On the Lusternik-Schnirelmann category of maps

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Abstract

We give conditions when $cat(f \times g) < cat(f) + cat(g)$. We apply our result to show that under suitable conditions for rational maps f, $mcat(f) < cat(f)$ is equivalent to $cat(f) = cat(f \times id_{S^n})$. Many examples with $mcat(f) < cat(f)$ satisfying our conditions are constructed. We also resolve one open case of Ganea's conjecture by constructing a space X such that $cat(X \times S^1) = cat(X) = 2$. In fact for every $Y \not\simeq *$, $cat(X \times Y) \leq cat(Y) + 1 < cat(Y) + cat(X)$. We show that this same X has the property $cat(X) = cat(X \times X) = cl(X \times X) = 2$. Finally we give an example of a CW complex Z such that $cat(Z) = 2$ but every skeleton of Z is of category 1.

1 Introduction

The Lusternik-Schnirelmann category of a space, $cat(X)$, (Definition 2.11) was introduced in the early 1930's [22], [21]. The category of a map, $cat(f)$, (Definition 2.12) was first defined by Fox [10] and seriously studied by Berstein and Ganea [2]. The notion of category of a map is strictly more general since we have that $cat(X) = cat(id_X)$. For an overview of the history of LS category we suggest the two survey articles of James [19] [20].

In this paper we study the relationship between $cat(f)$, $cat(g)$ and $cat(f \times g)$. It is well known that $cat(f \times g) \leq cat(f) + cat(g)$. Although examples where inequality holds have been known for a long time, it was thought that morally equality should hold. In fact no rational examples of inequality were known and actually if f and g are identity maps then Felix, Halperin and Lemaire [8] proved that equality holds. The counterexample of Iwase [17] to the long standing conjecture of Ganea that $cat(X \times S^n) = cat(X) + 1$ changes our perspective. We study the implication of this change on our knowledge of $cat(f \times g)$. We prove:

Theorem 1.1 *3.4 Let us be given a strictly commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \longrightarrow & B/A \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ F_n(X) & \longrightarrow & G_n(X) & \xrightarrow{p_n} & X \end{array}$$

where i is a cofibration and the bottom row is the n-th Ganea fibration for X (Definition 2.10). Assume $cat(f) = n + 1$ and $\Sigma^r h \simeq *$. Then for every map g such that $cat(g) \leq r > 0$, $cat(f \times g) \leq n + r$. In particular $cat(f) = cat(f \times id_{S^r}) = n + 1$.

Our interest in the theorem is due to two applications. The first area of application is in rational homotopy theory. In [28] Hans Scheerer and the author constructed an example of a rational map such that $cat(f \times id_{S^n}) = cat(f) = 2$. The proof was a direct calculation with Sullivan models. Here we show that many such examples can be constructed; for every r we construct maps f such that for every n, $cat(f \times id_{S^n}) = cat(f) = r$. The reason for the occurence of such counterexamples is essentially the same as the reason there are counterexamples to Ganea's conjecture: the instability of certain Hopf invariants. This same phenomenon also gives rise to examples of f such that $mcat(f) < cat(f)$ (see Definition 4.1). In fact we show:

Theorem 1.2 *4.9 Let $\Sigma W \longrightarrow X \xrightarrow{i} Y$ be a cofibration sequence of rational spaces and $f : Y \to Z$ a map of rational spaces . Assume that $cat(f) > cat(fi)$. Assume* dimension$(X) \leq 2cat(fi)($connectivity$Z + 1) - 2$. *Then for any n, $mcat(f) < cat(f)$ if and only if $cat(f \times id_{S^n}) = cat(f)$.*

The second application is to solve a case of Ganea's conjecture which was left open by Iwase (see [17] pg.2).

Theorem 1.3 *5.1 There exists a space X such that $cat(X) = cat(X \times S^1) = 2$*

In fact we show that for the X of the theorem and every $Y \not\simeq *$, $cat(X \times Y) \leq cat(Y) + 1 < cat(Y) + cat(X)$. In other words the *cat* of a nontrivial product with X is always less than what is "expected" using the product formula. This same X has another very interesting property.

Theorem 1.4 *Let X be the space of Theorem 5.1. Then $cat(X) = cat(X \times X) = cl(X \times X) = 2$.*

This is the first example of two p-local spaces whose product has LS category two less than the sum of their categories.

2 Notation and Background

This section contains some general results and definitions. After fixing some notation we prove a proposition (2.9) which describes a cone decomposition of a product in terms of the cone decomposition of the pieces. Next we define the LS category of spaces and maps (Definitions 2.11 and 2.12). Results which tell us if *cat* goes up when attaching a cone are then given (2.19, 2.20). This is determined by Hopf invariants (Definition 2.18) in the space case and by simple obstruction theory in the map case.

Let CG_* denote the category of pointed compactly generated Hausdorf spaces. For definitions and basic properties of CG_* see [32]. All of our spaces will be assumed to be in CG_*. All homotopies will be pointed and $[X, Y]$ denotes pointed homotopy classes of pointed maps. Except where we specifically say we are working in CG_* we will also assume that all our spaces have the homotopy type of pointed CW-complexes [24]. For our purposes the two categories are compatible since the Kellification functor (k in Definition 3.1 of [32]) does not effect CW-complexes and because none of our constructions produce non-Hausdorff spaces from Hausdorff ones. We choose CG_* over the categories of Vogt [38] because it is more familiar to a greater number of homotopy theorists. In CG_* let $\times$ denote the weak product. The only reason we use CG_* instead of all topological spaces is because we want our results to be general enough to handle products of spaces which are not locally compact. We also assume that the category we are working in is our category of spaces. This means that all objects, maps and diagrams will be of spaces unless otherwise indicated.

For a map $f : X \to Y$ in CG_*, we let $Y \cup_f CylX$ denote the reduced mapping cylinder on f. Explicitly

$$Y \cup_f CylX = (Y \cup X \times I)/(* \times I = *, (x, 1) = f(x))_{x \in X}$$

We let $C(f)$ denote the reduced cone on f. Explicitly

$$C(f) = (Y \cup_f CylX)/((x, 0) = *)_{x \in X}$$

We call $X \xrightarrow{f} Y \xrightarrow{g} Z$ together with a homeomorphism $Z \cong C(f)$ compatible with g and the inclusion $Y \to C(f)$ a cofibration sequence. Often we will not explicitly give the homeomorphism. This is consistent with standard practice. (For example it is usually ignored that pushouts are only defined up to isomorphism. This is because the isomorphism is canonical.) We call $F \longrightarrow E \xrightarrow{p} B$ a fibration sequence if p is a fibration and $F = p^{-1}(*)$. Observe that with our definitions fibration sequences and cofibration sequences are not quite dual notions.

For convenience we work localized at a prime or rationally. S^n and D^n refer to localized spheres and disks of dimension n. Let A be an object in any category.

In any category a diagram $\mathcal{C}$ is a functor from some small category into $\mathcal{C}$. This is sometimes refered to as a strictly commuting diagram. A diagram that commutes up to homotopy is a functor into the homotopy category of $\mathcal{C}$.

Lemma 2.1 *Let $f : X \to Y$ be a map. Then the following two conditions are equivalent:*

1) For every W, f induces a surjection $[\Sigma W, f] : [\Sigma W, X] \to [\Sigma W, Y]$

2) Ωf has a homotopy section.

Proof: The lemma follows since Ω and Σ are adjoint and preserve homotopies. □

The following lemma will be used a number of times. It's proof is an application of the coaction. (See [34] for example.)

Lemma 2.2 *Let $f : X \to Y$ be a map satisfying the equivalent conditions of the last lemma. Let $g : U \to A$ be any map. Let us also be given a (strictly commutative) solid arrow diagram:*

$$\begin{array}{ccccc} U & \xrightarrow{g} & A & \xrightarrow{i} & C(g) \\ & & {\scriptstyle h}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle h'} \\ & & X & \xrightarrow[f]{} & Y. \end{array}$$

Assume that $hg \simeq *$. Then there exists ϕ making the upper left triangle strictly commute and the bottom right triangle commute up to *homotopy. In particular if there exists a ϕ making the upper triangle strictly commute then there exists one making the upper triangle commute and the bottom triangle commute up to homotopy.*

Proof: The fact that $hg \simeq *$ implies there exists $\phi' : C(g) \to X$ such that $\phi' i = h$. Next we use [34] Proposition 2.48 i) and its notation. Let $\theta \in [\Sigma U, Y]$ be a map such that $\theta f \phi' = h'$. ($\theta f \phi'$ denotes the action of θ on $f\phi'$ via the coation.) Since $[\Sigma U, f]$ is surjective there exists $\theta' \in [\Sigma U, Y]$ such that $f\theta = \theta'$. Then by naturality of the coaction any representative of $\phi = \theta'\phi' \in [C(g), X]$ makes the diagram commute up to homotopy. □

Lemma 2.3 *For any map $f : \Sigma W \to X$ there is a cofibration sequence*

$$\Sigma W \to X \vee \Sigma W \to X \cup_f Cyl\Sigma W.$$

Proof: Looking at $I \times I$ we see that there is a cofibration sequence

$$S^1 \to S^1 \vee S^1 \to S^1 \cup_{id} CylS^1.$$

Since smashing with W preserves cofibration sequences we get a cofibration sequence

$$\Sigma W \to \Sigma W \vee \Sigma W \to \Sigma W \cup_{id} Cyl\Sigma W.$$

The lemma follows by taking a pushout. □

Lemma 2.4 *Let a homotopy commutative diagram*

$$\begin{array}{ccccc} \Sigma W & \xrightarrow{f} & X & \longrightarrow & C(f) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle g} \\ F & \xrightarrow{j} & E & \xrightarrow{p} & B \end{array}$$

be given. Assume that the bottom row is a fibration sequence and that Ωp splits. Then there exists a (strictly commutative) diagram

$$\begin{array}{ccccc} \Sigma W & \xrightarrow{i} & X \cup_f Cyl\Sigma W & \longrightarrow & C(f) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k\cup H} & & \downarrow{\scriptstyle g} \\ F & \xrightarrow{j} & E & \xrightarrow{p} & B \end{array} \qquad (+)$$

where i is the inclusion into the free end of the cylinder and $H : jh \simeq kf$ is a homotopy.

Proof: Start with the diagram (+) with any H. Then the left square stricly commutes but we know nothing about the right square. Since Ωp splits we can use Lemma 2.2 on the cofibration sequence

$$\Sigma W \to X \vee \Sigma W \to X \cup_f Cyl\Sigma W$$

which exists by Lemma 2.3. This gives us a diagram (+) in which the left square strictly commutes and the right square commutes up to a homotopy that fixes $X \vee \Sigma W$. Since p is a fibration and $X \vee \Sigma W \to X \cup_f Cyl\Sigma W$ is a cofibration we can adjust H not changing the ends of the cylinder so that both squares in diagram (+) commute exactly. □

The next three lemmas are preparation for Proposition 2.9.

Lemma 2.5 *Consider the following diagram in any category.*

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & D & \longrightarrow & F \end{array}$$

Assume that the left hand square is a pushout. Then the right hand square is a pushout if and only if the outside rectangle is a pushout.

Proof: Follows directly from the definition of pushout. □

Lemma 2.6 *In CG_* let the following diagram be a pushout.*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

Then for every X

$$\begin{array}{ccc} A \times X & \longrightarrow & B \times X \\ \downarrow & & \downarrow \\ C \times X & \longrightarrow & D \times X \end{array}$$

is also a pushout.

Proof: [32]. □

Definition 2.7 *For $i = 1, 2$, let $f(i) : A(i) \to B(i)$ be maps in CG_*. Then define $(Cf(1) \times Cf(2))^\bullet$ by letting the following diagram be a pushout.*

$$\begin{array}{ccc} B(1) \times B(2) & \longrightarrow & B(1) \times Cf(2) \\ \downarrow & & \downarrow \\ Cf(1) \times B(2) & \longrightarrow & (Cf(1) \times Cf(2))^\bullet \end{array}$$

Notice that when $B(i) = A(i)$ and $f(i) = id$ then we get $(Cf(1) \times Cf(2))^\bullet = A(1) * A(2)$, the join of the $A(i)$.

Lemma 2.8 *Let $A(i) \in CG_*$. Then there exists a homeomorphism ϕ such that*

$$\begin{array}{ccc} A(1) * A(2) & \xrightarrow{j'} & C(A(1) * A(2)) \\ & \searrow^{j} & \downarrow \phi \\ & & CA(1) \times CA(2) \end{array}$$

commutes. Where the two maps j and j' in the diagram are the usual inclusions. ϕ is natural in the $A(i)$. In other words for $i = 1, 2$, given maps $f(i) : A(i) \to B(i)$ in CG_* the following diagram commutes.

$$\begin{array}{ccc} C(A(1) * A(2)) & \xrightarrow{C(f(1)) * C(f(2))} & C(B(1) * B(2)) \\ \downarrow \phi & & \downarrow \phi \\ C(A(1)) \times C(A(2)) & \xrightarrow{C(f(1)) \times C(f(2))} & C(B(1)) \times C(B(2)) \end{array}$$

Proof:

$$C(A(1) * A(2)) = \{(a(1), a(2), s, t) | a(i) \in A(i), s, t \in [0, 1]\}$$

$$\cup \{(a(1), a(2), s', t) | a(i) \in A(i), s', t \in [0, 1]\} / \sim$$

Where $\sim$ is some equivalence relation. In particular $(a(1), a(2), s, t) \sim (a(1), a(2), s', t)$ if $s = s' = 0$. Also

$$CA(1) \times CA(2) = \{(a(1), a(2), t(1), t(2)) | a(i) \in A(i), t(i) \in [0, 1]\} / \sim'$$

Where $\sim'$ is some other equivalence relation. We then define ϕ to be the map induced by

$$(a(1), a(2), s, t) \mapsto (a(1), a(2), t + s(1 - t), t)$$

and

$$(a(1), a(2), s', t) \mapsto (a(1), a(2), t, t + s'(1-t)).$$

It is straightforward to check that ϕ is compatible with $\sim$ and $\sim'$ and is a homeomorphism. The naturality of ϕ is clear from the definition. $\square$

If $B(i) \simeq *$ then the following proposition is well known. Our proposition is stronger than that of Baues [1] since we have homeomorphisms where he has homotopy equivalences.

Proposition 2.9 *For $i = 1, 2$ let $f(i) : A(i) \to B(i)$ be maps in CG_*. Then there is a cofibration sequence*

$$A(1) * A(2) \to (C(f(1)) \times C(f(2)))^\bullet \to C(f(1)) \times C(f(2)).$$

This sequence is natural in both variables. In other words if for $i = 1, 2$ we have diagrams

$$\begin{array}{ccc} A(i) & \xrightarrow{f(i)} & B(i) \\ {\scriptstyle g(i)}\downarrow & & \downarrow \\ A'(i) & \xrightarrow{f'(i)} & B'(i) \end{array}$$

then we get a diagram

$$\begin{array}{ccccc} A(1) * A(2) & \longrightarrow & (C(f(1)) \times C(f(2)))^\bullet & \longrightarrow & C(f(1)) \times C(f(2)) \\ \downarrow & & \downarrow & & \downarrow \\ A'(1) * A'(2) & \longrightarrow & (C(f'(1)) \times C(f'(2)))^\bullet & \longrightarrow & C(f'(1)) \times C(f'(2)) \end{array}$$

where the maps $C(f(i)) \to C(f'(i))$ are the canonical extensions over the cone induced by $g \times id : A(i) \times I \to A'(i) \times I$.

Proof: In the following diagram

$$\begin{array}{ccccc} A(1) \times B(2) & \longrightarrow & B(1) \times B(2) & \longrightarrow & B(1) \times Cf(2) \\ \downarrow & & \downarrow & & \downarrow \\ CA(1) \times B(2) & \longrightarrow & Cf(1) \times B(2) & \longrightarrow & (Cf(1) \times Cf(2))^\bullet \end{array}$$

the left hand square is a pushout by Lemma 2.6 and the right hand square is by defintion. Therefore Lemma 2.5 implies that the outside square is a pushout.

Next consider the diagram

$$\begin{array}{ccccc}
A(1) \times B(2) & \longrightarrow & A(1) \times Cf(2) & \longrightarrow & B(1) \times Cf(2) \\
\downarrow & & \downarrow & & \downarrow \\
CA(1) \times B(2) & \longrightarrow & (CA(1) \times Cf(2))^{\bullet} & \longrightarrow & (Cf(1) \times Cf(2))^{\bullet} \\
 & & \downarrow & & \downarrow \\
 & & CA(1) \times Cf(2) & \longrightarrow & Cf(1) \times Cf(2).
\end{array}$$

The upper left square is a pushout by definition and we have just seen that the upper rectangle is a pushout. Therefore the upper right square is a pushout. Also the right rectangle is a pushout by Lemma 2.6. Therefore the bottom right square is a pushout. Using the same arguement again in the second variable we see that:

$$\begin{array}{ccc}
(CA(1) \times CA(2))^{\bullet} & \longrightarrow & (Cf(1) \times Cf(2))^{\bullet} \\
{\scriptstyle j}\downarrow & & \downarrow \\
CA(1) \times CA(2) & \longrightarrow & Cf(1) \times Cf(2)
\end{array} \qquad (*)$$

is a pushout. But $(CA(1) \times CA(2))^{\bullet} = A(1) * A(2)$ and by Lemma 2.8 $j' = \phi^{-1} j$. So replacing the map j by $j' : A(1) * A(2) \to C(A(1) * A(2))$ in diagram $(*)$ gives the same pushout. This is the first statement of the lemma. Naturality follows from the naturality of Lemma 2.8. $\square$

Observe that we get the pushout $(*)$ in any category where Lemma 2.6 holds. Therefore the proposition will hold in many model categories with monoidal structures.

We define a sequence of spaces using the fibre-cofibre construction of Ganea [11]. In this case the spaces coincide,up to homotopy, with the stages $E_n(\Omega X)$ of Milnor's classifying space construction for ΩX. The spaces are used to define category.

Definition 2.10 *Let X be a 0-connected space. We define fibration sequences.*

$$F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$$

Let $G_0'(X) = *$ and p_0' the inclusion. Let $G_n'(X) \longrightarrow G_n(X) \xrightarrow{p_n} X$ be a (functorial) factorization of p_n' into an acyclic cofibration followed by a fibration. (This is also refered to as turning p_n' into a fibration.)

Let $F_n(X) = Fib(p_n)$ and $G'_{n+1}(X) = C(i_n)$. We get the extension p'_{n+1} by mapping $F_n \times I$ to $*$. $G_n(X)$ is often refered to as the n-th Ganea space and p_n as the n-th Ganea fibration.

Notice that the fact that we are using a functorial factorization (as we get from the standard construction of turning a map into a fibration) means that the above construction is functorial. [24] Theorem 3 and [32] imply that these constructions keep us inside our category of spaces. It is shown in [11] Theorem 1.1 that $F_n(X) \simeq *_n\Omega X$, the n-fold join of ΩX with itself (this has n+1 copies of ΩX in it).

Definition 2.11 *We say a space 0-connected X has category n, $cat(X) = n$, if n is the least integer such that p_n has a section. If there does not exist such an n then we say $cat(X) = \infty$.*

We can also define category for maps [10] [2].

Definition 2.12 *We say a map $f : Y \to X$ of 0-connected spaces has category n, $cat(f) = n$, if n is the least integer such that there exists $g : Y \to G_n(X)$ such that $p_n g = f$. If there does not exist such an n then we say that $cat(X) = \infty$.*

Observe that $cat(id_X) = cat(X)$. Therefore the category of a map is strictly more general than the category of a space. It follows directly from the definitions and the homotopy invariance of the fibre-cofibre construction that $cat(f)$ and $cat(X)$ are homotopy invariant. The following concept was introduced by Scheerer-Tanré [30].

Definition 2.13 *Let $f : E \to X$ be a fibration of 0-connected spaces. Assume there exists maps $r : E \to G_n(X)$ and $s : G_n(X) \to E$ such that $p_n r = f$ and $fs = p_n$. Then we call f an n-LS fibration.*

Lemma 2.14 *Let $n > 0$ and $f : E \to X$ an n-LS fibration. Then Ωf has a section. In particular Ωp_n has a section.*

Proof: For p_n the lemma follows from [11] Proposition 1.5. For a general n-LS fibration it follows from the result for p_n and the definition of n-LS fibration. □

We remark that, as observed below in Definition 2.17, $G_1(X) \simeq \Sigma\Omega X$. Also e and p_1 are compatible with this equivalence. It follows that the splitting of Ωe gives a splitting of Ωp_1 and by composition of Ωp_n. This is another way to prove the result of the lemma.

The following proposition follows directly from the definition.

Proposition 2.15 *[30] Let $f : E \to X$ be an n-LS fibration. Then $cat(X) \le n$ if and only if f has a section.*

At times it can be more convenient to have some n-LS fibration rather than the Ganea fibration. One reason is because the n-LS fibration may be considerably smaller. For example it was shown in [31] that $(S^n)^l_{sn} \to (S^n)^l$ (that is the inclusion of the sn skeleton into $(S^n)^l$) turned into a fibration is an s-LS fibration. The following well known facts about the category of maps are generalizations of the corresponding facts about the category of spaces.

Proposition 2.16 *Let f and g be maps between 0-conected spaces. Then*

1) $cat(f \times g) \le cat(f) + cat(g)$

2) If f and g are composable then $cat(gf) \le \min\{cat(g), cat(f)\}$

3) If f is a homotopy equivalence then $cat(gf) = cat(g)$.

4) Let $h : X \to Y$ be any map and $f : Y \to C(h)$ be the inclusion. Then $cat(g) \le cat(gf) + 1$. Also $cat(C(h)) \le cat(Y) + 1$.

Proof: See [2] for a proof of 2). 1) follows from 2) and the product formula for spaces [10] Theorem 9. 3) is trivial. 4) follows from [2] Proposition 1.7. □

Definition 2.17 *There is a homotopy equivalence $\phi : \Sigma\Omega X = C\Omega X \cup_{\Omega X} C\Omega X \to G_1(X)$ induced by choosing a homotopy $H : i_0 \simeq *$. Since $G_0(X)$ is contractible the homotopy class of ϕ is independent of H. Let $f : \Sigma W \to X$ be any map and $f^a : W \to \Omega X$ denote its adjoint. Then Σf^a is a map $\Sigma W \to \Sigma\Omega X$. We also let Σf^a denote (the homotopy class of) the map $\phi\Sigma f^a$ and all further compositions with the inclusions into $G_n(X)$ for every n.*

We next define a kind of Hopf invariant. Our definition is equivalent to that of Bernstein-Hilton [3].

Definition 2.18 *Let X be a 0-connected space such that $cat(X) \le n \ne 0$ and $r : X \to G_nX$ a section. Let $f \in [\Sigma W, X]$. Define the Hopf invariant of f by $\mathcal{H}_r(f) = rf - \Sigma f^a$. Since $p_n\Sigma f^a \simeq f \simeq p_n r f$ we can, and will, consider $\mathcal{H}_r(f)$ to be in $[\Sigma W, F_n(X)]$. Observe that this homotopy lift is unique since the fibration p_n has a section.*

The next theorem gives a characterization of *cat* in terms of Hopf invariants. It is true both locally and integrally. In the theorem $dim(X)$ refers to the dimension of X. We define $dim(X)$ to be the dimension of the highest nontrivial cohomology class of X.

Theorem 2.19 *[31] Let X be a space that is simply connected. Assume that $dim(X) \leq l > 1$ and $cat(X) = n > 0$. Let $\vee f(i) : \vee_{i \in I} S^l \to X$ be a map. Then $cat(C(\vee f(i))) \leq n$ if and only if there exists a section $r : X \to G_n(X)$ such that for every i, $\mathcal{H}_r(f(i)) = *$.*

We can also characterize when extending over a cone causes the category of the map to go up. Since we are mapping into a fixed fibration the proof is easier then in the absolute case and follows directly from obstruction theory.

Proposition 2.20 *Let $f : W \to Y$ be a map of 0-connected spaces. Let $i : Y \to C(f)$ denote the inclusion. Let $p : E \to X$ be an n-LS fibration and F be the fibre of p. Let $g : C(f) \to X$ be a map. Then $cat(g) \leq n > 0$ if and only there exists a map $h : Y \to E$ such that $gi \simeq ph$ and such that the map $hf : W \to E$ is null homotopic. If $W = \Sigma W'$ then $cat(g) \leq n$ if and only if there exists $h : Y \to E$ such that $gi \simeq ph$ and such that the map $\Sigma W' \to F$ induced by h is null homotopic.*

Proof: Let us assume there exists h as in the proposition such that $hf \simeq *$. Then Lemmas 2.2 and 2.14 imply that there exists a map $\phi : C(f) \to E$ such that $p\phi \simeq g$. Therefore $cat(g) \leq n$. The other direction of the first statement is trivial.

The second part follows since Lemma 2.14 implies that when W is a suspension the induced map is uniquely determined and is inessential if and only if hf is. □

In the statement of the last proposition the condition $gi \simeq ph$ could be equivalently replaced by the condition $gi = ph$. This is because p is a fibration. Observe that the homotopic version allows us to replace $C(f)$ by any homotopy equivalent space. We would replace i by a corresponding map.

To demonstrate the deceptive power of this simple proposition we offer an example. The proposition will also we used for the results of the later sections. For a CW complex X let X_n denote the n-skeleton of X. This is the first example of a CW complex X such that $cat(X_n) \leq 1$ for every n but $cat(X) = 2$. Remember that a phantom map is an essential map that when precomposed with any map from a finite complex becomes trivial.

Theorem 2.21 *Let $f : CP^\infty \to S^3$ be any phantom map. (See [13] or [40] Theorem D for some examples.) Let $\eta : S^3 \to S^2$ denote the Hopf map. Then $cat(C(\eta f)) = 2$ but $cat(C(\eta f|_{CP^n})) = 1$ for every n.*

Proof: That $C(\eta f|_{CP^n}) = 1$ is clear since f being phantom implies that $f|_{CP^n} \simeq *$ and so $C(\eta f|_{CP^n}) \simeq S^2 \vee \Sigma CP^n$. Observe that $cat(C(\eta f)) = 1$ or 2 since it can be represented as a two-cone. Also observe that $\eta f \not\simeq *$ since if it were f would factor through $S^1 = H(\mathbb{Z}, 1)$. This can not happen since $H^1(CP^\infty) = 0$.

Let $g : C(\eta f) \to CP^\infty$ be a map which represents a generator of $H^2(C(\eta f)) = \mathbb{Z}$. Consider the following solid arrow diagram.

$$\begin{array}{ccc} S^2 & \xrightarrow{h} & G_1(CP^\infty) \simeq S^2 \\ \downarrow & \overset{\phi}{\nearrow} & \downarrow p_1 \\ C(\eta f) & \xrightarrow[g]{} & CP^\infty \end{array}$$

We can see by looking at cohomology that the only possible homotopy classes h making the diagram commute are id and $-id$. But then there does not exist ϕ making the diagram commute since h is a homotopy equivalence and $\eta f \not\simeq *$. Therefore by Proposition 2.20 $cat(g) \geq 2$. Therefore $cat(C(\eta f)) \geq 2$. So $cat(C(\eta f)) = 2$. □

Similarly by attaching with phantom maps to $T^n(S^l)$ we could construct CW complexes such that $cat(X_r) \leq n - 1$ for every r but $cat(X) = n$.

3 cat of Products of Maps

This section gives conditions when $cat(f \times g) < cat(f) + cat(g)$. We first prove a a general form of [17] Proposition 5.8. This is used to give conditions when maps have the property that for every g with $cat(g) \geq r$, $cat(f \times g) < cat(f) + cat(g)$ (Theorem 3.3). The next theorem (3.6) shows that spaces have a similar property whenever they are the cone on a map with an unstable Hopf invariant. The theorems will be applied in Sections 4 and 5.

For this section let us be given (strictly commuting) diagrams of 0-connected spaces of the following form for $i = 1, 2$.

$$\begin{array}{ccccc} W(i) & \xrightarrow{k(i)} & Y(i) & \longrightarrow & C(k(i)) \\ \downarrow l'(i) & & \downarrow l(i) & & \downarrow f(i) \\ F(i) & \xrightarrow{j(i)} & E(i) & \xrightarrow{p(i)} & B(i) \end{array}$$

Assume that the top row is a cofibration sequence and the bottom row is a fibration sequence. Let $g(i) : C(k(i)) \to C(j(i))$ and $p'(i) : C(j(i)) \to B(i)$ denote the canonical extensions over the cone of $l(i)$ and $p(i)$ respectively. The former induced by $l'(i) \times id$ and the latter induced by mapping the cone to $*$. We also assume that $f(i) = p'(i)g(i)$. In other words that $f(i)$ is the trivial extension over the cone of $p(i)l(i)$.

The proof of the following lemma uses a method of Iwase [17]. The arguement illustrates the phenomenon which gives rise to examples where $cat(f \times g) < cat(f) + cat(g)$. The same phenomenon is responsible for counterexamples to Ganea's conjecture.

Lemma 3.1 *Assume that* $l'(1) * l'(2) \simeq *$. *Then* $f(1) \times f(2) : C(k(1)) \times C(k(2)) \to B(1) \times B(2)$ *factors up to homotopy through* $(C(j(1)) \times C(j(2)))^\bullet$.

Proof: From Lemma 2.9 we get a solid arrow diagram

$$\begin{array}{ccccc}
W(1) * W(2) & \longrightarrow & (C(k(1) \times C(k(2)))^\bullet & \xrightarrow{\phi} & C(k(1)) \times C(k(2)) \\
\downarrow{\scriptstyle l'(1)*l'(2)} & & \downarrow{\scriptstyle (g(1)\times g(2))^\bullet} & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle g(1)\times g(2)} \\
F(1) * F(2) & \longrightarrow & (C(j(1)) \times C(j(2)))^\bullet & \xrightarrow{\phi'} & C(j(1)) \times C(j(2)).
\end{array}$$

Since $l'(1) * l'(2) \simeq *$ we get a map h such that $h\phi \simeq (g(1) \times g(2))^\bullet$. Since ϕ' splits after looping we can use Lemma 2.2 and assume that $\phi' h \simeq g(1) \times g(2)$. Since we also have a commutative diagram

$$\begin{array}{ccc}
C(k(1)) \times C(k(2)) & & \\
\downarrow{\scriptstyle g(1)\times g(2)} & \searrow{\scriptstyle f(1)\times f(2)} & \\
C(j(1)) \times C(j(2)) & \xrightarrow[p'(1)\times p'(2)]{} & B(1) \times B(2),
\end{array}$$

the lemma follows easily. □

Lemma 3.2 *Assume again that* $l'(1) * l'(2) \simeq *$ *and also that* $cat(E(i)) = n(i)$. *Then* $cat(f(1) \times f(2)) \leq n(1) + n(2) + 1$.

Proof: If $cat(E(i)) = n(i)$ then [35] Section 5 implies that there exists $W(i)$ such that $Cat(E(i) \vee \Sigma W(i) = n(i)$. Let $Z = ((C(j(1)) \vee \Sigma W(1)) \times (C(j(2)) \vee \Sigma W(2)))^\bullet$. It follows that $Cat(Z) \leq n(1) + n(2) + 1$. $f(1) \times f(2)$ factors through $(C(j(1)) \times C(j(2)))^\bullet$ and therefore through Z. So by 2.16 2) and since $cat \leq Cat$ the lemma follows. □

The last lemma in conjunction with Proposition 2.20 could easily be used to construct examples where $cat(f \times g) < cat(f) + cat(g)$. Next we use it to prove a theorem which is designed for the applications of the next two sections.

Theorem 3.3 *Assume $cat(f(1)) = n + 1$ and $E(1) \to B(1)$ is an n-LS fibration for $B(1)$ (for example the nth Ganea fibration for $B(1)$). Assume that $\Sigma^r(l'(1)) \simeq *$. Then $cat(f(1)) = cat(f(1) \times id_{S^r})$. Also for every map g such that $cat(g) \leq r > 0$, $cat(f(1) \times g) \leq n + r$.*

Proof: From the definition of n-LS fibration there is a commutative diagram.

$$\begin{array}{ccc} F(1) & \longrightarrow & F_n(B(1)) \\ \downarrow & & \downarrow \\ E(1) & \longrightarrow & G_n(B(1)) \\ \downarrow & & \downarrow \\ B(1) & \xrightarrow{=} & B(1) \end{array}$$

Therefore we can assume that $E(1) \to B(1)$ is $p_n : G_n(B(1)) \to B(1)$. First we show that for the Ganea fibration $p_r : G_r(X) \to X$ $cat(f(1) \times p_r) \leq n + r$. There exists a commutative diagram

$$\begin{array}{ccccc} F_{r-1}(X) & \longrightarrow & G_{r-1}(X) & \longrightarrow & G'_r(X) \\ \downarrow{=} & & \downarrow{=} & & \downarrow{p_r} \\ F_{r-1}(X) & \longrightarrow & G_{r-1}(X) & \xrightarrow{p_{r-1}} & X \end{array}$$

where the top row is a cofibration sequence. $F_{r-1}(X) \simeq \Sigma^{r-1}W$. Therefore $l'(1) * F_{r-1}(X) \simeq \Sigma^r l'(1) \wedge W' \simeq *$ by assumption. So since $cat(G_n(B(1)) \leq n$ and $cat(G_{r-1}(X)) \leq r - 1$ we can apply Lemma 3.2 to get that $cat(f(1) \times X) \leq n + r$.

Now let g be any map. Since $cat(g) \leq r$ there exists a factorization of g as $g'p_r : Y \to G_r(X) \to X$. But then Proposition 2.16 says that $cat(f(1) \times g) \leq cat(f(1) \times p_r) \leq n + r$.

$F_1(S^r) \simeq \Sigma\Omega S^r \simeq S^r \wedge Z$. So the same arguements show that $cat(f(1)) \leq cat(f(1) \times id_{S^r}) \leq cat(f(1))$. $\square$.

Corollary 3.4 *Let us be given a (strictly commutative) diagram of 0-connected spaces*

$$\begin{array}{ccccc} W(1) & \xrightarrow{k(1)} & Y(1) & \longrightarrow & Y(1)/W(1) \\ \Big\downarrow{\scriptstyle l'(1)} & & \Big\downarrow{\scriptstyle l(1)} & & \Big\downarrow{\scriptstyle f} \\ F(1) & \longrightarrow & E(1) & \xrightarrow[p(1)]{} & X(1) \end{array}$$

where $k(1)$ is a cofibration and the bottom row is an n-LS fibration for $X(1)$. Assume $cat(f) = n+1$ and $\Sigma^r l'(1) \simeq *$. Then for every g such that $cat(g) \leq r > 0$, $cat(f \times g) \leq n+r$. In particular $cat(f) = cat(f \times id_{S^n}) = n+1$.

Proof: When we let $f(1)$ be the trivial extension over the cone then we have the setup of the last theorem. Therefore for every map g such that $cat(g) \leq r > 0$, $cat(f(1) \times g) \leq n+r$. We have a commutative diagram

$$\begin{array}{ccc} C(k(1)) & \xrightarrow{\pi} & B/A \\ \Big\downarrow{\scriptstyle f(1)} & \swarrow{\scriptstyle f} & \\ X(1) & & \end{array}$$

Since $k(1)$ is a cofibration [39] chapter I section 5 implies that π is a homotopy equivalence. Therefore the theorem and Proposition 2.16 3) imply the corollary. □

We also give a more homotopic version of the above corollary. We do not use it but include it since it could be more convenient to apply in some situations.

Theorem 3.5 *Let $n > 0$. Let us be given a homotopy commutative diagram of 0-conected spaces*

$$\begin{array}{ccccc} \Sigma W & \xrightarrow{f} & X & \longrightarrow & C(f) \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle g} \\ F & \xrightarrow[j]{} & E & \xrightarrow[p]{} & B \end{array}$$

such that the bottom row is an n-LS fibration (for example the n-th Ganea fibration). Assume that $\Sigma^r h \simeq *$. Then $cat(g) = cat(g \times S^r)$. Also for every map g' such that $cat(g') \leq r > 0$, $cat(g \times g') \leq n+r$.

Proof: By Lemma 2.14 we can replace the diagram in the theorem by a strictly commuting one as in Lemma 2.4

$$\begin{array}{ccccc}
\Sigma W & \xrightarrow{i} & X \cup_f Cyl\Sigma W & \longrightarrow & C(f) \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k\cup H} & & \downarrow{\scriptstyle g} \\
F & \xrightarrow[j]{} & E & \xrightarrow[p]{} & B.
\end{array}$$

We are now in the situation of Corollary 3.4 and so the theorem follows. □

Next we prove a similar theorem but one which is sometimes more convenient to apply. We use it in Section 5 to construct examples.

Theorem 3.6 *Let $f : \Sigma W \to X$ be a map where X is 0-connected. Assume $cat(X) = n > 0$ and $cat(C(f)) = n+1$. Assume there exists a section of the Ganea fibration $s : X \to G_n(X)$ such that $\Sigma^r \mathcal{H}_s(f) \simeq *$. Then for every g such that $cat(g) = r$, $cat(id_{C(f)} \times g) \leq n + r < cat(C(f)) + cat(g)$.*

Proof: Since $G_n(i)(\Sigma f^a) \simeq *$ the following solid arrow diagram commutes up to homotopy even though adding the dashed arrow may cause commutativity to be lost.

$$\begin{array}{ccccc}
\Sigma W & \xrightarrow{f} & X & \xrightarrow{i} & C(f) \\
\downarrow{\scriptstyle \mathcal{H}_s(f)} & & \downarrow{\scriptstyle s} & & \\
F_n(X) & \dashrightarrow & G_n(X) & & \Big\| {\scriptstyle =} \\
\downarrow{\scriptstyle F_n(i)} & & \downarrow{\scriptstyle G_n(i)} & & \\
F_n(C(f)) & \xrightarrow{i_n} & G_n(C(f)) & \longrightarrow & C(f)
\end{array}$$

The theorem then follows directly from Theorem 3.5. □

4 Applications to Rational Homotopy

In this section we apply the results of the last section to rational homotopy theory. First we define *mcat* and *cat* in the rational context. We prove a result of Scheerer-Stelzer that *mcat* is determined by the existence of a certain CDGA map. We show how *mcat* of a map is determined by obstruction theory (Proposition 4.7). Next we prove Theorems 4.8 and 4.9 which demonstrate a connection between the statements $mcat(f) < cat(f)$ and $cat(f) = cat(f \times id_{S^n})$. This includes an equivalence of the two statements under certain hypotheses. Finally we construct some examples where $mcat(f) < cat(f)$ and $cat(f) = cat(f \times id_{S^n})$ both hold.

We work in the rational homotopy category represented by commutative differential graded algebras, CDGA's. For more information on CDGA's and rational homotopy theory we refer the reader to [14], [33] and [36]. ΛV refers to a CDGA which is free as a graded commutative algebra over some graded rational vector space V. $\Lambda V/\Lambda^{>n}V$ denotes ΛV modulo the ideal generated by all products of length greater than n. For this section all of our CDGA's and spaces will be simply connected and of finite type unless stated otherwise. A space is called rational if $\tilde{H}_*(X,\mathbb{Z})$ is a rational vector space. There is a rationalization functor from spaces to rational spaces. (See [4] for more details).

There are functors $F : CDGA \to CG_*$ and $A : CG_* \to CDGA$ which induce equivalences of rational homotopy category. (See [5]). The composition FA is equivalent to rationalization. (Actually the functors are into and from simplicial sets and not CG_*. We compose those functors with the singular simplices and realization functors to get the F and A above.)

For this section let

$$A \xrightarrow{j} \Lambda X \xrightarrow{p} \Lambda Y$$

be a fibration sequence in CDGA. In other words p is a surjection and $A = \ker p$. Also let

$$\begin{array}{ccc} \Lambda V & \longrightarrow & \Lambda V \otimes \Lambda W \\ & \searrow & \downarrow \pi \\ & & \Lambda V/\Lambda^{>n}V \end{array}$$

be a diagram such that π is a weak equivalence. Finally for this section we let the following be a diagram in CDGA.

$$\begin{array}{ccccc} \Lambda V & \longrightarrow & \Lambda V \otimes \Lambda W & \longrightarrow & \Lambda W \\ \downarrow f & & \downarrow g & & \downarrow h \\ A & \longrightarrow & \Lambda X & \longrightarrow & \Lambda Y. \end{array}$$

Unless otherwise specified the diagram is assumed to be strictly commutative. $\Lambda V \otimes \Lambda W$ has a differential such that $d(V) \subset \Lambda V$ and ΛW has the induced differential on the quotient.

The definition of LS category of CDGA's was made by Felix-Halperin in their pivital paper [7]. The definition of *mcat* is due to Halperin-Lemaire [15].

Definition 4.1 *[7][15] $cat(f) \leq n$ if and only if there exists a CDGA map h making the following diagram commute.*

$$\begin{array}{ccc} \Lambda V & \longrightarrow & \Lambda V \otimes \Lambda W \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle h} \\ & & A \end{array}$$

If no such n exists then $cat(f) = \infty$.

Similarly $mcat(f) \leq n$ if and only if there exists a ΛV module map h making the above diagram commute. If no such n exists then $mcat(f) = \infty$. If f is a map of spaces then $mcat(f)$ means the $mcat(A(f))$.

The equivalence of the algebraic and topological definitions of LS category for rational spaces was also shown in [7].

Theorem 4.2 *[7] $cat(f) = cat(F(f))$.*

We review the algebraic fibrewise Sp^∞ construction of Scheerer-Stelzer [29]. Let $(\Lambda V \otimes \Lambda W, d)$ be considered as a free ΛV module. Let $\overline{\Lambda W}$ denote the kernel of the augmentation $\Lambda W \to \mathbb{Q}$. Consider $\overline{\Lambda W}$ as a graded vector space. Define $M(\Lambda V \otimes \Lambda W)$ to be $\Lambda V \otimes \Lambda(\overline{\Lambda W})$ as an algebra with differential defined by the Leibniz law. Another way to describe the differential is as the unique one such that

$$\Lambda V \to M(\Lambda V \otimes \Lambda W)$$

is a KS extension and

$$\begin{array}{ccc} \Lambda V & \longrightarrow & \Lambda V \otimes \Lambda W \\ & \searrow & \downarrow {\scriptstyle i} \\ & & M(\Lambda V \otimes \Lambda W) \end{array}$$

is a diagram of ΛV modules. Clearly $M(\Lambda V \otimes \Lambda W)$ is a CDGA.

Proposition 4.3 *[29] For every map $f : \Lambda V \otimes \Lambda W \to \Lambda U$ of ΛV modules there exists a unique map $f' : M(\Lambda V \otimes \Lambda W) \to \Lambda U$ of CDGAs such that $f'i = f$.*

Proof: Straightforward. □

Applying the proposition to $id : \Lambda V \otimes \Lambda W \to \Lambda V \otimes \Lambda W$ we get that there exists a unique CDGA map $r : M(\Lambda V \otimes \Lambda W) \to \Lambda V \otimes \Lambda W$ such that $ri = id$. We apply the proposition to prove a result of Scheerer-Stelzer.

Theorem 4.4 *[29] Let $f : \Lambda V \to A$ be a map. Then $mcat(f) \leq n$ if and only if there exists a commutative diagram in CDGA*

$$\begin{array}{ccc} \Lambda V & \longrightarrow & M(\Lambda V \otimes \Lambda W) \\ & \searrow^{f} & \downarrow \\ & & A. \end{array}$$

Proof: Follows directly from Proposition 4.3. □

Next we describe a relationship between these ideas and the ideas of determining category by Hopf invariants. Let us translate a couple of results from the previous section into the language of Sullivan models. The translation of Theorem 3.3 gives us:

Theorem 4.5 *Assume $cat(f) = n + 1$ and $H_*(h)$ is trivial. Then for every r, $cat(f) = cat(F(f) \times id_{S^r})$. Also for every map g such that $cat(g) > 0$, $cat(f \times g) \leq (n + r) < cat(f) + cat(g)$.*

Proof: Apply F to the diagram of the section to get a strictly commuting diagram of spaces.

$$\begin{array}{ccccc} F(\Lambda Y) & \xrightarrow{F(p)} & F(\Lambda X) & \longrightarrow & F(A) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle F(g)} & & \downarrow{\scriptstyle F(h)} \\ F(\Lambda W) & \longrightarrow & F(\Lambda V \otimes \Lambda W) & \longrightarrow & F(\Lambda V) \end{array}$$

Replace $F(\Lambda X)$ by the mapping cylinder of $F(p)$ and $F(A)$ by $C(F(p))$ with the maps being the canonical extensions. Similarly replace $F(\Lambda V \otimes \Lambda W) \to F(\Lambda V)$ by a fibration and $F(\Lambda W)$ by its fibre. The new maps are the canonical liftings. [7] Theorem 4.7 implies that the topological realization of $\Lambda V \to \Lambda V / \Lambda^{>n} V$ is an n-LS fibration. Also $H_*(h)$ is trivial if and only if $\Sigma h \simeq *$ So we can use Theorem 3.3 to prove the theorem. □

An alternative proof of the above theorem would be given by a translation of the proof of Theorem 3.3. The translation of Proposition 2.20 says:

Proposition 4.6 *$cat(f) \leq n > 0$ if and only if there exists a g that makes our diagram commute up to homotopy and such that $pg \simeq *$. If $F(\Lambda Y)$ is a wedge of spheres then $cat(f) \leq n > 0$ if and only if there exists a g such that the induced map $h \simeq *$.*

There is also a version for *mcat*.

Proposition 4.7 *Consider homotopy commutative diagrams of the following form with f fixed.*

$$\begin{array}{ccccc} \Lambda V & \longrightarrow & M(\Lambda V \otimes \Lambda W) & \longrightarrow & \Lambda(\overline{\Lambda W}) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} \\ A & \xrightarrow{j} & \Lambda X & \xrightarrow{p} & \Lambda Y \end{array}$$

Then $mcat(f) \leq n > 0$ if and only if there exists g' making the diagram homotopy commute such that the $pg' \simeq *$. If $F(\Lambda Y)$ is a wedge of spheres then $mcat(f) \leq n > 0$ if and only if there exist a g' such that the induced map $h' \simeq *$.

Proof: Assume $mcat(f) \leq n$. Then by Theorem 4.4 there exists $\phi : M(\Lambda V \otimes \Lambda W) \to A$ such that

$$\begin{array}{ccc} \Lambda V & \longrightarrow & M(\Lambda V \otimes \Lambda W) \\ & {\scriptstyle f} \searrow & \downarrow \\ & & A \end{array}$$

commutes. Then define $g' = j\phi$ and let h' be any extension. Then $pg' = pj\phi = *$ and $h' = *$.

Now assume there exists g' such that $pg' \simeq *$. Then there exists $\phi : M(\Lambda V \otimes \Lambda W) \to A$ such that $j\phi \simeq g'$. Notice that $\Lambda V \to M(\Lambda V \otimes \Lambda W)$ is injective on the dual of homotopy (in other words it models a map that is surjective on homotopy). So ϕ can be adjusted using the action so that

$$\begin{array}{ccc} \Lambda V & \longrightarrow & M(\Lambda V \otimes \Lambda W) \\ & {\scriptstyle f} \searrow & \downarrow{\scriptstyle \phi} \\ & & A \end{array}$$

commutes up to homotopy. (The action exists for fibrations in any model category [27] Chapter I Section 3. To get the diagram to commute up to homotopy using more explicit methods of rational homotopy theory is also possible. A third way to get commutativity is to translate the problem to spaces, use the coaction (Lemma 2.2) and translate back to CDGAs.) ϕ can then be adjusted to make the diagram commute exactly since $\Lambda V \to M(\Lambda V \otimes \Lambda W)$ is a KS extension. Theorem 4.4 then shows $mcat(f) \leq n$.

The sequence $F(\Lambda V) \to F(M(\Lambda V \otimes \Lambda W)) \to F(\Lambda(\overline{\Lambda W}))$ splits after looping. So the statements when $F(\Lambda Y)$ is a wedge of spheres follow since in that case $pg' \simeq *$ if and only if $h' \simeq *$. □

We can also put the last two diagrams together:

$$\begin{array}{ccccc}
\Lambda V & \longrightarrow & M(\Lambda V \otimes \Lambda W) & \longrightarrow & \Lambda(\overline{\Lambda W}) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\
\Lambda V & \longrightarrow & \Lambda V \otimes \Lambda W & \longrightarrow & \Lambda W \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
A & \longrightarrow & \Lambda X & \longrightarrow & \Lambda Y
\end{array} \qquad (+)$$

and get the following:

Theorem 4.8 *Let $f : \Lambda V \to A$ be a map such that $cat(f) > cat(jf) = n > 0$. Assume we have a commutative diagram as throughout the section such that the composition $hr : \Lambda(\overline{\Lambda W}) \to \Lambda W \to \Lambda Y$ is null homotopic. Equivalently we can assume that $\Sigma F(h) \simeq *$. Then the following four statements hold:*
1) $cat(F(f) \times id_{S^r}) = cat(F(f))$ for some $r > 0$,
2) $cat(F(f) \times id_{S^r}) = cat(F(f))$ for all $r > 0$,
3) $cat(f \otimes g) \leq cat(f) + cat(g) - 1$ for all maps g,
4) $cat(f \otimes id_A) \leq cat(f) + cat(A) - 1$ for some CDGA A.
It $F(\Lambda Y)$ is a wedge of spheres then also:
5) $cat(f) > mcat(f)$

Proof: 5) follows directly from Proposition 4.7. 1), 2), and 4) are special cases of 3). Observe that $F(\Lambda(\overline{\Lambda W})) \simeq \Omega^\infty\Sigma^\infty F(\Lambda W)$ and that under this equivalence $E^\infty : F(\Lambda W) \to \Omega^\infty\Sigma^\infty F(\Lambda W)$ is equivalent to $F(r) : F(\Lambda W) \to F(\Lambda(\overline{\Lambda W}))$. Also rationally for any map, $\Sigma g \simeq *$ if and only if $\Sigma^\infty g \simeq *$. Therefore hr being null is equivalent to $\Sigma F(h) \simeq *$. So 3) follows from Theorem 4.5. □

The next theorem says that in a range all the five statements of the last theorem are equivalent. In the theorem $dim(\Lambda X)$ is the dimension of X. This is the dimension of the highest non-trivial homology class of ΛX.

Theorem 4.9 *Let $f : \Lambda V \to A$ be a map such that $cat(f) > cat(jf) = n > 0$. Assume that $dim(\Lambda X) \leq 2n(con(\Lambda V) + 1) - 2$. Also assume that $F(\Lambda Y)$ is a wedge of spheres. Then the following five statements are equivalent:*
1) $cat(F(f) \times S^r) = cat(F(f))$ for some $r > 0$,
2) $cat(F(f) \times S^r) = cat(F(f))$ for all $r > 0$,
3) $cat(f \times g) \leq cat(f) + cat(g) - 1$ for all maps g,
4) $cat(f \times A) \leq cat(f) + cat(A) - 1$ for some CDGA A,
5) $cat(f) > mcat(f)$.

Proof: Clearly 3) implies 1), 2) and 4). Since for every n, $mcat(S^n) = 1$, 1) implies 5) follows directly from the result of Parent [25] that for every f, g, $mcat(f \otimes g) = mcat(f) + mcat(g)$. So we just need to show that 5) implies 3). Assume 5) holds. Then by Proposition 4.7 there exists a diagram:

$$\begin{array}{ccccc} \Lambda V & \longrightarrow & M(\Lambda V \otimes \Lambda W) & \longrightarrow & \Lambda(\overline{\Lambda W}) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} \\ A & \longrightarrow & \Lambda X & \longrightarrow & \Lambda Y \end{array} \qquad (*)$$

such that $h' \simeq *$.

$con(\Lambda W) \geq n(con(\Lambda V) + 1) - 2$. So we see that $r : \Lambda(\overline{\Lambda W}) \to \Lambda W$ and hence $r : M(\Lambda V \otimes \Lambda W) \to \Lambda V \otimes \Lambda W$ induces an isomorphism on H_*. (indecomposables) of the minimal models in dimensions less than $2n(con(\Lambda V) + 1) - 3$. Therefore since $dim(\Lambda X) \leq 2(con(\Lambda V) + 1) - 2$ we can extend $(*)$ to get a diagram of the form $(+)$. We keep f fixed and changed g' by a homotopy that fixes ΛV. Since $F(\Lambda Y)$ is a wedge of spheres the new indced h' is homotopic to the old one. Therefore in the extended diagram $hr = h'$ is null homotopic. Therefore we can apply Theorem 4.8 to get 3). □

We beleive that the five statements of the corollary are equivalent for any map f. In particular we beleive that for any f, $mcat(f) < cat(f)$ if and only if $cat(f) = cat(f \times id_{S^r})$.

Examples: Let $n \geq 2$. Let $T^n(S^l) = \{(x_1, \ldots, x_n) \in (S^l)^n |$ for some i $x_i = *\}$ denote the fat wedge. Let $i : T^n(S^l) \to (S^l)^n$ denote the inclusion. Let F denote the fibre of i. Porter [26] shows that F is a wedge of spheres (In this case the result also follows easily from the cube theorem of Mather [23]. Also rationally it follows by direct calculation.) Let $g' : S^s \to F$ be any Whitehead product of the inclusions of two different spheres into F. Let $g : S^s \to T^n(S^l)$ be the composition of g' into $T^n(S^l)$. $f : C(g) \to (S^l)^n$ denote any extension of i. Then f satisfies all the hypothesis of Theorem 4.8 and Corollary 3.4. In particular for every $r > 0$, $cat(f \times S^r) = cat(f) = n$ and $mcat(f) \leq n-1$. (Remember $mcat(f)$ means $mcat$ of a model of the rationalization of f.)

Proof of Examples: We wish to apply Proposition 2.20 to show that $cat(f) > n-1$. Consider the diagram.

$$\begin{array}{ccccc} S^s & \xrightarrow{k} & T^n(S^l) \cup_g CylS^s & \xrightarrow{\pi} & C(g) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ F & \longrightarrow & \tilde{T}^n(S^l) & \xrightarrow{\tilde{i}} & (S^l)^n \end{array}$$

Let the bottom row be a fibration sequence and the map $\tilde{i}$ is the inclusion $i : T^n(S^l) \to (S^l)^n$ turned into a fibration. Let k denote the inclusion into the free end of the cylinder. Let h be an extension lift of i. This implies that h is a homotopy equivalence since $l \geq 2$ (remember we are assuming that our spaces are simply connected) implies that the connectivity of i is greater than the dimension of $T^n(S^l)$. g' is the lift of hk to the fibre. By [31] Lemma 6.9 $\tilde{i}$ is an (n-1)-LS fibration. Also since $\tilde{i}$ splits after looping the inclusion of F into $\tilde{T}^n(S^l)$ is injective on π_*. So $g \not\simeq *$ and Proposition 2.20 implies that $cat(f) > n-1$. But since $cat(T^n(S^l)) = n-1$, $cat(C(g)) \leq n$. Therefore $cat(f) \leq n$ and so $cat(f) = n$.

$\Sigma g' \simeq *$ since it is a Whitehead product. We can then apply Theorem 3.4 to see that for every r, $cat(f \times id_{S^r}) = cat(f) = n$ and Theorem 4.8 to see that $mcat(f) \leq n-1$. □

Notice that for our example we could have picked g to be any nontrivial homotopy class such that $i(g) \simeq *$ and such that the lift of g to F suspends to a null homotopic map.

5 An application to Ganea's conjecture

We give a counterexample to Ganea's conjecture (Theorem 5.1) for a case left open by [17]. Our example X is also interesting since $cat(X) = cat(X \times X) = cl(X \times X) = 2$ (Corollary 5.4). It is interesting to compare our example to the one of Fernandez [9]. Working at the prime 3 she shows a certain space Z has the property $cl(Z) = cl(Z \times Z) = 2$. However her Z has $cat(Z) = 1$.

For this section fix a prime $p > 2$. Let $\beta \in \pi_{4p-3}(S^3) \otimes \mathbb{Z}_{(p)}$ be a generator. (In fact $\beta = \alpha^2$ but we will not use this.) Let $X = (S^2 \vee S^3) \cup_{[\iota_2,\iota_3]\Sigma\beta} e^{4p-1}$. Let $Y = S^1 \vee S^2 \vee S^3 \cup_\beta e^{4p-2}$. ι_n always denotes the inclusion of a sphere of dimension n into a space.

Theorem 5.1 *$cat(X) = 2$ and for every $Z \not\simeq *$, $cat(X \times Z) < cat(Z) + cat(X)$. In particular $cat(X \times S^1) = cat(X)$.*

Proof: The only facts we use about β are that $\Sigma\beta \not\simeq *$ and $\Sigma^2\beta \simeq *$. The first fact follow since S^3 is an H space. The existence of β and the fact that $\Sigma^2\beta \simeq *$ were proved by Toda [37]. We must verify the hypotheses of Theorem 3.6.

To show $cat(X) = 2$ we use Proposition 2.20. Let $f : X \to S^2 \times S^3$ be any extension of the identity. Assume $cat(f) \leq 1$. Then by Proposition 2.20 there exists a diagram

$$\begin{array}{ccc} S^2 \vee S^3 & \xrightarrow{f'} & G_1(S^2 \times S^3) \\ \downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{f} & S^2 \times S^3 \end{array}$$

such that $f'[\iota_2, \iota_3]\Sigma\beta \simeq *$. $G_1(S^2 \times S^3) \simeq \Sigma\Omega(S^2 \times S^3) \simeq S^2 \vee S^3 \vee S^3 \vee$ higher spheres. (See [39] Chapter VII Section 2 for a proof of the second equivalence.) Since the diagram above commutes we see that f' is injective on π_*. This gives us a contradiction and so $cat(f) > 1$. Therefore $cat(X) > 1$ and so $cat(X) = 2$ by Propositon 2.16 since X can be represented as a two cone.

On the other hand $\Sigma^2\beta \simeq *$ so for every s

$$\Sigma\mathcal{H}_s([\iota_2, \iota_3]\Sigma\beta) = \Sigma\mathcal{H}_s([\iota_2, \iota_3])\Sigma^2\beta \simeq *.$$

So the hypotheses of Theorem 3.6 have been verified. □

One of the ingredients needed to make this example work was an unstable element in the homotopy groups of spheres. Since there are many unstable elements in the homotopy groups of spheres we could have chosen many other examples. We chose our example partially to demonstrate how easy Theorem 2.19 can be to use, even when there are many sections.

We procede to show another interesting property of the space X. We will show $cat(X) = cat(X \times X)$. First we need a preliminary lemma.

Lemma 5.2 *$Y * Y$ is a wedge of spheres.*

Proof:

$$\begin{aligned} Y * Y &\simeq \Sigma Y \wedge Y \\ &\simeq (S^2 \wedge (S^1 \vee S^2 \vee S^3 \cup_\beta e^{4p-2})) \\ &\vee (S^3 \wedge (S^1 \vee S^2 \vee S^3 \cup_\beta e^{4p-2})) \\ &\vee (S^4 \cup_{\Sigma\beta} e^{4p-1} \wedge (S^1 \vee S^2)) \\ &\vee (\Sigma(S^3 \cup_\beta e^{4p-2}) \wedge (S^3 \cup_\beta e^{4p-2})) \end{aligned}$$

Since $\Sigma^2\beta \simeq *$ all the pieces in the wedge decompoistion except $\Sigma(S^3 \cup_\beta e^{4p-2}) \wedge (S^3 \cup_\beta e^{4p-2})$ are esily seen to be wedges of spheres. Again since $\Sigma^2\beta \simeq *$ there is some f such that $\Sigma(S^3 \cup_\beta e^{4p-2}) \wedge (S^3 \cup_\beta e^{4p-2}) \simeq \Sigma(S^6 \vee S^{4p+1} \vee S^{4p+1} \cup_f e^{8p-4})$. Σf must be an element in $\pi_{8p-4}(S^7 \times S^{4p+2} \times S^{4p+2})$. Also $\Sigma^2 f \simeq *$ since $\Sigma^2\beta \simeq *$. So $\Sigma f \simeq *$ since S^7 is an H space (hence Σ induces an injection on π_*) and $\pi_{8p-4}(S^{4p+1})$ is already in the stable range. So $Y * Y$ is a wedge of spheres. $\square$

Theorem 5.3 *There exists a wedge of spheres W, a space $U \simeq X \times X$ and a cofibration sequence.*

$$W \to \Sigma Y \vee \Sigma Y \to U$$

Proof: From Lemma 2.9 we have a cofibration sequence.

$$Y * Y \xrightarrow{f} \Sigma Y \vee \Sigma Y \longrightarrow \Sigma Y \times \Sigma Y$$

Let $p : \Sigma Y \to X$ denote a map that sends S^4 to $[\iota_2, \iota_3]$, is the identity on $S^2 \vee S^3$ and is the canonical extension over the $4p-1$ cell. Clearly $H_*(p)$ is surjective.

Let $r : H_*(X \wedge X \to H_*(\Sigma Y \wedge \Sigma Y)$ be a splitting of $H_*(p \wedge p)$. Let Z be a wedge of spheres and $iZ \to Y * Y$ be a map such that there exists a homotopy equivalence $\phi : \Sigma Z \to X \wedge X$ and such that $H_*(\Sigma i) = rH_*(\phi)$. That there exists such an i follows from Lemma 5.2.

Next consider the following diagram.

$$
\begin{array}{ccccc}
Z & \xrightarrow{fi} & \Sigma Y \vee \Sigma Y & \longrightarrow & C(fi) \\
\big\downarrow{\scriptstyle i} & & \big\downarrow & & \big\downarrow{\scriptstyle g} \\
Y * Y & \xrightarrow{f} & \Sigma Y \vee \Sigma Y & \longrightarrow & \Sigma Y \times \Sigma Y \\
 & & \big\downarrow{\scriptstyle p\vee p} & & \big\downarrow{\scriptstyle p\times p} \\
 & & X \vee X & \longrightarrow & X \times X
\end{array}
$$

where g is the induced map between cofibres. Using the long exact sequence on homology and the fact that $p \wedge p\Sigma i : \Sigma Z \to X \wedge X$ is a homology equivalence we see that $H_*((p \times p)g)$ is surjective. Let $h : S^4 \vee S^4 \to \Sigma Y \vee \Sigma Y$ denote $(\iota_4 - [\iota_2, \iota_3]) \vee (\iota_4 - [\iota_2, \iota_3])$. Then since $(p \vee p)h \simeq *$ we get a diagram

$$
\begin{array}{ccccc}
Z \vee S^4 \vee S^4 & \xrightarrow{fi+h} & \Sigma Y \vee \Sigma Y & \longrightarrow & C(fi+h) \\
 & & \big\downarrow{\scriptstyle p\vee p} & & \big\downarrow{\scriptstyle \phi} \\
 & & X \vee X & \longrightarrow & X \times X
\end{array}
$$

where ϕ is an extension of $(p \times p)g$. ϕ is easily seen to be an H_* isomorphism and therefore a homotopy equivalence since $X \times X$ is a CW complex and all spaces are simply connected. $\square$

Recall that $cl(X)$ denotes the cone length of X. (See [31] Definition 2.9 for a definition.)

Corollary 5.4 $cat(X) = cat(X \times X) = cl(X) = 2$

Proof: $cat(X) = 2$ and so $cl(X) \geq 2$. But we have realized a space $U \simeq X \times X$ as a two cone. Therefore $cl(X) \leq 2$ and so $cl(X) = 2$. $\square$

More generally we beleive for every n there exists a simply connected space Z such that $cat(Z) = cat(Z^n) = n$. Perhaps an easier thing to construct would be an example of a space Z such that $cat(Z) = cat(Z \times (S^r)^{n-1}) = n$. Simpler still would be to construct a space Z with torsion free homology such that $cat(Z) = n$ but, for Z_0 denoting the rationalization of Z, $cat(Z_0) = 1$.

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