RATIONAL MORAVA E-THEORY AND DS^0

NEIL P. STRICKLAND AND PAUL R. TURNER

1. Introduction

The extended-power spectrum DS^0 has two coproducts and two products, which interact in an intricate way. Given an H_∞ ring spectrum E, the resulting algebraic structure on E^*DS^0 gives a framework in which to encode information about power operations. (However, we will not study power operations in this paper).

Fix a prime p and an integer $n > 0$. We shall take E to be a suitable completed and extended version of $E(n)$. To be more precise, we let W be the Witt ring of $\mathbb{F}_{p^n}$, and consider the following graded ring:

$$E^* = W[\![u_1, \dots, u_{n-1}]\!][u, u^{-1}]$$

The generators u_k have degree 0, and u has degree -2. We take $u_0 = p$ and $u_n = 1$ and $u_k = 0$ for $k > n$. There is a map $BP^* \to E^*$ sending v_k to $u^{p^k-1}u_k$. Using this, we define a functor from spectra to E_*-modules by

$$E_*(X) = E_* \otimes_{BP_*} BP_*(X).$$

The BP^*-module E^* is Landweber exact, so this functor is a homology theory, which we shall call Morava E-theory. It is represented by a spectrum which we shall also call E. It is known (by unpublished work of Miller and Hopkins) that E is an E_∞ ring spectrum (but we shall not use this fact).

In the present work, we discuss the ring $L(DS^0)$ obtained from $E^0(DS^0)$ by making a certain algebraic extension and inverting p. Let Λ be the group $(\mathbb{Q}_p/\mathbb{Z}_p)^n$, and $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Z}_p^n$ its dual. Write $\mathbb{B}$ for the Burnside semiring of Λ^*, in other words the semiring of isomorphism classes of finite sets with an action of Λ^*. Write $F(\mathbb{B}, L)$ for the set of functions from $\mathbb{B}$ to L. This has two coproducts and two products, as follows:

$$\begin{aligned}
(\psi_* f)(X,Y) &= f(X \sqcup Y) \\
(\psi_\circ f)(X,Y) &= f(X \times Y) \\
(f \times g)(X) &= \sum_{X = Y \sqcup Z} f(Y)f(Z) \\
(f \bullet g)(X) &= f(X)g(X)
\end{aligned}$$

Our central result is to give an isomorphism of $L(DS^0)$ with $F(\mathbb{B}, L)$, and show that this respects all structure in sight.

This result can be seen as an introduction (as well as technical input) to the more delicate integral analysis of $E^0(DS^0)$ and power operations in E-theory as studied by Mike Hopkins, Matthew Ando and the first author. The slogan is that in going from $L(X)$ to $E^0(X)$ one has to replace the discrete group Λ by the formal group associated to $E^0(\mathbb{C}P^\infty)$.

In the final section, we show how the same ideas give information about $L(\coprod_m BGL_m(k))$ (where k is a finite field) and $E^0(\coprod_m BU(m))$.

The motivation for this paper, as well a number of ideas used here, are due to Mike Hopkins.

2. The Extended Power Functor

We shall work in the category of spectra $\mathcal{S} = \mathcal{S}\mathbb{R}^\infty$ defined in [5]. Write h$\mathcal{S}$ for the associated homotopy category, and $\mathcal{W}$ for the category of spectra X such that X is homotopy equivalent to a CW spectrum. Recall from [5] that there is an extended power functor $D_k\colon \mathcal{S} \to \mathcal{S}$ defined by

$$D_k(X) = E\Sigma_k \ltimes_{\Sigma_k} X^{(k)}$$

(where $X^{(k)}$ is the k-fold external smash power). We also write

$$D(X) = \bigvee_{m\geq 0} D_m(X)$$

The basic properties of this functor are mostly stated in [1] and proved in [5]. It is a continuous functor, and it preserves $\mathcal{W}$ (see [5, prop. VI.5.2 and following remarks]).

There are fairly obvious maps

$$D_k(X \wedge Y) \to D_k(X) \wedge D_k(Y)$$

$$\bigvee_{k+l=m} D_k(X) \wedge D_l(Y) \to D_m(X \vee Y).$$

These assemble to give two maps

$$D(X \wedge Y) \to D(X) \wedge D(Y)$$

$$D(X) \wedge D(Y) \to D(X \vee Y).$$

The latter is an isomorphism [1, Theorem II.1.1].

If X is a space, then [1, Corollary I.2.2] states that

$$D_k\Sigma^\infty X = \Sigma^\infty((E\Sigma_k)_+ \wedge_{\Sigma_k} X^{(k)}).$$

Suppose that W is a real vector bundle over X with Thom spectrum X^W. Write V for the Σ_k-equivariant bundle over $E\Sigma_k$ corresponding to the usual representation of Σ_k on $\mathbb{R}^k$. Then by [5, Section IX.5], we see that

$$D_k(X^W) = (E\Sigma_k \times_{\Sigma_k} X^k)^{V\otimes_{\Sigma_k} W^k}.$$

Note that the stable pinch map $\Delta\colon X \to X \vee X$ gives rise to a map

$$D_m(X_+) \xrightarrow{D_m(\Delta)} D_m(X_+ \vee X_+) \simeq \bigvee_{m=k+l} D_k(X_+) \wedge D_l(X_+)$$

In particular, we have a component

$$\Sigma^\infty(E\Sigma_m \times_{\Sigma_m} X^m)_+ \to \Sigma^\infty((E\Sigma_k \times_{\Sigma_k} X^k) \times (E\Sigma_l \times_{\Sigma_l} X^l))_+$$

If we let $\Sigma_{k,l}$ be the evident copy of $\Sigma_k \times \Sigma_l$ in Σ_m, then this can be identified [1, Theorem II.1.5] with the transfer associated to the covering

$$\Sigma_m/\Sigma_{k,l} \to E\Sigma_m \times_{\Sigma_{k,l}} X^m \to E\Sigma_m \times_{\Sigma_m} X^m$$

3. The Category of Finite Sets

In this section, we discuss a different picture of DS^0, as the classifying space of a category. We would like to consider the category of all finite sets and bijective maps, but technical difficulties arise because this is not small. One way out is to consider the full subcategory $\mathcal{C}_0$ consisting of the sets $\underline{n} = \{0, 1, \dots, n-1\}$. Another way, which has some technical advantages, is to consider hereditarily finite sets. We define $V_0 = \emptyset$ and

$$V_{n+1} = V_n \cup \text{ power set of } V_n$$
$$\mathcal{C} = V_\omega = \bigcup_n V_n.$$

Then $\mathcal{C}$ is countable, and is equivalent to the category of all finite sets. It is also closed under products and coproducts, as defined in axiomatic set theory. Of course, $\mathcal{C}_0$ is equivalent to $\mathcal{C}$.

Let $B\mathcal{C}$ be the nerve of $\mathcal{C}$. This is homotopy equivalent to $B\mathcal{C}_0 = \coprod_{k\geq 0} B\Sigma_k$, from which it follows easily that

$$DS^0 = \Sigma^\infty B\mathcal{C}_+$$

We define several functors and maps as follows (the last of them being the diagonal):

$$\begin{array}{ccccc} \sqcup & : & \mathcal{C}^2 & \to & \mathcal{C} & \sigma = B\sqcup \\ \times & : & \mathcal{C}^2 & \to & \mathcal{C} & \mu = B\times \\ \Delta & : & \mathcal{C} & \to & \mathcal{C}^2 & \delta = B\Delta. \end{array}$$

The map σ is weakly equivalent to a finite covering. To make this precise, consider the category

$$\mathcal{C}' = \{(X, Y) \mid Y \subseteq X \in \mathcal{C}\}.$$

We will often write a typical object of $\mathcal{C}'$ as $(Y \subseteq X)$. The functor $\sqcup\colon \mathcal{C}^2 \to \mathcal{C}$ factors as

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\kappa} \mathcal{C}' \xrightarrow{\pi} \mathcal{C}$$

$$\kappa(X, Y) = (Y \subseteq X \sqcup Y) \qquad\qquad \pi(Y \subseteq X) = X.$$

Note that κ is an equivalence and that $B\pi\colon B\mathcal{C}' \to B\mathcal{C}$ is a covering space (with 2^n sheets over the n'th component of $B\mathcal{C}$). In fact, $B\pi$ is equivalent to the coproduct over k and l of the maps $B\Sigma_k \times B\Sigma_l \to B\Sigma_{k+l}$. We therefore get a stable transfer map $(B\pi)^!\colon \Sigma^\infty B\mathcal{C}_+ \to \Sigma^\infty B\mathcal{C}'_+$. Write

$$\theta = (B\sqcup)^! = (B\kappa)^{-1} \circ (B\pi)^!\colon DS^0 = \Sigma^\infty B\mathcal{C}_+ \to \Sigma^\infty B\mathcal{C}^2_+ = DS^0 \wedge DS^0.$$

Some of our maps can also be described in terms of the total extended power functor D. If we identify $DS^0 \wedge DS^0$ with $D(S^0 \vee S^0)$ and write $S^0 \xrightarrow{\Delta} S^0 \vee S^0 \xrightarrow{\nabla} S^0$ for the pinch and fold maps, we have $\sigma = D(\nabla)$ and $\theta = D(\Delta)$. The first of these claims is easy, and the second is essentially theorem II.1.5 of [1]. We also write $\chi = D(-1)\colon DS^0 \to DS^0$. In summary, we have

$$\begin{aligned} \sigma = B(\sqcup) = D(\nabla)\colon &\quad DS^0 \wedge DS^0 \to DS^0 \\ \mu = B(\times)\colon &\quad DS^0 \wedge DS^0 \to DS^0 \\ \theta = B(\sqcup)^! = D(\Delta)\colon &\quad DS^0 \to DS^0 \wedge DS^0 \\ \delta = B(\Delta)\colon &\quad DS^0 \to DS^0 \wedge DS^0. \end{aligned}$$

The maps σ and μ are commutative, associative products. They both have units, given by the maps $S^0 \simeq (B\Sigma_0)_+ \to DS^0$ and $S^0 \simeq (B\Sigma_1)_+ \to DS^0$ respectively. The maps θ and δ are cocommutative, coassociative coproducts. The counit for θ is the map $DS^0 \to S^0$ whose restriction to $(B\Sigma_k)_+$ is null for $k > 0$ and the identity for $k = 0$. The counit for δ is the map $DS^0 \to S^0$ whose restriction to $(B\Sigma_k)_+$ is the obvious projection.

We next need to analyse various identities satisfied the above maps. In this discussion, the additive structure of the stable category will not be relevant. Instead, we will consider certain kinds of ring objects in which the addition is given by a map $E \wedge E \to E$.

Definition 3.1. All the following definitions refer to the homotopy category $h\mathcal{S}$ of spectra.

A *coring spectrum* is a spectrum E equipped with a cocommutative, coassociative, counital coproduct $\delta\colon E \to E \wedge E$. For example, if X is a space then $\Sigma^\infty(X_+)$ is a coring in an obvious way. If E and F are coring spectra, we can make $E \wedge F$ into a coring spectrum, and it is the product of E and F in the category of coring spectra (cf. the case of coalgebras).

A *semiring* is the analogue of a ring in which we do not require additive inverses. A *Hopf semiring spectrum* is a semiring in the category of coring spectra. Equivalently, it is a spectrum E equipped with a commutative, associative, counital coproduct $\delta\colon E \to E \wedge E$ and two commutative, associative, unital products $\sigma, \mu\colon E \wedge E \to E$. We require that σ, μ and the corresponding unit maps are all maps of coring spectra, and also that the following distributivity diagram commutes:

$$\begin{array}{ccccc}
E \wedge (E \wedge E) & & \xrightarrow{1\wedge\sigma} & & E \wedge E \\
\downarrow{\scriptstyle \delta\wedge 1} & & & & \\
E \wedge E \wedge E \wedge E & & & & \downarrow{\scriptstyle \mu} \\
\downarrow{\scriptstyle 1\wedge \mathrm{twist}\wedge 1} & & & & \\
E \wedge E \wedge E \wedge E & \xrightarrow[\mu\wedge\mu]{} & E \wedge E & \xrightarrow[\sigma]{} & E
\end{array}$$

A *Hopf cosemiring spectrum* is the dual thing, with one product σ and two coproducts θ, δ making the dual diagram commute. A *Hopf coring spectrum* is a Hopf cosemiring spectrum E equipped with an antipode map $\chi\colon E \to E$, making the following diagram commute:

$$\begin{array}{ccccc}
S & \xrightarrow{\eta} & E & \xleftarrow{\sigma} & E \wedge E \\
\| & & & & \uparrow{\scriptstyle \chi\wedge 1} \\
S & \xleftarrow[\epsilon]{} & E & \xrightarrow[\theta]{} & E \wedge E
\end{array}$$

Here η and ϵ are the unit and counit for σ and θ respectively. Finally, a *Hopf ring spectrum* is a Hopf semiring spectrum equipped with an antipode making the appropriate diagram commute.

Given enough Künneth isomorphisms, applying a (co)homology theory to a Hopf (co)ring spectrum gives a Hopf ring. If E is a ring spectrum then $\Sigma^\infty(\Omega^\infty E)_+$ is a Hopf ring spectrum. This is essentially the usual source of Hopf rings.

We will want to prove that many diagrams involving DS^0 commute. We have three different techniques for this.

(1) We can apply the functor D to a commutative diagram of maps of finite wedges of zero-spheres, noting that $D(\bigvee_{i=1}^n S^0) = D(S^0)^{\wedge n}$. Note also that a map from the k-fold wedge to the l-fold wedge can be represented by a $k \times l$ matrix over $\mathbb{Z} = [S^0, S^0]$.
(2) We can apply the functor B to a diagram of categories, which commutes up to natural equivalence.
(3) We can use the Mackey property of transfer maps. Recall that a finite covering map $f\colon X \to Y$ gives rise to a stable transfer map $f^!\colon Y_+ \to X_+$. Moreover, a pullback square as shown on the left

(where f and g are finite coverings) gives a commutative square as shown on the right:

$$
\begin{array}{ccc}
V & \longrightarrow & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
W & \longrightarrow & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
V_+ & \longrightarrow & X_+ \\
{\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle g^!} \\
W_+ & \longrightarrow & Y_+
\end{array}
$$

Remark 3.1. We will use *ad hoc* methods to replace certain maps by equivalent maps which are finite coverings. For a more systematic approach, we can define a *quasi-covering* to be a map $f\colon X \to Y$ which behaves on homotopy groups (with any basepoint) as though it were a finite covering. We also require that Y be semi-locally 1-connected, so that honest coverings of Y can be classified in the usual way by subgroups of fundamental groups. It follows that quasi-coverings are precisely maps of the form $p \circ g$, where p is a finite covering and g is a weak equivalence. Moreover, composites and homotopy-pullbacks of quasi-coverings are quasi-coverings. We can define transfers for quasi-coverings by $f^! = \Sigma^\infty g^{-1} \circ p^!$, and then the Mackey property holds for homotopy pullbacks.

To apply this to spaces of the form $B\mathcal{C}$, it is useful to note the following fact. If $\mathcal{A} \xrightarrow{f} \mathcal{C} \xleftarrow{g} \mathcal{B}$ are functors of groupoids, and $\mathcal{D}$ is the category of triples $(A, B, fA \xrightarrow{u} gB)$, then the following square is a homotopy pullback:

$$
\begin{array}{ccc}
B\mathcal{D} & \longrightarrow & B\mathcal{A} \\
\downarrow & & \downarrow{\scriptstyle f} \\
B\mathcal{B} & \xrightarrow[g]{} & B\mathcal{C}
\end{array}
\qquad
\begin{array}{ccc}
(A, B, u) & \longmapsto & A \\
\downarrow & & \\
B & &
\end{array}
$$

To see this, first reduce to the case of groups. For any map $G \to H$ of groups, we can use the fibration $EG \times_G EH \to EH/H$ as a model for the map $BG \to BH$. With this model, the claim can be checked directly.

Theorem 3.1. DS^0 is a Hopf semiring spectrum with coproduct δ and products σ, μ.

Proof. It is clear that $\sqcup$ and $\times$ make $\mathcal{C}$ into a semiring object in the category of small categories and natural equivalence classes of functors. It follows that $B\mathcal{C}$ is a semiring object in the homotopy category of unbased spaces, and thus that $DS^0 = \Sigma^\infty B\mathcal{C}_+$ is a semiring object in the category of coring spectra. It is easy to check that the maps respect the grading in the required manner. □

Theorem 3.2. DS^0 is a Hopf coring spectrum with product σ, coproducts θ, δ, and antipode χ.

Proof. First, we need to prove that θ and δ are maps of ring spectra. The diagram for θ is as follows:

$$
\begin{array}{ccc}
DS^0 \wedge DS^0 & \xrightarrow{D(\nabla)} & DS^0 \\
{\scriptstyle D(\Delta)\wedge D(\Delta)}\downarrow & & \\
DS^0 \wedge DS^0 \wedge DS^0 \wedge DS^0 & & \downarrow{\scriptstyle D(\Delta)} \\
{\scriptstyle 1\times\mathrm{twist}\times 1}\downarrow & & \\
DS^0 \wedge DS^0 \wedge DS^0 \wedge DS^0 & \xrightarrow[D(\nabla)\wedge D(\nabla)]{} & DS^0 \wedge DS^0
\end{array}
$$

This is obtained by applying the functor D to the following commutative diagram of spectra:

$$\begin{array}{ccc}
S^0 \vee S^0 & \xrightarrow{(1\ 1)} & S^0 \\
{\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \\
S^0 \vee S^0 \vee S^0 \vee S^0 & \xrightarrow[\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}]{} & S^0 \vee S^0
\end{array}$$

Again, we leave discussion of units and counits to the reader. The antipode diagram is obtained by applying the functor D to the following visibly commutative diagram:

$$\begin{array}{ccccc}
0 & \xrightarrow{0} & S^0 & \xleftarrow{\nabla} & S^0 \vee S^0 \\
\| & & & & \uparrow {\scriptstyle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} \\
0 & \xleftarrow[0]{} & S^0 & \xrightarrow[\Delta]{} & S^0 \vee S^0
\end{array}$$

We also need to prove that δ is a map of ring spectra if we use σ as a product. It is equivalent to say that σ is a map of coring spectra if we use δ as a coproduct. This was proved as part of the previous theorem. Finally, we need to consider the distributivity diagram. For convenience, we will write $B\mathcal{C}$ instead of $\Sigma^\infty B\mathcal{C}_+$ and so on. We also write $t\colon B\mathcal{C}^2 \to B\mathcal{C}^2$ for the twist map. The diagram is as follows.

$$\begin{array}{ccccccc}
B\mathcal{C} & & & \xrightarrow{\delta} & & & B\mathcal{C}^2 \\
{\scriptstyle \theta} \downarrow & & & & & & \downarrow {\scriptstyle 1\times\theta} \\
B\mathcal{C}^2 & \xrightarrow[\delta\times\delta]{} & B\mathcal{C}^4 & \xrightarrow[1\times t\times 1]{} & B\mathcal{C}^4 & \xrightarrow[\sigma\times 1]{} & B\mathcal{C}^3
\end{array}$$

Recall that $\theta = \sigma^!$. The Mackey property of transfers will tell us that this diagram commutes, provided that we can show that the following diagram is equivalent to a pullback diagram in which the vertical maps are finite coverings.

$$\begin{array}{ccccccc}
B\mathcal{C} & & & \xrightarrow{\delta} & & & B\mathcal{C}^2 \\
{\scriptstyle \sigma} \uparrow & & & & & & \uparrow {\scriptstyle 1\times\sigma} \\
B\mathcal{C}^2 & \xrightarrow[\delta\times\delta]{} & B\mathcal{C}^4 & \xrightarrow[1\times t\times 1]{} & B\mathcal{C}^4 & \xrightarrow[\sigma\times 1]{} & B\mathcal{C}^3
\end{array}$$

The main rectangle of this diagram is obtained by applying B to the following system of functors.

$$\begin{array}{ccccc}
X \sqcup Y & & X \longmapsto (X, X) & & (X, Y \sqcup Z) \\
\uparrow & & & & \uparrow \\
(X, Y) & & (X, Y) \longmapsto (X \sqcup Y, X, Y) & & (X, Y, Z)
\end{array}$$

We can replace $\mathcal{C}^2$ by $\mathcal{C}'$, and $\mathcal{C}^3$ by $\mathcal{C} \times \mathcal{C}'$ to obtain an equivalent diagram of functors.

$$\begin{array}{ccc} X & X \longmapsto (X,X) & (Z,X) \\ \uparrow & & \uparrow \\ (Y \subseteq X) & (Y \subseteq X) \longmapsto (X, Y \subseteq X) & (Z, Y \subseteq X) \end{array}$$

One can check directly that this commutes on the nose, and is a pullback diagram of small categories. It therefore gives a pullback diagram of classifying spaces. The vertical maps are finite coverings, as required. □

4. Rational Morava E-theory

We now want to use the results of Hopkins-Kuhn-Ravenel [2, 3, 4] to describe the structure in the rational Morava E-theory of DS^0. In the introduction, we defined the spectrum E. This comes equipped with a map $BP \to E$, and thus a complex orientation in $\tilde{E}^2\mathbb{C}P^\infty$. We can divide this by u to get an orientation in degree zero, and thus a formal group law F defined over the ring $E^0 = W[\![u_1, \dots, u_{n-1}]\!]$. By the Weierstrass preparation theorem, there is a unique way to write $[p^m]_F(x) = g_m(x)u_m(x)$ with u_m invertible and g_m a monic polynomial of degree p^{nm} which reduces to $x^{p^{nm}}$ mod the maximal ideal. We let D_m be the ring obtained by adjoining to E a full set of roots for g_m. There is a natural map $D_m \to D_{m+1}$, and we write D_∞ for the colimit. We also write $L_m = p^{-1}D_m$ and $L = L_\infty$. The roots of g_m form a group $\Lambda(m) = (\mathbb{Z}/p^m)^n$ under the formal sum operation, and we write $\Lambda = \bigcup_m \Lambda(m) = (\mathbb{Q}_p/\mathbb{Z}_p)^n$. (If we examine the way in which D_m is constructed by successively adjoining roots of irreducible polynomials, we find that there are actually no choices involved, and that an isomorphism $\Lambda(m) = (\mathbb{Z}/p^m)^n$ is built in to the constuction). The dual group is $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathbb{Z}_p^n$. The group $\Gamma = \mathrm{Aut}(\Lambda^*) \simeq GL_n(\mathbb{Z}_p)$ acts on L, and the fixed subring is just $p^{-1}E$.

Now let G be a finite group. Write

$$\mathrm{Rep}(\Lambda^*, G) = \mathrm{Hom}(\Lambda^*, G)/\text{conjugacy}.$$

and

$$L(BG) = L \otimes_E E(BG).$$

Theorem 3.3.4 of [2] gives a character isomorphism

$$\tau \colon L(BG) \simeq F(\mathrm{Rep}(\Lambda^*, G), L)$$

(where the right hand side is just the set of functions from $\mathrm{Rep}(\Lambda^*, G)$ to L). (See also [4, theorem 5.2]). Given $u \in L(BG)$ and $\lambda \in \mathrm{Rep}(\Lambda^*, G)$ we write $\tau(u, \lambda) = \tau(u)(\lambda) \in L$ for the corresponding character value.

The functorality properties of the character isomorphism are as follows.

Proposition 4.1. Given $f \colon G \to H$ and $u \in L(BH)$ we have

$$\tau(f^*u, \lambda) = \tau(u, f \circ \lambda)$$

Write γ_g for the inner automorphism $x \mapsto g^{-1}xg$ of G. Given a subgroup $H \le G$ and $u \in L(BH)$ we have

$$\tau(\mathrm{tr}_H^G(u), \lambda) = \sum_{gH} \tau(u, \gamma_g \circ \lambda)$$

where the sum runs over those cosets gH such that the image of $\gamma_g \circ \lambda \colon \Lambda^* \to G$ actually lies in H.

Proof. The first statement is clear from the construction of τ, and the second is proposition 3.6.1 of [2]. □

More generally, let us say that a groupoid $\mathcal{G}$ is *point-finite* if $\mathrm{Aut}_{\mathcal{G}}(X)$ is finite for all X. If so, we can define $\mathrm{Rep}(\Lambda^*, \mathcal{G})$ to be the set of isomorphism classes of functors $\Lambda^* \to \mathcal{G}$, where Λ^* is regarded as a category with one object in the usual way. It turns out that $L \otimes_E E^0(B\mathcal{G})$ is not a convenient object to study; instead we perform a mild (if somewhat *ad hoc*) completion and define

$$L(B\mathcal{G}) = \prod_{\mathcal{F} \in \pi_0 \mathcal{G}} L \otimes_E E^0(B\mathcal{F}).$$

We next make $L(B\mathcal{G})$ into a topological L-module, and describe some constructions with such modules. We do not wish to emphasise the technical framework presented here; there is a more satisfactory theory for the more fundamental case of $E^0 B\mathcal{G}$.

We give L the discrete topology. We make $LB\mathcal{G}$ into a topological L-module by giving $L \otimes E(B\mathcal{F})$ the discrete topology, and $L(B\mathcal{G})$ the product topology. It is not hard to see that this is contravariantly functorial on the category of point-finite groupoids and natural equivalence classes of functors. We also write

$$L^\vee(B\mathcal{G}) = \mathrm{Hom}_L^{\mathrm{cts}}(LB\mathcal{G}, L) = \bigoplus_{\mathcal{F} \in \pi_0 \mathcal{G}} \mathrm{Hom}_L(L \otimes_E E^0(B\mathcal{F}), L) = \bigoplus_{\mathcal{F} \in \pi_0 \mathcal{G}} \mathrm{Hom}_{E^0}(E^0(B\mathcal{F}), L).$$

It follows from the character isomorphism for finite groups that

$$L(B\mathcal{G}) = F(\mathrm{Rep}(\Lambda^*, \mathcal{G}), L).$$

We now specialise to the case $\mathcal{G} = \mathcal{C}$, so that

$$L(DS^0) = L(B\mathcal{C}) = \prod_{k \geq 0} L(B\Sigma_k),$$

and

$$L^\vee(DS^0) = L^\vee(B\mathcal{C}) = \bigoplus_k \mathrm{Hom}_L(L(B\Sigma_k), L).$$

Note that this is very different from

$$L \otimes_E E_0(B\Sigma_k) = L \otimes_{\mathbb{Q}} H_{2*}(B\Sigma_k; \mathbb{Q}) = L.$$

It can be shown, however, that

$$\mathrm{Hom}_L(L(B\Sigma_k), L) = L \otimes_{E^0} \pi_0 L_{K(n)}(E \wedge B\Sigma_k).$$

Let $\mathbb{B}_m$ be the set of isomorphism classes of Λ^*-sets of order m, and put $\mathbb{B} = \coprod_{m \geq 0} \mathbb{B}_m$. This is a semiring, with addition given by disjoint union of Λ^*-sets, and multiplication by cartesian product. It is easy to see that $\mathbb{B} \simeq \mathrm{Rep}(\Lambda^*, \mathcal{C})$, and similarly that $\mathbb{B}_m \simeq \mathrm{Rep}(\Lambda^*, \Sigma_m)$. It follows that

$$L(B\Sigma_k) = F(\mathbb{B}_k, L)$$
$$L(DS^0) = F(\mathbb{B}, L).$$

Dually, we have

$$L^\vee(DS^0) = L[\mathbb{B}].$$

The right hand side is the semiring ring of $\mathbb{B}$, which is a free module over L with one generator $[X]$ for each isomorphism class of finite Λ^*-sets X. In particular, we write $[m]$ for the generator corresponding to a set of order m with trivial action.

We next want to describe $LB(\mathcal{G} \times \mathcal{H})$ as a functor of $LB\mathcal{G}$ and $LB\mathcal{H}$. We say that a topological L-module M is *linearly complete* if $M \simeq \varprojlim_\alpha M/M_\alpha$, where M_α runs over the open submodules of M. If $\{M_i\}$ is a family of finitely generated free modules with the discrete topology, one can check that $M = \prod_i M_i$ is linearly complete; in particular, this applies when $M = LB\mathcal{G}$. If M and N are linearly complete, we define

$$M \widehat{\otimes}_L N = \varprojlim_{\alpha, \beta} M/M_\alpha \otimes_L N/N_\beta.$$

One can check that this is again linearly complete. Moreover, given two families $\{M_i\}$ and $\{N_j\}$ of finitely generated free modules with the discrete topology, we have

$$\left(\prod_i M_i\right) \widehat{\otimes}_L \left(\prod_j N_j\right) = \prod_{i,j} M_i \otimes N_j.$$

In particular,

$$LB(\mathcal{G} \times \mathcal{H}) = L(B\mathcal{G} \times B\mathcal{H}) = LB\mathcal{G} \widehat{\otimes}_L LB\mathcal{H},$$

so

$$L(DS^0 \wedge DS^0) = F(\mathbb{B} \times \mathbb{B}, L) = L(DS^0) \widehat{\otimes}_L L(DS^0).$$

A similar argument gives

$$L^\vee(DS^0 \wedge DS^0) = L[\mathbb{B} \times \mathbb{B}] = L^\vee(DS^0) \otimes_L L^\vee(DS^0).$$

Because $LB\mathcal{G}$ is functorial for maps of groupoids, we get maps

$$\begin{array}{rll} \psi_*\colon & L(DS^0) \to L(DS^0)\widehat{\otimes}_L L(DS^0) & \text{induced by } \sigma \\ \psi_\circ\colon & L(DS^0) \to L(DS^0)\widehat{\otimes}_L L(DS^0) & \text{induced by } \mu \\ \bullet\colon & L(DS^0)\widehat{\otimes}_L L(DS^0) \to L(DS^0) & \text{induced by } \delta \end{array}$$

Moreover, the transfer map

$$\theta\colon B\Sigma_{k+} \to \bigvee_{k=l+m} B\Sigma_{l+} \wedge B\Sigma_{m+}$$

induces a map

$$\prod_{k=l+m} L(B\Sigma_l) \otimes_L L(B\Sigma_m) \to L(B\Sigma_k)$$

and thus a map

$$\times\colon L(DS^0)\widehat{\otimes}_L L(DS^0) = \prod_k \prod_{k=l+m} L(B\Sigma_l) \otimes_L L(B\Sigma_m) \to \prod_k L(B\Sigma_k).$$

We know from theorem 3.2 that DS^0 is a Hopf coring spectrum. It follows that $L(DS^0)$ becomes a Hopf ring (in the ungraded, completed sense) with products $\times$ and $\bullet$, and coproduct ψ_*. (For the definition of a Hopf ring, see for example [6]; we modify the definition there by dropping both gradings and using $\widehat{\otimes}_L$ instead of $\otimes_L$).

By duality, we get various maps of $L^\vee(DS^0)$:

$$\begin{array}{rll} *\colon & L^\vee(DS^0) \otimes_L L^\vee(DS^0) \to L^\vee(DS^0) & \text{induced by } \sigma \\ \circ\colon & L^\vee(DS^0) \otimes_L L^\vee(DS^0) \to L^\vee(DS^0) & \text{induced by } \mu \\ \psi_\times\colon & L^\vee(DS^0) \to L^\vee(DS^0) \otimes_L L^\vee(DS^0) & \text{induced by } \theta \\ \psi_\bullet\colon & L^\vee(DS^0) \to L^\vee(DS^0) \otimes_L L^\vee(DS^0) & \text{induced by } \delta \end{array}$$

We know by theorem 3.1 that DS^0 is a Hopf semiring spectrum, and by the above that $L^\vee(DS^0 \wedge DS^0) = L^\vee(DS^0) \otimes_L L^\vee(DS^0)$. It follows that we can make $L^\vee(DS^0)$ into a Hopf semiring (in the ungraded, uncompleted sense) using $*, \circ$ and $\psi_\bullet$.

Theorem 4.2. With our identification $L(DS^0) = F(\mathbb{B}, L)$, we have

$$\begin{aligned} (\psi_* f)(X,Y) &= f(X \sqcup Y) \\ (\psi_\circ f)(X,Y) &= f(X \times Y) \\ (f \times g)(X) &= \sum_{X = Y \sqcup Z} f(Y) g(Z) \\ (f \bullet g)(X) &= f(X) g(X) \\ (\chi f)(X) &= (-1)^{|X/\Lambda^*|} f(X) \end{aligned}$$

(In the third formula, the sum runs over Λ^*-equivariant partitions of X).

Proof. The $\bullet$-product is induced by the ordinary diagonal map δ, so it is the ordinary product in $L(B\mathcal{C})$. The character map is a ring homomorphism, which implies that $(f \bullet g)(X) = f(X)g(X)$. Next, let X and Y be Λ^*-sets, of order k and l respectively. Let $\rho_X\colon \Lambda^* \to \mathcal{C}$ be the functor classifying X, and similarly ρ_Y. It is easy to see that

$$\rho_{X\sqcup Y} = (\Lambda^* \xrightarrow{(\rho_X,\rho_Y)} \mathcal{C}^2 \xrightarrow{\sigma} \mathcal{C})$$
$$\rho_{X\times Y} = (\Lambda^* \xrightarrow{(\rho_X,\rho_Y)} \mathcal{C}^2 \xrightarrow{\mu} \mathcal{C})$$

It now follows from the naturality properties of τ that $(\psi_* f)(X,Y) = f(X\sqcup Y)$ and $(\psi_\circ f)(X,Y) = f(X\times Y)$. Finally, we need to show that $(f\times g)(X) = \sum_{X=Y\sqcup Z} f(Y)g(Z)$. We may assume for simplicity that f and g are "homogeneous" of degrees k and l, so that $f(U)=0$ unless $|U|=k$, and similarly for g. It is clear from the definitions that $f\times g$ is homogeneous of degree $k+l$. Suppose that $|X|=k+l$, so by choosing a bijection $X \simeq \{1,\dots,k+l\}$ we obtain a map $\rho_X\colon \Lambda^* \to \Sigma_{k+l}$. The cosets $g(\Sigma_k\times\Sigma_l)$ biject with partitions $X = Y\sqcup Z$ via $Y = \{g(1),\dots,g(k)\}$. This partition is Λ^*-equivariant iff $\gamma_g \circ \rho_X(\Lambda^*) \le \Sigma_k\times\Sigma_l$. Recalling that the $\times$-product comes from the transfer map, and using the formula given above for transferred characters, we see that $(f\times g)(X) = \sum_{X=Y\sqcup Z} f(Y)g(Z)$ as claimed.

Now define

$$\begin{aligned}
[0](X) &= \begin{cases} 1 & \text{if } X=\emptyset \\ 0 & \text{otherwise} \end{cases} \\
[1](X) &= 1 \\
[-1](X) &= (-1)^{|X/\Lambda^*|}
\end{aligned}$$

It is clear from our previous formulae that $[0]$ and $[1]$ are the units for $\times$ and $\bullet$ respectively. We claim that $[-1]\times[1] = [0]$. To see this, write X as a disjoint union of transitive Λ^*-sets, say $X = \coprod_{i\in I} X_i$. Then any decomposition $X = Y\sqcup Z$ has $Y = \coprod_{j\in J} X_j$ for some $J\subseteq I$. It follows that

$$([-1]\times[1])(X) = \sum_{J\subseteq I} (-1)^{|J|}.$$

It is easy to see that this is 1 if $I=\emptyset$ and 0 otherwise. The Hopf ring distributivity law now tells us that

$$\eta\epsilon(f) = [0]\bullet f = \sum([-1]\bullet f')\times([1]\bullet f'') = \sum([-1]\bullet f')\times f''$$

This is the characteristic property of the antipode in a Hopf algebra, proving that $\chi(f) = [-1]\bullet f$. This in turn implies the last formula in the theorem. □

Write $\mathbb{L}$ for the set of lattices (in other words, subgroups of finite index) in Λ^*. Given a lattice M, we write X_M for the Λ^*-set Λ^*/M, and $x_M = [X_M] \in L[\mathbb{B}]$. Any finite Λ^*-set X can be decomposed uniquely as the disjoint union of its orbits under the action of Λ^*, each of which is isomorphic to X_M for a unique lattice M (which is the stabiliser of any point in the orbit). It follows easily that with the product $[X]*[Y] = [X\sqcup Y]$, the ring $L[\mathbb{B}]$ is just the polynomial ring $L[x_M \mid M\in\mathbb{L}]$.

Corollary 4.3. We can identify $L^\vee(DS^0)$ with $L[\mathbb{B}]$, in such a way that

$$\begin{aligned}
[X]*[Y] &= [X\sqcup Y] \\
[X]\circ[Y] &= [X\times Y] \\
\psi_\times[X] &= \sum_{X=Y\sqcup Z} [Y]\otimes[Z] \\
\psi_\bullet[X] &= [X]\otimes[X]
\end{aligned}$$

The units for $*$ and $\circ$ are $[0]$ and $[1]$ respectively.

Proof. This follows from the previous theorem by duality. □

5. Ring Schemes

We first give a general discussion of ring schemes, being deliberately vague about completeness and continuity. We shall be more precise when we discuss specific examples. Let H be a Hopf ring over L, and A an L-algebra. Then $A \otimes_L H$ is a Hopf ring over A. We write

$$R(A) = \{x \in A \otimes_L H \mid \psi(x) = x \otimes x\ ,\ \epsilon(x) = 1\}$$

This is sometimes called the set of grouplike elements in $A \otimes_L H$. It is a semiring, with addition given by the $*$-product and multiplication by the $\circ$-product. It can also be described as

$$R(A) = \mathrm{Hom}_{L-\mathrm{Algebras}}(H^\vee, A)$$

where $H^\vee = \mathrm{Hom}_L(H, L)$. Thus R is a representable functor from L-algebras to semirings, or in other words a semiring scheme over L.

Theorem 5.1. The semiring scheme associated to the Hopf ring $L(DS^0)$ is given by $R(A) = F(\mathbb{L}, A)$ (considered as a ring under pointwise operations).

Proof. To be precise, the functor we consider sends A to the set of grouplike elements in $F(\mathbb{B}, A)$, which is a kind of completed tensor product $F(\mathbb{B}, L)\widehat{\otimes}_L A$. This can also be described as

$$R(A) = \mathrm{Hom}_{L-\mathrm{Alg}}(L^\vee(DS^0), A) = \mathrm{Hom}_{L-\mathrm{Alg}}(L[x_M \mid M \in \mathbb{L}], A)$$

Given an L-algebra map $f\colon L[x_M \mid M \in \mathbb{L}] \to A$, we define a map $f'\colon \mathbb{L} \to A$ by $f'(M) = f(x_M)$. Sending f to f' gives a bijection $R(A) \to F(\mathbb{L}, A)$. Suppose $f, g \in R(A)$. The addition in $\mathrm{Hom}_{L-\mathrm{Alg}}(L^\vee(DS^0), A)$ is induced from $\psi_\times$ which is dual to the $\times$-product in the Hopf ring $L(DS^0)$. We see, then, that the sum of f and g in $R(A)$ is the homomorphism $h\colon L^\vee(DS^0) = L[\mathbb{B}] \to A$ defined by

$$h([X]) = \sum_{X = Y \sqcup Z} f([Y])g([Z])$$

If $X = X_M$ we can only take $Y = \emptyset$ or $Y = X_M$, and $[\emptyset] = 1$. It follows that $h'(M) = f'(M) + g'(M)$, in other words that the addition operation on $R(A) \simeq F(\mathbb{L}, A)$ is the obvious one. Similarly, the product of f and g in $R(A)$ is the homomorphism $k\colon L^\vee(DS^0) = L[\mathbb{B}] \to A$ defined by $k([X]) = f([X])g([X])$. It is thus immediate that k' is the pointwise product $f'g'$, as required. □

Theorem 5.2. The semiring scheme associated to the Hopf semiring $L^\vee(DS^0)$ is just the constant scheme $\underline{\mathbb{B}}$ over L. In other words $R(A) = \mathbb{B}$ provided that A has no nontrivial idempotents.

Proof. To be precise, the functor which we consider sends A to the set of grouplike elements in $A \otimes_L L^\vee(DS^0) = A[\mathbb{B}]$. This can also be described as the set of homomorphisms $f\colon F(\mathbb{B}, L) \to A$ which factor through $F(S, L)$ for some finite subset $S \subset \mathbb{B}$, or as the set of homomorphisms which are continuous when we give L and A the discrete topology, and $F(\mathbb{B}, L)$ the product topology. Consider an element $a = \sum_X a_X[X] \in A[\mathbb{B}]$, so that $a_X = 0$ for almost all $X \in \mathbb{B}$. Then

$$\epsilon(a) = \sum_X a_X$$

$$\psi_\bullet(X) = \sum_X a_X[X] \otimes [X]$$

$$a \otimes a = \sum_{X,Y} a_X a_Y [X] \otimes [Y]$$

It follows that a is grouplike iff the elements a_X are idempotents with $a_X a_Y = 0$ whenever $X \neq Y$, and $\sum_X a_X = 1$. In particular, if A has no nontrivial idempotents then $a_X = 1$ for one X and $a_Y = 0$ for all $Y \neq X$. Thus $a \mapsto X$ gives a natural bijection $R(A) \simeq \mathbb{B}$, as claimed. □

Remark 5.1. The constant functor $A \mapsto \mathbb{B}$ is not a scheme. If we went to the trouble of setting up a good technical framework for schemes, then we would simply define the constant scheme $\underline{\mathbb{B}}$ to be $\mathrm{spec}(F(\mathbb{B}, L))$. The above argument is really a proof of the reasonableness of this definition.

6. Primitives and Indecomposables

Consider the Hopf ring $L(DS^0) = F(\mathbb{B}, L)$ (with coproduct ψ_* and products $\times, \bullet$). The augmentation map ϵ is the counit for ψ_*, which sends $f \in F(\mathbb{B}, L)$ to $f(\emptyset)$. Define $e_X \in F(\mathbb{B}, L)$ by $e_X(Y) = 1$ if $Y \simeq X$ and 0 otherwise. Using these functions as a basis, we see that f is decomposable iff $f(\emptyset) = 0$ and $f(X_M) = 0$ for all lattices M. It follows that we have an identification

$$\mathrm{Ind}(L(DS^0)) = F(\mathbb{L}, L)$$

The indecomposables in any Hopf ring form a ring under the second product. In the present case, this is just pointwise multiplication.

We next consider the space $\mathrm{Prim}(L(DS^0))$ of primitives. As in any Hopf ring, this is a module over the ring of indecomposables. The unit for the first product is just $e_\emptyset$. The primitives in $L(DS^0)$ are therefore the functions $f\colon \mathbb{B} \to L$ such that

$$\psi_*(f) = f \otimes e_\emptyset + e_\emptyset \otimes f$$

or equivalently

$$f(X \sqcup Y) = f(X)e_\emptyset(Y) + e_\emptyset(X)f(Y)$$

or equivalently, $f(X) = 0$ unless X is nonempty and transitive (ie $X = X_M$ for some M). Now define $c \in L(DS^0)$ by $c(X_M) = 1$ for all M and $c(X) = 0$ if X is empty or intransitive. Clearly, $\mathrm{Prim}(L(DS^0))$ is the free module over $\mathrm{Ind}(L(DS^0))$ on one generator c:

$$\mathrm{Prim}(L(DS^0)) = \mathrm{Ind}(L(DS^0))c$$

We next consider the Hopf ring $L^\vee(DS^0) = L[\mathbb{B}] = L[X_M \mid M \in \mathbb{L}]$, with coproduct $\psi_\bullet$ and products $*, \circ$. The augmentation is the counit for $\psi_\bullet$, which sends $[X]$ to 1 for all X, so that $[X] - 1 \in I = \ker(\epsilon)$. Moreover, it is easy to see (using $1 = [\emptyset]$) that

$$[X \sqcup Y] - 1 = ([X] - 1) + ([Y] - 1) \pmod{I^2}$$

$$([X] - 1) \circ ([Y] - 1) = [X \times Y] - 1$$

From this we conclude that $\mathrm{Ind}(L^\vee(DS^0))$ is the free module over L on the generators $X_M - 1$, and also (after making some obvious definitions for tensor products of semirings) that as semirings

$$\mathrm{Ind}(L^\vee(DS^0)) = L[\mathbb{L}] = L \otimes_{\mathbb{N}} \mathbb{B}.$$

On the other hand, because $\psi_\bullet([X]) = [X] \otimes [X]$, we see using the obvious bases that $\mathrm{Prim}(L^\vee(DS^0)) = 0$.

Let Γ^+ be the monoid of injective endomorphisms of Λ^*, and Γ the subgroup of automorphisms. There is an obvious action of Γ^+ on $\mathbb{L}$. It turns out that there is also a natural action of Γ^+ on L. These fit together to give an action of Γ^+ on $\mathrm{Ind}(L^\vee(DS^0)) = L[\mathbb{L}]$. This arises topologically from the H_∞ structure of E. This will be discussed in detail in future work.

Note that although the coproduct we use in $L(DS^0)$ is dual to the $*$-product in $L^\vee(DS^0)$, the primitives in $L(DS^0)$ are not directly dual to the indecomposables in $L^\vee(DS^0)$. The reason is that the definition of primitives involves a unit as well as a coproduct. The unit in $L(DS^0) = F(\mathbb{B}, L)$ is $e_\emptyset = [0]$, but the element of $L(DS^0)$ dual to the augmentation on $L^\vee(DS^0)$ is the constant function $1 = [1]$. The dual of the indecomposables in $L^\vee(DS^0)$ is naturally identified with

$$\mathrm{Prim}'(L(DS^0)) = \{f \in F(\mathbb{B}, L) \mid f(X \sqcup Y) = f(X) + f(Y)\}$$

Note that

$$([1] \times f)(X) = \sum_{Y \leq X} f(Y)$$

One can check easily that the map $f \mapsto [1] \times f$ is an isomorphism $\mathrm{Prim}(L(DS^0)) \simeq \mathrm{Prim}'(L(DS^0))$. If we let $R = \mathrm{spec}(L^\vee(DS^0))$ be the ring scheme corresponding to the Hopf ring $L(DS^0)$, then $\mathrm{Prim}(L(DS^0))$ and $\mathrm{Prim}'(L(DS^0))$ are the tangent spaces of R at 0 and 1. For any commutative algebraic group (such as the additive group of R), there is of course a canonical isomorphism between the tangent spaces at any two points.

7. Generalisations

Instead of $\mathcal{C}$, we can consider the category $\mathcal{V}$ of finite-dimensional vector spaces over a finite field k, so that

$$B\mathcal{V} = \coprod_{m \geq 0} BGL_m(k).$$

We replace the disjoint union by the direct sum and the product by the tensor product. The evident analogues of theorems 3.1 and 3.2 hold, except that we do not have an antipode map. We strongly suspect that an antipode map exists, but we do not have a construction as yet. The proofs for $\mathcal{V}$ are much the same as for $\mathcal{C}$, except that $\mathcal{C}'$ must be replaced by the following category:

$$\mathcal{V}' = \{(U; V, W) \mid V, W \leq U \ , \ V \cap W = 0 \text{ and } V + W = U\}$$

In this case, $\mathrm{Rep}(\Lambda^*, \mathcal{V})$ is the semiring $R_k^+(\Lambda^*)$ of isomorphism classes of finite-dimensional representations of Λ^* over k. Thus

$$LB\mathcal{V} = F(R_k^+(\Lambda^*), L)$$
$$L^\vee B\mathcal{V} = L[R_k^+(\Lambda^*)].$$

Moreover, the structure maps are described by evident analogues of theorem 4.2 and corollary 4.3.

Suppose that k has characteristic not equal to p. Then all finite-dimensional representations of Λ^* over k are completely reducible, and split over $\overline{k}$ as a direct sum of one-dimensional representations. These one-dimensional representations biject with the group C of continuous homomorphisms $\Lambda^* \to \overline{k}^\times$. Using these ideas one can show that $R_k^+(\Lambda^*)$ is a free Abelian monoid with the orbit set $C/\,\mathrm{Gal}(\overline{k}/k)$ as a basis.

As a different generalisation, we can consider the topological category $\mathcal{U}$ of finite-dimensional complex Hilbert spaces, so that

$$B\mathcal{U} = \coprod_{m \geq 0} BU(m).$$

In this case, we need to use a Becker-Gottlieb type transfer. For a version which works when the base is infinite (which we need), see chapter IV of [5]. We again get analogues of theorems 3.1 and 3.2. However, we cannot use generalised character theory to study $E^*B\mathcal{U}$ because the groups involved are not finite. There is a description in terms of divisors on the formal group associated to E, as discussed in [7].

References

1. R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ Ring Spectra and their Applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer–Verlag, 1986.
2. M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. Generalised group characters and complex oriented cohomology theories. To appear.
3. M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. Morava K-theories of classifying spaces and generalized characters for finite groups. In J. Aguadé, M. Castellet, and F. R. Cohen, editors, *Algebraic Topology: Homotopy and Group Cohomology*, volume 1509 of *Lecture Notes in Mathematics*, pages 186–209, New York, 1992. Springer-Verlag.
4. N. J. Kuhn. Character rings in algebraic topology. In S. M. Salamon, B. Steer, and W. A. Sutherland, editors, *Advances in Homotopy*, volume 139 of *London Mathematical Society Lecture Note Series*, pages 111–126, Cambridge, 1989. Cambridge University Press.
5. L. G. Lewis, J. P. May, and M. Steinberger. *Equivariant Stable Homotopy Theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer–Verlag, New York, 1986.
6. D. C. Ravenel and W. S. Wilson. The Hopf ring for complex cobordism. *Journal of Pure and Applied Algebra*, 9:241–280, 1977.
7. N. P. Strickland. Functorial philosophy for formal phenomena. in preparation, 1994.

Department of Mathematics, MIT, Cambridge MA 02139, USA
E-mail address: neil@math.mit.edu

Max-Planck-Institut-für-Mathematik, Gottfried-Claren-Strasse 26, 53225 Bonn, Germany
E-mail address: turner@mpim-bonn.mpg.de