Finite Complexes with Vanishing Lines of Small Slope

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§1 Introduction

The purpose of this paper is to construct finite CW–complexes whose mod–p cohomology have vanishing lines of small slope. A left module M over a connected $\mathbb{F}_p$–algebra A has a *vanishing line over A of slope m* if there is an intercept b such that $\operatorname{Ext}_A^{s,t}(M, \mathbb{F}_p) = 0$ for $s > m(t-s) + b$ (we are using Adams spectral sequence indexing).

To state the main result, give the dual Steenrod algebra the basis of monomials in Milnor's generators (see 3.2). Let P_t^s be dual to $\xi_t^{p^s}$ and (if $p \neq 2$) let Q_t be dual to τ_t. For a CW–complex X let H^*X be the mod–p cohomology of X, let H^{ev} denote the cohomology in even degrees, and let $X^{\wedge N}$ denote the N–fold smash power $X \wedge X \wedge \cdots \wedge X$. For an element a of the mod–p Steenrod algebra let $|a|$ denote the degree of a and let aH^*X denote the image of multiplication by a.

Theorem 1.1. *Let A be a sub–Hopf algebra of the mod–p Steenrod algebra and let X be a p–local CW–complex with H^*X a finite dimensional $\mathbb{F}_p$–vector space. Then there is an integer N_X (see 2.5) depending on H^*X as a graded vector space and:*

(1) *For $p = 2$, if $P_t^s H^*X \neq 0$ for all $P_t^s \in A$ with $s < t$, then $X^{\wedge N_X}$ has a non–trivial stable summand Y such that H^*Y is A–free.*
(2) *For $p \neq 2$, if $P_t^s H^{ev}X \neq 0$ for all $P_t^s \in A$ with $s < t$ and $Q_t H^*X \neq 0$ for all $Q_t \in A$, then $X^{\wedge N_X}$ has a non–trivial stable summand Y such that H^*Y is A–free.*
(3) *For $p = 2$, if $P_t^s H^*X \neq 0$ for all $P_t^s \in A$ such that $s < t$ and $|P_t^s| \leq d$, then $X^{\wedge N_X}$ has a non–trivial stable summand Y such that H^*Y has a vanishing line over A of slope $1/d$.*
(4) *For $p \neq 2$, if $P_t^s H^{ev}X \neq 0$ for all $P_t^s \in A$ such that $s < t$ and $p|P_t^s| \leq 2d$ and $Q_t H^*X \neq 0$ for all $Q_t \in A$ such that $|Q_t| \leq d$, then $X^{\wedge N_X}$ has a non–trivial stable summand Y such that H^*Y has a vanishing line over A of slope $1/d$.*

Next, we state some corollaries of the main result. For $p = 2$ let A_n be the sub–algebra of the mod–2 Steenrod algebra which is generated by $Sq^1, Sq^2, \ldots, Sq^{2^n}$ and for $p \neq 2$ let A_n be the subalgebra of the mod–p Steenrod algebra which is generated by $\beta, P^1, P^p, \ldots, P^{p^{n-1}}$.

Theorem 1.2.

(1) *For $p = 2$ and $n > 0$, if $P_n^0 H^*X \neq 0$ then $X^{\wedge N_X}$ has a stable summand Y such that H^*Y is A_{n-1}–free and therefore (see 4.1) H^*Y has a vanishing line over the Steenrod algebra of slope $\frac{1}{2(2^n-1)-1}$ for $n > 1$ and of slope $1/2$ for $n = 1$.*
(2) *For $p = 2$ and $n > 1$, if $P_n^0 H^*X \neq 0$ and $P_n^1 H^*X \neq 0$ then $X^{\wedge N_X}$ has a stable summand Y such that H^*Y has a vanishing line over the Steenrod algebra of slope $\frac{1}{2(2^n-1)}$.*

(3) *For $p \neq 2$ and $n > 0$, if $Q_{n-1}H^*X \neq 0$, then $X^{\wedge N_X}$ has a stable summand Y such that H^*Y is A_{n-1}–free and therefore (see 4.1) H^*Y has a vanishing line over the mod–p Steenrod algebra of slope $\frac{1}{2(p^n-1)}$.*

Let $k(n)$ denote the n'th connective Morava K–theory at the prime p. For a space X, the $k(n)$–homology $k(n)_*X$ is a module over the coefficient ring $k(n)_* = \mathbb{F}_p[v_n]$ which is the polynomial algebra on a generator of degree $2(p^n - 1)$.

THEOREM 1.3. *[**Mit**]For each prime p and integer $n \geq 1$ there is a finite CW–complex Y such that H^*Y has a vanishing line over the Steenrod algebra of slope $\frac{1}{2(p^n-1)}$ and $k(n)_*Y$ is v_n torsion free.*

Remark 1.4. The $k(n)$–homology of X is v_n–torsion free if and only if $Q_nH^*X = 0$ and the classical Adams Spectral Sequence converging to $\pi_*k(n) \wedge X$ collapses at E_2. On the other hand if H^*X has a vanishing line over the Steenrod algebra of slope less than $\frac{1}{2(p^n-1)}$ then $v_n\, k(n)_*X = 0$.

Let $\mathcal{A}_p$ denote the mod–p Steenrod algebra. For $p \neq 2$ there is an algebra splitting

$$\mathcal{A}_p = E[Q_0, Q_1, \ldots] \otimes \mathcal{P}_p$$

where $\mathcal{P}_p$ is the polynomial part of $\mathcal{A}_p$. Every $\mathcal{A}_p$–module is a $\mathcal{P}_p$–module by restriction. For $p = 2$ let

$$\mathcal{P}_2 = \mathcal{A}_2/\langle Sq^1\rangle$$

where $\langle Sq^1\rangle$ is the two sided ideal generated by Sq^1. Every $\mathcal{A}_2$–module M such that $Sq^1M = 0$ is a $\mathcal{P}_2$–module in a natural way. For $p \neq 2$, let P_n denote the subalgebra of $\mathcal{P}_p$ generated by $P^1, P^p, \ldots, P^{p^n}$ and for $p = 2$, let P_n be the quotient of the sub–algebra $A_{n+1} \subset \mathcal{A}_2$.

THEOREM 1.5. *For each prime p and integer $n \geq 0$ there is a finite CW–complex Y such that the integral cohomology of Y is torsion free and H^*Y is a free P_n–module.*

Remark. These finite complexes are of interest since they have an Adams–Novikov E_2–term with a vanishing line of small slope.

We also give a proof of the following algebraic result.

THEOREM 1.6. *[**Mit**] Let B denote one of the algebras A_n or P_n. Then B admits a left module structure over the mod–p Steenrod algebra extending its left B–module structure.*

Stable splittings of $X^{\wedge n}$ for X a p–local CW–complex can be constructed by a standard technique using idempotents in the group ring $Z_{(p)}S_n$. In §2 we construct idempotents in $\mathbb{Q}S_n$. In §3 we recall the idempotent splitting technique and prove the theorems stated above. The crux of the proof is Theorem 3.4 analyzing the summand $M^{\otimes N_M}e_M$ of $M^{\otimes N_M}$, for M a module over a sub–Hopf algebras of the Steenrod algebra and $e_M \in \mathbb{F}_pS_{N_M}$ an idempotent in the $\mathbb{F}_p$ group ring of the symmetric group. The proof of Theorem 3.4 is given in §4.

§2 The Idempotents

In this section we recall the construction of idempotents in the rational group ring of the symmetric group. The method is due to Young. For more details see [**J-K**].

For $n > 0$, let S_n denote symmetric group of permutations of the set $\{1, 2, \ldots, n\}$. For $A \subseteq \{1, 2, \ldots, n\}$ let S_A be the subgroup of permutations which leave the complement of A pointwise fixed.

A *partition of the positive integer* n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of integers such that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k > 0 \text{ and } \alpha_1 + \alpha_2 + \cdots + \alpha_k = n.$$

The *Young diagram* $[\alpha]$ of the partition is the array of n nodes $*$ arranged in k rows with α_i nodes in the i'th row and with all rows beginning in the first column. For example the Young diagram of $[4, 3, 1]$ is:

$$\begin{array}{cccc} * & * & * & * \\ * & * & * & \\ * & & & \end{array}$$

An α *tableau* t_α is an array constructed by replacing the nodes $*$ of the Young diagram $[\alpha]$ by the integers $1, 2, \ldots, n$. For example one of the 8! tableau on the Young diagram $[4, 3, 1]$ is:

$$\begin{array}{cccc} 1 & 5 & 8 & 7 \\ 4 & 6 & 3 & \\ 2 & & & \end{array} \tag{2.1}$$

The *row group* R_α *of the tableau* t_α is the subgroup of S_n consisting of those permutations for which the rows of t^α are invariant sets. Similarly, the *column group* C_α *of* t_α is the subgroup of S_n consisting of those permutations for which the columns of t^α are invariant sets. For example the row and column groups of tableau 2.1 are

$$\begin{aligned} R^{(4,3,1)} &= S_{\{1,5,7,8\}} \times S_{\{3,4,6\}} \times S_{\{2\}} \qquad \text{and} \\ C^{(4,3,1)} &= S_{\{1,2,4\}} \times S_{\{5,6\}} \times S_{\{3,8\}} \times S_{\{7\}} \end{aligned}$$

Let $\mathrm{sgn} : S_n \to \pm 1$ denote the sign homomorphism

Theorem 2.2. *(Young),[*__J-K__*, 3.1.10] Let R_α and C_α be the row and column groups of the Young tableau t_α. There is an integer k_α which depends only on the Young diagram $[\alpha]$ such that*

$$e_\alpha = \frac{1}{k_\alpha} \sum_{\substack{\sigma \in R^\alpha \\ \tau \in C^\alpha}} sgn(\tau)\sigma\tau$$

is an idempotent in the rational group ring QS_n.

Remark. The different tableau t_α on $[\alpha]$ give conjugate idempotents.

Next we recall the explicit formula for the integer k_α. The (i, j)*–node* of the Young diagram $[\alpha]$ is the node in the i'th row and j'th column. The (i, j)–hook is the set consisting of the (i,j)–node, all nodes to the right of it in the i'th row, and all nodes below it in the j'th column. The *hook length* $h(i,j)$ is the cardinality of the (i,j)–hook. For example, if each node of the Young diagram $(4,3,1)$ is replaced by its hook length the result is

$$\begin{array}{cccc} 6 & 4 & 3 & 1 \\ 4 & 2 & 1 & \\ 1 & & & \end{array}$$

THEOREM 2.3. *[**J-K**,2.3.2 and 3.1.10] The integer k_α is given by*

$$k_\alpha = \prod_{(i,j)\in[\alpha]} h(i,j)$$

For example

$$k_{[4,3,1]} = 6\cdot 4\cdot 4\cdot 3\cdot 3\cdot 2\cdot 1\cdot 1$$

COROLLARY 2.4. *If all the hook lengths of $[\alpha]$ are relatively prime to p then the idempotent e_α is in the p–local group ring $Z_{(p)}S_n$*

In the next section, we use idempotents in $Z_{(p)}S_n$ to construct stable splittings of $X^{\wedge n}$ for p–local CW– complexes X. The idempotent used for a particular X depends only on H^*X as a graded $\mathbb{F}_p$–vector space.

Definition 2.5. Let V be a finite dimensional graded $\mathbb{F}_p$–vector space. Let V_{ev} be the subspace of even degree elements and let V_{od} be the subspace of odd degree elements. Now define

(1) $n_V = \dim V_{ev} + \left[\frac{\dim V_{od}}{p-1}\right]$ where $[x]$ denotes the greatest integer function,
(2) $N_V = (p-1)\binom{n_V+1}{2}$,
(3) $N_X = N_{H^*X}$ for X a CW–complex with H^*X finite dimensional.
(4) α_V is the partition $(p-1)n, (p-1)(n-1), (p-1)(n-2), \ldots, p-1$ of N_V,
(5) t_V is the unique α_V–tableau such that the integers in each row are in increasing order from left to right and every integer in row i is less than every integer in row $i+1$.

For example, let $p=3$ and let V be concentrated in even degrees with $\dim V = 3$. Then

(1) $n_V = 3$
(2) $N_V = 12$
(3) $\alpha_V = (6,4,2)$
(4)

$$t_V = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & & \\ 11 & 12 & & & & \end{array}$$

PROPOSITION 2.6. *The idempotent e_{α_V} corresponding to the tableau t_V is in the group ring $Z_{(p)}S_{N_V}$*

PROOF: Use Corollary 2.4.

Definition 2.7. Let $e_V \in \mathbb{F}_pS_{N_V}$ be the idempotent which is the mod–p reduction of $e_{\alpha_V} \in Z_{(p)}S_{N_V}$.

§3 The Constructions

Idempotent splittings. We begin by recalling the construction of stable splittings using idempotents. Let $g : X \to X$ be a self map of a CW–complex X. The *mapping telescope* of g is the homotopy direct limit of the diagram

$$X \xrightarrow{g} X \xrightarrow{g} X \xrightarrow{g} \ldots.$$

Let $\overline{g} : X \to Xg$ denote the natural inclusion.

Proposition 3.1. *Assume that X is a double suspension, so that*

$$[X, X]$$

is naturally a ring, and let $g : X \to X$ be a homotopy idempotent. Then

(1) *$Id - g$ is homotopy idempotent, where Id is the identity map*
(2) *the natural map*

$$X \xrightarrow{\overline{g} \vee \overline{Id-g}} Xg \vee X(Id - g)$$

is a weak equivalence,

(3) *The induced map $g^* : H^*X \to H^*X$ is a projection,*
(4) *The splitting of X induces a splitting in cohomology as*

$$H^*(Xg) = H^*X\, g^* \oplus H^*X\, (Id - g^*)$$

If a finite group G acts on a double suspension X, there is a homomorphism $ZG \to [X, X]$. If X is p–local, this extends to a homomorphism $Z_{(p)}G \to [X, X]$. So any idempotent $e \in Z_{(p)}G$ gives a splitting

$$X \simeq Xe \vee X(1 - e).$$

Splitting the n–fold Smash. The symmetric group S_n acts on the n–fold smash power $X^{\wedge n}$ of any CW– complex X by permutting the factors. We use the right action

$$(x_1, x_2, \ldots, x_n)\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

For any graded vector space V, there is a (signed) permutation action on $V^{\otimes n}$. Again we use the right action

$$v_1 \otimes v_2 \otimes, \ldots, \otimes v_n \cdot \sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$

The permutation action of S_n on $X^{\wedge n}$ induces the (signed) permutation on $(H^*X)^{\otimes n}$.

If X is p–local, any idempotent $e \in Z_{(p)}S_n$ gives a splitting

$$\Sigma^2 X^{\wedge n} \simeq \Sigma^2 X^{\wedge n} e \vee \Sigma^2 X^{\wedge n}(1 - e)$$

and in cohomology

$$H^*(X^{\otimes n}e) = (H^*X)^{\otimes n}e$$

Dual Steenrod algebra 3.2. [**Mil**]Recall that the dual Steenrod algebra for $p = 2$ is

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots] \quad |\xi_i| = 2^i - 1$$

and for $p \neq 2$ is

$$\Lambda[\tau_0, \tau_1, \dots] \otimes \mathbb{F}_p[\xi_1, \xi_2, \dots] \quad |\tau_i| = 2p^i - 1 \quad |\xi_i| = 2(p^i - 1).$$

Let A be a sub–Hopf algebra of the Steenrod algebra. The dual, A_*, is a quotient of the dual Steenrod algebra and by [**A-M**] it is of the form

$$A_* = \mathcal{A}_2 / \langle \xi_1^{2^{e(1)}}, \xi_2^{2^{e(2)}}, \dots \rangle \qquad \text{for } p = 2$$

or

$$A_* = \mathcal{A}_p / \langle \xi_1^{p^{e(1)}}, \xi_2^{p^{e(2)}}, \dots, \tau_0^{k(0)}, \tau_1^{k(1)}, \dots \rangle \qquad \text{for } p \neq 2$$

where $0 \leq e(i) \leq \infty$ and (for $p \neq 2$) $0 \leq k(i) \leq 1$. The sequences e and k must satisfy certain conditions imposed by the diagonal of $\mathcal{A}_p$. The sub–Hopf algebra A_n corresponds to the sequence

$$e = (n + 1, n, \dots, 0, 0, \dots)$$

for $p = 2$, and for $p \neq 2$ to the sequences

$$\begin{aligned} &e = (n, n - 1, \dots, 1, 0, 0, \dots) \\ &k = (1, 1, \dots, 1, 0, 0, \dots) \qquad \text{with 1 repeated } n + 1 \text{ times} \end{aligned}$$

Give the dual Steenrod algebra the basis of monomials in the elements ξ_i and (for $p \neq 2$) τ_i. Then the Adams–Margolis elements are

$$P_t^s \text{ dual to } \xi_t^{p^s} \qquad \text{and for } (p \neq 2) \qquad Q_t \text{ dual to } \tau_t$$

Proposition 3.3.

(1) *For all primes p and $s < t$, $(P_t^s)^p = 0$.*
(2) *For $p \neq 2$, $Q_t^2 = 0$.*

Proof: It can be checked that $\mathbb{F}_p[\xi_t]/\xi_t^{p^t}$ is a quotient Hopf algebra of $\mathcal{A}_*$. So the elements P_s^t for $s < t$ lie in a divided polynomial sub–Hopf algebra of the Steenrod algebra, and therefore $(P_s^t)^p = 0$. For similar reasons $Q_t^2 = 0$.

Main Algebraic Result. The main algebraic result of this paper is

Theorem 3.4. *Let A be a sub–Hopf algebra of the Steenrod algebra and let M be an A–module which is finite dimensional over $\mathbb{F}_p$. Let N_M be the integer and let e_M be the idempotent given in Defintion 2.5.*

1. *Then $M^{\otimes N_M} e_M \neq 0$.*
2. *For $p = 2$, if $P^s_t M \neq 0$ for all $P^s_t \in A$ with $s < t$, then $M^{\otimes N_M} e_M$ is A–free.*
3. *For $p \neq 2$, if $P^s_t M_{ev} \neq 0$ for all $P^s_t \in A$ with $s < t$, and $Q_t M \neq 0$ for all $Q_t \in A$, then $M^{\otimes N_M} e_M$ is A–free.*
4. *For $p = 2$, if $P^s_t M \neq 0$ for all $P^s_t \in A$ such that $s < t$ and $|P^s_t| \leq d$, then $M^{\otimes N_M} e_M$ has a vanishing line of slope $1/d$*
5. *For $p \neq 2$, if $P^s_t M_{ev} \neq 0$ for all $P^s_t \in A$ such that $s < t$ and $p|P^s_t| \leq 2d$ and $Q_t M \neq 0$ for all $Q_t \in A$ such that $|Q_t| \leq d$, then $M^{\otimes N_M} e_M$ has a vanishing line of slope $1/d$*

The proof will be given in §4. As a corollary we have the

Proof of the Main Theorem: The theorem follows from Theorem 3.4 and the properties of idempotent splittings.

Remark. Notice that Theorem 3.4 applies to any sub–Hopf algebra of the Steenrod algebra so is stronger than is needed for the proof of 1.1.

Proof of 1.2. One has the following commutator relations in the Steenrod algebra

$$P^s_t = [P^{s+1}_{t-1}, P^s_1] = [P^{s+n-1}_1, P^s_{t-1}]$$
$$\text{For } p \neq 2 \quad Q_t = [Q_0, P^0_{t-1}] = [Q_{t-1}, P^{t-1}_1]$$

So for a module M over the Steenrod algebra, if $Q_{n-1}M \neq 0$ ($P^0_n M \neq 0$ for $p = 2$) then $P^s_t M \neq 0$ for all $P^s_t \in A_{n-1}$ with $s < t$ and (for $p \neq 2$) $Q_t M \neq 0$ for all $Q_t \in A_{n-1}$. Now use Theorem 1.1. This takes care of (1) and (3). Part (2) is similar.

Proof of Theorem 1.3. For $p \neq 2$, let

$$X = sk_{2p^{n-1}} BZ/pZ$$

$Q_{n-1}H^*X$ is non–trivial and therefore by 1.2 $X^{\wedge N_X}$ has a stable summand Y such that H^*Y has a vanishing line of slope $\frac{1}{2(p^n-1)}$. The $k(n)$–homology $k(n)_*X$ is v_n–torsion free for dimensional reasons. Therefore $k(n)_*X^{\otimes N_X}$ and $k(n)_*Y$ are v_n–torsion free.

For $p = 2$ let

$$X = RP^{2^n} \wedge CP^{2^n}$$

where RP^n is real projective space of dimension n and CP^n is complex projective space of complex dimension n. We have $P^0_n H^*X \neq 0$ and $P^1_n H^*X \neq 0$. Therefore by 1.2 $X^{\wedge N_X}$ has a stable summand Y such that H^*Y has a vanishing line of slope $\frac{1}{2(2^n-1)}$. The $k(n)$–homology $k(n)_*RP^{2^n}$ is v_n–torsion free for dimensional reasons and $k(n)_*CP^{2^n}$ is v_n–torsion free since all the cell of CP^{2^n} are in even degrees. Therefore $k(n)_*X^{\otimes N_X}$ and $k(n)_*Y$ are v_n–torsion free.

Proof of 1.5. For $p \neq 2$ let

$$X = CP^{p^n}$$

Clearly $P^s_t H^* X \neq 0$ for all $P^s_t \in \mathcal{P}_n$ with $s < t$. By 1.2 the p–localization of $X^{\otimes N_X}$ has a stable summand Y such that H^*Y is $\mathcal{P}_n$–free. Now Y is a p–local CW–complex and H^*Y is finite but Y need not be a finite CW–complex. By a variant of the CW–approximation theorem there is a finite CW–complex Y' and a map $Y' \to Y$ which induces an isomorphism in mod–p cohomology.

The case $p = 2$ is similar.

Proof of 1.6. Let C_n be the cyclic sub–module of

$$H^* sk_{2p^n} BZ/pZ$$

generated by a non-trivial class in degree 1. Then $Q_n C_n \neq 0$ (P^0_{n+1} for $p = 2$) and as in the proof of 1.2 it follows using 3.4 that

$$M = C_n^{\otimes N_{C_n}} e_{C_n}$$

is a non–trivial A_n–free module. The module C_n has the smallest possible vector space dimension for a module over the Steenrod algebra with a non–trivial Q_n multiplication. The proof is completed by showing that there is only one copy of A_n. Let *min* and *max* be the smallest the largest degrees for which the graded vector space V is non–zero. It follows from Proposition 4.3 (5) that the integer $max - min$ is equal to the top non–trivial degree of the graded algebra A_n. Thus there is a set of generators for the A_n–module M in degree *min*. But it also follows from Proposition 4.3 (5) that dim $M_{min} = 1$. The proof is finished.

§4 Proof of the Theorem 3.4

Theorem 3.4 follows by combining a result of Miller and Wilkerson with an algebraic Lemma.

Vanishing Lines. Miller and Wilkerson give criteria for freeness and for the existence of vanishing lines over sub–Hopf algebras of the mod–p Steenrod algebra.

An *elementary Hopf algebra* is a Hopf algebra having one of the following simple algebra structures:

$$\begin{array}{lll} TypeE: & E = E[x] = \mathbb{F}_p[x]/x^2 & |x| \text{ odd or } p = 2 \\ TypeD: & D = D[x] = \mathbb{F}_p[x]/x^p & |x| \text{ even and } p \neq 2 \end{array}$$

By Proposition 3.3 the Steenrod algebra has many elementary subalgebras. For $p = 2$ and $s < t$, let $E[P^s_t]$ be the type E elementary subalgebra generated by P^s_t. For $p \neq 2$ and $s < t$, let $D[P^s_t]$ be the type D elementary subalgebra generated by P^s_t and let $E[Q_t]$ be the elementary subalgebra generated by Q_t.

THEOREM 4.1. *[**A-D**][**M-P**][**M-W**]Let A be a finite sub–Hopf algebra of the mod–p Steenrod algebra and let M be a connective A–module.*

1. *For $p = 2$, if M is a free $E[P^s_t]$–module for all $P^s_t \in A$ with $s < t$, then M is A–free.*
2. *For $p \neq 2$, if M is a free $D[P^s_t]$–module for all $P^s_t \in A$ with $s < t$ and if M is a free $E[Q_t]$–module for all $Q_t \in A$, then M is A–free.*
3. *For $p = 2$, if M is a free $E[P^s_t]$–module for all $P^s_t \in A$ such that $s < t$ and $|P^s_t| \leq d$, then M has a vanishing line of slope $1/d$.*
4. *For $p \neq 2$, if M is a free $D[P^s_t]$–module for all $P^s_t \in A$ such that $s < t$ and $p|P^s_t| \leq 2d$ and M is a free $E[Q_t]$–module for all $Q_t \in A$ such that $|Q_t| \leq d$, then M has a vanishing line of slope $1/d$.*

Let $D^s[x]$ ($E^s[x]$ for $p = 2$) be the Hopf algebra dual of $\mathbb{F}_p[x]/x^{p^{s+1}}$ with $|x|$ even ($|x|$ arbitrary for $p = 2$). As an algebra,

$$\text{For } p \neq 2 \quad D^s[x] \cong \otimes_{i=0}^{s} D[x_i] \qquad \text{with } |x_i| = p^i|x| \text{ and}$$
$$\text{For } p = 2 \quad E^s[x] \cong \otimes_{i=1}^{s} E[x_i] \qquad \text{with } |x_i| = 2^i|x|$$

This brings us to the key technical point.

LEMMA 4.2. *For a finite dimensional graded vector space V.*

1. *The vector space $V^{\otimes N_V} e_V$ is non–zero.*
2. *If V is a $D^s[x]$–module ($E^S[x]$–module for $p = 2$) and $x_s V_{ev} \neq 0$ ($x_s V \neq 0$ for $p = 2$) then $V^{\otimes N_V} e_V$ is $D[x_s]$–free ($E[x_s]$–free for $p = 2$).*
3. *If V is an $E[x]$–module and $xV \neq 0$ then $V^{\otimes N_V} e_V$ is $E[x]$–free.*

Before proving this lemma we give the

PROOF OF 3.4: The vector space $V^{\otimes N_V} e_V$ is non– zero by proposition 4.2(1). We finish the proof by using Theorem 4.1

For a sub–Hopf algebra A of the Steenrod algebra, if $P^s_t \in A$ with $s < t$, then the dual of $\mathbb{F}_p[\xi]/\xi_t^{p^{s+1}}$ is a sub–Hopf algebra of A which is isomorphic to $D^s[x]$ ($E^s[x] for$p=2) with $x_s = P^s_t$. If $P^s_t V_{ev} \neq 0$ ($P^s_t V \neq 0$ for $p = 2$) then by lemma 4.2(2) $V^{\otimes N_V} e_V$ is $D[P^s_t]$–free ($E[P^s_t]$–free for $p = 2$). And (for $p \neq 2$) if $Q_t V \neq 0$ then $V^{\otimes N_V} e_V$ is $E[Q_t]$–free by 4.2(3).

Now use Theorem 4.1 to finish the proof of 3.4

PROOF OF LEMMA 4.2: Let

$$B_V = \{v_1, v_2 \ldots, v_d\}$$

be an ordered homogeneous basis of V. For a function

$$f : \{1, 2, \ldots, n\} \to B_V$$

let

$$\otimes f = f(1) \otimes f(2) \otimes \cdots \otimes f(n) \in V^{\otimes n}$$

The collection of all tensors of this form gives a basis of $V^{\otimes n}$. Notice that for $\sigma \in S_n$

$$\otimes f \cdot \sigma = \otimes f \circ \sigma$$

Now let $n = N_W$. A function

$$f : \{1, 2, \ldots, N_W\} \to B_V$$

is *standard* if

(1) for each integer i, the restriction of f to the set of integers in the i'th row of t_W is order preserving,
(2) for each integer j, the restriction of f to the set of integers in the j'th column of t_W is monotonic and order preserving,
(3) for each integer i and basis vector $v \in B_V$, if $|v|$ is even there are at most $p-1$ integers k in the i'th row of t_W with $f(k) = v$, and if $|v|$ is odd there is at most one integer k in the i'th row of t_W with $f(k) = v$.

For example, let $p = 3$, $V = W$, $\dim V_{ev} = 2$ and $\dim V_{od} = 2$. Let $B_V = \{v_1, v_2, v_3, v_4\}$ be a homogeneous basis of V with $|v_i|$ even for $i = 1, 2$ and odd for $i = 3, 4$. Replacing each integer of the tableau t_W by its image in B_V we display an example of a standard function.

$$\begin{matrix} v_1 & v_1 & v_2 & v_2 & v_3 & v_4 \\ v_2 & v_2 & v_3 & v_4 & & \\ v_3 & v_4 & & & & \end{matrix}$$

Proposition 4.3. *Let V and W be finite dimensional graded $\mathbb{F}_p$–vector spaces and let f be a function*

$$f : \{1, 2, \ldots, N_W\} \to B_V$$

where B_V is an ordered basis of V.

(1) *If there is an integer i and a basis vector $v \in B_V$ such that $|v|$ is even and there are p different integers k in the i'th row of t_W with $f(k) = v$ or such that $|v|$ is odd and there are two integers k in the i'th row of t_v with $f(k) = v$, then*

$$\otimes f e_W = 0.$$

(2) *If $n_V < n_W$,*

$$V^{\otimes N_W} e_W = 0.$$

(3) *If f is standard then*

$$\otimes f e_W \neq 0.$$

(4) *If $n_V \geq n_W$ then*

$$V^{\otimes N_W} e_W \neq 0.$$

(5) *Suppose that $n_V \geq n_W$. Choose a homogeneous basis*

$$B_V = \{v_1, v_2 \ldots, v_d\}$$

such that the map $i \mapsto |v_i|$ is order reversing and let the integers min and max be respectively the smallest and the largest degrees in which the graded vector space $V^{\otimes N_W} e_W$ is non–zero. Then the set

$$\{\otimes f e_W | f \text{standard and } |\otimes f| = min\}$$

is a basis of the degree min homogenous subspace of $V^{\otimes N_W}e_W$*. Similarly*

$$\{\otimes f e_W | f \textit{standard and } |\otimes f| = max\}$$

is a basis of the degree max homogenous subspace of $V^{\otimes N_W}e_W$*.*

PROOF: For (1), by hypothesis either, there is a set J of p different integers in the i'th row of t_W such that $f(J) = \{v\}$ with v an even degree element of the basis or there is a set J of two integers in the i'th row of t_W such that $f(J) = \{v\}$ with v an odd degree element of the basis. Then for $\sigma \in S_J$

$$\otimes f \cdot \sigma = \begin{cases} \otimes f & \text{for } |v| \text{ even} \\ \text{sgn}(\sigma) \otimes f & \text{for } |v| \text{ odd} \end{cases}$$

and it follows that

$$\otimes f \sum_{\sigma \in S_J} \sigma = 0$$

therefore

$$\otimes f \sum_{\sigma \in R_{\alpha_W}} \sigma = 0 \qquad \text{and}$$

$$\otimes f e_W = 0$$

For (2), notice that

$$(p-1)n_V \leq (p-1)\dim V_{ev} + \dim V_{od} < (p-1)(n_V + 1) \leq (p-1)n_W$$

and so by the pigeon hole principle, any function

$$f : \{1, 2, 3, \ldots, N_W\} \to B_V$$

satisfies the hypothesis of (1). Therefore

$$\otimes f\, e_W = 0$$

Part (2) now follows immediately.

For a standard function f, a permutation σ in the row group of t_W, and a permutation τ in the column group of t_W. If

$$\otimes f\, \sigma\tau = \otimes f$$

then $\tau = Id$ and σ is in the isotropy group of $\otimes f$. So the coefficient of the basis element $\otimes f$ in the sum $\otimes f e_W$ is $|G|$ where G is the group of permutations σ in the row group such that $\otimes f\sigma = \otimes f$. The order $|G|$ is relatively prime to p since G is a product of groups isomorphic to S_{p-1}. Therefore

$$\otimes f e_W \neq 0$$

If $n_V \geq n_W$ then standard functions exist, proving (4).

Part (5) is more of the same and is left to the reader.

Lemma 4.2(1) follows from proposition 4.3(4)

To prove lemma 4.2(2), let V be an $E[x]$–module with $xV \neq 0$ It follows that there is a splitting as $E[x]$–modules

$$V = E[x] \oplus K$$

There is an equivariant splitting

$$V^{\otimes N_V} = F \oplus K^{\otimes N_V}$$

as $E[x]$–modules and therefore a splitting

$$V^{\otimes N_V} e_V = F e_V \oplus K^{\otimes N_V} e_V.$$

of $E[x]$–modules. The module F is $E[x]$–free since it is a direct sum of modules of the form $E[x] \otimes W$. The summand Fe_V is $E[x]$–free since $E[x]$ is a local ring. But by Proposition 4.3(2)

$$K^{\otimes N_V} e_V = 0$$

since $e_K < e_V$.

We use a filtration argument to prove 4.3(3). We will construct a filtration of $V^{\otimes N_V}$

$$0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_m = V^{\otimes N_V}$$

by $D^s[x]$–modules ($E^s[x]$–modules for $p = 2$) such the associated graded module

$$E_0 V^{\otimes N_V} = \oplus_{i=1}^m W_i / W_{i-1}$$

is $D[x_s]$–free ($E[x_s]$–free for $p = 2$) which implies that $V^{\otimes N_V}$ is $D[x_s]$–free ($E[x_s]$–free for $p = 2$)

Let I be the kernel of the homomorphism $D^s[x] \to D[x_s]$ ($E^s[x] \to E[x_s]$ for $p = 2$). Let $v \in V_{ev}$ ($v \in V$ for $p = 2$) be an element such that $x_s v \neq 0$ and $(x_s)^2 v = 0$. Filter the $D^s[x]$–module ($E^s[x]$–module for $p = 2$) V by

$$\begin{aligned} W_1 &= Iv \\ W_2 &= D^s[x]v \qquad (E^s[x]v \text{for } p = 2) \\ W_3 &= V \end{aligned}$$

We have

$$W_2/W_1 = W = D[x^s]/(x^s)^2 \qquad (E[x_s] \text{ for } p = 2)$$

and the associated graded module is

$$E_0 V = W \oplus U \qquad \text{where } U = W_1 \oplus W_3/W_2$$

Then the tensor filtration of $V^{\otimes N_V}$ has associated graded module

$$E_0 V^{\otimes N_V} = (E_0 V)^{\otimes N_V} = (W \oplus U)^{\otimes N_V}$$

and we finish the argument by showing that

$$(W \oplus U)^{\otimes N_V} e_V \tag{4.4}$$

is $D[x_s]$–free ($E[x_s]$–free for $p = 2$).

The case $p = 2$ is covered by 4.2(2). We now assume $p \neq 2$. We need two Propositions.

Proposition 4.5. *Let V and W be $D^s[x]$–modules. If V is $D[x_s]$–free then $V \otimes W$ is $D[x_s]$–free.*

Proof: The statement is obvious when $W = \mathbb{F}_p$. Now proceed by induction on $\dim W$.

Proposition 4.6. *For a Hopf algebra A which is a local ring, let M be an A–module such that $M = F \oplus K$ and F is a free A–module. If $P : M \to M$ is an A–linear projection such that*

$$K \subseteq \operatorname{Ker} P$$

then the image of P is a free A–module.

Proof: We have

$$F = M/K = \mathrm{Im}P \oplus \mathrm{Ker}P/K$$

since $K \subseteq \mathrm{Ker}P$. So $\mathrm{Im}P$ is a summand of a free module and therefore it is free since A is a local ring.

Now we continue with the proof that 4.4 free.

Let $B_V = \{v_1, v_2 \ldots, v_d\}$ be a homogeneous basis of V such that $\{v_1, v_2\}$ is a basis of W and $\{v_3, \ldots, v_d\}$ is a basis of U. Let

$$R_1 = \{1, 2, \ldots, (p-1)n_V\}$$

be the set of integers in the first row of the tableau t_V. For $J \subset R_1$ let

$$U_J = \langle \{\otimes f | f : \{1, 2, \ldots, N_V\} \to B_V \text{ and } f^{-1}\{v_1, v_2\} \cap R_1 = J\} \rangle$$

where $\langle A \rangle$ denotes the vector space generated by the set A.

Proposition 4.7.

1. *For subsets J and K of R_1. If $J \neq K$ then $U_J \cap U_K = 0$*
2. *$V^{\otimes N_V} = \oplus_{J \subset R_1} U_J$*

If $|J| \geq p-1$, let sJ be the set consisting of the $p-1$ smallest integers in J and let

$$g_J = -\sum_{\sigma \in S_{sJ}} \sigma \in \mathbb{F}_p S_{N_V}.$$

The element g_J is an idempotent. Now define

$$K_J = \begin{cases} U_J & \text{if } |J| < p-1 \\ U_J(1 - g_J) & \text{if } |J| \geq p-1 \end{cases}$$

$$F_J = \begin{cases} 0 & \text{if } |J| < p-1 \\ U_J g_J & \text{if } |J| \geq p-1 \end{cases}$$

Proposition 4.8. *For $J \subset R_1$*

1. *$U_J = K_J \oplus F_J$*
2. *$K_J e_V = 0$*
3. *F_J is a free $D[x_s]$–module*

PROOF: Part (1) is clear since g_J is an idempotent. For $\sigma \in S_{R_1}$ we have

$$\sigma e_V = e_V \quad \text{and so} \quad g_J e_V = e_V$$

Then (2) follows from the identity $(1 - g_J)e_V = 0$.

The zero module is free and so for (3) we may assume that $|J| \geq p - 1$. Then there is a $D^s[x]$–module R such that

$$U_J = W^{\otimes p-1} \otimes R \qquad \text{as } D^s[x] text--modules.$$

The idempotent g_J acts on the factor W^{p-1} giving a splitting

$$W^{\otimes p-1} = W^{\otimes p-1} g_J \oplus W^{\otimes p-1}(1 - g_J)$$

Now $W^{\otimes p-1} g_J$ is a free $D[x_s]$–module on one generator. Then by Proposition 4.5

$$U_J g_J = W^{\otimes p-1} g_J \otimes R$$

is $D[x_s]$–free.

Now let

$$F = \sum_{J \subset R_1} F_J \qquad \text{and}$$
$$K = \sum_{J \subset R_1} K_J$$

The proof is finished by combining 4.6 with

PROPOSITION 4.9.

(1) *F is a free $D[x_s]$–module and*
(2) $K e_V = 0$

PROOF: F is free $D[x_s]$–module since it is a direct sum of free $D[x_s]$–modules. $K e_V$=0 since $K_J e_V = 0$ for all $J \subset R_1$.

This completes the proof that 4.4 is $D[x_s]$–free and the proof of Lemma 4.2.

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