)$ mapping to $D(\rho)$ through complex conjugation, which forgetting S^1-action has degree -1. □

We are now ready to prove the semi-free case of the geometric realization conjecture.

Theorem 3.11. *The following square is a pull-back square*

$$\begin{array}{ccc} \Omega_*^{SF} & \longrightarrow & MU_*^{SF} \\ \downarrow & & \lambda\downarrow \\ P_*^{SF} = MU_*[e_\rho^{-1}, e_{\rho^*}^{-1}, Z_{n,\rho}, Z_{n,\rho^*}] & \longrightarrow & MU_*[e_\rho^{\pm 1}, e_{\rho^*}^{\pm 1}, Z_{n,\rho}, Z_{n,\rho^*}]. \end{array}$$

Proof. We first go through the list of generators of MU_*^{SF}, determine which map to P_*, and show that those which do have geometric representatives.

By Proposition 3.9, the $\Gamma_\rho^i\Gamma_\rho^j(x)$ where $x = \mathbb{P}(\mathbb{C}^n\oplus\rho)$ or $\mathbb{P}(\mathbb{C}^n\oplus\rho^*)$ are in Ω_*^{SF}.

Next, by Lemma 3.7,

$$\lambda(\Gamma_\rho^i(e_{\rho^*})) = e_{\rho^*}e_\rho^{-i} + \sum \overline{\Gamma_\rho^i(e_{\rho^*})}e_\rho^{-m-i}, \tag{3}$$

which is not in P_*^{SF} as e_{ρ^*} appears with a positive power.

This leaves $\Gamma_\rho^i\Gamma_{\rho^*}^j(e_\rho)$, which if $i=0$ has image under λ which is not in P_* by a computation as in Equation 3. For $i>0$, note that relation 2 of Theorem 3.6 says that $\Gamma_\rho\Gamma_{\rho^*}(x) = \Gamma_{\rho^*}\Gamma_\rho(x)$ modulo $MU_*[\mathbb{P}(\mathbb{C}\oplus\rho)]$, which is of course in Ω_*^{SF}. Thus, modulo Ω_*^{SF}, we have

$$\Gamma_\rho^i\Gamma_{\rho^*}^j(e_\rho) = \Gamma_\rho^{i-1}\Gamma_{\rho^*}^{j-1}(\Gamma_\rho\Gamma_{\rho^*}(e_\rho)),$$

which by Lemma 3.10 is $\Gamma_\rho^{i-1}\Gamma_{\rho^*}^{j-1}(\mathbb{P}(\mathbb{C}\oplus\rho))$. Again applying Proposition 3.9, this class is in Ω_*^{SF}.

Theorem 3.6 gave an additive basis for MU_*^{SF} as given by monomials $M = \Gamma_\rho^i\Gamma_{\rho^*}^j(x)m$ where $x \in B = \{e_\rho, e_{\rho^*}, \mathbb{P}(\mathbb{C}^n\oplus\rho), \mathbb{P}(\mathbb{C}^n\oplus\rho^*)\}$ and m is a monomial in the $y \geq x$ in B. If $x = e_\rho$ and $i=0$ or if $x = e_{\rho^*}$ and $j=0$, then by a computation using Lemma 3.7 as above, $\lambda(M) \notin P_*^{SF}$, regardless of what m is, as e_ρ (respectively e_{ρ^*}) will appear with a positive power in the leading term of $\lambda(M)$. Otherwise, M is a product of generators which we have shown are in Ω_*^{SF}.

□

From the proof of Theorem 3.11, an explicit computation of Ω_*^{SF} including geometric representatives is immediate, since Ω_*^{SF} is just the sub-ring of MU_*^{SF} generated by the $\Gamma_\rho^i\Gamma_{\rho^*}^j\mathbb{P}(\mathbb{C}^n\oplus\rho)$ and $\Gamma_\rho^i\Gamma_{\rho^*}^j\mathbb{P}(\mathbb{C}^n\oplus\rho^*)$ for $n\geq 1$. Given the general complexities of equivariant bordism, in particular for the geometric theories, Ω_*^{SF} has a remarkably simple form.

Theorem 3.12. *Ω_*^{SF} is generated as an algebra over MU_* by the classes $\gamma^i\gamma_*^j\mathbb{P}(\mathbb{C}^n\oplus\rho)$ and $\gamma^i\gamma_*^j\mathbb{P}(\mathbb{C}^n\oplus\rho^*)$ where $n\geq 1$. Relations are*

1. $\gamma(x)(y-\bar{y}) = (x-\bar{x})\gamma(y)$*, and similarly for* γ_*,
2. $\gamma_*\gamma(x) = \gamma\gamma_*(x) + \overline{\gamma(x)}\mathbb{P}(\mathbb{C}\oplus\rho)$.

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