PRODUCTS ON MU-MODULES

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1. Introduction

In [2, Chapter V], Elmendorf, Kriz, Mandell and May (hereafter referred to as EKMM) use their new technology of modules over highly structured ring spectra to give new constructions of MU-modules such as BP, $K(n)$ and so on, which makes it much easier to analyse product structures on these spectra. Unfortunately, their construction only works in its simplest form for modules over $MU[\frac{1}{2}]_*$ that are concentrated in degrees divisible by 4; this guarantees that various obstruction groups are trivial. In the present paper we extend the EKMM results to the cases where $p = 2$ or the homotopy groups are allowed to be nonzero in all even degrees. In this context the obstruction groups are nontrivial. We shall show that there are never any obstructions to associativity. We prove in Section 7 that the obstructions to commutativity are given by a certain power operation; this was inspired by a parallel result of Mironov in Baas-Sullivan theory [6]. In Section 8 we shall use formal group theory to derive various formulae for this power operation. In Section 9 we deduce a number of results about realising 2-local MU_*-modules as MU-modules.

2. Statement of Results

We use the category $\mathcal{M} = \mathcal{M}^S$ constructed in [2, Section II.2]; we recall some details in Section 3. The main point is that $\mathcal{M}$ is a symmetric monoidal category with a closed model structure whose homotopy category is Boardman's homotopy category of spectra. In the language of [2], the objects of $\mathcal{M}$ are $\mathbb{L}$-spectra X for which $F_{\mathcal{L}}(S, X) = X$, but we shall simply call them spectra.

Because $\mathcal{M}$ is a symmetric monoidal category, it makes sense to talk about strictly commutative ring spectra; these are essentially equivalent to E_∞ ring spectra in earlier foundational settings. Let R be such an object, such that $R_* = \pi_* R$ is even (by which we mean, concentrated in even degrees). The main example of interest to us is $R = MU$ or $MU_{(p)}$ for some prime p. There are well-known constructions of MU as a spectrum in the earlier sense of Lewis and May [4], with an action of the E_∞ operad of complex linear isometries. Thus, the results of [2, Chapter II] allow us to construct MU as a strictly commutative ring spectrum. By [2, Theorem VIII.2.1], we can also realise $MU_{(p)}$.

One can define a category $\mathcal{M}_R$ of R-modules in the evident way, with all diagrams commuting at the geometric level. After inverting weak equivalences, we obtain a homotopy category $\mathcal{D} = \mathcal{D}_R$, referred to as the derived category of $\mathcal{M}_R$. We shall mainly work in this derived category, and the category $\mathcal{R} = \mathcal{R}_R$ of ring objects in $\mathcal{D}$ (referred to in [2] as R-ring spectra). All our ring objects are assumed to be associative and to have a two-sided unit. Thus, an object $A \in \mathcal{R}$ has an action $R \wedge A \to A$ which makes various diagrams commute at the geometric level, and a product $A \wedge A \to A$ that is geometrically compatible with the R-module structure, and is homotopically associative and unital. We also write $\mathcal{R}_*$ for the category of algebras over the discrete ring R_*. We write $\mathcal{R}_*^e$ for the category of even R_*-algebras, and $\mathcal{R}_*^c$ for the commutative ones, and similarly $\mathcal{R}_*^{ec}$, $\mathcal{R}^e$, $\mathcal{R}^c$ and $\mathcal{R}^{ec}$.

Definition 2.1. Let A_* be an even commutative R_*-algebra without 2-torsion. A *strong realisation* of A_* is a commutative ring object $A \in \mathcal{R}^{ec}$ with a given isomorphism $\pi_*(A) \simeq A_*$, such that the resulting map

$$\mathcal{R}(A, B) \to \mathcal{R}(A_*, \pi_*(B))$$

is an isomorphism whenever $B \in \mathcal{R}^{ec}$ and B_* has no 2-torsion. We say that A_* is *strongly realisable* if such a realisation exists.

Remark 2.2. It is easy to see that the category of strongly realisable R_*-algebras is equivalent to the category of those $A \in \mathcal{R}^{ec}$ for which $\pi_*(A)$ is strongly realisable. In particular, any two strong realisations of A_* are canonically isomorphic.

Our main aim is to prove that certain R_*-algebras are strongly realisable, and to prove some more *ad hoc* results for certain algebras over $MU_*/2$. The relationship between our results and the results of [2] in the same direction will be discussed after Theorem 2.14.

Definition 2.3. A *localised regular quotient (LRQ)* of R_* is an algebra A_* over R_* that can be written in the form $A_* = S^{-1}R_*/I$, where S is any set of (homogeneous) elements in R_* and I is an ideal which can be generated by a regular sequence.

Remark 2.4. If A_* is an LRQ of R_* and B_* is an arbitrary R_*-algebra then $\mathcal{R}_*(A_*, B_*)$ has at most one element.

We start by stating a result for odd primes, which is relatively easy.

Theorem 2.5. *If* 2 *is invertible in* R_* *then every LRQ of* R_* *is strongly realisable.*

This will be proved as Theorem 6.11.

Our main contribution is the extension to the case where 2 is not inverted. Our results involve a certain "commutativity obstruction" $\overline{c}(x) \in \pi_{2|x|+2}(R)/(2, x)$, which is defined in Section 4. In Section 7, we show that this arises from a power operation $\widetilde{P}\colon \pi_d(R) \to \pi_{2d+2}(R)/2$. This result was inspired by a parallel result of Mironov in Baas-Sullivan theory [6]. In Section 8 we show how to compute this power operation using formal group theory, at least in the case $R = MU$ or $MU_{(2)}$. The first steps in this direction were also taken by Mironov [6], but our results are much more precise.

The case where A_* has no 2-torsion is quite simple and similar to the case where 2 is inverted.

Theorem 2.6. *Let* $A_* = S^{-1}R_*/I$ *be an LRQ of* R_* *which has no* 2*-torsion. Suppose also that* $\widetilde{P}(I) \subseteq I \pmod 2$. *Then* A_* *is strongly realisable.*

This will be proved as Theorem 6.12.

We next recall the definitions of some algebras over MU_* which one might hope to realise as spectra using the above results. First, we have the rings

$$\begin{aligned} kU_* &= \mathbb{Z}[u] \qquad\qquad |u| = 2 \\ KU_* &= \mathbb{Z}[u^{\pm 1}] \\ H_* &= \mathbb{Z} \qquad \text{(in degree zero)} \\ H\mathbb{F}_* &= \mathbb{F}_p \qquad \text{(in degree zero).} \end{aligned}$$

These are LRQ's of MU_* in well-known ways. Next, we consider the Brown-Peterson ring

$$BP_* = \mathbb{Z}_{(p)}[v_k \mid k > 0] \qquad\qquad |v_k| = 2(p^k - 1).$$

We take $v_0 = p$ as usual. There is a p-typical formal group law F over this ring such that

$$[p]_F(x) = \exp_F(px) +_F \sum_{k>0}^{F} v_k x^{p^k}.$$

(Thus, our v_k's are Hazewinkel's generators rather than Araki's.) We make BP_* into an algebra over MU_* in the usual way. We define

$$\begin{aligned} P(n)_* &= BP_*/(v_i \mid i < n) = \mathbb{F}_p[v_j \mid j \geq n] \\ B(n)_* &= v_n^{-1}BP_*/(v_i \mid i < n) = v_n^{-1}\mathbb{F}_p[v_j \mid j \geq n] \\ k(n)_* &= BP_*/(v_i \mid i \neq n) = \mathbb{F}_p[v_n] \\ K(n)_* &= v_n^{-1}BP_*/(v_i \mid i \neq n) = \mathbb{F}_p[v_n^{\pm 1}] \\ BP\langle n\rangle_* &= BP_*/(v_i \mid i > n) = \mathbb{Z}_{(p)}[v_1, \dots, v_n] \\ E(n)_* &= v_n^{-1}BP_*/(v_i \mid i > n) = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}] \end{aligned}$$

These are all LRQ's of BP_*, and it is not hard to check that BP_* is an LRQ of $MU_{(p)*}$, and thus that all the above rings are LRQ's of $MU_{(p)*}$. We record a slightly sharper fact that is sometimes convenient; it surely counts as "well-known to the experts".

Proposition 2.7. *The image of the map $MU_* \to BP_*$ is $\mathbb{Z}[v_k \mid k > 0]$, and this is a regular quotient of MU_*.*

This will be proved as Proposition 8.15.

We also let $w_k \in \pi_{2(p^k-1)}MU$ denote the bordism class of a smooth hypersurface W_{p^k} of degree p in $\mathbb{C}P^{p^k}$. It is well-known that $I_n = (w_i \mid i < n)$ is the smallest ideal modulo which the universal formal group law over MU_* has height n, and that the image of I_n in BP_* is the ideal $(v_i \mid i < n)$. In fact, we have

$$\sum_{m>0} [W_m] x^m dx = [p]_F(x) d\log_F(x).$$

Moreover, the sequence of w_i's is regular, so that MU_*/I_n is an LRQ of MU_*.

One can also define LRQ's of $MU[\frac{1}{6}]_*$ giving rise to various versions of elliptic homology, but we refrain from giving details here. If we do not invert 6 then the relevant rings seem not to be LRQ's of MU_*. If we take $R = MU_p^\wedge$ then we can make $\mathbb{Z}_p[v_n]$ into an LRQ of R_* in such a way that the resulting formal group law is of the (non-p-typical) type considered by Lubin and Tate in algebraic number theory. We can also take $R = L_{K(n)}MU$ and consider $\widehat{E(n)}_*$ as an LRQ of R_* via the Ando orientation [1] rather than the more usual p-typical one. We leave the details of these applications to the reader.

The following proposition is immediate from Theorem 2.5.

Proposition 2.8. *If $R = MU_{(p)}$ with $p > 2$ then $kU_{(p)*}$, $KU_{(p)*}$, $H_{(p)*}$, $H\mathbb{F}_{p*}$, BP_*, $P(n)_*$, $B(n)_*$, $k(n)_*$, $BP\langle n\rangle_*$, $E(n)_*$ and MU_*/I_n are all strongly realisable.*

After doing some computations with the power operation $\widetilde{P}$, we will also prove the following.

Proposition 2.9. *If $R = MU$ then kU_*, KU_*, H_* and $H\mathbb{F}_*$ are strongly realisable. If $R = MU_{(2)}$ then $kU_{(2)*}$, $KU_{(2)*}$, $H_{(2)*}$ and BP_* are strongly realisable.*

The situation is less satisfactory for the rings $BP\langle n\rangle_*$ and $E(n)_*$ at $p = 2$. For $n > 1$, they cannot be realised as the homotopy rings of commutative ring objects in $\mathcal{D}$. However, if we kill off a slightly different sequence of elements instead of the sequence $(v_{n+1}, v_{n+2}, \dots)$, we get a quotient ring that is realisable. The resulting spectrum serves as a good substitute for $BP\langle n\rangle$ in almost all arguments.

Proposition 2.10. *If $R = MU_{(2)}$ and $n > 0$, there is a quotient ring $BP\langle n\rangle'_*$ of BP_* such that*

1. *The evident map*
$$\mathbb{Z}_{(2)}[v_1, \dots, v_n] \rightarrowtail BP_* \twoheadrightarrow BP\langle n\rangle'_*$$
is an isomorphism.
2. *$BP\langle n\rangle'_*$ is strongly realisable.*
3. *We have $BP\langle n\rangle'_*/I_n = k(n)_* = BP_*/(v_i \mid i \neq n)$ as MU_*-algebras.*

Moreover, the ring $E(n)'_* = v_n^{-1}BP\langle n\rangle'_*$ is also strongly realisable. If $n = 1$ then we can take $BP\langle 1\rangle'_* = BP\langle 1\rangle_*$.

This is proved in Section 9.

The situation for $MU_*/2$ and algebras over it is also more complicated than for odd primes.

Definition 2.11. Throughout this paper, we write τ for the twist map $X \wedge X \to X \wedge X$, for any object X for which this makes sense. We say that a ring map $f\colon A \to B$ in $\mathcal{R}$ is *central* if

$$\psi \circ \tau \circ (f \wedge 1) = \psi \circ (f \wedge 1)\colon A \wedge B \to B,$$

where $\phi\colon B \wedge B \to B$ is the product. We say that B is a *central A-algebra* if there is a given central map $A \to B$.

Theorem 2.12. *When $R = MU_{(2)}$, there is a ring $MU/I_n \in \mathcal{R}$ with $\pi_*(MU/I_n) = MU_*/I_n$, and derivations $Q_i \colon MU/I_n \to \Sigma^{2^{i+1}-1} MU/I_n$ for $0 \leq i < n$. If ψ is the product on MU/I_n we have*

$$\psi \circ \tau - \psi = w_n \psi \circ (Q_{n-1} \wedge Q_{n-1}).$$

There are actually many non-isomorphic rings with these properties. We will outline an argument that specifies one of them unambiguously.

This is proved in Section 9.

We get a sharper statement for algebras over $P(n)_*$.

Theorem 2.13. *When $R = MU_{(2)}$, there is a central BP-algebra $P(n) = BP \wedge MU/I_n \in \mathcal{R}$ and an isomorphism $\pi_* P(n) = P(n)_*$. This has derivations $Q_i \colon P(n) \to \Sigma^{2^{i+1}-1} P(n)$ for $0 \leq i < n$. If ψ is the product on $P(n)$ we have*

$$\psi \circ \tau - \psi = v_n \psi \circ (Q_{n-1} \wedge Q_{n-1}).$$

If B is another central BP-algebra such that

$$\pi_k B = \begin{cases} \{0,1\} & \text{if } k = 0 \\ 0 & \text{if } 0 < k < |v_n| \\ \{0, v_n\} & \text{if } k = |v_n| \end{cases}$$

then either there is a unique map $P(n) \to B$ of BP-algebras, or there is a unique map $P(n) \to B^{op}$. Analogous statements hold for $B(n)$, $k(n)$ and $K(n)$ with BP replaced by $v_n^{-1}BP$, $BP\langle n\rangle'$ and $E(n)'$ respectively.

This is also proved in Section 9. Related results were announced by Würgler in [10], but there appear to be some problems with the line of argument used there. A correct proof on similar lines has recently been given by Nassau [7, 8].

All of the results stated above rely on the following theorem, which analyses the possible products on a module of the form R/x.

Theorem 2.14. *Suppose that d is even and $x \in R_d$ is not a zero-divisor, and let R/x denote the cofibre of $x \colon \Sigma^d R \to R$. We consider only associative products on R/x for which the canonical map $R \to R/x$ is a two-sided unit.*

1. *The set of products on R/x is a principal homogeneous space for R_{2d+2}/x (in particular, it is nonempty).*
2. *There are commutative products on R/x if and only if $\widetilde{P}(x) = 0 \pmod{x}$.*
3. *If so, the commutative products form a principal homogeneous space for $\operatorname{ann}(2, R_{2d+2}/x)$.*

This is essentially proved in Section 4, except that we do not identify the commutativity obstruction as a power operation until Section 7.

We now explain the relationship between our results and those of [2, Chapter V] in the same direction. There EKMM prove that if A_* is an LRQ of $MU[\frac{1}{2}]_*$ which is concentrated in degrees divisible by 4, then there is a ring $A \in \mathcal{R}^{ec}$ with $\pi_*(A) = A_*$. The condition on degrees is necessary to ensure that a certain associativity obstruction group is zero. However, it turns out that the obstruction is zero even if the group is not; this is implicitly part of the theorem stated above. The corresponding result in Baas-Sullivan theory was already known (this is proved in [5] in a form which is valid when R_* need not be concentrated in even degrees). Apart from that, the main contributions of the present paper are to relate the commutativity obstruction to power operations and to compute the power operations using formal group theory. We deduce our realisation results from these calculations using methods that follow [2] quite closely.

3. Foundations

In this section, we recall some foundational material from [2]. There EKMM use the word "spectrum" in the sense defined by Lewis and May [4], rather than the sense we use elsewhere in this paper. They construct a category $\mathbb{L}\mathcal{S}$ of "$\mathbb{L}$-spectra". This depends on a universe $\mathcal{U}$, but

the functor $\mathcal{L}(\mathcal{U},\mathcal{V}) \ltimes_{\mathcal{L}(U)} (-)$ gives a canonical equivalence of categories from $\mathbb{L}$-spectra over $\mathcal{U}$ to $\mathbb{L}$-spectra over $\mathcal{V}$, so the dependence is only superficial. (Here $\mathcal{L}(\mathcal{U},\mathcal{V})$ is the space of linear isometries from $\mathcal{U}$ to $\mathcal{V}$.) We therefore take $\mathcal{U} = \mathbb{R}^\infty$. EKMM show that $\mathbb{L}\mathcal{S}$ has a commutative and associative smash product $\wedge_{\mathcal{L}}$, which is not unital. However, there is a sort of "pre-unit" object S, with a natural map $S \wedge_{\mathcal{L}} X \to X$. They define subcategories

$$\mathcal{M}_S = \{X \mid S \wedge_{\mathcal{L}} X = X\}$$

$$\mathcal{M}^S = \{X \mid X = F_{\mathcal{L}}(S, X)\}.$$

They also show that the functors $S \wedge_{\mathcal{L}} (-) \colon \mathcal{M}^S \to \mathcal{M}_S$ and $F_{\mathcal{L}}(S,-) \colon \mathcal{M}_S \to \mathcal{M}^S$ are equivalences. In this section we will refer to the objects of $\mathcal{M}^S$ as complete $\mathbb{L}$-spectra. EKMM prefer to study $\mathcal{M}_S$ but for a variety of technical reasons we prefer $\mathcal{M}^S$; in particular, rather more of the spectra occuring in nature lie in $\mathcal{M}^S$. To avoid cluttering the notation, we write $\mathcal{M}$ for $\mathcal{M}^S$. We also write $X \wedge Y = F_{\mathcal{L}}(S, X \wedge_{\mathcal{L}} Y)$, which is the natural smash product functor for $\mathcal{M}$.

We next give a brief outline of the properties of $\mathcal{M}$. Let $\mathcal{T}$ be the category of based spaces (all spaces are assumed to be compactly generated and weakly Hausdorff). We write 0 for the one-point space, or for the basepoint in any based space, or for the trivial map between based spaces.

We give $\mathcal{T}$ the usual Quillen model structure for which the fibrations are Serre fibrations. We write $h\mathcal{T}$ for the category with Hom sets $\pi_0 F(A,B) = \mathcal{T}(A,B)/\text{homotopy}$, and $\overline{h}\mathcal{T}$ for the category obtained by inverting the weak equivalences. We refer to $h\mathcal{T}$ as the strong homotopy category of $\mathcal{T}$, and $\overline{h}\mathcal{T}$ as the weak homotopy category.

The category $\mathcal{M}$ is a topological category: the Hom sets $\mathcal{M}(X,Y)$ are based spaces, and there are continuous composition maps

$$\mathcal{M}(X,Y) \wedge \mathcal{M}(Y,Z) \to \mathcal{M}(X,Z).$$

We again have a strong homotopy category $h\mathcal{M}$, with $h\mathcal{M}(X,Y) = \pi_0\mathcal{M}(X,Y)$; when we have defined homotopy groups, we will also define a weak homotopy category $\overline{h}\mathcal{M}$ in the obvious way.

$\mathcal{M}$ is a closed symmetric monoidal category, with smash product and function objects again written as $X \wedge Y$ and $F(X,Y)$. Both of these constructions are continuous functors of both arguments. The unit of the smash product is S.

There is a functor $\Sigma^\infty \colon \mathcal{T} \to \mathcal{M}$, such that

$$\begin{aligned} \Sigma^\infty S^0 &= S \\ \Sigma^\infty(A \wedge B) &= \Sigma^\infty A \wedge \Sigma^\infty B \\ \mathcal{M}(\Sigma^\infty A, \Sigma^\infty B) &= \mathcal{T}(A,B). \end{aligned}$$

This last equation shows that Σ^∞ is a full and faithful embedding of $\mathcal{T}$ in $\mathcal{M}$, so that all of unstable homotopy theory is embedded in the strong homotopy category $h\mathcal{M}$. In particular, $h\mathcal{M}$ is very far from Boardman's stable homotopy category $\mathcal{B}$. However, it turns out that the weak homotopy category $\overline{h}\mathcal{M}$ is equivalent to $\mathcal{B}$.

The objects of $\mathcal{M}$ are spectra in the sense of Lewis and May [4] with extra structure, so in particular there is a zero'th space functor $\Omega^\infty \colon \mathcal{M} \to \mathcal{T}$. This is represented by an object $\mathbb{L}S \in \mathcal{M}$, in the sense that $\mathcal{M}(\mathbb{L}S, X)$ is homeomorphic to $\Omega^\infty X$. (The zero'th space functor restricted to $\mathcal{M}_S$ is not representable, which is another reason to prefer $\mathcal{M} = \mathcal{M}^S$.)

To understand the properties of $\mathbb{L}S$, it is helpful to consider a larger class of objects. These are not mentioned explicitly in [2], but I find them conceptually illuminating. Let $\mathcal{U}$ be a universe. We will need to consider the functor $Q_{\mathcal{U}} \colon \mathcal{T} \to \mathcal{T}$ defined by

$$Q_{\mathcal{U}} A = \varinjlim_{U} \Omega^U \Sigma^U A,$$

where the colimit runs over finite-dimensional subspaces of $\mathcal{U}$. There is a natural way to make the Lewis-May spectrum $\Sigma^\infty \mathcal{L}(\mathcal{U},\mathbb{R}^\infty)_+$ into a $\mathcal{L}$-spectrum, which we call $S(\mathcal{U})$. It turns out that this lies in $\mathcal{M}$; this essentially reduces to the fact that the subspace of $Q_{\mathcal{V}\oplus\mathcal{W}}\mathcal{L}(\mathcal{U},\mathcal{V}\oplus\mathcal{W})_+$ invariant

under the action of $\mathcal{L}(\mathcal{W})$ is just $Q_{\mathcal{V}}\mathcal{L}(\mathcal{U},\mathcal{V})_+$. This gives a contravariant functor $S\colon \{\text{Universes}\} \to \mathcal{M}$, with properties as follows.

$$\begin{aligned} S(\mathbb{R}^\infty) &= \mathbb{L}S \\ S(\mathcal{U}) \wedge S(\mathcal{V}) &= S(\mathcal{U}\oplus\mathcal{V}) \\ \mathcal{M}(S(\mathcal{U}), \Sigma^\infty A) &= Q_{\mathcal{U}} A \\ \mathcal{M}(S(\mathcal{U}), S(\mathcal{V})) &= Q_{\mathcal{U}}\mathcal{L}(\mathcal{V},\mathcal{U})_+. \end{aligned}$$

Moreover, for any finite-dimensional subspace $U < \mathcal{U}$, there is a natural subobject $S(\mathcal{U},U) \leq S(\mathcal{U})$ and a canonical isomorphisms

$$\begin{aligned} S(\mathcal{U},U) \wedge S(\mathcal{U},V) &= S(\mathcal{U}\oplus\mathcal{V}, U\oplus V) \\ \Sigma^U S(\mathcal{U}, U\oplus V) &= S(\mathcal{U},V). \end{aligned}$$

This indicates that the objects $S(\mathcal{U},U)$ are in some sense stable. They can be defined as follows: take the Lewis-May spectrum $\Sigma^\infty_U S^0$ indexed on $\mathcal{U}$, and then take the twisted half smash product with the space $\mathcal{L}(\mathcal{U},\mathbb{R}^\infty)$ to get a Lewis-May spectrum $S(\mathcal{U},U)$ indexed on $\mathbb{R}^\infty$. This is easily seen to be an $\mathbb{L}$-spectrum in a natural way.

For any $n > 0$ and $d \geq 0$ we write

$$\begin{aligned} \mathcal{L}(n) &= \mathcal{L}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty) \\ S(n) &= S((\mathbb{R}^\infty)^n) = S(1)^{(n)} \\ S^d(1) &= \Sigma^d S(\mathbb{R}^\infty) \\ S^{-d}(1) &= S(\mathbb{R}^\infty, \mathbb{R}^d) \end{aligned}$$

We will also allow ourselves to write $S^d(n)$ for $\Sigma^k S((\mathbb{R}^\infty)^n, V)$ where V is a subspace of $(\mathbb{R}^\infty)^n$ of dimension $k-d$ and k and V are clear from the context.

Any object of the form $S^U \wedge S(\mathcal{V},V)$ is non-canonically isomorphic to $S^d(1)$, where $d = \dim(U) - \dim(V)$, but when one is interested in the naturality or otherwise of various constructions it is often a good idea to forget this fact. There are isomorphisms $S^n(1)\wedge S^m(1) \simeq S^{n+m}(1)$ that become canonical and coherent in the homotopy category. The homotopy groups of an object $X \in \mathcal{M}$ are defined by

$$\pi_n(X) = h\mathcal{M}(S^n(1), X).$$

We say that a map $f\colon X \to Y$ is a weak equivalence if it induces an isomorphism $\pi_*(X) \to \pi_*(Y)$, and we define the weak homotopy category $\overline{h}\mathcal{M}$ by inverting weak equivalences. We define a cell object to be an object of $\mathcal{M}$ that is built from the sphere objects $S^n(1)$ in the usual sort of way; the category $\overline{h}\mathcal{M}$ is then equivalent to the category of cell objects and homotopy classes of maps.

Now let R be a commutative ring object in $\mathcal{M}$, in other words an object equipped with maps $S \xrightarrow{\eta} R \xleftarrow{\mu} R\wedge R$ making the relevant diagrams geometrically (rather than homotopically) commutative. (The term "ring" is something of a misnomer, as there is no addition until we pass to homotopy.) We let $\mathcal{M}_R$ denote the category of module objects over R in the evident sense. This is again a topological model category with a closed symmetric monoidal structure. The basic cofibrant objects are the free modules $S^d(1)\wedge R$ for $d \in \mathbb{Z}$. The weak homotopy category $\overline{h}\mathcal{M}_R$ obtained by inverting weak equivalences is also known as the derived category of R, and written $\mathcal{D} = \mathcal{D}_R$; it is equivalent to the strong homotopy category of cell R-modules. It is not hard to see that $\mathcal{D}$ is a monogenic stable homotopy category in the sense of [3]; in particular, it is a triangulated category with a compatible closed symmetric monoidal structure.

For the rest of the paper, we fix a strictly commutative ring spectrum R and we work in $\mathcal{D}$ unless otherwise specified. In particular, $X \wedge Y$ will denote the smash product over R, and $[X,Y]$ will mean $\mathcal{D}(X,Y)$.

4. Products on R/x

Suppose that $x \in R_d$ is not a zero-divisor (so d is even). We then have a cofibre sequence in the triangulated category $\mathcal{D}$:

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho} R/x \xrightarrow{\beta} \Sigma^{d+1} R.$$

Because x is not a zero divisor, we have $\pi_*(R/x) = R_*/x$. In particular, $\pi_{d+1}R/x = 0$ (because $d+1$ is odd), and thus $\rho^* \colon [R/x, R/x] \simeq [R, R/x]$. It follows that R/x is unique up to unique isomorphism as an object under R.

We next set up a theory of products on objects of the form R/x. Apart from the fact that all such products are associative, our results are at most minor sharpenings of the those in [2, Chapter V].

Observe that $(R/x)^{(2)}$ is a cell R-module with one 0-cell, two $(d+1)$-cells and one $(2d+2)$-cell. We say that a map $\phi \colon (R/x)^{(2)} \to R/x$ is a *product* if it agrees with ρ on the bottom cell, in other words $\phi \circ (\rho \wedge \rho) = \rho \colon R \to R/x$. Products exist by [2, Theorem V.2.6].

The main result is as follows.

Proposition 4.1.
1. *All products are associative, and have ρ as a two-sided unit.*
2. *The set of products on R/x is a principal homogeneous space for R_{2d+2}/x (in particular, it is nonempty).*
3. *There is a naturally defined element $\overline{c}(x) \in \pi_{2d+2}(R)/(2, x)$ such that R/x admits a commutative product if and only if $\overline{c}(x) = 0$.*
4. *If so, the commutative products form a principal homogeneous space for* $\operatorname{ann}(2, R_{2d+2}/x)$.

We will identify $\overline{c}$ with a power operation in Section 7.

Proof. Part (1) is proved as Lemma 4.2 and Proposition 4.6. Part (2) is Lemma 4.5. Parts (3) and (4) form Corollary 4.8. □

Lemma 4.2. *If $\phi \colon (R/x)^{(2)} \to R/x$ is a product then*

$$\phi \circ (\rho \wedge 1) = \phi \circ (1 \wedge \rho) = 1 \colon R/x \to R/x.$$

Thus, our definition agrees with that in [2, Chapter V].

Proof. By hypothesis, $\phi \circ (\rho \wedge 1) \colon R/x \to R/x$ is the identity on the bottom cell of R/x. Because $[R/x, R/x] \simeq [R, R/x]$, we conclude that $\phi \circ (\rho \wedge 1) = 1$. Similarly $\phi \circ (1 \wedge \rho) = 1$. □

Corollary 4.3. *The map $x \colon \Sigma^d R/x \to R/x$ is zero.*

Proof. Choose a product ϕ on R/x. Using the unital property, this map can be rewritten as

$$\Sigma^d R \wedge R/x \xrightarrow{(x\rho)\wedge 1} (R/x)^{(2)} \xrightarrow{\phi} R/x,$$

and the map $x\rho \colon \Sigma^d R \to R/x$ is clearly zero. □

Lemma 4.4. *Let $A \in \mathcal{D}$ be such that $x \colon \Sigma^d A \to A$ is zero. Then the diagram*

$$R/x \vee R/x \xrightarrow{(\rho\wedge 1, 1\wedge\rho)} (R/x)^{(2)} \xrightarrow{\beta\wedge\beta} \Sigma^{2d+2} R$$

induces a left-exact sequence

$$[\Sigma^{2d+2} R, A] \rightarrowtail [(R/x)^{(2)}, A] \to [R/x \vee R/x, A].$$

Similarly, the evident analogous diagram gives an exact sequence

$$[\Sigma^{3d+3} R, A] \rightarrowtail [(R/x)^{(3)}, A] \to [(R/x)^{(2)} \vee (R/x)^{(2)} \vee (R/x)^{(2)}, A].$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & R/x \arrow[d, "\rho\wedge 1"] & \\
R/x \arrow[r, "1\wedge\rho"] \arrow[d, "\beta"'] & (R/x)^{(2)} \arrow[d, "\beta\wedge 1"] \arrow[dr, "\beta\wedge\beta"] & \\
\Sigma^{d+1}R \arrow[r, "\rho"'] & \Sigma^{d+1}R/x \arrow[r, "\beta"'] & \Sigma^{2d+2}R
\end{tikzcd}$$

We now apply the functor $[-,A]$ and make repeated use of the cofibration

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho} R/x \xrightarrow{\beta} \Sigma^{d+1}R.$$

The conclusion is that all maps involving β become monomorphisms, all maps involving ρ become epimorphisms, and the bottom row and the middle column become short exact. The first claim follows by diagram chasing. For the second claim, consider the diagram

$$\begin{tikzcd}
 & (R/x)^{(2)} \vee (R/x)^{(2)} \arrow[d, "(\rho\wedge 1\wedge 1, 1\wedge\rho\wedge 1)"] & \\
(R/x)^{(2)} \arrow[r, "1\wedge 1\wedge\rho"] \arrow[d, "\beta\wedge\beta"'] & (R/x)^{(3)} \arrow[d, "\beta\wedge\beta\wedge 1"] \arrow[dr, "\beta\wedge\beta\wedge\beta"] & \\
\Sigma^{2d+2}R \arrow[r, "\rho"'] & \Sigma^{2d+2}R/x \arrow[r, "\beta"'] & \Sigma^{3d+3}R
\end{tikzcd}$$

We apply the same logic as before, using the first claim (with A replaced by $F(R/x,A)$) to see that the middle column becomes left exact. □

We next determine how many different products there are on R/x.

Lemma 4.5. *The set of products on R/x is a principal homogeneous space for $\pi_{2d+2}(R)/x$. More precisely, if ϕ, ϕ' are two products on R/x then there is a unique element*

$$u \in \pi_{2d+2}R/x = [\Sigma^{2d+2}R, R/x]$$

such that $\phi' = \phi + u \circ (\beta \wedge \beta)$.

Proof. Using the unital properties of ϕ and ϕ' given by Lemma 4.2, we see that

$$(\phi' - \phi) \circ (\rho \wedge 1) = (\phi' - \phi) \circ (1 \wedge \rho) = 0.$$

Because of Corollary 4.3, we can apply Lemma 4.4 to see that $\phi' - \phi = u \circ (\beta \wedge \beta)$ for a unique element $u \in [\Sigma^{2d+2}R, R/x]$, as claimed. □

Proposition 4.6. *Any product on R/x is associative.*

Proof. Let ϕ be a product, and write

$$\delta = \phi \circ (\phi \wedge 1 - 1 \wedge \phi) \colon (R/x)^{(3)} \to R/x,$$

so the claim is that δ is nullhomotopic. Using the unital properties of ϕ we see that

$$\delta \circ (\rho \wedge 1 \wedge 1) = \delta \circ (1 \wedge \rho \wedge 1) = \delta \circ (1 \wedge 1 \wedge \rho) = 0.$$

Using Lemma 4.4, we conclude that $\delta = u \circ (\beta \wedge \beta \wedge \beta)$ for a unique element $u \in [\Sigma^{3d+3}R, R/x] = \pi_{3d+3}(R)/x = 0$ (because $3d+3$ is odd). Thus $\delta = 0$ as claimed. □

We now discuss commutativity.

Lemma 4.7. *There is a natural map c from the set of products to $\pi_{2d+2}R/x$ such that $c(\phi) = 0$ if and only if ϕ is commutative. Moreover,*

$$c(\phi + u \circ (\beta \wedge \beta)) = c(\phi) - 2u.$$

Proof. Let $\tau\colon (R/x)^{(2)} \to (R/x)^{(2)}$ be the twist map. Clearly, if ϕ is a product then so is $\phi\circ\tau$. Thus, there is a unique element $v \in \pi_{2d+2}R/x$ such that

$$\phi\circ\tau = \phi + v\circ(\beta\wedge\beta).$$

We define $c(\phi) = v$. Next, recall that the twist map on $\Sigma^{2d+2}R = \Sigma^{d+1}R\wedge\Sigma^{d+1}R$ is homotopic to (-1), because $d+1$ is odd. It follows by naturality that $(\beta\wedge\beta)\circ\tau = -\beta\wedge\beta$. Consider a second product $\phi' = \phi + u\circ(\beta\wedge\beta)$. We now see that

$$\phi'\circ\tau = \phi + v\circ(\beta\wedge\beta) - u\circ(\beta\wedge\beta) = \phi' + (v-2u)\circ(\beta\wedge\beta).$$

Thus $c(\phi') = c(\phi) - 2u$ as claimed. □

Corollary 4.8. *There is a naturally defined element $\overline{c}(x) \in \pi_{2d+2}(R)/(2,x)$ such that R/x admits a commutative product if and only if $\overline{c}(x) = 0$. If so, the commutative products form a principal homogeneous space for $\operatorname{ann}(2, \pi_{2d+2}(R)/x)$. In particular, if $\pi_*(R)/x$ has no 2-torsion then there is a unique commutative product.*

Proof. We choose a product ϕ on R/x and define $\overline{c}(x) = c(\phi) \pmod 2$. This is well-defined, by the lemma. If $\overline{c}(x) \neq 0$ then $c(\phi') \neq 0$ for all ϕ', so there is no commutative product. If $\overline{c}(x) = 0$ then $c(\phi) = 2w$, say, so that $\phi' = \phi + w\circ(\beta\wedge\beta)$ is a commutative product. In this case, the commutative products are precisely the products of the form $\phi' + z\circ(\beta\wedge\beta)$ where $2z = 0$, so they form a principal homogeneous space for $\operatorname{ann}(2, \pi_{2d+2}(R)/x)$. □

Next, we consider the Bockstein operation:

$$\overline{\beta} = \rho\beta\colon R/x \to \Sigma^{d+1}R/x.$$

Definition 4.9. Let $A \in \mathcal{R}$ be a ring, with product $\phi\colon A\wedge A \to A$. We say that a map $Q\colon A \to \Sigma^k A$ is a *derivation* if we have

$$Q\circ\phi = \phi\circ(Q\wedge 1 + 1\wedge Q)\colon A^{(2)} \to A.$$

Proposition 4.10. *The map $\overline{\beta}$ is a derivation with respect to any product ϕ on R/x.*

Proof. Write $\delta = \overline{\beta}\circ\phi - \phi\circ(\overline{\beta}\wedge 1 + 1\wedge\overline{\beta})$, so the claim is that $\delta = 0$. It is easy to see that $\delta\circ(\rho\wedge 1) = \delta\circ(1\wedge\rho) = 0$, so by Lemma 4.4 we see that δ factors through a unique map $\Sigma^{2d+2}R \to \Sigma^{d+1}R/x$. This is an element of $\pi_{d+1}(R)/x$, which is zero because $d+1$ is odd. □

We conclude this section by analysing maps out of the rings R/x.

Proposition 4.11. *Let $A \in \mathcal{R}^e$ be an even ring. If x maps to zero in π_*A then there is precisely one unital map $f\colon R/x \to A$, and otherwise there are no such maps. If f exists and ϕ is a product on R/x, then there is a naturally defined element $d_A(\phi) \in \pi_{2d+2}(A)$ such that*

(a) *$d_A(\phi) = 0$ if and only if f is a ring map with respect to ϕ.*
(b) *$d_A(\phi + u\circ(\beta\wedge\beta)) = d_A(\phi) + u$.*
(c) *If A is commutative then $2d_A(\phi) = c(\phi) \in \pi_{2d+2}A$.*

Proof. The statement about the existence and uniqueness of f follows immediately from the cofibration $\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho} R/x \xrightarrow{\beta} \Sigma^{d+1}R$, and the fact that $\pi_{d+1}A = 0$. Suppose that f exists; the argument of Corollary 4.3 shows that $x\colon \Sigma^d A \to A$ is zero. Now let ψ be the given product on A, and let ϕ be a product on R/x. Consider the map

$$\delta = \psi\circ(f\wedge f) - f\circ\phi\colon (R/x)^{(2)} \to A.$$

By the usual argument, we have $\delta = v\circ(\beta\wedge\beta)$ for a unique map $v\colon \Sigma^{2d+2}R \to A$. We define $d_A(\phi) = v \in \pi_{2d+2}A$. It is obvious that this vanishes if and only if f is a ring map, and that $d_A(\phi + u\circ(\beta\wedge\beta)) = d_A(\phi) + u$.

Now suppose that A is commutative, so $\psi = \psi\circ\tau$. On the one hand, using the fact that $(\beta\wedge\beta)\circ\tau = -\beta\wedge\beta$ we see that $\delta - \delta\circ\tau = 2d_A(\phi)\circ(\beta\wedge\beta)$. On the other hand, from the definition of δ and the fact that $\psi\circ\tau = \psi$, we see that

$$\delta - \delta\circ\tau = f\circ(\phi - \phi\circ\tau) = c(\phi)\circ(\beta\wedge\beta).$$

Because $(\beta \wedge \beta)^* \colon \pi_{2d+2}A \to [(R/x)^{(2)}, A]$ is a split monomorphism, we conclude that $2d_A(\phi) = c(\phi)$ in $\pi_{2d+2}A$. $\square$

5. Strictly unital products

In the previous section we worked in the derived category $\mathcal{D}$ of (strict) R-modules. This can be constructed from the category $\mathcal{M}_R$ of R-modules either by inverting the weak equivalences, or by restricting attention to cell R-modules and passing to homotopy. In this section we sharpen the picture slightly by working with modules with strict units. These are not cell R-modules, so we need to distinguish between $\overline{h}\mathcal{M}_R(X,Y) = \mathcal{D}(X,Y) = [X,Y]$ and $h\mathcal{M}_R(X,Y) = \pi_0\mathcal{M}_R(X,Y) = \mathcal{M}_R(X,Y)/\text{homotopy}$. Note that the latter need not have a group structure (let alone an Abelian one). However, most of the usual tools of unstable homotopy theory are available in $h\mathcal{M}_R$, because $\mathcal{M}_R$ is a topological category enriched over pointed spaces. In particular, we will need to use Puppe sequences.

As previously, we let x be a regular element in $\pi_d(R)$, so d is even. We regard x as an R-module map $S^d(1) \wedge R \to R$, and we write R/x for the cofibre. There is thus a pushout diagram

$$\begin{tikzcd} S^d(1) \wedge R \arrow[r] \arrow[d, "x"'] & I \wedge S^d(1) \wedge R \arrow[d] \\ R \arrow[r, "\rho"'] & R/x \end{tikzcd}$$

As R is not a cell R-module, the same is true of R/x. However, the map $\rho\colon R \to R/x$ is a cofibration. One can also see that $S^0(1) \wedge R/x$ is a cell R-module which is the cofibre in $\mathcal{D}$ of the map $x\colon \Sigma^d R \to R$, so it has the homotopy type referred to as R/x in the previous section. Moreover, the map $S^0(1) \wedge R/x \to R/x$ is a homotopy equivalence of underlying spectra, and thus a weak equivalence. It follows that our new R/x has the same weak homotopy type as in previous sections.

Let X be defined by the following pushout diagram:

$$\begin{tikzcd} R \arrow[r, "\rho"] \arrow[d, "\rho"'] & R/x \arrow[d, "i_0"] \\ R/x \arrow[r, "i_1"'] & X \end{tikzcd}$$

There is a unique map $\nabla\colon X \to R/x$ such that $\nabla i_0 = 1 = \nabla i_1$, and there is an evident cofibration

$$S^{2d+1}(2) \wedge R \to X \to (R/x)^{(2)}.$$

Here

$$S^{2d+1}(2) = \Sigma S^d(1) \wedge S^d(1) = \begin{cases} \Sigma^{2d+1} S(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) & \text{if } d \geq 0 \\ \Sigma S(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^{|d|} \oplus \mathbb{R}^{|d|}) & \text{if } d < 0. \end{cases}$$

We define a *strictly unital product* on R/x to be a map $\phi\colon (R/x)^{(2)} \to R/x$ of R-modules such that $\phi|_X = \nabla$. Let P be the space of strictly unital products, and let $\overline{P}$ be the set of products on R/x in the sense of section 4.

Proposition 5.1. *The evident map $\pi_0(P) \to \overline{P}$ is a bijection.*

Proof. The cofibration $S^{2d+1}(2) \wedge R \to X \to (R/x)^{(2)}$ gives a Puppe sequence

$$h\mathcal{M}(S^{2d+2}(2) \wedge R, R/x) \to h\mathcal{M}((R/x)^{(2)}, R/x) \to h\mathcal{M}(X, R/x) \to h\mathcal{M}(S^{2d+1}(2) \wedge R, R/x).$$

As usual in an unstable context, the first term is a group which acts on the second term. The orbit set maps bijectively to the preimage of zero under the third map. In our case, the first term is $\pi_{2d+2}(R/x)$ and the last one is $\pi_{2d+1}(R/x) = 0$. It follows that there exist R-module maps $\phi\colon (R/x)^{(2)} \to R/x$ whose restriction to X is homotopic to ∇, and that $\pi_{2d+2}(R/x)$ acts transitively on the set of homotopy classes of such maps. As $X \to (R/x)^{(2)}$ is a cofibration, this

set bijects with $\pi_0(P)$, so we have a transitive action of $\pi_{2d+2}(R)$ on $\pi_0(P)$. It is easy to see that this is compatible with the free and transitive action of $\pi_{2d+2}(R)$ on $\overline{P}$ given by Lemma 4.5, and thus that $\pi_0(P)$ maps isomorphically to $\overline{P}$. □

Remark 5.2. This gives another proof of associativity. Let Y be the union of all cells except the top one in $(R/x)^{(3)}$, so there is a cofibration $S^{3d+2}(3) \wedge R \to Y \to (R/x)^{(3)}$. If ϕ is a strictly unital product, it is easy to see that $\phi \circ (\phi \wedge 1)$ and $\phi \circ (1 \wedge \phi)$ have the same restriction to Y (on the nose). It follows using the Puppe sequence that they only differ (up to homotopy) by the action of the group $\pi_{3d+3}(R/x) = 0$.

6. Strong realisations

In this section we assemble the products which we have constructed on the R-modules R/x to get products on more general R_*-algebras. We will work entirely in the derived category $\mathcal{D}$, rather than the underlying geometric category. All the main ideas in this section come from [2, Chapter V].

We start with some generally nonsensical preliminaries.

Definition 6.1. Given a diagram $A \xrightarrow{f} C \xleftarrow{g} B$ in $\mathcal{R}$, we say that f commutes with g if and only if we have

$$\phi_C \circ (f \wedge g) = \phi_C \circ \tau \circ (f \wedge g) \colon A \wedge B \to C.$$

Note that this can be false when $f = g$; in particular A is commutative if and only if 1_A commutes with itself.

The next three lemmas become trivial if we replace $\mathcal{D}$ by the category of modules over a ring. The proofs in that context can easily be made diagrammatic and thus carried over to $\mathcal{D}$.

Lemma 6.2. *If A and B are rings in $\mathcal{R}$, then there is a unique ring structure on $A \wedge B$ such that the evident maps $A \xrightarrow{i} A \wedge B \xleftarrow{j} B$ are commuting ring maps. Moreover, this becomes the universal example of a commuting pair of maps out of A and B.* □

Lemma 6.3. *A map $f \colon A \wedge B \to C$ commutes with itself if and only if $f \circ i$ commutes with itself and $f \circ j$ commutes with itself. In particular, $A \wedge B$ is commutative if and only if i and j commute with themselves.* □

Lemma 6.4. *If A and B are commutative, then so is $A \wedge B$, and it is the coproduct of A and B in $\mathcal{R}^c$.* □

Corollary 6.5. *If A and B are strong realisations of A_* and B_*, and the natural map $A_* \otimes_{R_*} B_* \to \pi_*(A \wedge B)$ is an isomorphism, then $A \wedge B$ is a strong realisation of $A_* \otimes_{R_*} B_*$.* □

Proposition 6.6. *Let S be a set of homogeneous elements of R_*. Then $S^{-1}R_*$ has a strong realisation $S^{-1}R$, and $\pi_*(S^{-1}R \wedge M) = S^{-1}\pi_*(M)$ for all $M \in \mathcal{D}$.*

Proof. We know from [3, Theorem 3.3.7] that there is a smashing localisation functor $L \colon \mathcal{D} \to \mathcal{D}$ and a natural isomorphism $S^{-1}\pi_* M = \pi_*(LM) = \pi_*(LR \wedge M)$. We define $S^{-1}R = LR$, which is easily seen to be a commutative ring object [3, Proposition 3.1.8].

Consider an arbitrary ring $A \in \mathcal{R}$. Note that there is precisely one map $S^{-1}R_* \to A_*$ if S becomes invertible in A_*, and no maps otherwise.

If there is no map $S^{-1}R_* \to A_*$, then clearly there can be no map $S^{-1}R \to A$. On the other hand, suppose that there is a map $S^{-1}R_* \to A_*$, so that S becomes invertible in A_*. This means that $\pi_*(LA) = \pi_*(A)$, so $A = LA$, so $[S^{-1}R, A] = [LR, A] = [R, A]$. It follows easily that there is a unique ring map $S^{-1}R \to A$, as claimed. □

Now consider a sequence (x_i) in R_*, with products ϕ_i on R/x_i. Write $A_i = R/x_1 \wedge \ldots \wedge R/x_i$, and make this into a ring as in Lemma 6.2. There are evident maps $A_i \to A_{i+1}$, so we can form the telescope $A = \underrightarrow{\operatorname{holim}}_i A_i$.

Lemma 6.7. *If $M \in \mathcal{D}$ and $I = (x_1, x_2, \dots) \le R_*$ acts trivially on M and $r \ge 0$ then $[A^{(r)}, M] = \varprojlim_i [A_i^{(r)}, M]$.*

Proof. This will follow immediately from the Milnor sequence if we can show that $\varprojlim_i^1 [A_i^{(r)}, M]_* = 0$. For this, it suffices to show that the map $\rho^* \colon [B \wedge R/x_i, M] \to [B, M]$ is surjective for all B. This follows from the cofibration $\Sigma^{|x_i|} B \xrightarrow{x_i} B \to B \wedge R/x_i$ and the fact that x_i acts trivially on M. ☐

Proposition 6.8. *Let (x_i) be a sequence in R_*, and ϕ_i a product on R/x_i for each i. Let A be the homotopy colimit of the rings $A_i = R/x_1 \wedge \dots \wedge R/x_i$. Then there is a unique associative and unital product on A such that the evident maps $f_i \colon R/x_i \to A$ are commuting ring maps. This product is commutative if and only if each f_i commutes with itself. Ring maps from A to any ring B biject with systems of ring maps $g_i \colon R/x_i \to B$ such that g_i commutes with g_j for all $i \neq j$.*

Proof. Because R/x_i admits a product, we know that x_i acts trivially on R/x_i. Because A has the form $R/x_i \wedge B$, we see that x_i acts trivially on A. Thus I acts trivially on A, and Lemma 6.7 assures us that $[A^{(r)}, A] = \varprojlim_i [A_i^{(r)}, A]$.

Let ψ_i be the product on A_i. By the above, there is a unique map $\psi \colon A \wedge A \to A$ which is compatible with the maps ψ_i. It is easy to check that this is an associative and unital product, and that it is the only one for which the f_i are commuting ring maps. It is also easy to check that ψ is commutative if and only if each of the maps $A_i \to A$ commutes with itself, if and only if each f_i commutes with itself.

Now let B be any ring in $\mathcal{R}$. We may assume that each x_i maps to zero in $\pi_*(B)$, for otherwise the claimed bijection is between empty sets. As B is a ring, this means that each x_i acts trivially on B, so that $[A^{(r)}, B] = \varprojlim_i [A_i^{(r)}, B]$. The bijection follows easily. ☐

Corollary 6.9. *If each R/x_i is commutative, then A is the coproduct of the R/x_i in $\mathcal{R}^c$.* ☐

Remark 6.10. If the sequence (x_i) is regular, then it is easy to see that $\pi_*(A) = R_*/(x_1, x_2, \dots)$. Note also that ring maps out of R/x were analysed in Proposition 4.11.

We now restate and prove Theorems 2.5 and 2.6. Of course, the former is a special case of the latter, but it seems clearest to prove Theorem 2.5 first and then explain the improvements necessary for Theorem 2.6.

Theorem 6.11. *If 2 is invertible in R_*, then every LRQ of R_* is strongly realisable.*

Proof. Consider a localised regular quotient $A_* = S^{-1}R_*/I$, where I is generated by a regular sequence $(x_1, x_2, \dots)$. Using Proposition 6.6 and Corollary 6.5, we see that it suffices to give a strong realisation of R_*/I. We know from Proposition 4.1 that there is a unique commutative product ϕ_i on R/x_i. If $C \in \mathcal{R}^{ec}$ and $x_i = 0$ in $\pi_*(C)$ then in the notation of Proposition 4.11 we have $2d_C(\phi_i) = 0$ and thus $d_C(\phi_i) = 0$, so the unique unital map $R/x_i \to C$ is a ring map. It follows that R/x_i is a strong realisation of R_*/x_i, and thus that $A_i = R/x_1 \wedge \dots \wedge R/x_i$ is a strong realisation of $R_*/(x_1, \dots, x_i)$. Using Proposition 6.8, we get a ring A which is a strong realisation of R_*/I. ☐

We next address the case where 2 is not a zero-divisor, but is not invertible either.

Theorem 6.12. *Let $A_* = S^{-1}R_*/I$ be an LRQ of R_* which has no 2-torsion. Suppose also that $\widetilde{P}(I) \le I \pmod 2$. Then A_* is strongly realisable.*

Proof. Choose a regular sequence (x_i) generating I. As $\overline{c}(x_i) = \widetilde{P}(x_i) \in I \pmod 2$, we can choose a product ϕ_i on R/x_i such that $c(\phi_i) \in I$. We let A' be the "infinite smash product" of the R/x_i, as in Proposition 6.8, and we write $A = S^{-1}R \wedge A'$, so that $\pi_*(A) = A_*$. Because $c(\phi_i)$ maps to zero in $\pi_*(A')$, we see easily that the map $R/x_i \to A'$ commutes with itself. By Proposition 6.8, we conclude that A' is commutative, and thus that A is commutative. Clearly $\pi_*(A) = A_*$.

Let $B \in \mathcal{R}^{ec}$ be an even commutative ring, and that $\pi_*(B)$ has no 2-torsion. The claim is that $\mathcal{R}(A, B) = \mathcal{R}_*(A_*, \pi_*(B))$. The right hand side has at most one element, and if it is empty, then the left hand side is also. Thus, we may assume that there is a map $A_* \to \pi_*(B)$ of R_*-algebras, and we need to show that there is a unique ring map $A \to B$.

By Proposition 6.8, we know that ring maps $A' \to B$ biject with systems of ring maps $R/x_i \to B$ (which automatically commute as B is commutative). There is a unique unital map $f\colon R/x_i \to B$, and Proposition 4.11 tells us that the obstruction to f being a homomorphism satisfies $2d_B(\phi_i) = c(\phi_i) = 0 \in \pi_*(B)$. Because $\pi_*(B)$ has no 2-torsion, we have $d_B(\phi_i) = 0$, so there is a unique ring map $R/x_i \to B$, and thus a unique ring map $A' \to B$. As the elements of S become invertible in $\pi_*(B)$, this extends to give a unique ring map $A \to B$, as required. □

The following result is also useful.

Proposition 6.13. *Let A_* be a strongly realisable R_*-algebra, and let $A_* \to B_*$ be a map of R_*-algebras that makes B_* into a free module over A_*. Then B_* is strongly realisable.*

Proof. First, observe that if F and M are A-modules, there is a natural map

$$\mathrm{Hom}_A(F, M) \to \mathrm{Hom}_{A_*}(F_*, M_*),$$

which is an isomorphism if F is a wedge of suspensions of A (in other words, a free A-module).

Choose a homogeneous basis $\{e_i\}$ for B_* over A_*, where e_i has degree d_i. Define $B = \bigvee_i \Sigma^{d_i} A$, so that B is a free A-module with a given isomorphism $\pi_* B \simeq B_*$ of A_*-modules. Define $B_0 = A$ and $B_1 = B$ and

$$B_2 = \bigvee_{i,j} \Sigma^{d_i + d_j} A$$

$$B_3 = \bigvee_{i,j,k} \Sigma^{d_i + d_j + d_k} A.$$

The product map $\mu\colon A \wedge A \to A$ gives rise to evident maps $\phi_k\colon B^{(k)} \to B_k$ which in turn give isomorphisms $B^{\otimes_{A_*} k} = \pi_* B_k$ of A_*-modules. The multiplication map $B_* \otimes_{A_*} B_* \to B_*$ corresponds under the isomorphism $\mathrm{Hom}_A(B_2, B) = \mathrm{Hom}_{A_*}(\pi_* B_2, B_*)$ to a map $B_2 \to B$. After composing this with ϕ_2, we get a product map $\mu_B\colon B \wedge B \to B$. A similar procedure gives a unit map $A \to B$.

We next prove that this product is associative. Each of the two associated products $B^{(3)} \to B$ factors as ϕ_3 followed by a map $B_3 \to B$, corresponding to a map $B_*^{\otimes_{A_*} 3} \to A_*$. The two maps $B_*^{\otimes_{A_*} 3} \to A_*$ in question are just the two possible associated products, which are the same because B_* is associative. It follows that B is associative. Similar arguments show that B is commutative and unital.

Now consider an object $C \in \mathcal{R}$ equipped with a map $B_* \to C_*$ (and thus a map $A_* \to C_*$). As A is a strong realisation of A_*, there is a unique map $A \to C$ compatible with the map $A_* \to C_*$. This makes C into an A-module, and thus gives an isomorphism $\mathrm{Hom}_A(B, C) = \mathrm{Hom}_{A_*}(B_*, C_*)$. There is thus a unique A-module map $B \to C$ inducing the given map $B_* \to C_*$. It follows easily that B_* is a strong realisation of B_*. □

We will need to consider certain R_*-algebras that are not strongly realisable. The following result assures us that weaker kinds of realisation are not completely uncontrolled.

Proposition 6.14. *Let A_* be an LRQ of R_*, and let $A_1, A_2 \in \mathcal{R}$ be rings (not necessarily commutative) such that $\pi_*(A_1) = A_* = \pi_*(A_2)$. Then there is an isomorphism $f\colon A_1 \to A_2$ (not necessarily a ring map) that is compatible with the unit maps $A_1 \leftarrow R \to A_2$.*

Proof. Write $A_* = S^{-1} R_*/I$ and $I = (x_1, x_2, \dots)$ in the usual way. Let A' be the infinite smash product of the R/x_i's, and write $A = S^{-1} A'$ so that $\pi_*(A) = A_*$. It will be enough to show that there is a unital isomorphism $A \to A_1$. Moreover, any unital map $A \to A_1$ is automatically an isomorphism, just by looking at the homotopy groups.

There is a unique unital map $f_i\colon R/x_i \to A_1$. Write $A_i = R/x_1 \wedge \ldots \wedge R/x_i$, and let g_i be the map

$$A_i \xrightarrow{f_1\wedge\ldots\wedge f_i} A_1^{(i)} \to A_1,$$

where the second maps is the product. Because A_1 is a ring and each x_i goes to zero in $\pi_*(A_1)$, we can apply Lemma 6.7 to get a unital map $g\colon A' \to A_1$, and thus a unital map $S^{-1}A' \to S^{-1}A_1 = A_1$, as required. □

We conclude this section by investigating R-module maps $A \to A$ for various R-algebras $A \in \mathcal{R}$.

Proposition 6.15. *Let $\{x_1, x_2, \ldots\}$ be a regular sequence in R_*, let ϕ_i be a product on R/x_i, and let A be the infinite smash product of the rings R/x_i. Let $Q_i\colon A \to \Sigma^{|x_i|+1}A$ be obtained by smashing the Bockstein map $\overline{\beta}_{x_i}\colon R/x_i \to \Sigma^{|x_i|+1}R/x_i$ with the identity map on all the other R/x_j's. Then $\mathcal{D}(A,A)_*$ is isomorphic as an algebra over A_* to the completed exterior algebra on the elements Q_i.*

Proof. It is not hard to see that $Q_iQ_j = -Q_jQ_i$, with a sign coming from an implicit permutation of suspension coordinates. We also have $\overline{\beta}_i^2 = 0$ and thus $Q_i^2 = 0$. Given any finite subset $S = \{i_1 < \ldots < i_n\}$ of the positive integers, we define

$$Q_S = Q_{i_1}Q_{i_2}\ldots Q_{i_n}\colon A \to \Sigma^{d_S}A,$$

where $d_S = \sum_j(|x_{i_j}|+1)$. The claim is that one can make sense of homogeneous infinite sums of the form $\sum_S a_SQ_S$ with $a_S \in A_*$, and that any graded map $A \to A$ of R-modules is uniquely of that form.

Write $A_n = R/x_1 \wedge \ldots \wedge R/x_n$, and let $i_n\colon A_n \to A$ be the evident map. It is easy to check that $Q_S \circ i_n = 0$ if $\max(S) > n$, and a simple induction shows that $\mathcal{D}(A_n, A)_*$ is a free module over A_* generated by the maps $Q_S \circ i_n$ for which $\max(S) \leq n$. Moreover, Lemma 6.7 implies that $\mathcal{D}(A,A)_* = \varprojlim_n \mathcal{D}(A_n, A)_*$. The claim follows easily. □

The above result relies more heavily than one would like on the choice of a regular sequence generating the ideal $\ker(R_* \to A_*)$. We will use the following construction to make things more canonical.

Construction 6.16. Let $A \in \mathcal{R}^e$ be an even ring, with unit $\eta\colon R \to A$, and let I be the kernel of $\eta_*\colon R_* \to A_*$. Given a derivation $Q\colon A \to \Sigma^kA$, we define a function $d(Q)\colon I \to A_*$ as follows. Given $x \in I$, we have a cofibration

$$\Sigma^dR \xrightarrow{x} R \xrightarrow{\rho_x} R/x \xrightarrow{\beta_x} \Sigma^{d+1}R$$

as usual. Here x may be a zero-divisor in R_*, so we need not have $\pi_*(R/x) = \pi_*(R)/x$. Nonetheless, we see easily that there is a unique map $f_x\colon R/x \to A$ such that $f_x \circ \rho_x = \eta$. As Q is a derivation, one checks easily that $Q \circ \eta = 0$, so $(Q \circ f_x) \circ \rho_x = 0$, so $Q \circ f_x = y \circ \beta_x$ for some $y\colon \Sigma^{d+1}R \to \Sigma^kA$. Because x acts as zero on A, we see that y is unique. We can thus define $d(Q)(x) = y \in \pi_{d+1-k}A$.

Proposition 6.17. *Let $A \in \mathcal{R}^e$ be such that $\pi_*(A) = R_*/I$, where I can be generated by a regular sequence. Let $\mathrm{Der}(A)$ be the set of derivations $A \to A$. Then Construction 6.16 gives rise to a natural monomorphism $d\colon \mathrm{Der}(A) \to \mathrm{Hom}_{R_*}(I/I^2, A_*)$ (with degrees shifted by one).*

Proof. Choose a regular sequence $\{x_1, x_2, \ldots\}$ generating I. Write $A_n = R/x_1 \wedge \ldots \wedge R/x_n$, and let j_n be the map

$$R/x_1 \wedge \ldots R/x_n \xrightarrow{f_{x_1}\wedge\ldots\wedge f_{x_n}} A^{(n)} \xrightarrow{\text{product}} A.$$

It is easy to see that A is the homotopy colimit of the objects A_n (although there may not be a ring structure on A_n for which j_n is a homomorphism). We also write $A_{n,i}$ for the smash product of the R/x_j for which $j \leq n$ and $j \neq i$, and $j_{n,i}$ for the evident map $A_{n,i} \to A_n \xrightarrow{j_n} A$.

Consider a derivation $Q\colon A \to \Sigma^k A$, and write $b_i = d(Q)(x_i)$. Because Q is a derivation, we see that $Q \circ j_n$ is a sum of n terms, of which the i'th is b_i times the composite

$$A_n = A_{n,i} \wedge R/x_i \xrightarrow{1\wedge\beta_{x_i}} \Sigma^{|x_i|+1} A_{n,i} \xrightarrow{j_{n,i}} \Sigma^{|x_i|+1} A.$$

Now consider an element $x = \sum_{i=1}^n a_i x_i$ of I. It is easy to see that there is a unique unital map $f'_x\colon R/x \to A_n$, and that $j_n \circ f'_x = f_x$. Now consider the following diagram.

$$\begin{tikzcd}
\Sigma^d R \arrow[r, "x"] \arrow[d, "a_i"'] & R \arrow[r, "\rho_x"] \arrow[d, "1"'] & R/x \arrow[r, "\beta_x"] \arrow[d, "f'_x"] & \Sigma^{d+1} R \arrow[d, "a_i"] \\
\Sigma^{|x_i|} A_{n,i} \arrow[r, "x_i"'] & A_{n,i} \arrow[r, "1\wedge\rho_{x_i}"'] & A_n \arrow[r, "1\wedge\beta_{x_i}"'] & \Sigma^{|x_i|+1} A_{n,i}
\end{tikzcd}$$

The left hand square commutes because the terms $a_j x_j$ for $j \neq i$ become zero in $\pi_*(A_{n,i})$. It follows that there exists a map $R/x \to A_n$ making the whole diagram commute. However, f'_x is the *unique* map making the middle square commute, so the whole diagram commutes as drawn. Thus $j_{n,i}\circ(\beta_{x_i}\wedge 1)\circ f'_x = a_i \circ \beta_x$ (thinking of a_i as an element of $\pi_*(A)$). As $Q\circ j_n = \sum_i b_i.(j_{n,i}\circ(\beta_{x_i}\wedge 1))$, we conclude that $Q \circ f_x = Q \circ j_n \circ f'_x = (\sum_i a_i b_i) \circ \beta_x$. Thus $d(Q)(x) = \sum_i a_i b_i$.

This shows that $d(Q)$ is actually a homomorphism $I/I^2 \to A_*$. It is easy to check that the whole construction gives a homomorphism $d\colon \mathrm{Der}(A) \to \mathrm{Hom}_{A_*}(I/I^2, A_*)$. If $d(Q) = 0$ then all the elements b_i are zero, so $Q \circ j_n = 0$. As A is the homotopy colimit of the objects A_n, we conclude from Lemma 6.7 that $Q = 0$. Thus, d is a monomorphism. □

The meaning of the proposition is elucidated by the following elementary lemma.

Lemma 6.18. *If $\{x_1, x_2, \dots\}$ is a regular sequence in R_*, and I is the ideal that it generates, then I/I^2 is freely generated over R_*/I by the elements x_i.*

Proof. It is clear that I/I^2 is generated by the elements x_i. Suppose that we have a relation $\sum_{i=1}^n a_i x_i = 0$ in I (not I/I^2). We claim that $a_i \in I$ for all i. Indeed, it is clear that $a_n x_n \in (x_1, \dots, x_{n-1})$ so by regularity we have $a_n = \sum_{i=1}^{n-1} b_i x_i$ say; in particular, $a_n \in I$. Moreover, $\sum_{i=1}^{n-1}(a_i + b_i x_n)x_i = 0$, so by induction we have $a_i + b_i x_n \in I$ for $i < n$, and thus $a_i \in I$ as required.

Now suppose that we have a relation $\sum_i a_i x_i \in I^2$, say $\sum_i a_i x_i = \sum_{j\leq i} b_{ij} x_i x_j$. We then have $\sum_i (a_i - \sum_{j\leq i} b_j x_j) x_i = 0$, so by the previous claim we have $a_i - \sum_{j\leq i} b_j x_j \in I$, so $a_i \in I$. This shows that the elements x_i generate I/I^2 freely. □

Corollary 6.19. *In the situation of Proposition 6.15 the map $d\colon \mathrm{Der}(A) \to \mathrm{Hom}(I/I^2, A_*)$ is an isomorphism, and $\mathcal{D}(A,A)_*$ is the completed exterior algebra generated by* $\mathrm{Der}(A)$.

Proof. It is easy to see that Q_i is a derivation and that $d(Q_i)(x_j) = \delta_{ij}$ (Kronecker's delta). This shows that d is surjective, and the rest follows. □

7. Power operations

In this section, we identify the commutativity obstruction $\overline{c}(x)$ of Proposition 4.1 with a kind of power operation. This is parallel to a result of Mironov in Baas-Sullivan theory, although the proofs are independent. The proof given here seems distressingly complicated, but I have not been able to find a simpler one.

We need some geometric preliminaries. Because R^* is concentrated in even degrees, we know that the Atiyah-Hirzebruch spectral sequence converging to $R^*\mathbb{C}P^\infty$ collapses and thus that R is complex orientable. We choose a complex orientation once and for all, taking the obvious one if R is (a localisation of) MU. This gives Thom classes for all complex bundles.

We will need to consider the bundle $V^d = \mathbb{R}^{2d} \times_{C_2} S^2$ over $S^2/C_2 = \mathbb{R}P^2$. Here C_2 is acting on $\mathbb{R}^d \times \mathbb{R}^d$ by the twist map, and antipodally on S^2. When d is even, this can be regarded as $\mathbb{C} \otimes_{\mathbb{R}} V^{d/2}$, so we have a Thom class $u \in \widetilde{R}^{2d}(S^2_+ \wedge_{C_2} S^{2d})$ which generates $\widetilde{R}^*(S^2_+ \wedge_{C_2} S^{2d})$ as a free module over $R^*\mathbb{R}P^2 = R^*[\epsilon]/(2\epsilon, \epsilon^2)$.

Next, observe that $V^d = \mathbb{R}^d \oplus H^d$, where H is the Hopf bundle. We can regard S^2 as the space of unit pure-imaginary quarternions, and let C_2 act antipodally on $\mathbb{H}$ and S^2. Observe that there is a trivialisation $\mathbb{H}\times\mathbb{R}P^2 \to \mathbb{H}\times_{C_2} S^2$ sending $(u,[v])$ to $[u,uv]$. This shows that $H^4 = \mathbb{H}\otimes_{\mathbb{R}} H$ is trivial. Thus, when d is even the bundle $V^{2d} = \mathbb{H}\otimes_{\mathbb{R}} V^{d/2}$ is trivial, and we have an equivalence of Thom spectra $(\mathbb{R}P^2)^{-V^d} = \Sigma^{-4d}(\mathbb{R}P^2)^{V^d}$. We will implicitly use this to analyse power operations on negative-dimensional elements.

Let X be a space such that R_*X is a free module over R_*, concentrated in even degrees. The usual quadratic construction gives a non-additive power operation

$$P\colon R^{2n}X \to R^{4n}(X\times\mathbb{R}P^2) = (R^*X[\epsilon]/(2\epsilon,\epsilon^2))^{4n}$$

with the following properties:

$$\begin{aligned}
P(1) &= 1\\
P(xy) &= P(x)P(y)\\
P(x+y) &= P(x)+P(y)+txy\\
P(x) &= x^2 \pmod{\epsilon}.
\end{aligned}$$

Here t is the image of $1\in R^0(S^2)$ under the transfer map associated to the covering $S^2\to\mathbb{R}P^2$. In view of the last property, we can define a map $\widetilde{P}\colon R^{2n}(X)\to R^{2n-2}(X)/2$ by

$$P(x) = x^2 + \epsilon\widetilde{P}(x).$$

In the case $R = MU$, it is well-known that the transfer associated to $S^\infty \to \mathbb{R}P^\infty$ sends 1 to $[2](x)/x = 2\pm w_1x \pmod{x^2}$, where x is the Euler class of the complexified Hopf bundle. The restriction of x to $\mathbb{R}P^2$ is ϵ, so $t = 2+w_1\epsilon$. It also follows easily from the standard construction of the E_∞ structure on MU that whenever x is the Euler class of a complex line bundle we have

$$P(x) = x(x+_F\epsilon).$$

As usual, we let $x\in\pi_dR$ be a regular element, so that d is even. The main result of this section is as follows.

Theorem 7.1. *The commutativity obstruction $\overline{c}(x)$ is given by $\overline{c}(x) = \widetilde{P}(x) \pmod{\epsilon}$.*

The rest of this section will constitute the proof. As it is rather intricate, we start with an outline. Suppose that $d\geq 0$, for simplicity. In that case, we consider the following spaces

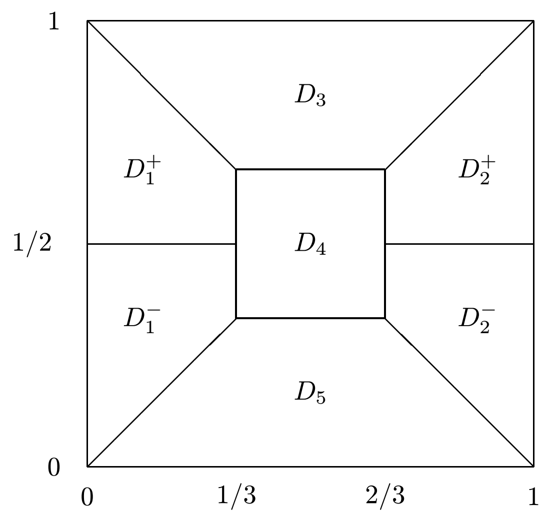
$$\begin{aligned}
M_2 &= \mathcal{M}(S(2),R)\\
N_2 &= \mathcal{M}(S(2),R/x).
\end{aligned}$$

Thus, $\pi_{2d+2}(R) = \pi_{2d+2}M_2$ and so on. We will need to consider RP^2 and various Thom spaces over it. Our model for $\mathbb{R}P^2$ will be the quotient of the unit square I^2 where (s,t) is identified with $(1-s,1-t)$ whenever (s,t) lies on the boundary. We also use the analogous model for the Thom space $(\mathbb{R}P^2)^{V_d}$, viz. $I^2\times S^{2d}/\sim$, where $(s,t,a\wedge b)\sim(1-s,1-t,b\wedge a)$ on the boundary and $I^2\times\infty$ is collapsed to a base point. (When we define maps out of this Thom space, it will always be obvious that $I^2\times\infty$ is sent to the base point, so we will not bother to mention this explicitly.)

We start by setting up a rather elaborate dissection of the unit square, whose use we have unfortunately not been able to avoid. Using this dissection, and a chosen strictly unital product on R/x, we define a map $\zeta\colon I^2\times S^{2d}\to N_2$ which factors through a map $S^{2d+2}\to N_2$. We will be able to see quite directly that this represents $c(\phi)\in\pi_{2d+2}(R/x)$, and thus $\overline{c}(x)\in\pi_{2d+2}(R)/(2,x)$.

We next work through the definition of power operations to come up with a map $\chi\colon I^2\times S^{2d}\to M_2$ which factors through a map $(\mathbb{R}P^2)^{V^d}\to M_2$, which represents $P(x)$. Thus, the composite $\chi'_1 = ((\mathbb{R}P^2)^{V^d}\to M_2\to N_2)$ represents $P(x) \pmod{x}$. It is easy to see that the restriction of this to $(\mathbb{R}P^1)^{V^d}$ becomes nullhomotopic. We will write down an explicit homotopy χ'_r such that χ'_0 is actually zero on the boundary of I^2, so that χ'_0 corresponds to a map $S^{2d+2}\to N_2$. We will then show that χ'_0 is homotopic to ζ. After untangling the definitions, this shows that $P(x) = \epsilon\overline{c}(x) \pmod{x}$, as claimed.

The dissection we need is as follows.



We write $D_1 = D_1^+ \cup D_1^-$ and $D_2 = D_2^+ \cup D_2^-$. We will use the following three functions $u, v, w \colon I^2 \to I$.

	u	v	w
D_1^+	$3s$	$3s - \frac{(2t-1)(3s-1)}{(1-2s)}$	0
D_1^-	$3s + (2t-1)(3s-1)/(1-2s)$	$3s$	0
D_2^+	$3 - 3s$	$3 - 3s + \frac{(2t-1)(2-3s)}{(1-2s)}$	1
D_2^-	$3 - 3s - \frac{(2t-1)(2-3s)}{(1-2s)}$	$3 - 3s$	1
D_3	$3 - 3t$	1	$\frac{(s-1+t)}{(2t-1)}$
D_4	1	1	$3s - 1$
D_5	1	$3t$	$\frac{(s-t)}{(1-2t)}$

The salient features are as follows. The map $(u, v) \colon I^2 \to I^2$ collapses the right hand edge of D_1 to a point, but otherwise induces an orientation-preserving homeomorphism from D_1 to I^2. Similarly, it induces an orientation-reversing homeomorphism from D_2 with the left hand edge collapsed, to I^2. It sends $D_3 \cup D_4 \cup D_5$ to $\partial(I^2)$ by a map that is constant on horizontal lines. Everywhere on the boundary of I^2 we have either $u = 0$ or $v = 0$. We also have

$$(u(1-s, 1-t), v(1-s, 1-t)) = (v(s,t), u(s,t)).$$

The map w is zero on D_1 and one on D_2, and it increases linearly from zero to one along horizontal lines in $D_3 \cup D_4 \cup D_5$.

We now examine the commutativity obstruction $\overline{c}(x)$ and the power operation $P(x)$ more explicitly.

As in section 5, we let R/x denote the cofibre of the map $x \colon S^d(1) \wedge R \to R$, so that R/x is not cofibrant as an R-module, but it admits a strictly unital product $\phi \colon (R/x)^{(2)} \to R/x$.

Write

$$T = \begin{cases} S(1) & \text{if } d \geq 0 \\ S^{-2|d|}(1) & \text{if } d < 0. \end{cases}$$

Either way, x can be regarded as a map of $\mathbb{L}$-spectra $S^{|d|} \wedge T \to R$.

We consider the following spaces:

$$\begin{aligned} M_1 &= \mathcal{M}(T, R) \\ M_2 &= \mathcal{M}(T \wedge T, R) \\ N_1 &= \mathcal{M}(T, R/x) \\ N_2 &= \mathcal{M}(T \wedge T, R/x). \end{aligned}$$

Thus, x can be further reinterpreted as a map of spaces $S^{|d|} \to M_1$. As the composite $S^{|d|} \wedge T \xrightarrow{x} R \xrightarrow{\rho} R/x$ is null, there is a continuous map $\gamma\colon I \times S^{|d|} \to N_1$ with $\gamma(0, a) = 0$ and $\gamma(1, a) = \rho \circ x(a)$.

Using our strictly unital product ϕ, we define a map

$$\psi\colon I^2 \times S^{2|d|} \to N_2$$

by

$$\psi(s, t, a \wedge b) = \phi \circ (\gamma(s, a) \wedge \gamma(t, b))\colon T \wedge T \to R/x.$$

We find that

$$\begin{aligned} \psi(0, t, a \wedge b) &= \psi(s, 0, a \wedge b) = 0 \\ \psi(1, t, a \wedge b) &= \psi(t, 1, b \wedge a) \circ \tau \\ \psi(1, 1, a \wedge b) &= \rho \circ \mu \circ (x(a) \wedge x(b)). \end{aligned}$$

It follows that when $(s, t) \in \partial(I^2)$ we have

$$\psi(s, t, a \wedge b) = \psi(t, s, b \wedge a) \circ \tau.$$

We can thus glue the maps $\psi(s, t, a \wedge b)$ and $\psi(t, s, b \wedge a) \circ \tau$ along $\partial(I^2)$ to get a map $S^2 \to N_2$, corresponding to an element of $\pi_{2d+2}(R/x)$. It is easy to see that this is the commutativity obstruction $c(\phi)$. To ease the comparison with $P(x)$, it is convenient to distort our two copies of I^2 and introduce a collar between them. This leads us to consider the map $\zeta\colon I^2 \times S^{2|d|} \to N_2$ defined by

$$\zeta(s, t, a \wedge b) = \begin{cases} \psi(v, u, b \wedge a) \circ \tau & \text{if } (s, t) \notin D_1 \\ \psi(u, v, a \wedge b) & \text{if } (s, t) \notin D_2. \end{cases}$$

Here of course $u = u(s, t)$ and $v = v(s, t)$. We leave it to the reader to check that this is well-defined and continuous, and that it sends $\partial(I^2) \times S^{2|d|} \cup I^2 \times \infty$ to zero, so that it induces a map $S^{2|d|+2} \to N_2$, or equivalently an element of $\pi_{2d+2}(R/x)$. This is just $c(\phi)$. (The two distorted squares are D_1 and D_2, and $D_3 \cup D_4 \cup D_5$ forms the collar.)

We next need to describe the power operation. We first define a map $y\colon S^{2|d|} \to M_2$ by $y(a \wedge b) = \mu \circ (x(a) \wedge x(b))$, where $\mu\colon R \wedge R \to R$ is the product. Thus, y corresponds to x^2 under the evident identification $\pi_{2|d|} M_2 = \pi_{2d} R$. Because μ is commutative on the nose, we find that $y(b \wedge a) = y(a \wedge b) \circ \tau$. Next, if $d \geq 0$ we let E be the space of linear isometries of $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$, so that E is a contractible monoid which acts contravariantly on $T \wedge T = S(2)$. If $d < 0$ then we recall that $T = S(\mathbb{R}^\infty, \mathbb{R}^{2|d|})$. We think of $\mathbb{R}^{2|d|}$ as $\mathbb{H}^{|d|/2}$, and we let E be the space of linear isometries of $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$ that send $\mathbb{H}^{|d|/2} \oplus \mathbb{H}^{|d|/2}$ to itself by an $\mathbb{H}$-linear isomorphism. In this case, E still acts contravariantly on $T \wedge T$ and it is homotopy equivalent to $Sp(|d|)$ and thus is 2-connected.

In either case, we can choose a path $f\colon I \to E$ with $f(0) = 1$ and $f(1) = \tau$. We then define a map $f\colon \partial(I^2) \to E$ by

$$\begin{aligned} f(s, 0) &= f(s) \\ f(0, t) &= 1 \\ f(1, t) &= \tau \\ f(s, 1) &= f(1 - s) \circ \tau. \end{aligned}$$

As E is 2-connected, we can extend this to get a map $f\colon I^2 \to E$. Note that $f(1 - s, 1 - t) = f(s, t) \circ \tau$ for all $(s, t) \in \partial(I^2)$. We also define $g(s, t) = S(f(s, t))\colon T \wedge T \to T \wedge T$, so that $g(1 - s, 1 - t) = \tau \circ g(s, t)$ on the boundary.

Using this, we define $\chi\colon I^2 \times S^{2|d|} \to M_2$ by

$$\chi(s,t,a\wedge b) = y(a\wedge b)\circ g(s,t).$$

On the boundary, this satisfies

$$\chi(1-s,1-t,b\wedge a) = \chi(s,t,a\wedge b).$$

The quotient of I^2 in which (s,t) is identified with $(1-s,1-t)$ on the boundary, is just $\mathbb{R}P^2$. Using this, we see that χ can be regarded as a map from the Thom space $(\mathbb{R}P^2)^{V^{|d|}} \to M_2$, and thus as an element of $\widetilde{R}^0((\mathbb{R}P^2)^{V^d})$, which is identified with $R^{2d}\mathbb{R}P^2$ under the Thom isomorphism. This element is just $P(x)$.

However, we are really interested in $P(x)$ modulo x, or equivalently the composite $\chi' = \rho_*\circ\chi\colon (\mathbb{R}P^2)^{V^{|d|}} \to N_2$. We know that $P(x) = x^2+\epsilon z$ for some $z = \widetilde{P}(x)\in\pi_{2d+2}(R)/2$, so $P(x) = \epsilon z \pmod{x}$. This just means that χ' factors up to homotopy as $(\mathbb{R}P^2)^{V^{|d|}} \to S^{2|d|+2} \xrightarrow{z} N_2$, where the first map is the collapse onto the top cell, and z is unique mod 2. We will exhibit a family of maps $\chi'_r\colon I^2\times S^{2|d|}\to N_2$ (for $0\le r\le 1$) with the following properties:

$$\begin{aligned}
\chi'_r(1-s,1-t,b\wedge a) &= \chi'_r(s,t,a\wedge b)\\
\chi'_1 &= \chi'\\
\chi'_0(s,t,a\wedge b) &= 0 \qquad \text{if} \qquad (s,t)\in\partial(I^2)
\end{aligned}$$

Because of the first property, these maps can be regarded as maps $(\mathbb{R}P^2)^{V^{|d|}} \to N_2$. As χ'_0 factors through the top cell, it follows that $z = \chi'_0 \pmod 2$.

To define the maps χ'_r, we need the map $h\colon I^2\to\mathcal{M}(T\wedge T, T\wedge T)$ defined by

$$h(s,t) = \begin{cases} 1 & \text{if } (s,t)\in D_1\\ \tau & \text{if } (s,t)\in D_2\\ g(w(s,t),1) & \text{if } (s,t)\in D_3\\ g(w(s,t),3t-1) & \text{if } (s,t)\in D_4\\ g(w(s,t),0) & \text{if } (s,t)\in D_5.\end{cases}$$

We leave it to the reader to check that this is continuous and that $h(s,t) = \tau\circ h(1-s,1-t)$ for all (s,t) outside the central square D_4. Using this, we define

$$\xi(s,t,a\wedge b) = \zeta(s,t,a\wedge b)\circ h(s,t)$$

and

$$\chi'_r(s,t) = \xi((1-2r/3)s + r/3, (1-2r/3)t + r/3).$$

In other words, χ'_r is just ξ restricted to a subsquare of I^2 and rescaled. It is easy to check that this has the required properties. Thus, $z = \chi'_0 = \xi \pmod 2$. On the other hand, we can deform h to the constant map with value 1, and thus deform ξ to ζ through maps which send $\partial(I^2)\times S^{2|d|}$ to the basepoint. Thus, as elements of $\pi_{2d+2}(R)/(2,x)$ we have $z = \xi = \zeta = c(\phi) = \overline{c}(x)$, as claimed.

8. Formal group theory

In this section, we take $R = MU$ or $MU_{(2)}$, and let F be the usual formal group law over R^*. In this context the power operation has the following properties, as discussed in Section 7.

$$\begin{aligned}
P(1) &= 1\\
P(xy) &= P(x)P(y)\\
P(x+y) &= P(x)+P(y)+(2+\epsilon w_1)xy\\
P(x) &= x(x+_F\epsilon) \text{ if } x \text{ is the Euler class of a line bundle.}
\end{aligned}$$

In places it will be convenient to use cohomological gradings; we recall the convention $R_* = R^{-*}$. We will write q for the usual map $R^*\to BP^*$, and note that $q(w_1) = v_1 \pmod 2$.

To handle the nonadditivity of P, we make the following construction. For any R^*-algebra A^*, we define

$$T(A^*) = \{(a,b) \in A^*/2 \times A^*[\epsilon]/(2,\epsilon) \mid b = a^2 \pmod{\epsilon}\}.$$

Given $a, b \in A^*$ (with $|b| = 2|a| - 2$) we define $[a,b] = (a, a^2 + \epsilon b) \in T(A^*)$. We make $T(A^*)$ into a ring by defining

$$(a,b) + (c,d) = (a+c, b+d+\epsilon w_1 ac)$$

$$(a,b).(c,d) = (ac, bd)$$

or equivalently

$$[a,b] + [c,d] = [a+c, b+d+w_1 ac]$$

$$[a,b].[c,d] = [ac, a^2 d + bc^2].$$

Note that $2[a,b] = [0, w_1 a^2]$ and $4[a,b] = 0$, so $4T(A^*) = 0$. If we define $Q(a) = (a, P(a))$, then Q gives a ring map $R^*X \to T(R^*X)$.

Now suppose that A^* is an LRQ of R^*, and let $f\colon R^* \to A^*$ be the unit map. We say that A^* has an induced power operation if there is a ring map $\overline{Q}\colon A^* \to T(A^*)$ making the following diagram commute:

$$\begin{tikzcd} R^* \arrow[r, "Q"] \arrow[d, "f"'] & T(R^*) \arrow[d, "T(f)"] \\ A^* \arrow[r, "\overline{Q}"'] & T(A^*). \end{tikzcd}$$

Because A^* is an LRQ, we know that such a map is unique if it exists. If a is invertible in A^* then $Q(a)$ is the invertible element (a, a^2) plus a nilpotent, so it is again invertible. It follows easily that an induced power operation exists if and only if $\widetilde{P}(I) \leq I + (2)$, where I is the kernel of the map $R^* \to A^*$. This is of course the condition appearing in Theorem 2.6. Our main task in this section is to prove that there is an induced power operations on BP_*, and to compute it. We also do the same for kU_* (which is easy) and H_* and $H\mathbb{F}_*$ (which are trivial).

Proposition 8.1. *There is an induced power operation on BP^*.*

This is proved after Lemma 8.7.

Definition 8.2. Given a subset $J = \{j_1 < \ldots < j_r = n\} \subseteq \{1, \ldots, n\}$, we define

$$u_J = v_{j_1+1} \prod_{k=1}^{r-1} (v_1 v_{j_k})^{2(2^{j_{k+1}-j_k}-1)} \in \pi_{|v_{n+1}|} BP$$

and $u_n = \sum_J u_J$, where J runs over subsets of $\{1, \ldots, n\}$ that contain n. Note that we have a recurrence relation

$$u_n = v_{n+1} + \sum_{j=1}^{n-1} (v_1 v_{n-j})^{2(2^j-1)} u_{n-j}.$$

Proposition 8.3. *The induced power operation on BP^* is given by*

$$\overline{Q}(v_n) = \begin{cases} [0, v_1] & \text{if } n = 0 \\ [v_1, v_2] & \text{if } n = 1 \\ [v_n, v_1 v_n^2 + u_n] & \text{if } n > 1 \end{cases}$$

Moreover, we have $u_n = v_{n+1} \pmod{v_1^2}$.

This is proved after Corollary 8.9.

We now start working towards Proposition 8.1. Given an element $x \in A^2$, we write $Z(x) = (x, x(x +_F \epsilon)) \in T(A^*)$, so that $Q(x) = Z(x)$ if $x \in R^2X$ is the Euler class of a complex line bundle.

Lemma 8.4. *The formal group law QF over $T(R^*)$ satisfies*

$$Z(x) +_{QF} Z(y) = Z(x +_F y).$$

Moreover, QF is the unique formal group law over $T(R^*)$ with this property.

Note that Z is not a homomorphism of formal groups in the usual sense, because $Z(x) \in T(BP^*[\![x]\!])$ is not a power series in $x \in BP^*[\![x]\!]$.

Proof. Let x and y be the usual generators of $R^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Then x, y and $x +_F y$ are Euler classes, so that $Q(x) = Z(x)$, $Q(y) = Z(y)$ and $Q(x +_F y) = Z(x +_F y)$. On the other hand, Q is a ring map, so $Q(x +_F y) = Q(x) +_{QF} Q(y) = Z(x) +_{QF} Z(y)$. Thus $Z(x +_F y) = Z(x) +_{QF} Z(y)$. It is easy to see that x and y are algebraically independent over R^*, and $Z(x)$ and $Z(y)$ are algebraically independent over $T(R^*)$. The lemma follows. □

Definition 8.5. For the rest of this section, we will write

$$z = z(x) = \sum_{k \geq 0} v_1^{2^k} x^{2^k} \in \mathbb{F}_2[v_1][\![x]\!].$$

Note that

$$z^2 = z + v_1 x$$

so

$$z/(v_1 x) = 1/(1+z).$$

Lemma 8.6. *We have*

$$x +_F \epsilon = x + (1+z)\epsilon \qquad \textit{in} \qquad BP^*[\epsilon][\![x]\!]/(2\epsilon, \epsilon^2).$$

Proof. Working rationally and modulo ϵ^2, we have

$$\log_F(\epsilon) = \epsilon$$

so

$$x +_F \epsilon = \exp_F(\log_F(x) + \epsilon) = x + \exp_F'(\log_F(x))\epsilon = x + \epsilon/\log_F'(x).$$

We next recall the formula for $\log_F(x)$ given in [9, Section 4.3]. We consider sequences $I = (i_1, \dots, i_l)$ with $l \geq 0$ and $i_j > 0$ for each j. We write $|I| = l$ and $\|I\| = i_1 + \cdots + i_l$. We also write

$$v_I = v_{i_1}^{m_1} \dots v_{i_l}^{m_l}$$

where

$$m_j = 2^{\sum_{k<j} i_k}.$$

The formula is

$$\log_F(x) = \sum_I v_I x^{2^{\|I\|}}/2^{|I|}.$$

The only terms which contribute to $\log_F'(x)$ modulo 2 are those for which $\|I\| = |I|$, so $i_j = 1$ for all j. If I has this form and $|I| = k$ then $v_I = v_1^{2^k - 1}$. Thus

$$\log_F'(x) = \sum_k v_1^{2^k - 1} x^{2^k - 1} = z/(v_1 x) \pmod 2.$$

As remarked in Definition 8.5, we have $z/(v_1 x) = 1/(1+z)$, so

$$x +_F \epsilon = x + \epsilon/\log_F'(x) = x + (1+z)\epsilon$$

as claimed. □

Lemma 8.7. *In $T(BP^*[\![x,y]\!])$ we have*

$$[0,x] +_{QF} [0,y] = [0,x+y]$$

and

$$[x,y] = [x,0] +_{QF} \left[0, \sum_{k\geq 0} (v_1x)^{2(2^k-1)}y\right].$$

In particular, we have

$$Z(x) = [x,0] +_{QF} [0,z/v_1] = [x,0] +_{QF} \left[0, \sum_{k\geq 0} v_1^{2^k-1}x^{2^k}\right].$$

Proof. The first statement is clear, just because $[0,x][0,y]=0$. For the second statement, write $X=[x,0]$ and

$$w = \sum_k (v_1x)^{2(2^k-1)}y = y(z/v_1x)^2 = y/(1+z)^2,$$

and $W=[0,w]$. Let $a_{ij}\in MU_{2(i+j-1)}$ be the coefficient of x^iy^j in $x+_F y$. Because $W^2=0$ we have $X+_{QF}W = X+W+\sum_{j>0} Q(a_{1j})X^jW$, and

$$Q(a_{1j})X^jW = [a_{1j}, \widetilde{P}(a_{1j})][x^j,0][0,w] = [0, a_{1j}^2x^{2j}w].$$

This expression is to be interpreted in $T(BP^*[\![x,y]\!])$, so we need to interpret a_{1j} in $BP^*/2$. Thus Lemma 8.6 tells us that $a_{10}=1$ and $a_{1,2^k}=v_1^{2^k}$ and all other a_{1j}'s are zero. Thus

$$X +_{QF} W = X+W+\sum_{k\geq 0}[0,(v_1x)^{2^{k+1}}w] = \left[x, w\left(1+\sum_{k\geq 0}(v_1x)^{2^{k+1}}\right)\right] = [x, w(1+z^2)] = [w,y]$$

as claimed.

For the last statement, Lemma 8.6 gives

$$Z(x) = (x, x(x+_F\epsilon)) = (x, x^2 + x(1+z)\epsilon) = [x, x(1+z)].$$

By the previous paragraph, this can be written as $[x,0]+_{QF}[0,x(1+z)/(1+z)^2] = [x,0]+_{QF}[0,z/v_1]$. □

Proof of Proposition 8.1. To show that $\overline{Q}$ exists, it is enough to show that the formal group law on $T(BP^*)$ obtained from the map $MU^* \xrightarrow{Q} T(MU^*) \xrightarrow{T(q)} T(BP^*)$ is 2-typical. Let p be an odd prime, so the associated cyclotomic polynomial is $\Phi_p(t) = 1+t+\dots+t^{p-1}$. We need to show that

$$X +_{QF} \Omega X +_{QF} \dots +_{QF} \Omega^{p-1}X = 0 \qquad \text{in} \qquad B^* = T(BP^*)[\Omega][\![X]\!]/\Phi_p(\Omega).$$

(This is just the definition of 2-typicality for formal groups over rings which may have torsion.) Consider the ring $C^* = T(BP^*[\omega][\![x]\!]/\Phi_p(\omega))$. Write $\Omega=[\omega,0]$ and $X=[x,0]$, so that $\Omega, X\in C^*$. Note that

$$\Phi_p(\Omega) = \sum_{i=0}^{p-1}[\omega^i,0] = \left[\Phi_p(\omega), v_1\sum_{0\leq i<j<p}\omega^{i+j}\right].$$

By assumption $\Phi_p(\omega)=0$. Moreover, the last term in the above equation is the second symmetric function of the roots of t^p-1, and thus is zero. We thus have $\Phi_p(\Omega)=0$, so there is an obvious ring map $B^*\to C^*$. We claim that this is injective. Indeed, it is easy to see that $\{\omega,\omega^2,\dots,\omega^{p-1}\}$ is a basis for $\mathbb{Z}[\omega]/\Phi_p(\omega)$, and that $\alpha\mapsto\alpha^2$ is a permutation of this basis. Suppose that we have

$$\sum_{i=1}^{p-1}\sum_{j\geq 0}[a_{ij},b_{ij}]\Omega^iX^j = 0 \text{ in } C^*.$$

Using the evident map $C^* \to BP^*[\omega][\![x]\!]/\Phi_p(\omega)$, we see that $a_{ij} = 0$ for all i, j. As $[0, b]\Omega^i X^j = [0, \omega^{2i} X^{2j} b]$, we see that

$$\sum_{i=1}^{p-1} \sum_{j \geq 0} b_{ij} \omega^{2i} X^{2j} = 0.$$

As the elements ω^{2i} are a permutation of the elements ω^i, we see that $b_{ij} = 0$ for all i, j. We may thus regard B^* as a subring of C^*.

Next, we know that

$$Z(x) +_{QF} Z(\omega x) +_{QF} \cdots +_{QF} Z(\omega^{p-1} x) = Z(x +_F \omega x +_F \cdots +_F \omega^{p-1} x) = 0, \tag{1}$$

because F is 2-typical over BP_*. By Lemma 8.7, we also know that

$$Z(\omega^i x) = \Omega^i X +_{QF} [0, w_i], \tag{2}$$

where $w_i = \sum_k v_1^{2^k - 1} \omega^{2^k i} x^{2^k}$. It is easy to see that $[0, w_i][0, w_j] = 0$, so that $[0, w_i] +_{QF} [0, w_j] = [0, w_i + w_j]$. We also have $\sum_{i=0}^{p-1} \omega^{2^k i} = 0$ for all k. This means that

$$\sum_i^{QF} [0, w_i] = [0, \sum_k v_1^{2^k - 1} x^{2^k} \sum_{i=0}^{p-1} \omega^{2^k i}] = 0. \tag{3}$$

By combining equations (1) to (3), we see that

$$\sum_i^{QF} \Omega^i X = 0$$

as required. □

We now start working towards the proof of Proposition 8.3.

Lemma 8.8. *We have* $\exp_F(2x) = 2z/v_1$ *in* $BP^*[\![x]\!]/4$.

Proof. Using Ravenel's formulae as in the proof of Lemma 8.6, we have

$$\log_F(2x)/2 = \sum_I 2^{2^{\|I\|} - |I| - 1} v_I x^{2^{\|I\|}}.$$

When $k \geq 0$ we have $2^k \geq k + 1$, with equality only when $k = 0$ or $k = 1$. It follows easily that

$$\log_F(2x)/2 = x + v_1 x^2 \pmod 2.$$

By inverting this, we find that

$$\exp_F(2x)/2 = \sum_{k \geq 0} v_1^{2^k - 1} x^{2^k} = z/v_1 \pmod 2,$$

and thus that $\exp_F(2x) = 2z/v_1 \pmod 4$. □

Because $T(BP^*)$ is a torsion ring, the formal group law QF has no exp series. Nonetheless, $\exp_F(2X)$ is a power series over BP^*, so we can apply Q to the coefficients to get a power series over $T(BP^*)$ which we call $\exp_{QF}(2X)$. This makes perfect sense even though $\exp_{QF}(X)$ does not.

Corollary 8.9. *In* $T(BP^*)[\![X]\!]$, *we have*

$$\exp_{QF}(2X) = \sum_{k \geq 0} [0, v_1^{2^{k+1} - 1}] X^{2^k}.$$

By taking $X = Z(x) \in T(BP^*[\![x]\!])$, *we get*

$$\exp_{QF}(2Z(x)) = \left[0, \sum_{j > 0} v_1^{2^j - 1} x^{2^j} \right] = [0, z/v_1 + x].$$

Proof. Because $4=0$ in $T(BP^*)$, it follows immediately from the lemma that $\exp_{QF}(2X)=\sum_k 2\overline{Q}(v_1)^{2^k-1}X^{2^k}$. Using $\overline{Q}(v_1)=[v_1,\widetilde{P}(v_1)]$, we see that $2\overline{Q}(v_1)^{2^k-1}=[0,v_1^{2^{k+1}-1}]$, and the first claim follows. If we now put $X=Z(x)=[x,x(1+z)]$ then $[0,v_1^{2^{k+1}-1}]X^{2^k}=[0,v_1^{2^{k+1}-1}x^{2^{k+1}}]$, and the second claim follows. □

Proof of Proposition 8.3. Write $p_k=q(\widetilde{P}(v_k))\in BP^*/2$ and $V_k=\overline{Q}(v_k)=[v_k,p_k]\in T(BP^*)$. Recall that the Hazewinkel generators v_k are characterised by the formula

$$[2]_F(x)=\exp_F(2x)+_F\sum_{k>0}^{F}v_kx^{2^k}\in BP^*[\![x]\!].$$

By applying the ring map $\overline{Q}$ and putting $X=Z(x)$ we obtain

$$[2]_{QF}(Z(x))=\exp_{QF}(2Z(x))+_{QF}\sum_{k>0}^{QF}V_kZ(x)^{2^k}\in T(BP^*[\![x]\!]).$$

The first term can be evaluated using Corollary 8.9. For the remaining terms, we have

$$V_kZ(x)^{2^k}=[v_k,p_k][x^{2^k},0]=[v_kx^{2^k},p_kx^{2^{k+1}}].$$

We can use Lemma 8.7 to rewrite this as

$$\begin{aligned}V_kZ(x)^{2^k}&=[v_kx^{2^k},0]+_{QF}\left[0,\sum_{l>0}(v_1v_kx^{2^k})^{2^l-2}p_kx^{2^{k+1}}\right]\\&=[v_kx^{2^k},0]+_{QF}\left[0,\sum_{l>0}(v_1v_k)^{2^l-2}p_kx^{2^{k+l}}\right].\end{aligned}$$

After using the formula $[0,b]+_{QF}[0,c]=[0,b+c]$ to collect terms, we find that

$$[2]_{QF}(Z(x))=\left[0,\sum_{l>0}v_1^{2^l-1}x^{2^l}+\sum_{k,l>0}(v_1v_k)^{2^l-2}p_kx^{2^{k+l}}\right]+_{QF}\sum_{k>0}^{QF}[v_kx^{2^k},0]. \tag{4}$$

On the other hand, we know that

$$\begin{aligned}[2]_{QF}(Z(x))&=Z([2]_F(x))\\&=Z\left(\exp_F(2x)+_F\sum_{k>0}^{F}v_kx^{2^k}\right)\\&=Z(\exp_F(2x))+_{QF}\sum_{k>0}^{QF}Z(v_kx^{2^k}).\end{aligned}$$

The first term is zero because $\exp_F(2x)$ is divisible by 2. For the remaining terms, Lemma 8.7 gives

$$Z(v_kx^{2^k})=[v_kx^{2^k},0]+_{QF}[0,\sum_{j\geq 0}v_1^{2^j-1}v_k^{2^j}x^{2^{k+j}}].$$

Thus, we have

$$[2]_{QF}(Z(x))=\left[0,\sum_{k>0}\sum_{l\geq 0}v_1^{2^l-1}v_k^{2^l}x^{2^{k+l}}\right]+_{QF}\sum_{k>0}^{QF}[v_kx^{2^k},0]. \tag{5}$$

By comparing this with equation (4) and equating coefficients of $x^{2^{n+1}}$, we find that

$$v_1^{2^{n+1}-1}+\sum_{j=1}^{n}(v_1v_{n+1-j})^{2^j-2}p_{n+1-j}=\sum_{j=0}^{n}v_1^{2^j-1}v_{n+1-j}^{2^j}.$$

After some rearrangement and reindexing, this becomes

$$p_n + v_1 v_n^2 = v_1^{2^{n+1}-1} + v_{n+1} + \sum_{j=1}^{n-1} (v_1 v_{n-j})^{2(2^j-1)} (p_{n-j} + v_1 v_{n-j}^2).$$

In particular, we have $p_1 = v_2$. We now define

$$p'_n = \begin{cases} v_1 & \text{if } n = 0 \\ v_2 & \text{if } n = 1 \\ v_1 v_n^2 + u_n & \text{if } n > 1 \end{cases}$$

The claim of the proposition is just that $p_n = p'_n$ for all $n \geq 0$. Using the recurrence relation given in definition 8.2, one can check that for all $n > 0$ we have

$$p'_n + v_1 v_n^2 = v_1^{2^{n+1}-1} + v_{n+1} + \sum_{j=1}^{n-1} (v_1 v_{n-j})^{2(2^j-1)} (p'_{n-j} + v_1 v_{n-j}^2).$$

In particular, we have $p'_1 = v_2 = p_1$, and it follows inductively that $p_n = p'_n$ for all $n > 0$. We also have

$$Q(v_0) = Q(1) + Q(1) = [1, 0] + [1, 0] = [0, v_1]$$

so $p_0 = v_1 = p'_0$. □

Remark 8.10. The first few cases are

$$\begin{aligned} p_0 &= v_1 \\ p_1 &= v_2 \\ p_2 &= v_1^4 v_2 + v_1 v_2^2 + v_3 \\ p_3 &= v_1^{12} v_2 + v_1^6 v_2^3 + v_1^2 v_2^2 v_3 + v_1 v_3^2 + v_4. \end{aligned}$$

In particular, we find that $p_3 \notin (v_k \mid k \geq 3)$, which shows that there is no commutative product on $BP\langle 2\rangle$, considered as an object of $\mathcal{D}$. This problem does not go away if we replace the Hazewinkel generator v_k by the corresponding Araki generator, or the bordism class w_k of a smooth quadric hypersurface in $\mathbb{C}P^{2^k}$. However, it is possible to choose a more exotic sequence of generators for which the problem does go away, as indicated by the next result.

Proposition 8.11. *Fix an integer $n > 0$. There is an ideal $J \leq BP^*$ such that*

1. *The evident map*

$$\mathbb{Z}_{(2)}[v_1, \dots, v_n] \to BP^* \to BP^*/J$$

   is an isomorphism.
2. $\widetilde{P}(J) \leq J \pmod 2$.
3. $I_n + J = I_n + (v_k \mid k > n) = (v_k \mid k \neq n)$.

The proof will construct an ideal explicitly, but it is not the only one with the stated properties. If $n = 1$ we can take $J = (v_k \mid k > n)$, but in general this violates condition (2).

Proof. First consider the case $n = 1$, and write $J = (v_k \mid k > 1)$. By inspecting definition 8.2, we see that $u_n \in J$ for all $n > 1$, and thus Proposition 8.3 tells us that $\widetilde{P}(J) \leq J \pmod 2$. We may thus assume that $n > 1$. Write $B^* = \mathbb{Z}_{(2)}[v_1, \dots, v_n]$, thought of as a subring of BP^*. We will recursively define a sequence of elements x_k for $k > n$ such that

(a) $x_k \in v_k + v_1^2 B^*$
(b) $\widetilde{P}(x_j) \in (x_{n+1}, \dots, x_k) \pmod 2$ if $j < k$.

It is clear that we can then take $J = (x_k \mid k > n)$. We start by putting $x_{n+1} = v_{n+1}$. Suppose that we have defined $x_{n+1}, \dots, x_k$ with the stated properties. There is an evident map

$$\mathbb{F}[v_1, \dots, v_n, v_{k+1}] \xrightarrow{f} BP_*/(2, x_{n+1}, \dots, x_k),$$

which is an isomorphism in degree $2(2^{k+1}-1)=|v_{k+1}|$. Let $\overline{p}_k$ be the image of $\widetilde{P}(x_k)$ in $BP_*/(2,x_{n+1},\dots,x_k)$, and write $\overline{x}_{k+1}=f^{-1}(\overline{p}_{k+1})$. We can lift this to get an element x_{k+1} of $\mathbb{Z}_{(2)}[v_1,\dots,v_n,v_{k+1}]$ such that $\overline{x}_{k+1}=x_{k+1} \pmod 2$ and every coefficient in x_{k+1} is 0 or 1. It is easy to see that condition (b) is satisfied, and that $x_{k+1}\in v_{k+1}+B^*$. However, we still need to show that $x_{k+1}-v_{k+1}$ is divisible by v_1^2. By assumption we have $x_k=v_k+v_1^2b$ for some $b\in B^*$. Recall from Proposition 8.3 that $\widetilde{P}(v_k)=v_{k+1}+v_1v_k^2 \pmod{2,v_1^2}$. It follows after a small calculation that $\widetilde{P}(x_k)=v_{k+1}+v_1v_k^2 \pmod{2,v_1^2}$ also. Moreover, we have $v_k^2=v_1^4b^2 \pmod{2,x_k}$, so $\overline{p}_{k+1}=v_{k+1} \pmod{2,v_1^2}$. It follows easily that $x_{k+1}=v_{k+1} \pmod{v_1^2}$, as required. □

This completes our consideration of power operations in BP. We next consider K-theory. It is known that the Todd genus gives an H_∞ map $MU\to kU$, which implies the next result, but we give a self-contained proof anyway.

Proposition 8.12. *Recall that $kU^*=\mathbb{Z}[u]$, considered as an algebra over MU^* via the Todd genus. There is an induced power operation on kU^*, given by $\overline{Q}(u)=[u,u^3]$.*

Proof. Let $f\colon MU^*\to kU^*$ be the Todd genus, so that f_*F is just the multiplicative formal group law $x+y+uxy$. We define a map $\overline{Q}\colon kU^*\to T(kU^*)$ by $\overline{Q}(u)=U=[u,u^3]$, and we need to check that $\overline{Q}\circ f=T(f)\circ Q\colon MU^*\to T(kU^*)$, or equivalently that the two formal group laws $F_1=\overline{Q}_*f_*F$ and $F_2=T(f)_*Q_*F$ are the same. It is easy to see that $F_1(X,Y)=X+Y+UXY$. Note that

$$T(q)Z(x)=T(q)(x,x(x+_F\epsilon))=(x,x^2+x\epsilon+ux^2\epsilon)=[x,x+ux^2].$$

Write $X=[x,x+ux^2]$ and $Y=[y,y+uy^2]$ so that $X,Y\in T(kU^*[\![x,y]\!])$. Using Lemma 8.4, we see that

$$X+_{F_2}Y=[x+_Fy,x+_Fy+u(x+_Fy)^2]=[x+y+uxy,x+y+u(x^2+xy+y^2)+u^3x^2y^2].$$

We need only check that this is the same as $X+Y+UXY$, which is a direct calculation. □

The case of ordinary homology is trivial and is left to the reader.

Proposition 8.13. *There is an induced power operation on $\mathbb{F}_2$, considered as an algebra over MU^* in the evident way, and similarly for $\mathbb{Z}$.* □

We give one further calculation, closely related to Proposition 8.3.

Proposition 8.14. *Recall that $I_k=(w_0,\dots,w_{k-1})<MU^*$, where w_i is the bordism class of a smooth quadric hypersurface in $\mathbb{C}P^{2^i}$. We have $\widetilde{P}(I_{k-1})\leq I_k$, and $\widetilde{P}(w_{k-1})=w_k \pmod{I_k}$.*

Proof. If $k=1$ we have $I_0=0$ and $w_0=2$, so $P(w_0)=P(1)+P(1)+(2+w_1\epsilon)=w_1\epsilon \pmod 2$, as required. Thus, we may assume that $k>1$, and it follows easily from the formulae for $P(x+y)$ and $P(xy)$ that P induces a ring map $MU^*\to B^*=(MU^*/I_k)[\epsilon]/\epsilon^2$. Note that $[2]_F(x)=w_kx^{2^k}+O(x^{2^k+1})$ over B^*. Write $X=x(x+_F\epsilon)\in MU^*[\epsilon][\![x]\!]/(I_k,\epsilon^2)$. Arguing in the usual way, we see that

$$[2]_{P_*F}(X)=[2]_F(x)([2]_F(x)+_F\epsilon)=\epsilon w_kx^{2^k}+O(x^{2^k+1}).$$

It follows easily that we must have

$$[2]_{P_*F}(X)=\epsilon w_kX^{2^{k-1}}+O(X^{2^{k-1}+1}).$$

It follows that $P(w_i)=0\in B^*$ for $i<k-1$, and that $P(w_{k-1})=\epsilon w_k\in B^*$, as required. □

Finally, we prove a background result that was promised in Proposition 2.7.

Proposition 8.15. *The image of the map $MU_*\to BP_*$ is $\mathbb{Z}[v_k \mid k>0]$, and this is a regular quotient of MU_*.*

Proof. Write $B_* = \mathbb{Z}[v_k \mid k > 0]$, and let C_* be the image of MU_* in BP_*, so the claim is that $B_* = C_*$. Note that C_* is the subring generated by the coefficients of the formal group law. Using Ravenel's formulae [9, Section 4.3] as in the proof of Lemma 8.6, we see that the coefficients of $\log_F(x)$ lie in $\mathbb{Z}[\frac{1}{p}][v_k \mid k > 0]$, and thus that the same is true of the coefficients of the formal group law. On the other hand, these coefficients also lie in $BP_* = \mathbb{Z}_{(p)}[v_k \mid k > 0]$. They therefore lie in $B_* = \mathbb{Z}[v_k \mid k > 0]$, so $C_* \subseteq B_*$.

Next, over MU_* we have $\log_F(x) = \sum_i [\mathbb{C}P^i] x^{i+1}/(i+1)$, and over BP_* we have $\log_F(x) = \sum_i \lambda_i x^{p^i}$. It follows that $p^i \lambda_i \in C_*$, and thus that

$$p^{-1} \log_F(px) = \sum_i p^{p^i - 1} \lambda_i x^{p^i} \in C_*[\![x]\!].$$

By inverting this, we see that $p^{-1} \exp_F(px) \in C_*[\![x]\!]$, and thus that $[p](x) -_F \exp_F(px) \in C_*[\![x]\!]$. A simple induction now shows that $v_k \in C_*$ for all k, and thus that $B_* = C_*$.

Now let $q\colon MU_* \twoheadrightarrow B_*$ be the evident map, and choose $x_k \in MU_*$ with $q(x_k) = v_k$. It is clear that x_k generates the group of indecomposables in $MU_{2(p^k-1)}$. Now choose a sequence of elements $y_1, y_2, \ldots$ of increasing degrees such that MU_* is the polynomial algebra generated by the x's and y's. After adjusting y_i by a decomposable element of $\mathbb{Z}[x_k \mid k > 0]$, we may clearly assume that $q(y_i) = 0$. It then follows that the y's form a regular sequence that generates the kernel of q, as required. $\square$

9. Applications to MU

Proof of Proposition 2.9. All the claims for $R = MU_{(p)}$ with $p > 2$ follow immediately from Theorem 2.5. Except for the case of $H\mathbb{F}$, all the claims for $R = MU$ or $MU_{(2)}$ follow from Theorem 2.6 combined with Propositions 8.1 to 8.13. The case of $H\mathbb{F}$ is handled in the same way as Theorem 2.6, but is easier because all the obstruction groups that occur are trivial. $\square$

Proof of Proposition 2.10. Choose an ideal J as in Proposition 8.11 and set $BP\langle n\rangle'_* = BP_*/J$. Everything then follows from Theorem 2.6. $\square$

We now take $R = MU_{(2)}$ and turn to the proof of Theorem 2.13. As previously, we let $w_k \in \pi_{2^{k+1}-2} R$ denote the bordism class of the quadric hypersurface

$$W_{2^k} = \{[z_0 : \ldots : z_{2^k}] \in \mathbb{C}P^{2^k} \mid \sum_i z_i^2 = 0\}.$$

Recall that the image of w_k in BP_* is v_k modulo $I_k = (v_0, \ldots, v_{k-1})$, and thus $P(n)_* = BP_*/(w_0, \ldots, w_{n-1})$.

We next choose a product ϕ_k on R/w_k for each k. For $k = 0$ we choose there are two possible products, and we choose one of them randomly. (It is possible to specify one of them precisely using Baas-Sullivan theory, but that would lead us too far afield.) For $k > 0$, we recall from Proposition 8.14 that $\widetilde{P}(w_k) = w_{k+1} \pmod{I_{k+1}}$. It follows easily that there is a product ϕ_k such that $c(\phi_k) = w_{k+1} \pmod{w_1, \ldots, w_k}$, and that this is unique up to a term $u \circ (\beta \wedge \beta)$ with $u \in (w_1, \ldots, w_k)$. From now on, we take ϕ_k to be a product with this property. It is easy to see that the resulting product $R/w_0 \wedge \ldots \wedge R/w_{n-1}$ is independent of the choice of ϕ_k's (except for ϕ_0).

Definition 9.1. We write

$$R/I_n = R/w_0 \wedge \ldots \wedge R/w_{n-1},$$

made into a ring as discussed above. For $i < n$, we define

$$Q_i \colon R/I_n \to \Sigma^{2^{i+1}-1} R/I_n$$

by smashing the Bockstein map $\overline{\beta}\colon R/w_i \to \Sigma^{2^{i+1}-1}R/w_i$ with the identity on the other factors. We also define

$$\begin{aligned} P(n) &= BP \wedge R/I_n \\ B(n) &= w_n^{-1}P(n) \\ k(n) &= BP\langle n\rangle' \wedge R/I_n \\ K(n) &= w_n^{-1}k(n). \end{aligned}$$

It is clear that $\pi_*(R/I_n) = R_*/I_n$ and $\pi_*(P(n)) = P(n)_*$ and $\pi_*(B(n)) = B(n)_*$. Condition (2) in Proposition 2.10 assures us that $\pi_*k(n) = k(n)_*$ and $\pi_*K(n) = K(n)_*$ as well. As BP and $BP\langle n\rangle'$ are commutative, it is easy to see that $P(n)$, $B(n)$, $k(n)$ and $K(n)$ are central algebras over BP, $v_n^{-1}BP$, $BP\langle n\rangle'$ and $E(n)'$ respectively. The derivations Q_i on MU/I_n clearly induce compatible derivations on $P(n)$, $B(n)$, $k(n)$ and $K(n)$.

Proposition 9.2. *The product ϕ on R/I_n satisfies*

$$\phi - \phi\circ\tau = w_n\phi\circ(Q_{n-1}\wedge Q_{n-1}).$$

Similarly for $P(n)$, $B(n)$, $k(n)$ and $K(n)$.

Proof. This follows easily from the fact that $c(\phi_{k-1}) = w_k \pmod{I_k}$, given by Proposition 8.14. □

Proposition 9.3. *Let A be a central BP-algebra such that $\pi_0(A) = \{0,1\}$, $\pi_{2^{n+1}-2}(A) = \{0, v_n\}$ and $\pi_k(A) = 0$ for $0 < k < 2^{n+1}-2$. Then either there is a unique map $P(n) \to A$ of BP-algebras, or there is a unique map $P(n) \to A^{op}$ (but not both). Analogous statements hold for $B(n)$, $k(n)$ and $K(n)$ with BP replaced by $v_n^{-1}BP$, $BP\langle n\rangle'$ and $E(n)'$ respectively.*

Proof. We treat only the case of $P(n)$; the other cases are essentially identical. Any ring map $R/I_n \to A$ commutes with the given map $BP \to A$, because the latter is central. It follows that maps $P(n) \to A$ of BP-algebras biject with maps $R/I_n \to A$ of rings, which biject with systems of commuting ring maps $R/w_i \to A$ for $0 \le i < n$. For $i < n-1$ we have $\pi_{2|w_i|+2}(A) = 0$, so Proposition 4.11 tells us that the unique unital map $f_i\colon R/w_i \to A$ is a ring map. This remains the case if we replace the product ψ on A by $\psi\circ\tau$, or in other words replace A by A^{op}. There is an obstruction $d_A(\phi_{n-1}) \in \pi_{2^{n+1}-2}(A) = \{0, v_n\}$ which may prevent f_{n-1} from being a ring map. If it is nonzero, we have

$$d_A(\phi_{n-1}\circ\tau) = d_A(\phi_{n-1} + \widetilde{P}(w_{n-1})\circ(\beta\wedge\beta)) = d_A(\phi_{n-1}) + v_n = 0$$

This shows that $f_{n-1}\colon R/w_{n-1} \to A^{\mathrm{op}}$ is a ring homomorphism. After replacing A by A^{op} if necessary, we may thus assume that all the $f_i\colon R/w_i \to A$ are ring maps.

The obstruction too f_i commuting with f_j lies in $\pi_{|w_i|+|w_j|+2}(A)$. If i and j are different then at least one is strictly less than $n-1$; it follows that $|w_i| + |w_j| + 2 < 2^{n+1} - 2$ and thus that the obstruction group is zero. Thus f_i commutes with f_j when $i \ne j$, and we get a unique induced map $R/I_n \to A$, as required. □

References

[1] M. Ando. Isogenies of formal group laws and power operations in the cohomology theories E_n. *Duke Mathematical Journal*, 79(2), 1995.

[2] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, Modules and Algebras in Stable Homotopy Theory*, volume 47 of *Amer. Math. Soc. Surveys and Monographs*. American Mathematical Society, 1996.

[3] M. Hovey, J. H. Palmieri, and N. P. Strickland. *Axiomatic stable homotopy theory.* Memoirs Of The American Mathematical Society. American Mathematical Society, 1996.

[4] L. G. Lewis, J. P. May, M. Steinberger, and (with contributions by Jim E. McClure). *Equivariant Stable Homotopy Theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer–Verlag, New York, 1986.

[5] O. K. Mironov. Existence of multiplicative structures in the theory of cobordism with singularities. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(5):1065–1092,1219, 1975.

[6] O. K. Mironov. Multiplicativity in the theory of cobordism with singularities and the Steenrod-Tom Dieck operations. *Math USSR Izvestiya*, 13(1):89–107, 1979.

[7] C. Nassau. On the structure of $P(n)_*(P(n))$ for $p = 2$. preprint, 1996.

[8] C. Nassau. *Eine nichtgeometrische Konstruktion der Spektren $P(n)$, Multiplikativen und antimultiplikativen Automorphismen von $K(n)$*. Diplomarbeit, Johann Wolfgang Goethe-Universität Frankfurt, October 1995.
[9] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press, 1986.
[10] U. Würgler. Commutative ring–spectra in characteristic 2. *Commentarii Mathematici Helvetici*, 61:33–45, 1986.

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