AXIOMATIC STABLE HOMOTOPY — A SURVEY

N. P. STRICKLAND

Abstract. We survey various approaches to axiomatic stable homotopy theory, with examples including derived categories, categories of (possibly equivariant or localized) spectra, and stable categories of modular representations of finite groups. We focus mainly on representability theorems, localisation, Bousfield classes, and nilpotence.

1. Introduction

Axiomatic stable homotopy theory is the study of triangulated categories formally similar to the homotopy category of spectra in the sense of Boardman [1, 49]. While various authors have used different systems of axioms, there is broad agreement about the main examples, of which the following are a sample:

(1) Boardman's category itself, which we denote by $\mathcal{B}$.
(2) The subcategories of $E(n)$-local and $K(n)$-local spectra [66, 36].
(3) The homotopy category $\mathcal{B}_G$ of G-spectra (indexed on a complete universe), where G is a compact Lie group [48, 50].
(4) The homotopy category of A-modules, where A is a commutative ring spectrum. (Here and throughout this paper, the phrase "ring spectrum" refers to a strictly associative monoid in a suitable geometric category of spectra, such as that defined in [23].)
(5) The derived category $\mathcal{D}_R$ of R-modules, for a commutative ring R. If we let HR denote the associated Eilenberg-MacLane ring spectrum, then $\mathcal{D}_R \simeq \mathcal{D}_{HR}$ (by [23, Theorem IV.2.4]), so this is a special case of the previous example.
(6) The stable category Stab_{kG} of kG-modules and projective equivalence classes of homomorphisms, where G is a finite group and k is a field [70]. This occurs as a quotient of the category $\mathcal{D}_{kG}$, in which the morphism sets are given by group cohomology; the morphism sets in Stab_{kG} itself are more closely related to Tate cohomology.
(7) The derived category (in a suitable sense) of MU_*MU-comodules [33].

There are some further examples that satisfy some authors' axioms but not others, or where the axioms have not yet been checked (to the best of my knowledge):

(1) The derived category of modules over a noncommutative ring.
(2) The derived category of quasicoherent sheaves over a nonaffine scheme.
(3) The homotopy category of G-spectra indexed by an incomplete G-universe.
(4) Various versions of the category of motivic spectra. (Motivic spaces are discussed in [54]; at the time of writing, there is no published account of the corresponding category of spectra.)

It seems an important problem to decide which of the usual axioms apply to the motivic stable category, and to see what the axiomatic literature teaches us about this example.

The main topics that have been discussed from an axiomatic point of view are as follows.

(a) Theorems saying that certain (covariant or contravariant) functors are representable, generalising the Brown representability theorems for (co)homology theories on (finite or infinite) spectra.
(b) Phantom maps.

Date: July 10, 2003.

1991 *Mathematics Subject Classification.* 55U35.

Key words and phrases. axiomatic stable homotopy,triangulated category,Bousfield class .

(c) Various kinds of localisation, generalising Bousfield's theory of localisation with respect to a homology theory. Special cases such as finite, cofinite, algebraic or smashing localisations.
(d) The lattice of Bousfield classes, and various related lattices (some of them conjecturally identical to the Bousfield lattice).
(e) Nilpotence theorems in the spirit of Devinatz, Hopkins and Smith.
(f) Picard groups, Grothendieck rings, and Euler characteristics [28, 51, 52, 46].
(g) Projective classes, and generalisations of the Adams spectral sequence [15].
(h) Duality theorems generalising those of Verdier and Gross-Hopkins [57, 24].

For (f) to (h) we refer the reader to the cited papers and their bibliographies. This survey will concentrate on (a) to (e).

The relevant literature consists partly of papers that are explicitly axiomatic, and partly of papers that are nominally restricted to some particular category, but whose methods allow straightforward generalisation to other examples. Some authors are as follows:

(1) Margolis's book [49] treats $\mathcal{B}$ from an axiomatic point of view; this was an important inspiration for much of the later work. Earlier still, there were relevant papers by Freyd, Heller and Joel Cohen.
(2) Neeman has written extensively, particularly on questions related to representability and localisation [13, 17, 59, 56, 62, 55, 61, 58, 63]. Some papers are restricted to the case of $\mathcal{D}_R$; often R need not be commutative or noetherian. When working axiomatically, he has generally assumed that his triangulated category $\mathcal{C}$ is "compactly generated", but not that $\mathcal{C}$ has a symmetric monoidal structure. The class of categories considered is thus rather large, but unfortunately it is not closed under Bousfield localisation. Recently he has introduced the more complex notion of a "well-generated" triangulated category to repair this problem.
(3) Krause has also written extensively on representability, localisation, and versions of the Bousfield lattice [11, 3, 39, 40, 43, 42, 41, 44, 45, 46, 47].
(4) Beligiannis has written a long paper [2] covering many themes in axiomatic stable homotopy, considered as an analog of relative homological algebra in the context of triangulated categories.
(5) Benson, Carlson, Rickard and Gnacadja (working in various combinations) have proved many results about the categories Stab_{kG} and $\mathcal{D}_{kG}$, often using methods that transfer easily to an axiomatic setting [4, 6, 7, 8, 9, 10, 5, 11, 3, 72, 70, 68, 71, 69]. Benson and Wheeler have interpreted the Green correspondence in this context [12].
(6) Hovey, Schwede and Shipley have worked in a more rigid context, studying Quillen model categories $\mathcal{C}_0$ such that the homotopy category $\mathrm{Ho}(\mathcal{C}_0)$ is triangulated [34, 73, 75, 76].
(7) May and coauthors have studied the equivariant stable categories $\mathcal{B}_G$, often using methods that transfer easily to an axiomatic setting. This applies particularly to their work on duality, traces, and Picard groups [25, 52, 51, 48].
(8) Hovey and Palmieri and Strickland wrote a memoir [35] on axiomatic stable homotopy theory. We assumed much more than Neeman, and thus could obtain results closer to those previously known for $\mathcal{B}$. In particular, we assume that $\mathcal{C}$ has a closed symmetric monoidal structure.

2. Axioms

We next discuss the various axioms that have been used. We start with a category $\mathcal{C}$.

2.1. **Basics.** The category $\mathcal{C}$ should be triangulated, and should have coproducts for all families of objects (indexed by a set). These are core axioms, used by almost all authors. Existence of coproducts should be seen as an important test of the correctness of the technical details of the definition of $\mathcal{C}$. Boardman's category itself came after several attempts to define a good category of spectra, and it was the first to be triangulated and coproduct-complete; it rapidly became clear that Boardman's version was much more convenient than all the others. Similarly, the earliest versions of $\mathcal{D}_R$ incorporated various boundedness conditions, and so were not coproduct-complete. Bokstedt and Neeman [13] adjusted the definitions to remove this problem, and this allowed much smoother comparisons between $\mathcal{D}_R$ and $\mathcal{B}$.

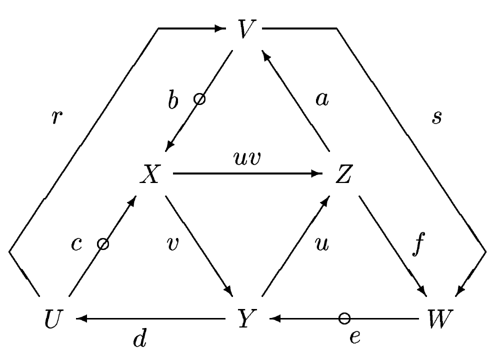
We recall the definition of a triangulation:

Definition 2.1. A *triangulation* of an additive category $\mathcal{C}$ is an additive (*suspension*) functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ giving an automorphism of $\mathcal{C}$, together with a collection $\triangle$ of diagrams, called *distinguished triangles* or *cofibre sequences*, of the form

$$X \to Y \to Z \to \Sigma X$$

such that

1. Any diagram isomorphic to a cofibre sequence is a cofibre sequence.
2. Any diagram of the following form is a cofibre sequence:
$$0 \to X \xrightarrow{1} X \to 0$$
3. The first of the following diagrams a cofibre sequence iff the second is a cofibre sequence:
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$
$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y.$$
4. For any map $f\colon X \to Y$, there is a cofibre sequence of the following form:
$$X \xrightarrow{f} Y \to Z \to \Sigma X$$
5. Suppose we have a diagram as shown below (with h missing), in which the rows are cofibre sequences and the rectangles commute. Then there exists a (nonunique) map h making the whole diagram commutative.
$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, dashed, "h"] & \Sigma U \arrow[d, "\Sigma f"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X \end{tikzcd}$$
6. Verdier's octahedral axiom holds: Suppose we have maps $X \xrightarrow{v} Y \xrightarrow{u} Z$, and cofibre sequences (X, Y, U), (X, Z, V) and (Y, Z, W) as shown in the diagram. (A circled arrow $U \multimap X$ means a map $U \to \Sigma X$.) Then there exist maps r and s as shown, making (U, V, W) into a cofibre sequence, such that the following commutativities hold:
$$au = rd \qquad es = (\Sigma v)b \qquad sa = f \qquad br = c$$



(If u and v are inclusions of CW spectra, this essentially just says that $(Z/X)/(Y/X) = Z/Y$. The diagram can be turned into an octahedron by lifting the outer vertices and drawing an extra line from W to U.)

Following the standard topological notation, we write $[X, Y]$ for the set $\mathcal{C}(X, Y)$ (of morphisms in $\mathcal{C}$ from X to Y). We also put $[X, Y]_n = [\Sigma^n X, Y]$.

If $\mathcal{C}_0$ is a pointed Quillen model category, then the homotopy category $\mathcal{C} = \mathrm{Ho}(\mathcal{C}_0)$ automatically has structure close to that described above, except that the functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ need not be an equivalence. Following Hovey, we say that $\mathcal{C}_0$ is a *stable model category* if Σ is an equivalence; if so, one can show that $\mathcal{C}$ is triangulated [34, Chapter 7]. Similar results appear in [51]. Schwede and Shipley have shown [75] that most stable model categories are Quillen-equivalent to $\mathcal{D}_A$ for some ring spectrum A, and that the general case is only a little more general.

Various modifications and refinements of triangulated categories have been considered by Neeman [55], May [51], Franke [26] and probably others. It seems likely that these all follow from the existence of an underlying model category $\mathcal{C}_0$ as above. For the categories $\mathcal{C}$ occurring in practice, it seems that there is always an underlying model category, and that any two natural choices are Quillen equivalent. It thus seems reasonable to assume whenever convenient that one is given a category $\mathcal{C}_0$.

2.2. **Smash products.** In [35], we assume that our categories $\mathcal{C}$ come equipped with a symmetric monoidal product. We use notation coming from topology, and thus write $X \wedge Y$ for the monoidal product of X and Y, and S for the unit, so $S \wedge X = X = X \wedge S$. We also assume that there are adjoint function objects $F(Y, Z)$, so $[X, F(Y, Z)] \simeq [X \wedge Y, Z]$, naturally in all variables. We write $S^n = \Sigma^n S$, so $S^n \wedge S^m = S^{n+m}$ and $\Sigma^n X = S^n \wedge X$. We also put $\pi_n X = [S^n, X]$.

This structure certainly exists in all the categories mentioned so far, except for the category $\mathcal{D}_A$ when A is not commutative. However, the theory of blocks in group algebras decomposes Stab_{kG} as a product of smaller categories, which need not have a symmetric monoidal structure. There may be similar examples related to $\mathcal{B}_G$.

It is natural to require that the smash product be compatible with the triangulation. In [35], we wrote down the most obvious compatibility conditions:

(1) The smash product commutes with suspension, so $\Sigma(X \wedge Y) \simeq X \wedge \Sigma Y$.
(2) The functors $X \wedge (-)$ and $F(X, -)$ preserve cofibre triangles. The contravariant functors $F(-, Y)$ preserve cofibre triangles up to a sign change.
(3) The twist map $S^1 \wedge S^1 \to S^1 \wedge S^1$ is multiplication by -1.

However, May has given strong evidence that further conditions should be added. To explain this, consider a map $f\colon X \to X$. Under suitable finiteness conditions, this has a trace $\tau(f)\colon S \to S$. If we have a cofibre sequence

$$X_0 \to X_1 \to X_2 \to \Sigma X_0$$

and compatible maps $f_i\colon X_i \to X_i$, it is natural to hope that $\tau(f_0) - \tau(f_1) + \tau(f_2) = 0$; this is suggested by the theory of Lefschetz numbers, among other things. It turns out that the statement must be adjusted slightly: given f_0 and f_1, one can choose a compatible f_2 such that $\tau(f_0) - \tau(f_1) + \tau(f_2) = 0$, but this may not be the case for all compatible f_2's. This (and various extensions) can be proved in Boardman's category, but the proof cannot be transferred to the axiomatic setting without adding some more conditions. In outline, consider two cofibre sequences

$$X_0 \to X_1 \to X_2 \to \Sigma X_0$$
$$Y_0 \to Y_1 \to Y_2 \to \Sigma Y_0.$$

From these we obtain a 4×4 diagram with vertices $X_i \wedge X_j$, in which all squares commute, except that one square anticommutes. By writing in the diagonal composite in each square, we get 18 commutative triangles. Each commutative triangle fits in an octahedron, as in Verdier's axiom. The 18 resulting octahedra have many vertices and edges in common, and one can hope to add in some extra vertices and edges making everything fit together more coherently. May [51] has formulated three axioms about this situation, and explained how they can be checked when $\mathcal{C} = \mathrm{Ho}(\mathcal{C}_0)$ for some Quillen model category $\mathcal{C}_0$.

There are many interesting cases of noncommutative rings (or ring spectra) R and R' for which $R \not\simeq R'$ but $\mathcal{D}_R \simeq \mathcal{D}_{R'}$; this is a natural extension of Morita theory. Examples come from Koszul duality, the Fourier-Mukai transform for sheaves on abelian varieties, tilting complexes in representation theory, and so on [68, 69, 14, 20, 74]. Nonetheless, it seems that there are no examples of this type where the categories involved have smash products and the equivalence respects them. We know of no rigorous results in this direction, however.

2.3. **Generation.** A key feature of Boardman's category is that every spectrum X has a cell structure, and (essentially equivalently) that if $[S^0, X]_* = 0$ then $X = 0$. More generally, suppose we have a set $\mathcal{G}$ of objects in a triangulated category $\mathcal{C}$. We say that $\mathcal{G}$ *generates* $\mathcal{C}$ if there is no proper triangulated subcategory $\mathcal{C}' \subset \mathcal{C}$ closed under all coproducts such that $\mathcal{G} \subseteq \mathcal{C}'$. We also say that $\mathcal{G}$ *detects* $\mathcal{C}$ if every object $X \in \mathcal{C}$ with $[A, X]_* = 0$ for all $A \in \mathcal{G}$ actually has $X = 0$. (By taking $\mathcal{C}' = \{A \mid [A, X]_* = 0\}$, we see that generation implies detection, and the converse holds under suitable finiteness conditions.) Generating sets for the main examples are as follows.

(1) $\{S^0\}$ generates $\mathcal{B}$, and $L_E S^0$ generates the subcategory of E-local spectra [35, Section 3.5].
(2) If X is a finite spectrum of type n (in the usual chromatic sense) then $L_{K(n)}X$ generates the category of $K(n)$-local spectra [36, Theorem 7.3], and this is generally a better choice of generator than $L_{K(n)}S^0$.
(3) The G-spectra G/H_+ (as H runs over conjugacy classes of closed subgroups) generate $\mathcal{B}_G$.
(4) If R is a ring, then R generates $\mathcal{D}_R$. This also works for (strictly associative) ring spectra.
(5) The nonprojective simple kG-modules generate Stab_{kG}.

All authors in axiomatic stable homotopy theory assume that $\mathcal{C}$ is generated by some set $\mathcal{G}$ of objects, and impose some smallness conditions on $\mathcal{G}$. The details vary between authors, however.

One popular condition is as follows. We say that an object $A \in \mathcal{C}$ is *small* (or *compact*) if the natural map

$$\bigoplus_{i \in I} [A, X_i] \to [A, \bigoplus_i X_i]$$

is an isomorphism for all families of objects $\{X_i\}$. For example:

(1) In $\mathcal{B}$, the small objects are those of the form $\Sigma^d X$ where $d \in \mathbb{Z}$ and X is a finite CW complex.
(2) In the category of $E(n)$-local spectra, the small objects are those that can be written as a retract of $\Sigma^d L_{E(n)}X$ for some $d \in \mathbb{Z}$ and some finite CW complex X.
(3) In the category of $K(n)$-local spectra, the small objects are those that can be written as a retract of $\Sigma^d L_{K(n)}X$ for some $d \in \mathbb{Z}$ and some finite CW complex X of type n.
(4) In $\mathcal{B}_G$, the small objects are those that can be written as a retract of $\Sigma^d X$ for some $d \in \mathbb{Z}$ and some finite G-CW complex X.
(5) In $\mathcal{D}_R$, the small objects are the finite complexes of finitely generated projective modules.
(6) In Stab_{kG}, the small objects are the kG-modules M that are finite-dimensional over k.

(Most of these facts are proved in [35], for example.)

We say that $\mathcal{C}$ is *compactly generated* if there is a set $\mathcal{G}$ of small objects that generates $\mathcal{C}$. In the terminology of [35], a stable homotopy category is *algebraic* iff it is compactly generated. This is a very convenient condition, and is often satisfied; in particular, the generators listed above for $\mathcal{B}$, $\mathcal{B}_G$, $\mathcal{D}_R$ and Stab_{kG} are all small. In the case of the $K(n)$-local category, the obvious generator $L_{K(n)}S^0$ is *not* small, but $L_{K(n)}X$ is small whenever X is finite of type n, so the category is nonetheless compactly generated. For a simpler example of the same phenomenon, let $\mathcal{C}$ be the p-completion of the category $\mathcal{D}_{\mathbb{Z}}$; then the obvious generator is $\mathbb{Z}_p$ (which is not small), but the object $\mathbb{Z}/p$ is also a generator, and is small. (There is a well-understood axiomatic framework covering both of these examples: see [35, Section 3.3], and Section 7 of the present paper.) More seriously, there are many known spectra E for which the category $\mathcal{C}_E$ of E-local spectra has no nontrivial small objects, so in particular, $\mathcal{C}_E$ is not compactly generated [60][36, Appendix B]. For example, this applies with $E = BP$ or $E = H$ or $E = \bigvee_{0 \leq n < \infty} K(n)$.

Another useful condition is *dualisability*. To formulate this, we need to assume that $\mathcal{C}$ has a symmetric monoidal smash product, as in the previous section. We write $DA = F(A, S)$, where S is the unit object for the smash product. We say that A is *dualisable* if the natural map $DA \wedge A \to F(A, A)$ is an isomorphism; this implies that we have $DA \wedge B = F(A, B)$ for all B.

The category of dualisable objects is formally very similar to the category of finite dimensional vector spaces over a field.

In Boardman's category $\mathcal{B}$, or in the derived category $\mathcal{D}_R$, it is known that an object is dualisable iff it is small. However, $L_{K(n)}S^0$ is dualisable but not small in the $K(n)$-local category. A number of interesting things are known about $K(n)$-locally dualisable spectra:

(1) A $K(n)$-local spectrum X is dualisable iff $\dim_{K(n)_*} K(n)_*X < \infty$ (proved in [36]).
(2) If X is a finite complex, then $L_{K(n)}\Sigma^\infty X$ is easily seen to be dualisable.
(3) If X is a connected space with $|\prod_{k>0}\pi_k X| < \infty$, then it is probably true that $K(n)_*X$ is finite-dimensional and so $L_{K(n)}\Sigma^\infty X$ is dualisable. The results in the literature involve some additional conditions, however; for example, the claim is true if $X = BG$ with G finite [65] (in which case $L_{K(n)}\Sigma^\infty BG$ turns out to be self-dual [77]), or if X is a double loop space [31].

In [35], we assume that our generators are dualisable, but not that they are small. This theory has the advantage that any localisation of a category satisfying our axioms, again satisfies our axioms. The disadvantages are:

(i) We need a smash product to formulate the definition of dualisability, and this is absent or unnatural in many examples, such as $\mathcal{D}_R$ when R is not commutative.
(ii) If $\mathcal{C} = \mathrm{Ho}(\mathcal{C}_0)$ for some Quillen model category $\mathcal{C}_0$, then there are natural conditions on objects in $\mathcal{C}_0$ guaranteeing that they are small in $\mathcal{C}$. This is not the case for dualisability: we need special geometric arguments to show that G/H_+ is dualisable in $\mathcal{B}_G$, for example.
(iii) There are naturally occurring cases where the generators are small but not dualisable, for example the category of G-spectra based on an incomplete universe, and possibly also derived categories for nonaffine schemes.

In [63], Neeman introduces the notion of a *well-generated* triangulated category; the definition is explained and simplified in [42]. To explain the nature of this concept, we recall some generalisations of Quillen's small object argument. Quillen originally considered a category $\mathcal{E}$ closed under limits and colimits, and looked for objects A that were small in the sense that the functor $\mathcal{E}(A,-)$ preserves filtered colimits. Later, it was realised that one can fix a large cardinal κ and say that A is *κ-small* if $\mathcal{E}(A,-)$ preserves colimits of sequences indexed by ordinals larger than κ. In many categories, every object is κ-small for some κ, and in many applications related to localisation, this is an adequate substitute for smallness. Neeman works with triangulated categories, which typically do not have colimits for most diagrams. Thus, the above cannot be applied directly, but a somewhat more elaborate argument leads to Neeman's well-generated categories. It is shown in [62] that the derived category of any Grothendieck abelian category is well-generated.

2.4. **Representability.** A *cohomology functor* on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to the category Ab of abelian groups, that converts coproducts to products and cofibre sequences to exact sequences. For any fixed object Z, it is well-known that the representable functor $X \mapsto [X,Z]$ is a cohomology functor. We say that *the representability theorem holds for $\mathcal{C}$* if the converse is true, so that every cohomology theory on $\mathcal{C}$ is representable. Brown proved the representability theorem for $\mathcal{B}$, and the same proof works for any compactly generated triangulated category. By much more elaborate arguments, Neeman has extended this to all well-generated triangulated categories [63], and similar results have been obtained by Krause [44, 45] and Franke [27].

It is also easy to see that if the representability theorem holds for $\mathcal{C}$, then it holds for any localisation of $\mathcal{C}$.

In [35], we take the representability theorem as an axiom; this gives a convenient way to treat compactly generated categories and their localisations in parallel. All other authors assume axioms that turn out to imply the representability theorem.

2.5. **Extra axioms.** We now discuss some possible additional assumptions. No author takes any these as a standard axiom, but they define special classes of examples with usefully simplified behaviour.

(a) Let $\mathcal{C}$ be an abelian category in which all monomorphisms split (and thus all epimorphisms split), and suppose we have an equivalence $\Sigma\colon \mathcal{C} \to \mathcal{C}$. We can give $\mathcal{C}$ a triangulation by declaring that all all triangles of the form

$$A \oplus B \xrightarrow{f} B \oplus C \xrightarrow{g} C \oplus \Sigma A \xrightarrow{h} \Sigma A \oplus \Sigma B$$

(with $f(a, b) = (b, 0)$ and so on) are cofibre sequences. We call this the *abelian case*. Under mild finiteness conditions, one can show that $\mathcal{C}$ is the category of graded A_*-modules, for some graded ring A_* that is a finite product of graded division rings. This situation is discussed in [35, Section 8].

Examples include the category of rational G-spectra for any finite group G, or the category Stab_{kG} when $|G|$ is invertible in k.

(b) Suppose that $\mathcal{C}$ has a symmetric monoidal structure, and that $\mathcal{C}$ is generated by the single object S (the unit for the smash product). We then have a graded-commutative ring $\pi_* S$ defined by $\pi_n S = [\Sigma^n S, S]$. If this is noetherian, we say that we are in the *noetherian case*; this is discussed in [35, Section 6]. Examples include $\mathcal{D}_R$ where R is commutative and noetherian, and $\mathcal{D}_{kG}$.

(c) Suppose again that $\mathcal{C}$ has a symmetric monoidal structure, and that $\mathcal{C}$ is generated by the single object S. If $\pi_n S = 0$ for $n < 0$, then we are in the *connective case*; this is discussed in [35, Section 7].

We will make a number of remarks about the noetherian case below. Beyond that, we refer the reader to [35] for further discussion.

3. Functors on small objects

Let $\mathcal{F}$ be the category of small objects in $\mathcal{C}$, and consider the category $\mathcal{A} = [\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}]$ of additive contravariant functors from $\mathcal{F}$ to the category of abelian groups. This is a bicomplete abelian category satisfying the AB5 condition (filtered colimits are exact). The functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ induces a functor $\Sigma\colon \mathcal{A} \to \mathcal{A}$. If $\mathcal{C}$ has a good symmetric monoidal structure, then so does $\mathcal{A}$. If the objects of $\mathcal{F}$ are strongly dualisable, then we have $\mathcal{F} \simeq \mathcal{F}^{\mathrm{op}}$ and so $\mathcal{A} \simeq [\mathcal{F}, \mathrm{Ab}]$. One can think of $\mathcal{F}$ as a "ring with many objects", and regard $\mathcal{A}$ as its module category.

There is a Yoneda functor $h\colon \mathcal{C} \to \mathcal{A}$ sending X to the functor $h_X(A) = [A, X]$ (for $A \in \mathcal{F}$). The structure of $\mathcal{A}$ and the behaviour of h have proved to be very useful in the study of $\mathcal{C}$, at least when $\mathcal{C}$ is compactly generated. If $\mathcal{C}$ is merely well-generated, then Neeman has developed a partially parallel theory based on more complicated functor categories [63, Chapter 6]. In the compactly generated case, Beligiannis [2] has considered categories of the form $[\mathcal{G}^{\mathrm{op}}, \mathrm{Ab}]$ where $\mathcal{G}$ is an arbitrary triangulated subcategory of $\mathcal{F}$.

Let $\mathcal{E} \subseteq \mathcal{A}$ be the category of *exact* functors: those that send cofibre sequences in $\mathcal{F}$ to exact sequences in Ab. It is standard that $h_X \in \mathcal{E}$ for all X. In good cases, $\mathcal{E}$ is the category of objects of finite projective (or injective) dimension in $\mathcal{A}$, and $h\colon \mathcal{C} \to \mathcal{E}$ is close to being an equivalence; see Section 9 for more discussion.

The functor $h\colon \mathcal{C} \to \mathcal{A}$ always preserves coproducts and sends cofibre sequences to exact sequences. In other words, it is an $\mathcal{A}$-valued homology theory on $\mathcal{C}$. A morphism $u\colon X \to Y$ in $\mathcal{C}$ is said to be *phantom* if $h_u\colon h_X \to h_Y$ is zero.

4. Types of subcategories

Let R be a commutative noetherian ring, and let $\mathrm{zar}(R)$ denote the space of prime ideals in R, with the Zariski topology. (We do not use the notation $\mathrm{spec}(R)$, to avoid conflicting uses of the word "spectrum".) It turns out [30, 56] that we can recover $\mathrm{zar}(R)$ from the category $\mathcal{D}_R$, in several slightly different ways. More precisely, one can recover the lattice of radical ideals in R, which is well-known to be anti-isomorphic to the lattice of closed subsets of $\mathrm{zar}(R)$, and this lattice determines $\mathrm{zar}(R)$ itself [37]. The key is to study various lattices of subcategories of $\mathcal{D}_R$. Using parallel constructions in the category Stab_{kG}, we can recover the space $\mathrm{zar}(H^*(G; \mathbb{F}_p))$. In the case of Boardman's category, this study makes contact with the chromatic approach to stable homotopy theory, and the nilpotence theorems of Devinatz, Hopkins and Smith. This has many important applications that are not visible in the purely algebraic examples. For example, suppose we want to prove that all finite spectra X have some property $P(X)$. Suppose we can show that

- Whenever we have a cofibre sequence $X \to Y \to Z$ in which two terms have property P, then the third also has property P
- Whenever $P(X \vee Y)$ holds, so do $P(X)$ and $P(Y)$
- There exists a finite spectrum X such that $H_*(X;\mathbb{Q}) \neq 0$ and $P(X)$ holds.

Then one can show using the subcategory classification theorems that $P(X)$ holds for all X.

The basic definitions are as follows.

Definition 4.1. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. For simplicity, we assume that any object in $\mathcal{C}$ that is isomorphic to an object in $\mathcal{D}$, is itself in $\mathcal{D}$. Let $\mathcal{A}$ be an arbitrary collection (possibly a proper class) of objects in $\mathcal{C}$.

(a) $\mathcal{D}$ is *thick* if
  (i) The zero object lies in $\mathcal{D}$.
  (ii) Any retract of any object in $\mathcal{D}$, again lies in $\mathcal{D}$.
  (iii) Whenever $X \to Y \to Z$ is a cofibre sequence with two terms in $\mathcal{D}$, the third term is also in $\mathcal{D}$.
(b) If $\mathcal{C}$ has a symmetric monoidal structure, we say that $\mathcal{D}$ is an *ideal* if $X \wedge Y \in \mathcal{D}$ whenever $X \in \mathcal{D}$. Dually, we say that $\mathcal{D}$ is a *coideal* if $F(Y,Z) \in \mathcal{D}$ whenever $Z \in \mathcal{D}$.
(c) $\mathcal{D}$ is a *localising subcategory* if it is thick, and closed under (possibly infinite) coproducts. Dually, $\mathcal{D}$ is a *colocalising subcategory* if it is thick, and closed under (possibly infinite) products.
(d) A *(co)localising (co)ideal* is a (co)localising subcategory that is also a (co)ideal.
(e) A *bilocalising subcategory* is a subcategory that is both a localising subcategory and a colocalising subcategory. A *biideal* is a subcategory that is both an ideal and a coideal.

If $\mathcal{C}$ is monoidal and the unit object $S \in \mathcal{C}$ is small and generates $\mathcal{C}$, then every (co)localising subcategory is a (co)ideal. This holds in the following cases:

(1) $\mathcal{C} = \mathcal{B}$ (but not $\mathcal{C} = \mathcal{B}_G$ for general G)
(2) $\mathcal{C} = \mathcal{D}_R$, where R is commutative
(3) $\mathcal{C} = \mathrm{Stab}_{kG}$, where k has characteristic p and G is a p-group.

If we have a functor F between triangulated categories that preserves cofibre sequences, then $\ker(F) := \{X \mid FX = 0\}$ is evidently a thick subcategory. Similarly, if F is a functor from a triangulated category to an abelian category, and F converts cofibre sequences to exact sequences, then $\ker(F)$ will again be a thick subcategory. Under various auxiliary conditions, we can conclude that $\ker(F)$ is a (co)localising subcategory or a (co)localising ideal.

Little is known about classification of subcategories that are not ideals. The examples studied in [8, Section 6] suggest that there is no simple and general picture.

On the other hand, there are good classification results for many of our central examples; the main method of proof will be discussed in Section 8. To state the results, it is convenient to introduce one more definition. Given an object $A \in \mathcal{F}$, we write $\mathrm{thickid}\langle A\rangle$ for the smallest thick ideal containing A. We then say that a thick ideal $\mathcal{I}$ is *finitely generated* if it has the form $\mathrm{thickid}\langle A\rangle$ for some A (this makes sense because the thick ideal generated by $A_1,\dots,A_r$ is also generated by the single object $A = \bigvee_i A_i$). A classification of finitely generated thick ideals (or thick subcategories) in $\mathcal{F}$ extends in a fairly obvious way to give a classification of all ideals (or thick subcategories) in $\mathcal{F}$.

(a) Let R be a commutative noetherian ring, and put $\mathcal{C} = \mathcal{D}_R$. Then the localising subcategories of $\mathcal{C}$ biject with the colocalising subcategories, and with the subsets of $\mathrm{zar}(R)$. The finitely generated thick subcategories of $\mathcal{F}$ biject with closed subsets of $\mathrm{zar}(R)$. On the other hand, Neeman has considered the nonnoetherian ring

$$R = k[x_2, x_3, x_4, \dots]/(x_2^2, x_3^3, x_4^4, \dots),$$

where k is a field. This has only one prime ideal, but $\mathcal{D}_R$ has an enormous collection of localising subcategories [61].
(b) Let k be a field, let G be a finite group, and put $\mathcal{C} = \mathrm{Stab}_{kG}$. It is proved in [8] that the finitely generated thick ideals in $\mathcal{F}$ biject with the closed subsets of the projective scheme

proj$(H^*(G;k))$. One can also show that the (co)localising (co)ideals in $\mathcal{C}$ biject with all subsets of proj$(H^*(G;k))$.

(c) In the category of $E(n)$-local spectra, the (co)localising subcategories in $\mathcal{C}$ biject with subsets of $\{0,1,\ldots,n\}$, and the thick subcategories of $\mathcal{F}$ biject with the subsets of the form $\{m,m+1,\ldots,n\}$ for some m. All the relevant subcategories are (co)ideals [36, Theorem 6.14].

(d) Now let $\mathcal{C}$ be the category of $K(n)$-local spectra. Then 0 and $\mathcal{C}$ are the only localising subcategories of $\mathcal{C}$, and also the only colocalising subcategories of $\mathcal{C}$ [36, Theorem 7.5]. Similarly, 0 and $\mathcal{F}$ are the only thick subcategories of $\mathcal{F}$.

(e) Finally, let $\mathcal{C}$ be the category of p-local spectra. The thick subcategories of $\mathcal{F}$ are then the categories $\mathcal{F}_n := \{X \mid K(m)_*X = 0 \text{ for all } m < n\}$, where $0 \leq n \leq \infty$; this was proved in [32]. The theory of Bousfield classes gives many known examples of (co)localising subcategories and inclusions between them. However, almost nothing is known about the collection of *all* localising subcategories (which might even be a proper class).

The classification results in Examples (a) and (b) have a partial generalisation that applies in the noetherian case. A strong but technically complex statement is proved in [35, Section 6.3]; given some additional hypotheses (conjecturally always satisfied) this implies the evident analog of (a) and (b).

So far we have only discussed results about ideals in $\mathcal{F}$; we next consider results about (co)localising subcategories or (co)ideals in $\mathcal{C}$. On the one hand, given $\mathcal{D} \subseteq \mathcal{C}$ we can certainly consider the thick subcategory $\mathcal{D} \cap \mathcal{F} \subseteq \mathcal{F}$; if we have a good understanding of $\mathcal{F}$ then this will be a useful invariant, but rather a coarse one. On the other hand, given a thick subcategory $\mathcal{A} \subseteq \mathcal{F}$ we can consider the category

$$\mathcal{A}^{\perp} := \{X \mid [A,X]_* = 0 \text{ for all } A \in \mathcal{A}\},$$

which is easily seen to be a bilocalising subcategory. The *telescope conjecture* for $\mathcal{C}$ is closely related to the statement that every bilocalising subcategory is of the form $\mathcal{A}^{\perp}$ for some $\mathcal{A}$. This is known to hold in $\mathcal{D}_R$ when R is noetherian and commutative, and also in Stab_{kG} when k has characteristic p and G is a p-group. It is believed to be false in Boardman's category, although many years of study have still not produced a watertight argument.

5. Quotient categories and Bousfield localisation

Let $\mathcal{C}$ be a triangulated category. Given a thick subcategory $\mathcal{D}$, we can look for a triangulated category $\mathcal{C}'$ and an exact functor $Q\colon \mathcal{C} \to \mathcal{C}'$ that sends all objects in $\mathcal{D}$ to zero. It turns out that there is an initial example of such a functor, whose target we call $\mathcal{C}/\mathcal{D}$. To be more precise, we say that a map $s\colon X \to Y$ in $\mathcal{C}$ is a $\mathcal{D}$-equivalence if the cofibre of s lies in $\mathcal{D}$. The class of $\mathcal{D}$-equivalences has a number of useful properties:

- Any isomorphism is a $\mathcal{D}$-equivalence.
- Given morphisms $X \xrightarrow{s} Y \xrightarrow{t} Y$, if any two of $\{s,t,ts\}$ are $\mathcal{D}$-equivalences then so is the third.
- Given maps $X \xrightarrow{f} Y \xleftarrow{s} Z$ in which s is a $\mathcal{D}$-equivalence, there is a commutative square

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in which t is a $\mathcal{D}$-equivalence.

We then define $\mathcal{C}/\mathcal{D}$ as follows: the objects are the same as in $\mathcal{C}$, and the morphisms from X to Y are equivalence classes of "formal fractions" gt^{-1}, where g and t fit in a diagram of the shape $X \xleftarrow{t} W \xrightarrow{g} Y$, and t is a $\mathcal{D}$-equivalence. The properties listed above allow us to compose and manipulate fractions in a natural way.

Krause has considered some more delicate notions of quotient categories, which are important in the study of smashing localisations; but we will not discuss these here.

As is well-known, there is a potential problem with the above construction, which is of great importance in some applications. We always assume implicitly that the morphism sets $\mathcal{C}(X,Y)$ are genuine sets; but as defined above, $(\mathcal{C}/\mathcal{D})(X,Y)$ might be a proper class. There are a number of techniques that can be used in different circumstances to show that this problem does not arise. As far as I know, no one has looked systematically for examples where proper classes do arise; it is possible that (some version of) our standing axioms are enough to prevent this.

If $\mathcal{C}/\mathcal{D}$ has small Hom sets, then for any $Y \in \mathcal{C}$ we have a functor from $\mathcal{C}$ to Ab given by $X \mapsto (\mathcal{C}/\mathcal{D})(X,Y)$. The Representability Theorem shows that this is representable, so we have an object $LX \in \mathcal{C}$ and an isomorphism $\mathcal{C}(X, LY) \simeq (\mathcal{C}/\mathcal{D})(X,Y)$, naturally in X. A standard argument shows that L can be regarded as a functor $\mathcal{C}/\mathcal{D} \to \mathcal{C}$, right adjoint to the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{D}$. If we put

$$\mathcal{D}^{\perp} = \{Y \mid \mathcal{C}(X,Y) = 0 \text{ for all } X \in \mathcal{D}\},$$

we find that L actually gives an equivalence $\mathcal{C}/\mathcal{D} \simeq \mathcal{D}^{\perp}$. We also use the letter L for the composite functor $\mathcal{C} \to \mathcal{C}/\mathcal{D} \to \mathcal{D}^{\perp} \subseteq \mathcal{C}$; in this guise, it is left adjoint to the inclusion of $\mathcal{D}^{\perp}$ in $\mathcal{C}$.

The functors $L\colon \mathcal{C} \to \mathcal{C}$ arising in this way can be characterised by certain well-known properties: they are exact functors, equipped with a natural map $i_X\colon X \to LX$ such that $Li_X\colon LX \to L^2X$ is an equivalence, and $i_X^*\colon [LX, LY] \to [X, LY]$ is an isomorphism for all Y. We call such a pair (L,i) a *Bousfield localisation functor*. We can recover $\mathcal{D}$ as the category $\ker(L) = \{X \mid LX = 0\}$.

The above discussion shows that quotients are really localisations. Of course, the converse is also true: to invert a class of maps $\mathcal{E}$ is the same as to quotient out the localising subcategory generated by the cofibres of the maps in $\mathcal{E}$.

In [35], it is assumed that $\mathcal{C}$ is symmetric monoidal and that $\mathcal{D}$ is a localising ideal. In this case, the quotient category $\mathcal{C}/\mathcal{D}$ (or equivalently, the category $\mathcal{D}^{\perp}$) inherits a symmetric monoidal structure.

Given any localisation functor L, there is another functor C and natural transformations $CX \xrightarrow{q_X} X \xrightarrow{i_X} LX \xrightarrow{d_X} \Sigma CX$ giving a cofibre sequence for all X. The theory can be set up in such a way that C and L play precisely dual rôles. In any case, the pair (L,i) determines (C,q) (up to an obvious notion of equivalence) and *vice versa*.

Given two localisation functors (L,i) and (L',i'), there is at most one morphism $u\colon L \to L'$ with $ui = i'$. We write $L \geq L'$ if such a morphism exists. This gives a partial order on the collection of isomorphism classes of localisation functors. It is not known whether this collection is a set or a proper class.

6. Versions of the Bousfield lattice

In this section, we assume that $\mathcal{C}$ has a symmetric monoidal structure. Without such a structure, one could set up a formal theory along the same lines, but it seems hard to analyze any examples explicitly.

We can now define various partially ordered sets Λ_i; some of them may actually be proper classes, but we will suppress this from the terminology. An optimistic conjecture would be that they are all the same; this is known to be true in the noetherian case. The general case appears to be open (related work of Gutierrez and Casacuberta turns out not to provide a counterexample).

Definition 6.1.
- Λ_0 is the class of all colocalising coideals (ordered by inclusion). For any class $\mathcal{A}$ of objects in $\mathcal{C}$, we write $\operatorname{colocid}\langle\mathcal{A}\rangle$ for the intersection of all colocalising ideals containing $\mathcal{A}$. The poset Λ_0 is actually a lattice, with meet operation $\mathcal{D} \wedge \mathcal{D}' = \mathcal{D} \cap \mathcal{D}'$, and join $\mathcal{D} \vee \mathcal{D}' = \operatorname{colocid}\langle\mathcal{D} \cup \mathcal{D}'\rangle$.
- Λ_1 is the class of all localising ideals, ordered by *reverse* inclusion. This is a lattice by a dual argument.
- For any class $\mathcal{A}$ of objects in $\mathcal{C}$, we put

$$\mathcal{A}^{\perp} = \{X \mid F(A,X) = 0 \text{ for all } A \in \mathcal{A}\}$$
$$^{\perp}\mathcal{A} = \{X \mid F(X,A) = 0 \text{ for all } A \in \mathcal{A}\}.$$

It is easy to see that $\mathcal{A}^\perp \in \Lambda_0$ and ${}^\perp\mathcal{A} \in \Lambda_1$. We say that a colocalising coideal $\mathcal{D}$ is *closed* if it has the form $\mathcal{A}^\perp$ for some $\mathcal{A}$, or equivalently if $\mathcal{D} = ({}^\perp\mathcal{D})^\perp$; we write Λ_2 for the set of closed colocalising coideals, so $\Lambda_2 \subseteq \Lambda_0$.

- Dually, we say that a localising ideal $\mathcal{E}$ is *closed* if it has the form ${}^\perp\mathcal{A}$ for some $\mathcal{A}$, or equivalently if $\mathcal{E} = {}^\perp(\mathcal{E}^\perp)$. We write Λ_3 for the set of closed localising ideals, so that $\Lambda_3 \subseteq \Lambda_1$.
- There are order-preserving maps $\Lambda_0 \to \Lambda_3$ and $\Lambda_1 \to \Lambda_2$, given by $\mathcal{D} \mapsto {}^\perp\mathcal{D}$ and $\mathcal{E} \mapsto \mathcal{E}^\perp$. A purely formal argument shows that these give an isomorphism $\Lambda_2 \simeq \Lambda_3$.
- We say that a colocalising coideal $\mathcal{D}$ is *reflective* if the inclusion $\mathcal{D} \to \mathcal{C}$ has a left adjoint; one can show that this implies that $\mathcal{D}$ is closed, so the coreflective coideals give a subset $\Lambda_4 \subseteq \Lambda_2$. Dually, we say that a localising ideal $\mathcal{E}$ is *coreflective* if the inclusion has a right adjoint, and these ideals give a subset $\Lambda_5 \subseteq \Lambda_3$. The bijection $\Lambda_2 \simeq \Lambda_3$ restricts to give a bijection $\Lambda_4 \simeq \Lambda_5$. If $\mathcal{D}$ and $\mathcal{E}$ correspond under this bijection, then there is a pair of functors (L, C) as in Section 5, with

$$\mathcal{D} = \mathrm{image}(L) = \ker(C) = \ker(L)^\perp = \mathrm{image}(C)^\perp$$
$$\mathcal{E} = \mathrm{image}(C) = \ker(L) = {}^\perp\ker(L) = {}^\perp\mathrm{image}(C).$$

It follows that Λ_4 and Λ_5 are equivalent to the poset of localisation functors L for which $\ker(L)$ is an ideal, or to the poset of colocalisation functors C for which $\ker(C)$ is a coideal.
- We say that a localising ideal $\mathcal{E}$ is *principal* if $\mathcal{E} = \mathrm{locid}\langle\{E\}\rangle$ for some object E. Note that if $\mathcal{E} = \mathrm{locid}\langle\{E_i\}_{i\in I}\rangle$ (where I is a set, not a proper class) then we also have $\mathcal{E} = \mathrm{locid}\langle\{E\}\rangle$ where $E = \bigvee_{i\in I} E_i$, so $\mathcal{E}$ is principal. In Boardman's category, it is known that principal ideals are coreflective, so they form a subset $\Lambda_6 \subseteq \Lambda_5$. It is not clear in what generality this argument works. If $\mathcal{E} = \mathrm{locid}\langle\{E\}\rangle$ then the corresponding localisation functor is called *stable E-nullification*, and written $P_{\Sigma^* E}$. (Confusingly, it was called *colocalisation* in Bousfield's original papers.)
- We say that a localising ideal $\mathcal{E}$ is a *Bousfield class* if it has the form $\mathcal{E} = \langle E\rangle = \{X \mid E \wedge X = 0\}$ for some $E \in \mathcal{C}$. We write Λ_7 for the collection of all Bousfield classes. In Boardman's category, this is contained in Λ_6; it is not clear how far this fact can be generalised. It is also known that Λ_7 is a set rather than a proper class [64, 22]. To see this, for any finite spectrum A and any element $x \in E_*A$ we let $\mathrm{ann}_A(x)$ be the set of maps $f\colon A \to B$ in $\mathcal{F}$ such that $(E_*f)(x) = 0$. We then write

$$\langle\langle E\rangle\rangle = \{\mathrm{ann}_A(x) \mid A \in \mathcal{F}\ ,\ x \in E_*A\},$$

and call this the *Ohkawa class* of E. As $\mathcal{F}$ has small Hom sets and only a set of isomorphism classes, we see that there is only a set of possible Ohkawa classes. One can check that $\langle\langle E\rangle\rangle$ determines $\langle E\rangle$, so there is only a set of Bousfield classes.

7. Special types of localisation

In this section, we assume that $\mathcal{C}$ is compactly generated.

Definition 7.1. A Bousfield localisation functor $L\colon \mathcal{C} \to \mathcal{C}$ is *smashing* if $\mathrm{image}(L)$ (which is automatically a colocalising subcategory) is closed under coproducts (and so is also a localising subcategory). In the monoidal case, this implies that there is a natural equivalence $LS \wedge X \to LX$, and the corresponding colocalisation functor C also satisfies $CX = CS \wedge X$. The category $\mathcal{D} = \mathrm{image}(L) = \ker(C) = \ker(L)^\perp = \mathrm{image}(C)^\perp$ is then both a localising ideal and a colocalising coideal. We put $\mathcal{U} = {}^\perp\mathcal{D}$ and $\mathcal{V} = \mathcal{D}^\perp$, and then $\widehat{C}X = F(LS, X)$ and $\widehat{L}X = F(CS, X)$. It turns out that $\widehat{L}$ is a localisation functor, and $\widehat{C}$ is the corresponding colocalisation. Moreover, we have

$$\begin{aligned}
\mathcal{U} &= \mathrm{image}(C) = \ker(L) = \langle LS\rangle\\
\mathcal{V} &= \mathrm{image}(\widehat{L}) = \ker(\widehat{C})\\
\mathcal{D} &= \mathrm{image}(\widehat{C}) = \ker(\widehat{L}) = \mathrm{image}(L) = \ker(C)\\
&= \mathcal{U}^\perp = {}^\perp\mathcal{V} = \langle CS\rangle.
\end{aligned}$$

It follows that $C\widehat{C} = 0 = \widehat{L}L$, so $\widehat{L}C \simeq \widehat{L}$ and $C\widehat{L} \simeq C$. This implies that $\widehat{L}\colon \mathcal{U} \to \mathcal{V}$ and $C\colon \mathcal{V} \to \mathcal{U}$ are mutually inverse equivalences.

Apart from the finite localisations discussed below, the most important examples are the localisations with respect to the Johnson-Wilson spectra $E(n)$. It is a highly nontrivial theorem [67, Chapter 8] that these are smashing.

Krause [40] has shown that L is determined by the set $\mathrm{ann}(L|_{\mathcal{F}})$ of morphisms $u\colon A \to B$ in $\mathcal{F}$ for which $Lu = 0$. This means in particular that there is only a set of smashing localisations.

Definition 7.2. A *finite localisation* is a localisation functor $L\colon \mathcal{C} \to \mathcal{C}$ where $\ker(L) = \mathrm{loc}\langle\mathcal{A}\rangle$ for some thick subcategory $\mathcal{A} \subseteq \mathcal{F}$. Functors of this type are always smashing [53][35, Section 3.3]. One formulation of the *telescope conjecture* for $\mathcal{C}$ is the statement that every smashing localisation is a finite localisation. This is known to be true in many noetherian cases, but believed to be false in Boardman's category. Keller [38] has provided a counterexample in $\mathcal{D}_R$ for a certain ring R (but his framework of definitions is slightly different from ours, and we have not pinned down the precise relationship).

An important example of finite localisation is as follows. Let R be a noetherian ring, and put $\mathcal{C} = \mathcal{D}_R$. Fix an ideal I, and let $\mathcal{A}$ consist of the objects $X \in \mathcal{C}$ for which π_*X is an I-torsion module. Here the category $\mathcal{U} = \ker(L)$ consists of the I-torsion objects in $\mathcal{C}$, and $\mathcal{V} = \ker(\widehat{C})$ consists of I-complete objects in a suitable sense. Thus, the equivalence $\mathcal{U} \simeq \mathcal{V}$ shows that the torsion category and the complete category are essentially the same. All this is closely related to the theory of local (co)homology [29]. See [21] and [35, Section 3.3] for other perspectives.

Definition 7.3. Suppose that $\mathcal{C}$ is symmetric monoidal, and is generated by the unit object S. Given a set T of homogeneous elements in the graded ring π_*S, we let $\mathcal{A}$ denote the thick subcategory of $\mathcal{F}$ generated by the cofibres of the maps in T, and then let L be the corresponding finite localisation functor. One can show that $\pi_*LX = (\pi_*X)[T^{-1}]$. Functors of this type are called *algebraic localisations*; in the special case where $T \subseteq \mathbb{Z}$, they are called *arithmetic localisations.*

8. Nilpotence

Our understanding of Boardman's category relies heavily on the nilpotence theorem of Devinatz, Hopkins and Smith [19] and its consequences [32, 67]. We next explain the formal parts of this story that are amenable to axiomatic generalisation [78][35, Section 5]. We will assume here that $\mathcal{C}$ is compactly generated and has a symmetric monoidal structure, and that all objects of $\mathcal{F}$ are strongly dualisable.

We say that an object $I \in \mathcal{F}$ equipped with a map $i\colon I \to S$ is an *ideal* if the map $i \wedge 1\colon I \wedge S/I \to S/I$ is null. (Here S/I denotes the cofibre of i.) We write $I \leq J$ if the map $I \to S \to S/J$ is zero. It turns out that if $I \xrightarrow{i} S$ and $J \xrightarrow{j} S$ are ideals, then so is $I \wedge J \xrightarrow{i \wedge j} S$; we will just write IJ for this, making the set of isomorphism classes of ideals into a commutative monoid. We say that I and J are *radically equivalent* if for large n we have $I^n \leq J$ and $J^n \leq I$. We write $\overline{\mathrm{Id}}(S)$ for the set of radical equivalence classes of ideals. Given any $A \in \mathcal{F}$, the fibre of the unit map $S \to F(A, A)$ is an ideal, and we write $\mathrm{ann}(A)$ for its equivalence class. One can show that the rule $\mathrm{ann}(A) \leftrightarrow \mathrm{thickid}\langle A\rangle$ gives a well-defined bijection between $\overline{\mathrm{Id}}(S)$ and the set of finitely generated thick ideals in $\mathcal{F}$.

Next, we say that a map $u\colon A \to B$ in $\mathcal{F}$ is *smash-nilpotent* if the m'th smash power $u^{(m)}\colon A^{(m)} \to B^{(m)}$ is zero for $m \gg 0$. One checks that $I^m \leq J$ for some m iff the composite $I \to S \to S/J$ is smash-nilpotent.

Now suppose we are given a set N and a collection of objects $K(n) \in \mathcal{C}$ for each $n \in N$. For any object $X \in \mathcal{C}$ we put $\mathrm{supp}(X) = \{n \mid K(n) \wedge X \neq 0\}$. Similarly, given a thick ideal $\mathcal{A} \subseteq \mathcal{F}$ we put $\mathrm{supp}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \mathrm{supp}(A)$.

We say that the $K(n)$'s *detect ideals* if whenever $A, B \in \mathcal{F}$ and $\mathrm{supp}(A) \subseteq \mathrm{supp}(B)$, we have $\mathrm{thickid}\langle A\rangle \subseteq \mathrm{thickid}\langle B\rangle$. This implies that the map $\mathcal{A} \to \mathrm{supp}(\mathcal{A})$ gives an embedding of the lattice of thick ideals in the lattice of subsets of N. (Except in the noetherian case, we know of no general method to determine the image of this map.)

We next explain two versions of what it might mean for the $K(n)$'s to *detect nilpotence*. It is a key theorem that if the $K(n)$'s detect nilpotence, then they also detect ideals.

For the most algebraically natural version, we need auxiliary hypotheses. First, we assume that each $K(n)$ has a commutative ring structure, and that every nonzero homogeneous element in the coefficient ring $K(n)_*$ is invertible (so $K(n)_*$ is a graded field). We also assume that the resulting Künneth maps

$$K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \to K(n)_*(X \wedge Y)$$

are isomorphisms for all X and Y (this is not automatic unless $\mathcal{C}$ is generated by $\{S\}$). Thus, we can regard $K(n)$ as giving a monoidal functor from $\mathcal{F}$ to the category of finite-dimensional vector spaces over $K(n)_*$. We say that a map $u\colon A \to B$ in $\mathcal{F}$ is $K(*)$-null if the induced map $K(n)_*A \to K(n)_*B$ is zero for all n. We say that the $K(n)$'s *detect smash-nilpotence* if every $K(*)$-null map is smash-nilpotent. Assuming this, a straightforward argument (based on our discussion of $\overline{\mathrm{Id}}(S)$) shows that the $K(n)$'s detect ideals.

In a general stable homotopy category $\mathcal{C}$, it is very hard to produce ring objects $K(n)$ such that $K(n)_*$ is a graded field. However, one can use another line of argument with rather different hypotheses. First, we say that the $K(n)$'s *detect rings* if for every nonzero ring object R we have $K(n) \wedge R \neq 0$ for some n. (This will obviously hold if $\bigvee_n \langle K(n) \rangle = \langle S \rangle$ in the Bousfield lattice.) Suppose in addition that whenever $A \in \mathcal{F}$ and $K(n) \wedge A \neq 0$ we have $\langle K(n) \wedge A \rangle = \langle K(n) \rangle$. We claim that the $K(n)$'s detect ideals. To see this, consider a thick ideal $\mathcal{A} \subseteq \mathcal{F}$, and let L be the finite localisation functor with $\ker(L) = \mathrm{loc}\langle \mathcal{A} \rangle$ (and thus $\ker(L) \cap \mathcal{F} = \mathcal{A}$). Given $X \in \mathcal{F}$ with $\mathrm{supp}(X) \subseteq \mathrm{supp}(\mathcal{A})$, we must show that $X \in \mathcal{A}$. It turns out to be equivalent to say that the ring object $R = F(X, X) \wedge LS$ is zero, so it will suffice to show that $K(n) \wedge R = 0$ for all n, and this is easy.

The main examples are as follows.

(a) In the motivating example [19, 32, 67], $\mathcal{C}$ is the category of p-local spectra, N is $\mathbb{N} \cup \{\infty\}$, and $K(n)$ is the n'th Morava K-theory (which is well-known to be a field theory, up to a slight adjustment of the definition at the prime 2). The proof that these theories detect nilpotence is a *tour de force* of stable homotopy theory, using methods very far from those surveyed in this paper. It follows that they also detect smash nilpotence. It is deduced in [32] that they detect nilpotence in various other senses, and that they detect rings. Part of this argument can be axiomatised (at least in a connective stable homotopy category) but we shall not attempt that here.

(b) Let G be a finite group, and let $\mathcal{C}$ be the category of p-local G-spectra. We then let N be the set of pairs (H, n), where H is a (representative of a) conjugacy class of subgroups of G, and $n \in \mathbb{N} \cup \{\infty\}$. We take $K(H, n)$ to be the representing object for the cohomology theory $X \mapsto K(n)^* \Phi^H X$, where Φ^H is the geometric fixed point functor, and $K(n)$ is the usual nonequivariant Morava K-theory. These representing objects can be made quite explicit, but we shall not give the details. It follows quite easily from the previous example that they detect smash-nilpotence, and thus that they detect ideals [78].

(c) Let R be a noetherian ring, and put $\mathcal{C} = \mathcal{D}_R$. We then take N to be the set of prime ideals in R, and let $K(\mathfrak{p})$ be the field of fractions of $R/\mathfrak{p}$. Here it is not hard to show that $\bigvee_{\mathfrak{p}} \langle K(\mathfrak{p}) \rangle = \langle S \rangle$ and that $\langle K(\mathfrak{p}) \rangle$ is a minimal Bousfield class; it follows that these objects detect nilpotence, and also that they detect ideals [56][35, Section 6].

(d) Now consider the case $\mathcal{C} = \mathrm{Stab}_{kG}$, where k is a field of characteristic p and G is a finite p-group. Take N to be the set of homogeneous prime ideals in $H^*(G; k)$. Next, fix an algebraically closed field L of infinite transcendence degree over k. For any $\mathfrak{p} \in N$, the theory of "shifted subgroups" gives an algebra $A \leq LG$ isomorphic to $L[u]/u^p$ and an object $K(\mathfrak{p}) \in \mathcal{C}$ such that $K(\mathfrak{p}) \wedge M = 0$ iff $L \otimes_k M$ is free as a module over A. It follows easily from the infinite version of Dade's Lemma [7] that $\bigvee_{\mathfrak{p}} \langle K(\mathfrak{p}) \rangle = \langle S \rangle$. Morever, we see from [7, Theorem 10.8] that $\langle K(\mathfrak{p}) \wedge M \rangle = \langle K(\mathfrak{p}) \rangle$ whenever $K(\mathfrak{p}) \wedge M \neq 0$, so the $K(\mathfrak{p})$'s detect ideals.

9. Brown representability

For all authors, it is either an axiom or a theorem that cohomology functors defined on the whole category $\mathcal{C}$ are representable. It follows easily that the Yoneda functor is an equivalence between $\mathcal{C}$ and the category of cohomology functors defined on $\mathcal{C}$. This is a very satisfactory result, with many applications (existence of infinite products, existence of Bousfield localisations, Brown-Comenetz duality, and so on).

It is desirable to extend this result to various subcategories $\mathcal{D} \subseteq \mathcal{C}$. If there is an exact localisation functor $L\colon \mathcal{C} \to \mathcal{D}$ (as in Section 5), then this is easy. In some other cases, it can be proved using Neeman's theory of well-generated categories [63, Chapter 8].

Similarly, it would be helpful to have a dual theorem. This should say that any product-preserving exact covariant functor $\mathcal{C} \to \mathrm{Ab}$ has the form $Y \mapsto [X, Y]$ for some representing object X. This has also been proved by Neeman [59, 63], under some additional hypotheses.

Next, let $\mathcal{F} \subseteq \mathcal{C}$ be the subcategory of small objects, and suppose that $\mathcal{F}$ generates $\mathcal{C}$. Given a cohomology functor $H\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$, it is natural to ask whether there is an object $Z \in \mathcal{C}$ and a natural isomorphism $HX = [X, Z]$ for $X \in \mathcal{F}$. It is equivalent to ask whether H can be extended to a cohomology functor defined on all of $\mathcal{C}$.

We first observe that in the case $\mathcal{C} = \mathcal{B}$, this reduces to a more familiar question. In that context, the Spanier-Whitehead duality functor $D\colon X \mapsto F(X, S)$ gives an equivalence $\mathcal{F}^{\mathrm{op}} \simeq \mathcal{F}$, so the covariant functor $H' = H \circ D\colon \mathcal{F} \to \mathrm{Ab}$ is homological. A natural isomorphism $HX = [X, Z]$ (for all $X \in \mathcal{F}$) is thus the same as a $H'X = \pi_0(X \wedge Z)$, and Brown's homological representability theorem (in the version proved by Adams) says that such an isomorphism can always be found. Adams's proof used some countability arguments, and implicitly relied on the existence of an underlying model category, so it could not directly be transferred to our axiomatic setting. Margolis [49] and Neeman [58] independently gave reformulations that do not use model categories. Neeman also showed, however, that the countability hypothesis is essential.

To explain this and related results (mostly distilled from [58, 17, 2]), it is convenient to use the category $\mathcal{A} = [\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}]$, the subcategory $\mathcal{E} \subseteq \mathcal{A}$ of exact functors, and the Yoneda functor $h\colon \mathcal{C} \to \mathcal{E}$, as discussed in Section 3. Brown's theorem says that when $\mathcal{C} = \mathcal{B}$, the functor $h\colon \mathcal{C} \to \mathcal{E}$ is full and essentially surjective. The work of Margolis and Neeman says that the same holds whenever

(a) $\mathcal{C}$ is compactly generated; and
(b) $\mathcal{F}$ has only countably many isomorphism classes, and $\mathcal{F}(A, B)$ is countable for all $A, B \in \mathcal{F}$.

Now consider the case $\mathcal{C} = D(k[x, y])$, where k is a field. Neeman has shown that if $|k| \geq \aleph_2$ then h is not full, and if $|k| \geq \aleph_3$ then h is not essentially surjective. These examples are obtained from more general and more complicated statements of two different types. Firstly, there are results relating properties of h to homological algebra in $\mathcal{A}$; secondly, there are relations between homological algebra in $\mathcal{A}$ and in the category $\mathcal{M}_R$ of R-modules, in the case where $\mathcal{C} = \mathcal{D}_R$.

For the first step, we define $\mathrm{pgldim}(\mathcal{C})$ to be the supremum of the projective dimensions in $\mathcal{A}$ of all the objects in $\mathcal{E}$. Even though h need not be essentially surjective, this is known to be the same as the supremum of the projective dimensions of objects in the image of h. It is also known that

(a) $\mathrm{pgldim}(\mathcal{C}) \leq 1$ iff h is full.
(b) If $\mathrm{pgldim}(\mathcal{C}) \leq 2$, then h is essentially surjective.
(c) Thus, if h is full, then it is essentially surjective.

For the second step, we recall some additional definitions. An R-module P is said to be *pure-projective* if it is a retract of a (possibly infinite) direct sum of finitely presented modules. The *pure-projective dimension* of a module M is the minimum possible length for a pure projective resolution of M. The *pure global dimension* of R (written $\mathrm{pgldim}(R)$) is the supremum of the pure-projective dimensions of all R-modules. The ring R is said to be *hereditary* if every submodule of a projective module is again projective. It is known that $\mathrm{pgldim}(R) \leq \mathrm{pgldim}(\mathcal{D}_R)$; the inequality is an equality when R is hereditary, but can be strict in more general cases. Using this and some additional arguments, one proves the following result:

Theorem 9.1. *Suppose that $\mathcal{C} = \mathcal{D}_R$, where R is hereditary. Then the functor $h\colon \mathcal{C} \to \mathcal{A}$ is*

(a) *full iff* $\operatorname{pgldim}(R) \leq 1$
(b) *essentially surjective iff* $\operatorname{pgldim}(R) \leq 2$.

Benson and Gnacadja [9, 10] have proved similar results for the case $\mathcal{C} = \mathrm{Stab}_{kG}$, involving questions of purity for kG-modules. In particular, they show that the following are equivalent:

(a) h is full and essentially surjective
(b) $\operatorname{pgldim}(kG) \leq 1$
(c) Either k is countable, or the Sylow p-subgroup of G is cyclic (where p is the characteristic of k).

They also give a number of intricate examples related to these results.

Now consider the case where h is full and essentially surjective, as with the original case of Boardman's category of spectra. We then say that $\mathcal{C}$ is a *Brown category.* This has a number of useful consequences [35, 18, 2]. Firstly, for $F \in \mathcal{A}$, the following are equivalent:

(a) F has finite projective dimension in $\mathcal{A}$
(b) F has projective dimension at most one
(c) F has finite injective dimension
(d) F has injective dimension at most one
(e) $F \in \mathcal{E}$
(f) F is in the image of h.

Next, we say that a map v in $\mathcal{C}$ is *phantom* if $h(v) = 0$. In a Brown category, the composite of any two phantom maps is zero, so the phantoms form a square-zero ideal. (Benson [5] has shown that this can fail when $\mathcal{C}$ is not a Brown category; in particular, it fails when $\mathcal{C} = \mathrm{Stab}_{kG}$, k is an uncountable field of characteristic p, and the p-rank of G is at least two.) Some further properties of phantom maps are studied in [16].

Finally, consider a diagram $X\colon I \to \mathcal{C}$, where I is a filtered category. A *weak colimit* for the diagram consists of a object U and compatible maps $X_i \to U$ for $i \in I$, such that the induced map $[U, Y] \to \varprojlim_I [X_i, Y]$ is surjective for all $Y \in \mathcal{C}$. Such a weak colimit is *minimal* if the induced map $\varinjlim_i [Z, X_i] \to [Z, U]$ is a bijection for all $Z \in \mathcal{F}$. In a Brown category, it is known that

(a) Every filtered diagram of small objects has a minimal weak colimit, which is a retract of any other weak colimit.
(b) Every object can be expressed as the minimal weak colimit of a filtered diagram of small objects.

References

[1] J. F. Adams. *Stable Homotopy and Generalised Homology.* University of Chicago Press, Chicago, 1974.
[2] A. Beligiannis. Relative homological algebra and purity in triangulated categories. *J. Algebra*, 227(1):268–361, 2000.
[3] D. Benson and H. Krause. Pure injectives and the spectrum of the cohomology ring of a finite group. *J. Reine Angew. Math.*, 542:23–51, 2002.
[4] D. J. Benson. Infinite dimensional modules for finite groups. In *Infinite length modules (Bielefeld, 1998)*, Trends Math., pages 251–272. Birkhäuser, Basel, 2000.
[5] D. J. Benson. Phantom maps and purity in modular representation theory. III. *J. Algebra*, 248(2):747–754, 2002.
[6] D. J. Benson, J. F. Carlson, and J. Rickard. Complexity and varieties for infinitely generated modules. *Math. Proc. Cambridge Philos. Soc.*, 118(2):223–243, 1995.
[7] D. J. Benson, J. F. Carlson, and J. Rickard. Complexity and varieties for infinitely generated modules. II. *Math. Proc. Cambridge Philos. Soc.*, 120(4):597–615, 1996.
[8] D. J. Benson, J. F. Carlson, and J. Rickard. Thick subcategories of the stable module category. *Fund. Math.*, 153(1):59–80, 1997.
[9] D. J. Benson and G. P. Gnacadja. Phantom maps and purity in modular representation theory. I. *Fund. Math.*, 161(1-2):37–91, 1999. Algebraic topology (Kazimierz Dolny, 1997).

[10] D. J. Benson and G. P. Gnacadja. Phantom maps and purity in modular representation theory. II. *Algebr. Represent. Theory*, 4(4):395–404, 2001.
[11] D. J. Benson and H. Krause. Generic idempotent modules for a finite group. *Algebr. Represent. Theory*, 3(4):337–346, 2000. Special issue dedicated to Klaus Roggenkamp on the occasion of his 60th birthday.
[12] D. J. Benson and W. W. Wheeler. The Green correspondence for infinitely generated modules. *J. London Math. Soc. (2)*, 63(1):69–82, 2001.
[13] M. Bökstedt and A. Neeman. Homotopy limits in triangulated categories. *Compositio Math.*, 86(2):209–234, 1993.
[14] T. Bridgeland. Equivalences of triangulated categories and Fourier-Mukai transforms. *Bull. London Math. Soc.*, 31(1):25–34, 1999.
[15] J. D. Christensen. Ideals in triangulated categories: phantoms, ghosts and skeleta. *Adv. Math.*, 136(2):284–339, 1998.
[16] J. D. Christensen and M. Hovey. Phantom maps and chromatic phantom maps. *Amer. J. Math.*, 122(2):275–293, 2000.
[17] J. D. Christensen, B. Keller, and A. Neeman. Failure of Brown representability in derived categories. *Topology*, 40(6):1339–1361, 2001.
[18] J. D. Christensen and N. P. Strickland. Phantom maps and homology theories. *Topology*, 37(2):339–364, 1998.
[19] E. S. Devinatz, M. J. Hopkins, and J. H. Smith. Nilpotence and stable homotopy theory I. *Annals of Mathematics*, 128:207–242, 1988.
[20] D. Dugger and B. Shipley. K-theory and derived equivalences, arXiv:math.KT/0209084.
[21] W. G. Dwyer and J. P. C. Greenlees. Complete modules and torsion modules. *Amer. J. Math.*, 124(1):199–220, 2002.
[22] W. G. Dwyer and J. H. Palmieri. Ohkawa's theorem: there is a set of Bousfield classes. *Proc. Amer. Math. Soc.*, 129(3):881–886, 2001.
[23] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, Modules and Algebras in Stable Homotopy Theory*, volume 47 of *Amer. Math. Soc. Surveys and Monographs*. American Mathematical Society, 1996.
[24] H. Fausk, P. Hu, , and J. May. Isomorphisms between left and right adjoints, arXiv:math.AT/0206079.
[25] H. Fausk, L. G. Lewis, Jr., and J. P. May. The Picard group of equivariant stable homotopy theory. *Adv. Math.*, 163(1):17–33, 2001.
[26] J. Franke. Uniqueness theorems for certain triangulated categories posessing an Adams spectral sequence. Preprint, 1996.
[27] J. Franke. On the Brown representability theorem for triangulated categories. *Topology*, 40(4):667–680, 2001.
[28] P. Freyd. The Grothendieck group for stable homotopy is free. *Bull. Amer. Math. Soc.*, 73:84–86, 1967.
[29] J. P. C. Greenlees and J. P. May. Derived functors of I-adic completion and local homology. *Journal of Algebra*, 149(2):438–453, 1992.
[30] M. J. Hopkins. Global methods in homotopy theory. In E. Rees and J. D. Jones, editors, *Homotopy Theory — Proceedings of the Durham Symposium*, pages 73–96. Cambridge University Press, 1987. LMSLN 117.
[31] M. J. Hopkins, D. C. Ravenel, and W. S. Wilson. Morava Hopf algebras and spaces $K(n)$ equivalent to finite Postnikov systems. In S. O. Kochman and P. Selick, editors, *Stable and unstable homotopy*, Fields Institute Communications. AMS, 1998. To appear.
[32] M. J. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.
[33] M. Hovey. Homotopy theory of comodules over a Hopf algebroid, arXiv:math.AT/0301229.
[34] M. Hovey. Model category structures on chain complexes of sheaves. *Trans. Amer. Math. Soc.*, 353(6):2441–2457 (electronic), 2001.
[35] M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.
[36] M. Hovey and N. P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):104, 1999.
[37] P. T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1982.
[38] B. Keller. A remark on the generalized smashing conjecture. *Manuscripta Math.*, 84(2):193–198, 1994.
[39] H. Krause. Decomposing thick subcategories of the stable module category. *Math. Ann.*, 313(1):95–108, 1999.
[40] H. Krause. Smashing subcategories and the telescope conjecture—an algebraic approach. *Invent. Math.*, 139(1):99–133, 2000.
[41] H. Krause. Brown representability and flat covers. *J. Pure Appl. Algebra*, 157(1):81–86, 2001.
[42] H. Krause. On Neeman's well generated triangulated categories. *Doc. Math.*, 6:121–126 (electronic), 2001.
[43] H. Krause. The spectrum of a module category. *Mem. Amer. Math. Soc.*, 149(707):x+125, 2001.
[44] H. Krause. A Brown representability theorem via coherent functors. *Topology*, 41(4):853–861, 2002.
[45] H. Krause. Coherent functors in stable homotopy theory. *Fund. Math.*, 173(1):33–56, 2002.
[46] H. Krause and U. Reichenbach. Endofiniteness in stable homotopy theory. *Trans. Amer. Math. Soc.*, 353(1):157–173 (electronic), 2001.
[47] H. Krause and G. Zwara. Stable equivalence and generic modules. *Bull. London Math. Soc.*, 32(5):615–618, 2000.

[48] L. G. Lewis, J. P. May, and M. S. (with contributions by Jim E. McClure). *Equivariant Stable Homotopy Theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer–Verlag, New York, 1986.
[49] H. R. Margolis. *Spectra and the Steenrod Algebra*. North-Holland, 1983.
[50] J. P. May. *Equivariant homotopy and cohomology theory*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.
[51] J. P. May. The additivity of traces in triangulated categories. *Adv. Math.*, 163(1):34–73, 2001.
[52] J. P. May. Picard groups, Grothendieck rings, and Burnside rings of categories. *Adv. Math.*, 163(1):1–16, 2001.
[53] H. R. Miller. Finite localizations. *Boletin de la Sociedad Matematica Mexicana*, 37:383–390, 1992. This is a special volume in memory of José Adem, and is really a book. The editor is Enrique Ramírez de Arellano.
[54] F. Morel and V. Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.
[55] A. Neeman. Some new axioms for triangulated categories. *Journal of Algebra*, 139:221–255, 1991.
[56] A. Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt.
[57] A. Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *Journal of the American Mathematical Society*, 9:205–236, 1996.
[58] A. Neeman. On a theorem of Brown and Adams. *Topology*, 36(3):619–645, 1997.
[59] A. Neeman. Brown representability for the dual. *Invent. Math.*, 133(1):97–105, 1998.
[60] A. Neeman. Non-compactly generated categories. *Topology*, 37(5):981–987, 1998.
[61] A. Neeman. Oddball Bousfield classes. *Topology*, 39(5):931–935, 2000.
[62] A. Neeman. On the derived category of sheaves on a manifold. *Doc. Math.*, 6:483–488 (electronic), 2001.
[63] A. Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.
[64] T. Ohkawa. The injective hull of homotopy types with respect to generalized homology functors. *Hiroshima Mathematical Journal*, 19(3):631–639, 1989.
[65] D. C. Ravenel. Morava k–theories and finite groups. In S. Gitler, editor, *Symposium on Algebraic Topology in Honor of José Adem*, Contemporary Mathematics, pages 289–292. American Mathematical Society, 1982.
[66] D. C. Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106:351–414, 1984.
[67] D. C. Ravenel. *Nilpotence and Periodicity in Stable Homotopy Theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, 1992.
[68] J. Rickard. Morita theory for derived categories. *J. London Math. Soc. (2)*, 39(3):436–456, 1989.
[69] J. Rickard. Splendid equivalences: derived categories and permutation modules. *Proc. London Math. Soc. (3)*, 72(2):331–358, 1996.
[70] J. Rickard. Idempotent modules in the stable category. *J. London Math. Soc. (2)*, 56(1):149–170, 1997.
[71] J. Rickard. Some recent advances in modular representation theory. In *Algebras and modules, I (Trondheim, 1996)*, volume 23 of *CMS Conf. Proc.*, pages 157–178. Amer. Math. Soc., Providence, RI, 1998.
[72] J. Rickard. Bousfield localization for representation theorists. In *Infinite length modules (Bielefeld, 1998)*, Trends Math., pages 273–283. Birkhäuser, Basel, 2000.
[73] S. Schwede. The stable homotopy category has a unique model at the prime 2. *Adv. Math.*, 164(1):24–40, 2001.
[74] S. Schwede and B. Shipley. Classification of stable model categories, arXiv:math.AT/0108143.
[75] S. Schwede and B. Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003.
[76] B. Shipley. Monoidal uniqueness of stable homotopy theory. *Adv. Math.*, 160(2):217–240, 2001.
[77] N. P. Strickland. $K(N)$-local duality for finite groups and groupoids. *Topology*, 39(4):733–772, 2000.
[78] N. P. Strickland. Equivariant Bousfield classes. In preparation, 2002.

Department of Pure Mathematics, University of Sheffield, Sheffield S3 7RH, UK
E-mail address: N.P.Strickland@sheffield.ac.uk