UPPER TRIANGULAR TECHNOLOGY AND THE ARF-KERVAIRE INVARIANT

VICTOR P. SNAITH

Abstract. This paper introduces the upper triangular technology (UTT) into classical homotopy theory. This is a new and easy to use method to calculate the effect of the left unit map in 2-adic connective K-theory; the map which is the basis for operations in bu-theory. By way of application, UTT is used to give a new, very simple proof of a conjecture of Barratt-Jones-Mahowald, which rephrases K-theoretically the existence of framed manifolds of Arf-Kervaire invariant one.

1. Introduction

1.1. The upper triangular technology (UTT) referred to in the title consists of the following two results. Let bu and bo denote the stable homotopy spectra representing 2-adically completed unitary and orthogonal connective K-theory respectively. Thus the smash product, $bu \wedge bo$ is a left bu-module spectrum and so we may consider the ring of left bu-module endomorphisms of degree zero in the stable homotopy category of spectra [2], which we shall denote by $End_{left-bu-mod}(bu \wedge bo)$. The group of units in this ring will be denoted by $Aut_{left-bu-mod}(bu \wedge bo)$, the group of homotopy classes of left bu-module homotopy equvialences and let $Aut^0_{left-bu-mod}(bu \wedge bo)$ denote the subgroup of left bu-module homotopy equivalences which induce the identity map on $H_*(bu \wedge bo; \mathbb{Z}/2)$.

Let $U_\infty \mathbb{Z}_2$ denote the group of infinite, invertible upper triangular matrices with entries in the 2-adic integers. That is, $X = (X_{i,j}) \in U_\infty \mathbb{Z}_2$ if $X_{i,j} \in \mathbb{Z}_2$ for each pair of integers $0 \leq i, j$ and $X_{i,j} = 0$ if $j < i$ and $X_{i,i}$ is a 2-adic unit.

Theorem 1.2. *([20] §2.1)*
There is an isomorphism of the form

$$\Psi : Aut^0_{left-bu-mod}(bu \wedge bo) \xrightarrow{\cong} U_\infty \mathbb{Z}_2.$$

Let $\psi^3 : bo \longrightarrow bo$ denote the Adams operation.

This isomorphism Ψ is defined up to inner automorphisms of $U_\infty \mathbb{Z}_2$. Given an important automorphism in $\mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo)$ one is led to ask what is its conjugacy class in $U_\infty \mathbb{Z}_2$. By far the most important such automorphism is $1 \wedge \psi^3$.

Date: 26 September 2006.

Theorem 1.3. *([10] §1.1)*

Under the isomorphism Ψ the automorphism $1 \wedge \psi^3$ corresponds to an element in the conjugacy class of the matrix

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 9 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9^2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right).$$

To be precise, the proof in ([10] §1.1) shows that $1 \wedge \psi^3$ can be conjugated to this form in its first N columns for arbitarily large N. In Example 3.2 I shall explain what this means in practice.

The main purpose of this paper is to illustrate UTT at work in an application. By way of illustration I shall use UTT to give an elementary proof of the following result, whose terminology and proof will be given in §3.

Theorem 1.4.

Let m be a positive integer and let $\Theta_{8m-2} : S^{8m-2} \longrightarrow \mathbb{RP}^{8m-2}$ be a morphism in the 2-local stable homotopy category. Then the bo e-invariant of Θ_{8m-2} is $(3^{4m}-1)/4$ (modulo 2^{4m-1}) if and only if $m = 2^q$ and Θ_{8m-2} is detected by the Steenrod operation $Sq^{2^{q+2}}$.

Any 2-adic stable homotopy class $\tilde{\Theta} \in \pi_{8m-2}(S^0) \otimes \mathbb{Z}_2$ lifts canonically (via the Kahn-Priddy Theorem [3] [12]) to an element of the stable homotopy of $\mathbb{RP}^\infty$ and thence to Θ_{8m-2}. Detection by a primary mod 2 cohomology operation can only occur if m is a power of 2 and is equivalent to $\tilde{\Theta}$ being represented by a framed manifold of Arf-Kervaire invariant one ([8], [11], [18], [21]). The existence or otherwise of framed manifolds of Arf-Kervaire invariant one is a classical unsolved problem in homotopy theory. As explained in [13], the alternative formulation of Theorem 1.1 is equivalent to a conjecture of [7] which was first proved in [14] by a very difficult study of Ad-theory and is proved in [21] by a straightforward but brutally long-winded use of BP-operations. The attempted proof of [13] contains a gap caused by lack of control of the filtration in an Adams spectral sequence. Intuitively, the UTT proof of Theorem 1.1 is conceptually simple because it amounts to inflicting the relevant mod 2 Adams spectral sequence with a "mixed Hodge structure"; that is, a direct sum decomposition (corresponding to the entries in $U_\infty \mathbb{Z}_2$) compatible with the usual Adams filtration.

Incidentally Theorem 1.4 poses the existence of framed manifolds of Arf-Kervaire invariant one in a form which is similar to the formulation of the existence of framed manifolds of Hopf invariant one in terms of

$\Theta_{2m-1} : S^{2m-1} \longrightarrow \mathbb{RP}^{2m-1}$ (see [19]). These only exist for dimensions $1, 3, 7$ and currently framed manifolds of Arf-Kervaire invariant one have been constructed in dimensions $2, 6, 14, 30, 62$ (see [18] and [15] – I believe that [16] has a gap in its construction). Accordingly the following conjecture seems reasonable:

Conjecture 1.5.
Framed manifolds of Arf-Kervaire invariant one exist at most in dimensions $2, 6, 14, 30$ *and* 62.

The paper is organised in the following manner. The basis of Theorems 1.2 and 1.3 is the left bu-module splitting of $bu \wedge bo$ into a sum of spectra of the form $bu \wedge (F_{4k}/F_{4k-1})$ for $k \geq 0$. In §2 the mod 2 Adams spectral sequence for the homotopy of $bu \wedge (F_{4k}/F_{4k-1}) \wedge \mathbb{RP}^{2m}$ is described. The crucial result is the multiplicative structure stated in Theorem 2.6 and proved in §2.8. Combined with results from [10], Theorem 2.6 yields Proposition 2.10, which evaluates the effect on homotopy of the maps corresponding to the super-diagonal entries of an upper-triangular matrix. These maps correspond to the 1's in the matrix for $1 \wedge \psi^3$ in Theorem 1.3. In §3 Theorem 1.3 is combined with Proposition 2.10 to transform expressions for the bo e-invariant of Θ_{8m-2} into a series of 2-adic equations in Example 3.4 (and Propositions 3.5 and 3.6) from which Theorem 1.4 is easily deduced in §3.8.

I am very grateful for help and advice to Huajian Yang, my postdoc at McMaster University in 1996-98, to Jonathan Barker, my PhD student at the University of Southampton in 2003-2006 and to Francis Clarke.

2. K-theory examples

2.1. Let $bu_*(X)$ (resp. $KU_*(X)$) denote the reduced, connective (resp. periodic) complex K-theory of a (based) CW complex X. When X equals the zero-dimensional sphere we have $bu_*(S^0) \cong \mathbb{Z}[u]$ and $KU_*(S^0) \cong \mathbb{Z}[u^{\pm 1}]$ where $\deg(u) = 2$. Let $\mathbb{RP}^n$ denote n-dimensional real projective space. Let $\mathbb{Z}/t\langle w \rangle$ denote a cyclic group of order t with generator w The following result is well known.

Proposition 2.2. *For* $1 \leq m \leq \infty$

$$bu_j(\mathbb{RP}^{2m}) = \begin{cases} 0 & \text{if } j \text{ is even,} \\ \mathbb{Z}/2^i\langle v_{2i-1} \rangle & \text{if } 1 \leq j = 2i-1 < 2m, \\ \mathbb{Z}/2^m\langle v_{2m-1}u^{i-m} \rangle & \text{if } 2m < j = 2i-1. \end{cases}$$

In addition, the generators may be chosen to satisfy $uv_{2i-1} = 2v_{2i+1}$ *for* $1 \leq i \leq m-1$.

Proof

The Atiyah-Hirzebruch spectral sequences for computing $bu_*(\mathbb{RP}^{2m})$ and $KU_*(\mathbb{RP}^{2m})$ collapse for dimensional reasons. This implies that $bu_j(\mathbb{RP}^{2m})$ has the correct order. It also implies the injectivity of the canonical maps

$$h_j(\mathbb{RP}^{2m}) \longrightarrow h_j(\mathbb{RP}^{2m+2}) \ (h = bu, KU), \ \lambda_* : bu_j(\mathbb{RP}^{2m}) \longrightarrow KU_j(\mathbb{RP}^{2m}).$$

However, by the universal coefficient theorem for KU and the results of ([5] p.107) we have $KU_{2i-1}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^\infty$ so each $bu_{2i-1}(\mathbb{RP}^{2m})$ is cyclic. The relation $uv_{2i-1} = 2v_{2i+1}$ follows from Bott periodicity and the fact that the injection λ_* commutes with multiplication by u. $\square$

2.3. $\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)$

Let $B = E(Sq^1, Sq^{0,1})$ denote the exterior subalgebra of the mod 2 Steenrod algebra $\mathcal{A}$ [22] generated by Sq^1 and $Sq^{0,1} = [Sq^1, Sq^{0,1}]$. There is an isomorphism of bigraded algebras $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b]$, the polynomial algebra on a and b with $\mathrm{bideg}(a) = (1, 1), \mathrm{bideg}(b) = (1, 3)$. Also $\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2) = \langle x, x^2, \ldots \rangle/(x^{2m+1})$ with $Sq^1(x^n) = nx^{n+1}, Sq^{0,1}(x^n) = nx^{n+3}$.

Consider the bigraded $\mathbb{Z}/2[a, b]$-module $\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)$. Denote the non-zero element of $\mathrm{Ext}_B^{0,2i-1}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)$ by $\tilde{v}_{2i-1}$ for $1 \leq i \leq m$.

Proposition 2.4. *For $1 \leq m \leq \infty$ the bigraded $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$-module $\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)$ is equal to*

$$\frac{\mathbb{Z}/2[a, b]\langle \tilde{v}_1, \tilde{v}_3, \ldots, \tilde{v}_{2m-1} \rangle}{\{a^i \tilde{v}_{2i-1}, \ b\tilde{v}_{2i-1} - a\tilde{v}_{2i+1}\}}.$$

Proof

We prove this by induction on m. When $m = 1$ we have

$$\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^2; \mathbb{Z}/2), \mathbb{Z}/2) \cong \mathrm{Ext}_B^{*,*}(E(Sq^1)[1], \mathbb{Z}/2) \cong \mathbb{Z}/2[b]\langle \tilde{v}_1 \rangle$$

where $X[n]$ denotes X with a dimension shift by n so that $X[1] = \Sigma X$ in the notation of [2] and [20]. We have a short exact sequence of B-modules

$$0 \longrightarrow \tilde{H}^*(\mathbb{RP}^{2m-2}; \mathbb{Z}/2)[-2] \stackrel{(x^2 \cdot -)}{\longrightarrow} \tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2) \longrightarrow \tilde{H}^*(\mathbb{RP}^2; \mathbb{Z}/2) \longrightarrow 0.$$

By induction, for each non-negative integer r the resulting long exact sequence yields an upper bound for the sums of $\mathbb{F}_2$- dimensions

$$\sum_{s=0}^{\infty} \sum_{t-s=r} \dim_{\mathbb{F}_2}(\mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)) \leq d_r$$

where $d_r = 0$ if r is even, $d_{2i-1} = i$ for $1 \leq i \leq m$ and $d_{2i-1} = m$ for $m \leq i$. On the other hand, if $\mathbb{Z}_2$ denotes the 2-adic integers, the Adams spectral sequence ([2]) for $\pi_*(bu \wedge \mathbb{RP}^{2m}) \otimes \mathbb{Z}_2 = bu_*(\mathbb{RP}^{2m}) \otimes \mathbb{Z}_2$ has the form ([10]; [20])

$$E_2^{s,t} = \mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow bu_{t-s}(\mathbb{RP}^{2m}) \otimes \mathbb{Z}_2$$

and collapses for dimensional reasons, being concentrated where $t-s$ is odd. Therefore Proposition 2.2 shows that d_r is also a lower bound. The relations folllow from the fact that a and b represent 2 and u respectively in the Adams spectral sequence for $bu_*(S^0) \otimes \mathbb{Z}_2$. □

2.5. $\mathrm{Ext}_B^{*,*}(\tilde{H}^*(X \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2)$

Consider the second loopspace of the 3-sphere, $\Omega^2 S^3$. There exists a model for $\Omega^2 S^3$ which is filtered by finite complexes ([9],[17])

$$S^1 = F_1 \subset F_2 \subset F_3 \subset \ldots \subset \Omega^2 S^3 = \bigcup_{k \geq 1} F_k$$

and there is a stable homotopy equivalence, an example of the so-called Snaith splitting, of the form $\Omega^2 S^3 \simeq \vee_{k \geq 1} F_k/F_{k-1}$.

Consider the finite complexes F_{4k}/F_{4k-1} with the convention that $F_0/F_{-1} = S^0$, the 0-sphere. Let $\alpha(n)$ denote the number of 1's in the dyadic expansion of the positive integer n. The results of Adams-Margolis ([2], [4]; see also [10] and [20]) yield $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$-module isomorphisms of the form

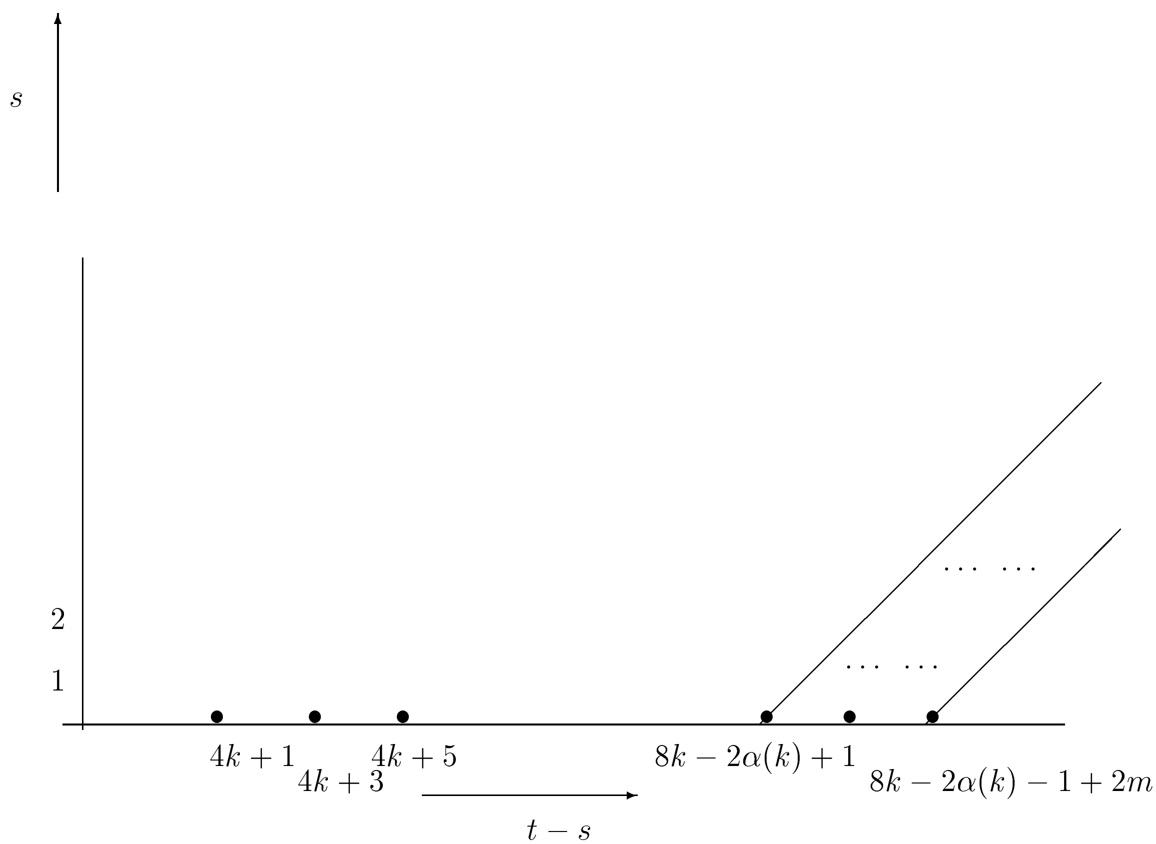
$$\mathrm{Ext}_B^{s,t}(\tilde{H}^*(X \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2)$$

$$\cong \mathrm{Ext}_B^{s+2k-\alpha(k), t-2k-\alpha(k)}(\tilde{H}^*(X; \mathbb{Z}/2), \mathbb{Z}/2)$$

for all $s > 0$. We shall need this isomorphism in the case where X is either a real projective space or a sphere. The case when X is a sphere is described extensively in [20] in connection with the left bu-module equivalence of 2-local spectra (see also ([10] §2)

$$\hat{L} : \vee_{k \geq 0}\, bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo.$$

The groups $\mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{2m} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2)$, when depicted in the traditional Adams spectral sequence manner with s along the vertical and $t-s$ along the horizontal axis, looks as in the figure below. The figure is interpreted as follows: the groups are $\mathbb{F}_2$-vector spaces which are possibly non-zero only when $s = 0$ and $t - s \geq 4k+1$ or a copy of $\mathbb{Z}/2$ at each point with $(s, t-s) = (v, 8k - 2\alpha(k) - 1 + 2w + 2v)$ with $v = 1, 2, 3, \ldots$ and $1 \leq w \leq m$.



We have the following result describes the important aspects of the module structure over the bigraded algebra $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b]$. Let $\tilde{v}_{2i-1} \in \mathrm{Ext}_B^{0,2i-1}(\tilde{H}^*(\mathbb{RP}^{2m};\mathbb{Z}/2),\mathbb{Z}/2)$ be as in Proposition 2.4 and let $\hat{z}_{4k} \in \mathrm{Ext}_B^{0,4k}(\tilde{H}^*(F_{4k}/F_{4k-1};\mathbb{Z}/2),\mathbb{Z}/2)$ be element represented as a homomorphism on mod 2 cohomology by the inclusion of the bottom cell of F_{4k}/F_{4k-1} ([10] Theorem 2.12). Hence we have a (non-zero) external product $\tilde{v}_{2i-1}\hat{z}_{4k} \in \mathrm{Ext}_B^{0,4k+2i-1}(\tilde{H}^*(\mathbb{RP}^{2m} \wedge (F_{4k}/F_{4k-1});\mathbb{Z}/2),\mathbb{Z}/2)$ for $1 \le i \le m$.

Theorem 2.6.
In $\mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{2m} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2)$ *we have:*
(i) $\tilde{v}_i \hat{z}_{4k} \neq 0$ *for each* $1 \leq i \leq m$,
(ii) $b\tilde{v}_i \hat{z}_{4k} = 0 = a\tilde{v}_i \hat{z}_{4k}$ *for* $i = 1, \ldots, 2k - \alpha(k)$,
(iii) $b^e \tilde{v}_i \hat{z}_{4k} \neq 0$ *for* $e \geq 1$ *and* $i = 2k - \alpha(k) + 1, \ldots, m$.

The proof of Theorem 2.6 will be given in §2.8 after some preliminaries concerning B-resolutions.

2.7. Resolutions

For $1 \leq m \leq \infty$, the B-action on $\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2) = \langle x, x^2, \ldots, x^{2m} \rangle$ is given by $Sq^1(x^i) = ix^{i+1}$ and $Sq^{0,1}(x^i) = ix^{i+3}$. The beginning of a free B-resolution

$$\ldots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} \tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2) \longrightarrow 0$$

may be given by $P_0 = B\langle \sigma_1, \sigma_3, \ldots, \sigma_{2m-1} \rangle$ and $P_1 = B\langle \nu_5, \nu_4, \nu_6, \ldots, \nu_{2m} \rangle$ where $\deg(\sigma_i) = i$, $\epsilon(\sigma_i) = x^i$ and $d(\nu_{2t}) = Sq^1 \sigma_{2t-1} + Sq^{0,1} \sigma_{2t-3}$ and $d(\nu_5) = Sq^1 Sq^{0,1} \sigma_1$. By Proposition 2.4 $b\tilde{v}_{2i-1}$ is the only non-zero element in the group $\mathrm{Ext}_B^{1,2i+2}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)$ for $1 \leq i \leq m$. Therefore it must be represented by homomorphism $h_i \in \mathrm{Hom}_B(P_1, \mathbb{Z}/2)$ given by $h_i(\nu_{2i+2}) \equiv 1$ (modulo 2) and $\mathrm{h}_i(\nu_j) \equiv 0$ otherwise.

Let $H(k)$ be the graded $\mathbb{F}_2$-vector space with basis

$$y_{k,2^k}, y_{k,2^k+2}, y_{k,2^k+4}, \ldots, y_{k,2^{k+1}-2}, y_{k,2^k+3}, y_{k,2^k+5}, \ldots, y_{k,2^{k+1}-1}$$

where $\deg(y_i) = i$, with the "lightning flash" B-module structure given by

$$Sq^{0,1} y_{k,2^k} = y_{k,2^k+3} = Sq^1 y_{k,2^k+2}, \ \ldots, Sq^{0,1} y_{k,2^{k+1}-4} = y_{k,2^{k+1}-1} = Sq^1 y_{k,2^{k+1}-2}.$$

We define the start of a free B-resolution

$$\ldots \longrightarrow R_{k,1} \xrightarrow{d(k)} R_{k,0} \xrightarrow{\epsilon(k)} H(k) \longrightarrow 0$$

by $R_{k,0} = B\langle \Sigma_{k,0,2^k}, \Sigma_{k,0,2^k+2}, \ldots, \Sigma_{k,0,2^{k+1}-2} \rangle$ where $\deg(\Sigma_{k,0,i}) = i$ and $R_{k,1} = B\langle \Sigma_{k,1,2^k+1}, \Sigma_{k,1,2^k+3}, \ldots, \Sigma_{k,1,2^{k+1}+1} \rangle$ where $\deg(\Sigma_{k,1,i}) = i$. Also $\epsilon(k)$ and $d(k)$ are given by $\epsilon(k)(\Sigma_{k,0,2i}) = y_{k,2i}$ and

$$d(k)(\Sigma_{k,1,2^k+1}) = Sq^1 \Sigma_{k,0,2^k},$$

$$d(k)(\Sigma_{k,1,2^k+3}) = Sq^1 \Sigma_{k,0,2^k+2} + Sq^{0,1} \Sigma_{k,0,2^k},$$

$$d(k)(\Sigma_{k,1,2^k+5}) = Sq^1 \Sigma_{k,0,2^k+4} + Sq^{0,1} \Sigma_{k,0,2^k+2},$$

$$\vdots \quad \vdots \quad \vdots$$

$$d(k)(\Sigma_{k,1,2^{k+1}+1}) = Sq^{0,1} \Sigma_{k,0,2^{k+1}-2}.$$

We are now ready to embark on the proof of Theorem 2.6.

2.8. Proof of Theorem 2.6

Part (i) follows since the exterior product of two non-zero B-homomorphisms to $\mathbb{Z}/2$ is also non-zero and part (ii) follows because the elements in question lie in groups which are zero, by Proposition 2.4 and the discussion of §2.5. Part (iii) is more substantial. By naturality it suffices to work with $m = \infty$.

Let $k = 2^{\epsilon_1} + 2^{\epsilon_2} + \ldots + 2^{\epsilon_t}$ with $0 \leq \epsilon_1 < \epsilon_2 < \ldots < \epsilon_t$ so we are interested in $i \geq 2^{\epsilon_1+1} + 2^{\epsilon_2+1} + \ldots + 2^{\epsilon_t+1} - t + 1$ and $4k = 2^{\epsilon_1+2} + 2^{\epsilon_2+2} + \ldots + 2^{\epsilon_t+2}$. From ([2] pp.341-2) or ([20] p.1267) $\tilde{H}^*(F_{4k}/F_{4k-1};\mathbb{Z}/2) \cong \otimes_{j=1}^t H(\epsilon_j + 2)$. We have a free B-resolution given by the tensor product

$$\ldots \to \otimes_{j=1,\ \sum a_j=1}^t R_{\epsilon_j+2,a_j} \stackrel{d}{\to} \otimes_{j=1}^t R_{\epsilon_j+2,0} \to \tilde{H}^*(F_{4k}/F_{4k-1};\mathbb{Z}/2) \to 0.$$

We introduce the convention that

$$\Sigma_{\epsilon_j+2,1,2s+1} = 0 = \Sigma_{\epsilon_j+2,0,2s} \text{ if } s \leq 2^{\epsilon_j+1} \text{ or } 2^{\epsilon_j+2} \leq s.$$

With this convention the differential has the form

$$d(\epsilon_j + 2)(\Sigma_{\epsilon_j+2,1,2s+1}) = Sq^1\Sigma_{\epsilon_j+2,1,2s} + Sq^{0,1}\Sigma_{\epsilon_j+2,1,2s-2}.$$

The element $\hat{z}_{4k}$ is represented by the B-homomorphism

$$g_k \in \mathrm{Hom}_B(\otimes_{j=1}^t R_{\epsilon_j+2,0}, \mathbb{Z}/2)$$

given by $g_k(\otimes_{j=1}^t \Sigma_{\epsilon_j+2,0,2^{\epsilon_j+2}}) \equiv 1$ (modulo 2) and $g_k(\otimes_{j=1}^t \Sigma_{\epsilon_j+2,0,w_j}) \equiv 0$ otherwise.

We must show that there does not exist a B-homomorphism

$$f \in \mathrm{Hom}_B(P_0 \otimes (\otimes_{j=1}^t R_{\epsilon_j+2,0}), \mathbb{Z}/2)$$

such that $f \cdot d = (0, h_i \otimes g_k)$ in the group of B-homomorphisms

$$\mathrm{Hom}_B(P_0 \otimes (\otimes_{j=1,\ \sum a_j=1}^t R_{\epsilon_j+2,a_j}) \oplus P_1 \otimes (\otimes_{j=1}^t R_{\epsilon_j+2,0}), \mathbb{Z}/2)$$

when $m = \infty$ and i lies in the range $i \geq 2k - \alpha(k)$. This will show that $b\tilde{v}_i\hat{z}_{4k} \neq 0$ from which $b^e\tilde{v}_i\hat{z}_{4k} \neq 0$ for $e \geq 1$ follows because the isomorphism of §2.5 commutes with multiplication by b when $s > 0$.

I shall first give the argument to prove that f does not exist and finally I shall explain where the $i \geq 2k - \alpha(k)$ is necessary.

In degree $4k + 2i + 2$, suppose that we have the relation $(0, h_i \otimes g_k) = f \cdot d$. Then we shall apply $f \cdot d$ to all the B-basis elements in $P_* \otimes (\otimes_{j=1}^t R_{(\epsilon_j+2,*)})$ in resolution degree 1 and homological degree $4k + 2i + 2$ and add the results in two ways to get a contradiction. The basis elements in question are

$$\{\nu_{2q+2} \otimes (\otimes_j \Sigma_{\epsilon_j+2,0,2s_j}) \mid 2q + 2 + \sum_j 2s_j = 4k + 2i + 2\ \}$$

and (where $0 \leq a_j \leq 1$ and $\sum_j a_j = 1$)

$$\{\sigma_{2q+1} \otimes (\otimes_j \Sigma_{\epsilon_j+2,a_j,2s_j+a_j}) \mid 2q + 2 + \sum_j 2s_j + a_j = 4k + 2i + 2\ \}$$

disregarding, of course, the ones of this list which are zero by the convention introduced above.

We have

$$f(d(\nu_{2q+2} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})))$$

$$= f(Sq^1\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})) + f(Sq^{0,1}\sigma_{2q-1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j}))$$

and (where $0 \leq a_j \leq 1$ and $\sum_j a_j = 1$)

$$f(d(\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,a_j,2s_j+a_j})))$$

$$= f(\sigma_{2q+1} \otimes \ldots \otimes Sq^1\Sigma_{\epsilon_j+2,0,2s_j} \otimes \ldots)$$

$$+ f(\sigma_{2q+1} \otimes \ldots \otimes Sq^{0,1}\Sigma_{\epsilon_j+2,0,2s_j-2} \otimes \ldots)$$

where in the last expression the Sq's appear precisely in the unique factor for which a_j was equal to one.

Now fix a $t+1$-tuple $(q, s_1, \ldots, s_t)$ such that $4k + 2i + 2 = 2q + 2 + \sum_j \ 2s_j$ and consider the sum

$$f(Sq^1\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})) + \sum_{j=1}^t \ f(\sigma_{2q+1} \otimes \ldots \otimes Sq^1\Sigma_{\epsilon_j+2,0,2s_j} \otimes \ldots)$$

$$= Sq^1(f(\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})))$$

$$= 0$$

because Sq^1 acts trivially on $\mathbb{Z}/2$ for dimensional reasons. Similarly

$$f(Sq^{0,1}\sigma_{2q-1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j}))$$

$$+ \sum_{j=1}^t \ f(\sigma_{2q+1} \otimes \ldots \otimes Sq^{0,1}\Sigma_{\epsilon_j+2,0,2s_j} \otimes \ldots)$$

$$= 0.$$

Therefore applying $f \cdot d$ to each of the basis elements listed above and adding the results yields zero modulo 2.

Now consider what happens if we apply $(0, h_i \otimes g_k)$ to each of the basis elements listed above and add the results. The sum equals 1 because the map is zero on $P_0^* \otimes (\otimes_{j=1, \ \sum \ a_j=1}^t R_{\epsilon_j+2,a_j})$ and is also zero on $\nu_{2q+2} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})$ unless $q = i$ and $2s_j = 2^{\epsilon_j+2}$ for $j = 1, \ldots, t$.

This contradiction completes the proof of part (iii) except that it remains to explain why we need the condition that

$$i \geq 2^{\epsilon_1+1} + 2^{\epsilon_2+1} + \ldots + 2^{\epsilon_t+1} - t + 1.$$

We require that i be large enough so that all the elements $\nu_{2q+2} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})$ and $\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,a_j,2s_j+a_j})$ over which we want to sum are permissible within homological degree $2i + 2 + 4k$. However if $i \geq 2^{\epsilon_1+1} + 2^{\epsilon_2+1} + \ldots + 2^{\epsilon_t+1} - t + 1$ then

$$2i + 2 + 4k \geq 2^{\epsilon_1+3} - 2 + 2^{\epsilon_2+3} - 2 + \ldots + 2^{\epsilon_t+3} - 2 + 2 \geq \sum_{j=1}^t \ 2s_j$$

for all possible choices of the s_j's involved in the sum. □

2.9. The maps $\iota_{k,l}$

As in §2.5 let bu and bo denote the 2-localised, connective unitary and orthogonal K-theory spectra, respectively. Consider a left-bu-module spectrum map

$$\iota : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1}).$$

This map is determined up to homotopy by its restriction, via the unit of bu, to (F_{4k}/F_{4k-1}). By S-duality this restriction is equivalent to a map of the form

$$S^0 \longrightarrow D(F_{4k}/F_{4k-1}) \wedge bu \wedge (F_{4l}/F_{4l-1}),$$

which DX denotes the S-dual of X. Maps of this form are studied by means of the (collapsed) Adams spectral sequence (see [20] §3.1)

$$E_2^{s,t} = Ext_B^{s,t}(\tilde{H}^*(D(F_{4k}/F_{4k-1}); \mathbb{Z}/2) \otimes \tilde{H}^*(F_{4l}/F_{4l-1}; \mathbb{Z}/2), \mathbb{Z}/2)$$

$$\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2$$

where $\mathbb{Z}_2$ denotes the 2-adic integers. It is shown in [20] that such maps ι are trivial when $l < k$ and form a copy of the $\mathbb{Z}_2$ when $l \geq k$. Following [10] and [20] we choose left-bu-module spectrum maps

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

to satisfy $\iota_{k,k} = 1$, $\iota_{k,l} = \iota_{l+1,l}\iota_{l+2,l+1} \ldots \iota_{k,k-1}$ for all $k - l \geq 2$ and each $\iota_{t+1,t}$ is a $\mathbb{Z}_2$-module generator of the group of such left-bu-module maps.

Let $\tilde{z}_{4k} \in \pi_{4k}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2$ denote the element represented by the smash product of the unit η of the bu-spectrum with the inclusion of the bottom cell j_k into F_{4k}/F_{4k-1} (see [10] §2.12)

$$S^0 \wedge S^{4k} \xrightarrow{\eta \wedge j_k} bu \wedge F_{4k}/F_{4k-1}$$

and let $v_{2i-1} \in \pi_{2i-1}(bu \wedge \mathbb{RP}^\infty) \otimes \mathbb{Z}_2 = bu_{2i-1}(\mathbb{RP}^\infty)$ be as in Proposition 2.2. Then we have the exterior product

$$v_{2i-1}\tilde{z}_{4k} \in \pi_{4k+2i-1}(bu \wedge \mathbb{RP}^\infty \wedge (F_{4k}/F_{4k-1}) \otimes \mathbb{Z}_2$$

which is non-zero and is represented by $\tilde{v}_{2i-1}\hat{z}_{4k}$ in the collapsed Adams spectral sequence whose E_2-term is described in §2.5.

The following formula is central to the proof in §3.8 of our main theorem.

Proposition 2.10.

For $l < k$, for some 2-adic unit $\mu_{4k,4l}$,

$$(\iota_{k,l})_*(v_{2i-1}\tilde{z}_{4k}) = \mu_{4k,4l}2^{4k-4l-\alpha(k)+\alpha(l)}v_{2i+4k-4l-1}\tilde{z}_{4l}$$

Proof

Since $\iota_{k,l}$ is a left-b-module map we have $(\iota_{k,l})_*(v_{2i-1}\tilde{z}_{4k}) = v_{2i-1}(\iota_{k,l})_*(\tilde{z}_{4k})$ and, by ([10] Proposition 3.2) $(\iota_{k,l})_*(\tilde{z}_{4k}) = \mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}\tilde{z}_{4l}$ for some 2-adic unit $\mu_{4k,4l}$. The result follows since, by Proposition 2.2,

$$v_{2i-1}\mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}\tilde{z}_{4l} = v_{2i+4k-4l-1}\mu_{4k,4l}2^{4k-4l-\alpha(k)+\alpha(l)}\tilde{z}_{4l}.$$

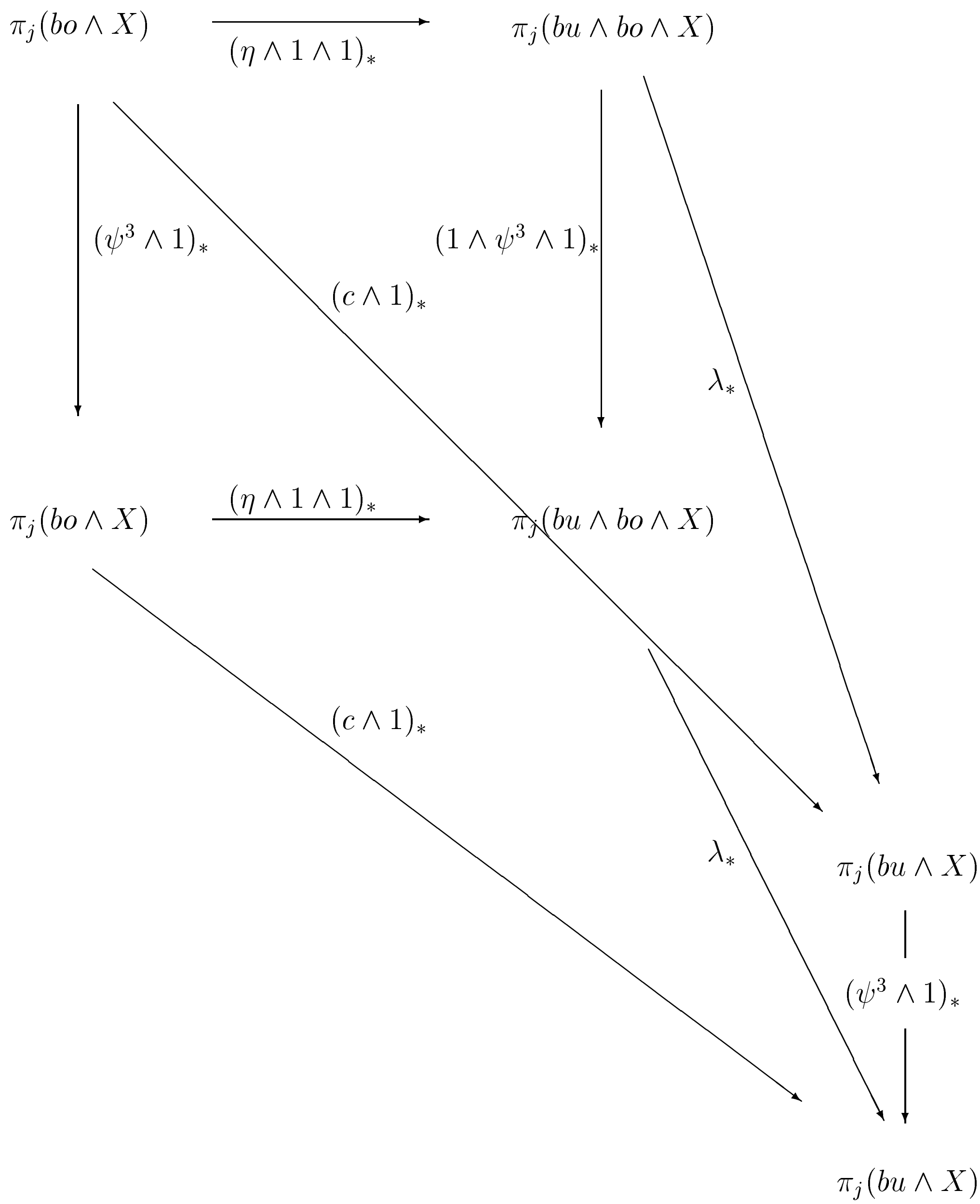
□

3. Applications

3.1. The main diagram

In this section we are going to apply the results of the previous section together with the upper triangular yoga of [20] and [10] to the following *partially* commutative diagram to prove Theorem 1.4. In the diagram η is the unit of bu, c is complexification, μ is the bu-multiplication and ψ^3 is the Adams operation. The homomorphism λ_* is equal to $(\mu \wedge 1)_* \cdot (1 \wedge c \wedge 1)_*$.

The diagram does *not* commute because the right-hand oblique vertical rectangle does not commute. However the upper and lower triangles, the back rectangle and the front left-hand oblique vertical rectangle do commute.



Example 3.2. $\mathbb{RP}^{8m-1}$

Let $\tilde{KU}$ denote reduced 2-local periodic unitary K-theory and let bu, bo denote the associated connective K-theories. In this example, by way of illustration, we shall show how to use the results of [20] and [10] to calculate the map

$$(\eta \wedge 1 \wedge 1)_* : bo_{8m-1}(\mathbb{RP}^{8m-1}) \longrightarrow \pi_{8m-1}(bu \wedge bo \wedge \mathbb{RP}^{8m-1}).$$

From [5] we have $\tilde{KU}^0(\mathbb{RP}^{2t-1}) \cong \tilde{KU}^0(\mathbb{RP}^{2t-2}) \cong \mathbb{Z}/2^{t-1}$ and $\tilde{KU}^1(\mathbb{RP}^{2t-1}) \cong \mathbb{Z}_2$. The KU-theory universal coefficient theorem (proved by the method of [6]) shows that

$$\tilde{KU}_1(\mathbb{RP}^{2t-1}) \cong \mathbb{Z}_2\langle F_2\rangle \oplus \mathbb{Z}/2^{t-1}\langle F_1\rangle,\ \tilde{KU}_0(\mathbb{RP}^{2t-1}) = 0.$$

The Adams operation ψ^3 gives a stable operation on 2-local KU-homology and in the book review [19] a (then) new, one-line proof of the non-existence of maps of Hopf invariant one based on the formula

$$\psi^3(F_2) = ((3^t-1)/2)F_1 + 3^t F_2,\ \psi^3(F_1) = F_1.$$

The canonical map from $bu_{2t-1}(\mathbb{RP}^{2t-1})$ to $\tilde{KU}_{2t-1}(\mathbb{RP}^{2t-1})$ is an isomorphism commuting with ψ^3 so that

$$bu_{2t-1}(\mathbb{RP}^{2t-1}) \cong \mathbb{Z}_2\langle F_2\rangle \oplus \mathbb{Z}/2^{t-1}\langle F_1\rangle$$

with the ψ^3 acting on the generators by the formulae

$$\psi^3(F_2) = F_2 + u_t((3^t-1)/2)F_1, \quad \psi^3(F_1) = F_1,$$

where u_t is an odd integer. When $t = 4m$ the complexification map is an isomorphism giving, in the notation of Proposition 2.2,

$$bo_{8m-1}(\mathbb{RP}^{8m-1}) \cong bu_{8m-1}(\mathbb{RP}^{8m-1}) \cong \mathbb{Z}_2\langle \iota_{8m-1}\rangle \oplus \mathbb{Z}/2^{4m-1}\langle v_{8m-3}u\rangle$$

where the second summand is $bo_{8m-1}(\mathbb{RP}^{8m-2}) \cong bu_{8m-1}(\mathbb{RP}^{8m-2})$ and

$$\psi^3(\iota_{8m-1}) = \iota_{8m-1} + u_{4m}((3^{4m}-1)/2)v_{8m-3}u, \quad \psi^3(v_{8m-3}u) = v_{8m-3}u.$$

Proposition 2.4, the discussion of §2.5 and Theorem 2.6 easily imply (c.f. [2] Lemma 17.12) that the Adams spectral sequence

$$\mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow bu_{t-s}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}))$$

collapses and that for $1 \leq k \leq 2m-1$ and $4m \geq 4k - \alpha(k) + 1$

$$\begin{aligned} bu_{8m-1}(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1})) &\cong bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})) \\ &\cong V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle \end{aligned}$$

where V_k is a finite-dimensional $\mathbb{F}_2$-vector space consisting of elements which are detected in mod 2 cohomology (i.e. in Adams filtration zero) in the spectral sequence. If $8m-1 \leq 8k - 2\alpha(k) + 1$ then the group is a $\mathbb{Z}/2$-vector space of the form

$$bu_{8m-1}(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1})) \cong V_k \oplus \mathbb{Z}/2\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle$$

entirely in Adams filtration zero and if $k \geq 2m$ the group is zero.

By means of the 2-local equivalence $\hat{L}$ of §2.5 we have a direct sum decomposition

$$\hat{L}_* :\ \oplus_{k \geq 0}\ bu_*(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1})) \xrightarrow{\cong} \pi_*(bu \wedge bo \wedge \mathbb{RP}^{8m-1})$$

and by means of this identification we may write the element $(\eta\wedge 1\wedge 1)_*(\iota_{8m-1}) \in \pi_{8m-1}(bu\wedge bo\wedge \mathbb{RP}^{8m-1})$ as a vector $(w_0, w_1, \ldots, w_{2m-1})$ with $w_k \in bu_{8m-1}(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1}))$. In the diagram of §3.1 with $X = \mathbb{RP}^{8m-1}$ and $j = 8m-1$ the map $(c\wedge 1)_*$ is an isomorphism which sends ι_{8m-1} to itself. Hence $w_0 = \iota_{8m-1}$.

According to the main theorem of [20] a left-bu-module self-equivalence of $bu\wedge bo$ inducing the identity on mod 2 homology determines a unique conjugacy class in the upper triangular group with entries in the 2-adic integers. According to the main theorem of [10] the conjugacy class associated to the map $1\wedge\psi^3$ is equal to

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In practical terms this means that, for any positive integer N, we may choose $\hat{L}$ in §2.5 so that, for all $k \leq N$, $1\wedge\psi^3$ maps the wedge summand $bu\wedge(F_{4k}/F_{4k-1})$ to itself by 9^k times the identity map, to $bu\wedge(F_{4k-4}/F_{4k-5})$ by $\iota_{k,k-1}$ and to all other wedge summands $bu\wedge(F_{4t}/F_{4t-1})$ trivially if $t \leq N$. If we choose $\hat{L}$ in this manner, taking N very much larger than $8m-1$, we have

$$(1\wedge\psi^3\wedge 1)_*((\eta\wedge 1\wedge 1)_*(\iota_{8m-1}))$$

$$= (1\wedge\psi^3\wedge 1)_*(\iota_{8m-1}, w_1, w_2, \ldots, w_{2m-1})$$

$$= (\iota_{8m-1} + (\iota_{1,0})_*(w_1), 9w_1 + (\iota_{2,1})_*(w_2), 9^2 w_2 + (\iota_{3,2})_*(w_3), \ldots, 9^{2m-1}w_{2m-1}).$$

On the other hand this element is equal to

$$(\eta\wedge 1\wedge 1)_*((\psi^3\wedge 1)_*(\iota_{8m-1}))$$

$$= (\eta\wedge 1\wedge 1)_*(\iota_{8m-1} + u_{4m}((3^{4m}-1)/2)v_{8m-3}u)$$

$$= (\iota, w_1, w_2, \ldots, w_{2m-1}) + u_{4m}((3^{4m}-1)/2)(v_{8m-3}u, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{2m-1})$$

where $(\eta\wedge 1\wedge 1)_*(v_{8m-3}u) = (v_{8m-3}u, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{2m-1})$.

Equating coordinates we obtain a string of equations

$$(\iota_{1,0})_*(w_1) = u_{4m}((3^{4m}-1)/2)v_{8m-3}u \in bu_{8m-1}(\mathbb{RP}^{8m-1}),$$

$$(9-1)w_1 + (\iota_{2,1})_*(w_2)$$

$$= u_{4m}((3^{4m}-1)/2)\tilde{w}_1 \in bu_{8m-1}(bu \wedge \mathbb{RP}^{8m-2} \wedge (F_4/F_3)),$$

$$(9^2-1)w_2 + (\iota_{3,2})_*(w_3)$$

$$= u_{4m}((3^{4m}-1)/2)\tilde{w}_2 \in bu_{8m-1}(bu \wedge \mathbb{RP}^{8m-2} \wedge (F_8/F_7)),$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$(9^k-1)w_k + (\iota_{k+1,k})_*(w_{k+1})$$

$$= u_{4m}((3^{4m}-1)/2)\tilde{w}_k \in bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})),$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots .$$

There is a relation between w_i and $\tilde{w}_i$ of the form $2w_i = r_{8m-1}\tilde{w}_i$ for all i where r_{8m-1} is an odd integer. For we have a cofibration

$$S^{8m-1} \xrightarrow{\pi} \mathbb{RP}^{8m-1} \xrightarrow{\phi} \mathbb{RP}^{8m}$$

in which π is the canonical projection. Also $\phi_*(\iota_{8m-1})$ generates $bu_{8m-1}(\mathbb{RP}^{8m})$ so, by Proposition 2.2, $\phi_*(2\iota_{8m-1} - r_{8m-1}v_{8m-3}u) = 0$ for some odd integer r_{8m-1}. Therefore $2\iota_{8m-1} - r_{8m-1}v_{8m-3}u$ originates in $bu_{8m-1}(S^{8m-1})$ and

$$(1 \wedge \eta \wedge 1)_*(2\iota_{8m-1}) = (2\iota_{8m-1} - r_{8m-1}v_{8m-3}u, 0, 0, 0, \ldots).$$

If we write u'_{4m} for the 2-adic unit $u_{4m}r_{8m-1}^{-1}$ our string of equations simplifies to

$$(\iota_{1,0})_*(w_1) = u_{4m}((3^{4m}-1)/2)v_{8m-3}u \in bu_{8m-1}(\mathbb{RP}^{8m-1}),$$

$$(9-1)w_1 + (\iota_{2,1})_*(w_2) = 2u'_{4m}((3^{4m}-1)/2)w_1,$$

$$(9^2-1)w_2 + (\iota_{3,2})_*(w_3) = 2u'_{4m}((3^{4m}-1)/2)w_2,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$(9^k-1)w_k + (\iota_{k+1,k})_*(w_{k+1}) = 2u'_{4m}((3^{4m}-1)/2)w_k,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots .$$

We conclude the discussion of this example with the following result concerning the homomorphism $(\iota_{k,k-1})_*$

$$bu_{8m-1}(\mathbb{RP}^{8m-2}\wedge(F_{4k}/F_{4k-1}))\otimes\mathbb{Z}_2 \longrightarrow bu_{8m-1}(\mathbb{RP}^{8m-2}\wedge(F_{4k-4}/F_{4k-5}))\otimes\mathbb{Z}_2.$$

Proposition 3.3.

In the notation of Proposition 2.10 and Example 3.2

(i) $(\iota_{1,0})_*(v_{8m-5}\tilde{z}_4) = \mu_{4,0}2^2 v_{8m-3}u$,

(ii) If $2 \leq k \leq 2m-1$ *and* $4m \geq 4k - \alpha(k) + 1$ *then* $(\iota_{k,k-1})_*$

$$V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle \longrightarrow V_{k-1}\oplus\mathbb{Z}/2^{4m-4k+4+\alpha(k-1)}\langle v_{8m-4k+3}\tilde{z}_{4k-4}\rangle$$

satisfies $(\iota_{k,k-1})_*(v_{8m-4k-1}\tilde{z}_{4k}) = \mu_{4k,4k-4}2^{4-\alpha(k)+\alpha(k-1)}v_{8m-4k+3}\tilde{z}_{4k-4}$ *where* $\mu_{4k,4k-4}$ *is a 2-adic unit.*

In particular, $(\iota_{k,k-1})_*$ *is injective on* $\mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle$ *in cases (i) and (ii).*

Proof

These formulae follow from those of Proposition 2.10, concerning $\mathbb{RP}^\infty$ together with the injectivity of the map from $bu_{8m-1}(\mathbb{RP}^{8m-2}\wedge(F_{4k}/F_{4k-1}))$ to $bu_{8m-1}(\mathbb{RP}^\infty\wedge(F_{4k}/F_{4k-1}))$, which follows from the Adams spectral sequence via Proposition 2.4 and §2.5. The formulae make sense because $\alpha(k-1) = \alpha(k) - 1 + \nu_2(k)$. □

Example 3.4. The maps Θ_{2j}

In this example we shall study stable homotopy classes of maps of the form $\Theta_{2j} : S^{2j} \longrightarrow \mathbb{RP}^{2j}$ with mapping cone $C(\Theta_{2j})$ such that, on 2-local connective K-theory

$$bu_{2j+1}(C(\Theta_{2j})) \cong \mathbb{Z}_2\langle\iota_{2j+1}\rangle \oplus \mathbb{Z}/2^j\langle v_{2j-1}u\rangle$$

for some 2-adic unit u_j

$$(\psi^3\wedge 1)_*(\iota_{2j+1}) = \iota_{2j+1} + u_j((3^j-1)/4)v_{2j-1}u.$$

In other words, the ψ^3 e-invariant (see [1]) of $C(\Theta_{2j})$ is half that of $\mathbb{RP}^{2j+1}$ in Example 3.2, which is the mapping cone of θ in the canonical cofibre sequence

$$\mathbb{RP}^{2j} \longrightarrow \mathbb{RP}^{2j+1} \longrightarrow S^{2j+1} \stackrel{\theta}{\longrightarrow} \Sigma\mathbb{RP}^{2j}.$$

For simplicity we shall restrict ourselves to the case when $j = 4m-1$. In this case there are isomorphisms of 2-local K-groups

$$bo_{8m-1}(C(\Theta_{8m-2})) \cong bu_{8m-1}(C(\Theta_{8m-2})) \cong bu_{8m-1}(\mathbb{RP}^{8m-1})$$

and for each $k \geq 1$

$$bu_{8m-1}(C(\Theta_{8m-2})\wedge(F_{4k}/F_{4k-1})) \cong bu_{8m-1}(\mathbb{RP}^{8m-2}\wedge(F_{4k}/F_{4k-1})).$$

Furthermore if $(\eta \wedge 1 \wedge 1)_*(\iota_{8m-1}) = (\iota_{8m-1}, w'_1, w'_2, \ldots , w'_{2m-1})$ the e-invariant condition yields, as in Example 3.2, a string of equations

$$(\iota_{1,0})_*(w'_1) = u''_{4m}((3^{4m} - 1)/4)v_{8m-3}u \in bu_{8m-1}(\mathbb{RP}^{8m-1}),$$

$$(9 - 1)w'_1 + (\iota_{2,1})_*(w'_2)$$

$$= u''_{4m}((3^{4m} - 1)/4)\tilde{w}_1 \in bu_{8m-1}(bu \wedge \mathbb{RP}^{8m-2} \wedge (F_4/F_3)),$$

$$(9^2 - 1)w'_2 + (\iota_{3,2})_*(w'_3)$$

$$= u''_{4m}((3^{4m} - 1)/4)\tilde{w}_2 \in bu_{8m-1}(bu \wedge \mathbb{RP}^{8m-2} \wedge (F_8/F_7)),$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$(9^k - 1)w'_k + (\iota_{k+1,k})_*(w'_{k+1})$$

$$= u''_{4m}((3^{4m} - 1)/4)\tilde{w}_k \in bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})),$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots.$$

where u''_{4m} is a 2-adic unit and the $\tilde{w}_k$'s are the *same* as in Example 3.2.

Suppose that we have an element

$$w \in V_i \oplus \mathbb{Z}/2^{4m-4i+\alpha(i)}\langle v_{8m-4i-1}\tilde{z}_{4i}\rangle$$

we shall write $w \simeq 2^N$ if $w = (x, 2^N(2t + 1)v_{8m-4i-1}\tilde{z}_{4i})$ for some integers t and $N < 4m - 4i + \alpha(i)$.

Also we observe that in Proposition 3.3(i) or (ii) we may choose V_k so that $(\iota_{k,k-1})_*(V_k) = 0$. Choosing V_k in this manner will simplify our subsequent calculations.

Proposition 3.5.

Let $m = (2p + 1)2^q$ with $p \geq 1$. In the notation of Examples 3.2 and 3.4

(i) For $2 \leq k \leq 2^{q+1}$, $\iota_{k,k-1}(w_k) \simeq 2^{4+q}$ and $\iota_{k,k-1}(w'_k) \simeq 2^{3+q}$.

(ii) $w_{2^{q+1}} \simeq 1$.

(iii) Under these hypotheses Θ_{8m-2} does not exist.

Proof

First we observe that $k \leq 2^{q+1}$ implies that $4m = p2^{q+3}+2^{q+2} \geq 4k-\alpha(k)+1$ so that we may apply Proposition 3.3. Therefore the relations $(\iota_{1,0})_*(w'_1) = u''_{4m}((3^{4m} - 1)/4)v_{8m-3}u$ and $\nu_2((9^{2m} - 1)/4) = 3 + \nu_2(2m) - 2 = 2 + q$ implies $w'_1 \simeq 2^q$ and similarly $w_1 \simeq 2^{1+q}$ (once we observe that the condition $1 + q < 4m - 4 + \alpha(1)$ is fulfilled). The relation

$$(9 - 1)w_1 + (\iota_{2,1})_*(w_2) = u_{4m}((3^{4m} - 1)/2)\tilde{w}_1 \in V_1 \oplus \mathbb{Z}/2^{4m-3}\langle v_{8m-5}\tilde{z}_4\rangle$$

implies that $(\iota_{2,1})_*(w_2) \simeq 2^{4+q}$ and similarly $(\iota_{2,1})_*(w'_2) \simeq 2^{3+q}$, which starts an induction on k.

Suppose $2 \leq k < 2^{q+1}$ that

$$\iota_{k,k-1}(w_k) \simeq 2^{4+q} \in \mathbb{Z}/2^{4m-4k+4+\alpha(k-1)}.$$

Therefore $w_k \in \mathbb{Z}/2^{4m-4k+\alpha(k)}$ satisfies $w_k \simeq 2^{\alpha(k)-\alpha(k-1)+q} = 2^{q+1-\nu_2(k)}$. Then, since $\nu_2(9^k-1) = 3+\nu_2(k)$,

$$(9^k-1)w_k + \iota_{k+1,k}(w_{k+1}) = (9^{2m}-1)w_k$$

implies that $\iota_{k+1,k}(w_{k+1}) \simeq 2^{4+q}$, as required. Similarly $\iota_{k+1,k}(w'_{k+1}) \simeq 2^{3+q}$.

Since $(\iota_{2^{q+1},2^{q+1}-1})_*(v_{8m-2^{q+3}-1}\tilde{z}_{2^{q+3}}) = \mu_{2^{q+3},2^{q+3}-4}2^{q+4}v_{8m-2^{q+3}+3}\tilde{z}_{2^{q+3}-4}$ we see that $w_{2^{q+1}} \simeq 1$ and that $w'_{2^{q+1}}$ cannot exist. □

Proposition 3.6.

Let $m = 2^q$. In the notation of Examples 3.2 and 3.4

(i) For $2 \leq k \leq 2^q$, $(\iota_{k,k-1})_*(w_k) \simeq 2^{4+q}$ and $(\iota_{k,k-1})_*(w'_k) \simeq 2^{3+q}$.

(ii) In $\mathbb{Z}/2 \oplus V_{2^q}$, $w'_{2^q} \simeq 1$.

Proof

This time we observe that $k \leq 2^q$ implies $4m = 2^{q+2} \geq 4k - \alpha(k) + 1$ so that Proposition 3.3 applies and therefore part (i) follows as in Proposition 3.5. For part (ii) we have

$$(\iota_{2^q,2^q-1})_*(v_{8m-2^{q+2}-1}\tilde{z}_{2^{q+2}}) = \mu_{2^{q+2},2^{q+2}-4}2^{3+q}v_{8m-2^{q+2}+3}\tilde{z}_{2^{q+2}-4}$$

and the result follows. □

Remark 3.7. (i) In Proposition 3.6 $(\iota_{2^q,2^q-1})_*$ has the form

$$\mathbb{Z}/2 \oplus V_{2^q} \longrightarrow \mathbb{Z}/2^{q+4} \oplus V_{2^q-1}$$

so that the first component of $(\iota_{2^q,2^q-1})_*(w_{2^q})$ is zero and so is that of w_{2^q}.

(ii) When $m = 2^q$ we have shown that $w'_k \simeq 2^{q-\nu_2(k)}$ for $1 \leq k \leq 2^q$.

3.8. Proof of Theorem 1.4

Let $m = 2^q$ then we have $\iota_{8m-1} \in bo_{8m-1}(C(\Theta_{8m-2}))$ giving a stable homotopy class $\iota_{8m-1} : S^{2^{q+3}-1} \longrightarrow bo \wedge C(\Theta_{8m-2})$. Let $\iota : bu \longrightarrow H\mathbb{Z}/2$ be the canonical cohomology class. Then, if $h_{8m-1} \in H_{2^{q+3}-1}(bo \wedge C(\Theta_{8m-2}); \mathbb{Z}/2)$ is the mod 2 Hurewicz image of ι_{8m-1}, it is represented by either of the compositions

$$S^{2^{q+3}-1} \stackrel{\iota_{8m-1}}{\longrightarrow} bo \wedge C(\Theta_{8m-2}) \stackrel{(\eta\wedge 1\wedge 1)}{\longrightarrow} bu \wedge bo \wedge C(\Theta_{8m-2}) \stackrel{(\iota\wedge 1\wedge 1)}{\longrightarrow} H\mathbb{Z}/2 \wedge bo \wedge C(\Theta_{8m-2})$$

or

$$S^{2^{q+3}-1} \stackrel{\iota_{8m-1}}{\longrightarrow} S^0 \wedge bo \wedge C(\Theta_{8m-2}) \stackrel{(\tilde{\eta}\wedge 1\wedge 1)}{\longrightarrow} H\mathbb{Z}/2 \wedge bo \wedge C(\Theta_{8m-2})$$

where $\tilde{\eta}$ is the unit for $H\mathbb{Z}/2$.

We have an isomorphism of $\mathbb{Z}/2$-vector spaces

$$H_*(bo; \mathbb{Z}/2) \cong \oplus_{k\geq 0} H_*(F_{4k}/F_{4k-1}; \mathbb{Z}/2)$$

and

$$H_j(C(\Theta_{8m-2});\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2\langle \iota_{8m-1}\rangle & \text{if } j = 8m-1, \\ \mathbb{Z}/2\langle v_j\rangle & \text{if } 1 \leq j = 8m-2, \\ 0 & \text{otherwise.} \end{cases}$$

If the e-invariant is correct then we have shown that

$$h_{8m-1} \in H_*(bo;\mathbb{Z}/2) \otimes H_*(C(\Theta_{8m-2});\mathbb{Z}/2)$$

has the form

$$h_{8m-1} = \tilde{z}_0 \otimes \iota_{2^{q+3}-1} + \tilde{z}_{2^{q+2}} \otimes v_{2^{q+2}-1} + \sum_{j=1}^{2^{q+2}-2} x_{8m-1-j} \otimes v_j$$

with $x_{8m-1-j} \in H_*(F_{2^{q+2}+t}/F_{2^{q+2}-1+t};\mathbb{Z}/2)$ with $t \geq 0$.

If $X \in \mathcal{A}^{2^{q+2}}$ is an element of the mod 2 Steenrod algebra of degree 2^{q+2} we write X_* for the dual homomorphism on mod 2 homology, which decreases dimensions by 2^{q+2}. Since $H^*(bo;\mathbb{Z}/2)$ is a cyclic $\mathcal{A}$-module generated by 1 in dimension zero there exists an X such that $X_*(\tilde{z}_{2^{q+2}}) = \tilde{z}_0$. In fact, since the Hurewicz image of the bottom cell in $H_{4k}(F_{4k}/F_{4k-1};\mathbb{Z}/2)$ equals ξ_1^{4k}, unravelling the homology isomorphism between $H_*(bu;\mathbb{Z}/2) \otimes H_*(bo;\mathbb{Z}/2)$ and $H_*(bu;\mathbb{Z}/2) \otimes H_*(\wedge_{l\geq 0}\ F_{4l}/F_{4l-1};\mathbb{Z}/2)$ described in [20] one can show that $Sq_*^{4k}(\tilde{z}_{4k}) = \tilde{z}_0$.

Since h_{8m-1} is stably spherical we have $0 = Sq_*^{2^{q+2}}(h_{8m-1})$ so that

$$\begin{aligned} &\tilde{z}_0 \otimes Sq_*^{2^{q+2}}(\iota_{2^{q+3}-1}) \\ &= \tilde{z}_0 \otimes v_{2^{q+2}-1} + \tilde{z}_{2^{q+2}} \otimes Sq_*^{2^{q+2}}(v_{2^{q+2}-1}) \\ &\quad + \textstyle\sum_{j=1}^{2^{q+2}-1} Sq_*^j(\tilde{z}_{2^{q+2}}) \otimes Sq_*^{2^{q+2}-j}(v_{2^{q+2}-1}) + \sum_{j=1}^{2^{q+2}-2}\ x_{8m-1-j} \otimes Sq_*^{2^{q+2}}(v_j) \\ &\quad + \textstyle\sum_{a=1}^{2^{q+2}-1}\ \sum_{j=1}^{2^{q+2}-2}\ Sq_*^a(x_{8m-1-j}) \otimes Sq_*^{2^{q+2}-a}(v_j) \end{aligned}$$

which implies, comparing coefficients of $\tilde{z}_0$, that $Sq_*^{2^{q+2}}(\iota_{2^{q+3}-1}) = v_{2^{q+2}-1}$ so that ι_{8m-1} is detected by $Sq^{2^{q+2}}$ on its mapping cone, as required. That is:–

$$H_{2^{q+3}-1}(C(\Theta_{2^{q+3}-2});\mathbb{Z}/2) \stackrel{\cong}{\longrightarrow} H_{2^{q+3}-1}(S^{2^{q+3}-1};\mathbb{Z}/2),$$

$$Sq_*^{2^{q+2}} : H_{2^{q+3}-1}(C(\Theta_{2^{q+3}-2});\mathbb{Z}/2) \stackrel{\cong}{\longrightarrow} H_{2^{q+2}-1}(C(\theta_{2^{q+3}-2});\mathbb{Z}/2),$$

$$H_{2^{q+2}-1}(\Sigma\mathbb{RP}^{2^{q+3}-2};\mathbb{Z}/2) \stackrel{\cong}{\longrightarrow} H_{2^{q+2}-1}(C(\Theta_{2^{q+3}-2});\mathbb{Z}/2).$$

Conversely, if this detection by $Sq^{2^{q+2}}$ is correct then $w'_{2^q} \simeq 1$ and therefore the e-invariant is right.

When m is not a power of 2, Proposition 3.5(iii) shows that Θ_{8m-2} with this e-invariant cannot exist. For mod 2 cohomology one can appeal to a theorem of Browder [8] or alternatively well-known formulae for the $\mathcal{A}$-module action on the mod 2 cohomology of real projective space show that Θ_{8m-2} cannot be detected by a primary operation on $H^*(C(\Theta_{8m-2});\mathbb{Z}/2)$. $\square$

Department of Pure Mathematics, University of Sheffield,
Sheffield S3 7RH, England.
v.snaith@sheffield.ac.uk

References

[1] J.F. Adams: J(X) I-IV; Topology 2 (1963) 181-195, Topology 3 (1964) 137-171, Topology 3 (1964) 193-222, Topology 5 (1966) 21-71.

[2] J.F. Adams: *Stable Homotopy and Generalised Homology*; University of Chicago Press (1974).

[3] J.F. Adams: The Kahn-Priddy Theorem; Proc. Camb. Phil. Soc. (1973) 45-55.

[4] J.F. Adams and H.R. Margolis: Modules over the Steenrod algebra; Topology 10 (1971) 271-282.

[5] M.F. Atiyah: *K-Theory*; Benjamin, New York (1967).

[6] M.F. Atiyah: Vector bundles and the Künneth formula; Topology 1 (1962) 245-248.

[7] M.G. Barratt, J.D.S. Jones and M. Mahowald: The Kervaire invariant and the Hopf invariant; Proc. Conf. Algebraic Topology Seattle Lecture Notes in Math. #1286 Springer Verlag (1987).

[8] W. Browder: The Kervaire invariant of framed manifolds and its generalizations; Annals of Math. (2) 90 (1969) 157-186.

[9] E.H. Brown and F.P. Peterson: On the stable decomposition of $\Omega^2 S^{r+2}$; Trans. Amer. Math. Soc. 243 (1978) 287-298.

[10] J. Barker and V.P. Snaith: ψ^3 as an upper triangular matrix; K-theory 36 (2005) 91-114.

[11] J.D.S. Jones and E.G. Rees: Kervaire's invariant for framed manifolds; *Algebraic and geometric topology*, Proc. Symp. Pure Math XXXII (Part 2) Amer. Math. Soc. (1978) 111-117.

[12] D. S. Kahn and S. B. Priddy: The transfer and stable homotopy theory; Proc. Camb. Phil. Soc. 83 (1978) 103-112 (See also: Applications of the transfer to stable homotopy theory; Bull. A.M.Soc. 741 (1972) 981-987).

[13] J. Klippenstein and V.P. Snaith: A conjecture of Barratt-Jones-Mahowald concerning framed manifolds having Kervaire invariant one; Topology (4) 27 (1988) 387-392.

[14] K. Knapp: Im(J)-theory and the Kervaire invariant; Math. Zeit. 226 (1997) 103-125.

[15] S.O. Kochman: *Stable Homotopy Groups of Spheres*; Lecture Notes in Math. #1423, Springer-Verlag (1990).

[16] R.J. Milgram: Symmetries and operations in homotopy theory; Proc. Symp. Pure Math XXII Amer. Math. Soc. (1971) 203-210.

[17] V.P. Snaith: A stable decomposition of $\Omega^n S^n X$; J. London Math. Soc. 2 (1974) 577-583.

[18] V.P. Snaith and J. Tornehave, On $\pi_*(BO)$ and the Arf invariant of framed manifolds; Contemporary Mathematics 12 (1989) 299-313.

[19] V.P. Snaith: Review of *Some applications of topological K-theory* by N. Mahammed, R. Piccinini and U. Suter; Bull.A.M.Soc. v.8 (1) 117-120 (1983).

[20] V.P. Snaith: The upper triangular group and operations in algebraic K-theory; Topology 41 (2002) 1259-1275.

[21] V.P. Snaith: Hurewicz images in BP and the Arf-Kervaire invariant; Glasgow J.Math. 44 (2002) 9-27.

[22] N.E. Steenrod: *Cohomology operations*; Annals Math. Studies #50 (written and revised by D.B.A. Epstein) Princeton Univ. Press (1962).