THE $BP\langle N\rangle$ COHOMOLOGY OF ELEMENTARY ABELIAN GROUPS

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1. Introduction

In this paper we define three elements of a certain generalised cohomology ring $BP\langle m,n\rangle^* BV_k$. Here m, n and k are nonnegative integers with $k+m\leq n+1$, there is a fixed prime p not exhibited in the notation, and V_k is an elementary Abelian p-group of rank k. We show that these elements are equal; this is striking, because the three definitions are very different. The significance of our equation is not yet entirely clear, but it makes contact with other work in the literature in a number of fascinating ways.

1. In the case where $n=1$, $m=0$ and $k=2$, our result is closely related to [1, Theorem 4.2], which exhibits a relation in the connective complex K-theory group $kU^6 B(\mathbb{Z}/k)^2$ for all k. That result is in turn a key part of a proof (at least in special cases) of the main result of [5], which relates the Witten genus for spin manifolds [17] to the theory of elliptic curves and elliptic spectra. The work described in the present paper started with an attempt to generalise this theorem in connective K-theory.
2. Our methods also give an interesting filtration of the ring $BP\langle n,m\rangle^* BV_k$. Again in the case $n=1$, $m=0$, $k=2$ our filtration is compatible with a splitting of $kU\wedge BV_2$ discovered earlier by Ossa [11]. This was also a motivation for our investigations. It would clearly be interesting to improve our filtration to some kind of stable splitting in the general case, but we have not succeeded as yet.
3. One of our three definitions involves an iterated Bockstein map
$$q'_n := (BP\langle 0\rangle \xrightarrow{q_1} \Sigma^{2p-1} BP\langle 1\rangle \xrightarrow{q_2} \dots \xrightarrow{q_n} \Sigma^{2(p^{n+1}-1)/(p-1)-n-2} BP\langle n\rangle).$$
There is an analogous operation in Beilinson's motivic cohomology, which plays a central rôle in Voevodsky's proof of the Milnor conjecture in algebraic K-theory [15]. We have no idea whether this fact is significant or not.
4. By applying the $(n+2)$'th space functor to the map q'_n of spectra, we get a map
$$r\colon K(\mathbb{Z}_{(p)},n+2)\to \underline{BP\langle n\rangle}_{2(p^{n+1}-1)/(p-1)}.$$
This map has appeared in a number of other places, for example the proof that suspension spectra are harmonic [6]. Moreover, Ravenel, Wilson and Yagita have shown that BP^*r is surjective, and thus given a nice description of $BP^*K(\mathbb{Z}_{(p)},n+2)$ [12]. Our results shed some interesting light on the nature of the map r.
5. Our other two definitions make contact with the classical theory of Dickson invariants in $H^*(BV_k;\mathbb{F}_p)$ (see [16] for an exposition).
6. Our proofs use the theory of multiple level structures [4], which generalise Drinfel'd level structures on elliptic curves [2]. This theory was also central to the results of [4], which give a crude generalisation of the Hopkins-Kuhn-Ravenel generalised character theory for the rationalised Morava cohomology of classifying spaces of finite groups.

There are other approaches to $BP\langle n\rangle_* BV_k$ and $BP\langle n\rangle^* BV_k$, related to the Steinberg idempotents and the Conner-Floyd conjecture [10, 7, 8]. We have not yet managed to find any fruitful interaction between our methods and these, but it is clearly an interesting project to look for one.

2. Statement of results

We now state our results in more detail.

Fix a prime p and integers $m \leq n$, and define $w = n+1-m$. There is a BP-algebra spectrum $E = BP\langle m,n\rangle$ with homotopy ring $E^* = \mathbb{F}_p[v_m,\dots,v_n]$ (or $E^* = \mathbb{Z}_{(p)}[v_1,\dots,v_n]$ if $m=0$). (We will recall a construction in Section 3.) Note that w is the Krull dimension of E^*. We use cohomological gradings, so that $|v_k| = -2(p^k-1) < 0$. Write V_k for the elementary Abelian group $\mathbb{F}_p^k$. We will give a filtration of E^*BV_k by ideals, such that each quotient is in a natural way a finitely generated free module over a regular local ring. We will show that E^*BV_k has no v_m-torsion when $k < w$. When $k = w$, we show that the v_m-torsion is annihilated by the ideal $I_{n+1} = (v_m,\dots,v_n)$, and that it is a free module on one generator over the ring $\mathbb{F}_p[\![x_0,\dots,x_{w-1}]\!]$. We give three very different formulae for this generator.

Recall that the *depth* of a module M^* over R^* is the largest d such that there exists a sequence $\{a_0,\dots,a_{d-1}\}$ in R^* which is regular on M^*. We will need an analogous but different notion, which we now define.

Definition 2.1. If M^* is a module over E^*, we define the *chromatic depth* of M^* to be the largest integer $d = \operatorname{cdepth}(M^*)$ such that the sequence $(v_m, v_{m+1},\dots,v_{m+d-1})$ is regular on M^*. Clearly we have $0 \leq \operatorname{cdepth}(M^*) \leq w$.

We next introduce some abbreviated notation for rings which will appear many times.

Definition 2.2. Given any algebra R^* over BP^*, and integers $0 \leq i \leq j \leq k$, we write

$$P(R^*;j) = R^*[\![x_t \mid 0 \leq t < j]\!]/([p](x_t) \mid 0 \leq t < j)$$
$$P(R^*;i,j) = R^*[\![x_t \mid i \leq t < j]\!]/([p](x_t) \mid i \leq t < j)$$
$$P(R^*;i,j;k) = R^*[\![x_t \mid i \leq t < k]\!]/([p](x_t) \mid i \leq t < j).$$

Thus $P(R^*;j) = P(R^*;0,j)$ and $P(R^*;i,j) = P(R^*;i,j;j)$.

The most basic fact is as follows.

Proposition 2.3. *If $k \leq w$ then we have*

$$E^*BV_k = P(E^*;k) = E^*[\![x_0,\dots,x_{k-1}]\!]/([p](x_0),\dots,[p](x_{k-1}))$$

and this has chromatic depth $w-k$.

This is proved in Corollary 5.2 and Corollary 5.3.

In the extreme case $k = w$ we see that E^*BV_w has chromatic depth 0, so it must have some v_m-torsion. It is thus of interest to see what the torsion subgroup is. In order to state the answer, we need some definitions.

Definition 2.4. Given $\underline{\lambda} \in \mathbb{F}_p^j$, we write $[\underline{\lambda}](\underline{x}) = [\lambda_0](x_0) +_F \cdots +_F [\lambda_{j-1}](x_{j-1})$. We write

$$\phi_j(t) = \prod_{\underline{\lambda}\in\mathbb{F}_p^j} (t -_F [\underline{\lambda}](\underline{x}))^{p^m}.$$

We also define $\alpha = \alpha(m,n) = \prod_{j=0}^{w-1} \phi_j(x_j) \in E^*BV_w$. Note that α is the product of the terms $[\underline{\lambda}](\underline{x})^{p^m}$, where $\underline{\lambda}$ runs over elements of $\mathbb{F}_p^w$ whose last nonzero entry is one. The degree of α is $2p^m(p^w-1)/(p-1)$.

In the following lemma, we regard $BP\langle k\rangle^* = \mathbb{Z}_{(p)}[v_1,\dots,v_k]$ as a subring of $BP\langle n\rangle^*$ in the obvious way.

Lemma 2.5. *There are unique series $\pi_k(t) \in BP\langle k\rangle^*[\![t]\!]$ such that $[p](t) = \sum_{k=0}^n v_k\pi_k(t) \in BP\langle n\rangle^*[\![t]\!]$. Moreover, the series $\pi_k(t)$ is divisible by t^{p^k}, and is equal to t^{p^k} modulo (p, t^{p^k+1}) or modulo I_{n+1}.*

Proof. We give t degree two, which makes everything homogeneous. Write $\pi_0(t) = t$. We know that $[p](t) = pt \pmod{t^2}$ so $[p](t) - v_0\pi_0(t) = \sum_{k>1} a_k t^k$ say. Each a_k has strictly negative cohomological degree (because $|t| = |[p](t)| = 2$) so it lies in the ideal $(v_1,\dots,v_n)$. Let $f_k(t)$ be the sum of those monomials in $[p](t) - v_0\pi_0(t)$ that lie in $BP\langle k\rangle^*[\![t]\!]$, but do not lie in $BP\langle k-1\rangle^*[\![t]\!]$.

It is clear that $f_k(t)$ is divisible by v_k, say $f_k(t) = v_k\pi_k(t)$. This gives series $\pi_k(t) \in BP\langle k\rangle^*[\![t]\!]$ with $[p](t) = \sum_{k=0}^n v_k\pi_k(t)$, as required. It is easy to see that they are unique.

Note that the degree of $v_k\pi_k(t)$ is the same as the degree of $[p](t)$, which is 2, so $\pi_k(t)$ has degree $2p^k$. As $|t| = 2$ and $|v_i| < 0$ for $i > 0$ we see that $\pi_k(t)$ is divisible by t^{p^k}. We thus have $\pi_k(t) = b_k t^{p^k} \pmod{t^{p^k+1}}$, say. One checks that $|b_k| = 0$, so that $b_k \in \mathbb{Z}_{(p)}$. Given that $[p](t) = v_k t^{p^k} \pmod{v_0, \dots, v_{k-1}, t^{p^k+1}}$, we see that $b_k = 1 \pmod p$, and thus $\pi_k(t) = t^{p^k} \pmod{p, t^{p^k+1}}$.

Similarly, the image of $\pi_k(t)$ modulo I_{n+1} is a series in $\mathbb{F}_p[\![t]\!]$ of degree $2p^k$, so it has the form $c_k t^{p^k}$ for some $c_k \in \mathbb{F}_p$. Because $[p](t) = v_k t^{p^k} \pmod{v_0, \dots, v_{k-1}, t^{p^k+1}}$, we see that $c_k = 1$ and $\pi_k(t) = t^{p^k} \pmod{I_{n+1}}$. □

Definition 2.6. We define $\alpha' = \alpha'(m, n)$ to be the determinant of the square matrix with entries $\pi_i(x_j)$ for $m \le i \le n$ and $0 \le j < w$.

Definition 2.7. We shall see in Section 3 that there are cofibrations of spectra

$$\Sigma^{2p^n-2} BP\langle m, n\rangle \xrightarrow{v_n} BP\langle m, n\rangle \to BP\langle m, n-1\rangle \xrightarrow{q_n} \Sigma^{2p^n-1} BP\langle m, n\rangle.$$

This also works for $n = m$ if we interpret $BP\langle m, m-1\rangle$ as the mod p Eilenberg-MacLane spectrum $H\mathbb{F}_p$. We let $a_0, \dots, a_{w-1}$ be the usual generators of $H^1(BV_w; \mathbb{F}_p)$ and we define

$$\alpha'' = \alpha''(m, n) = q_n \cdots q_{m+1} q_m (a_0 a_1 \dots a_{w-1}).$$

It is not hard to see that when $g \in \mathrm{Aut}(V_w)$ we have $\det(g) \in \mathbb{F}_p^\times$ and $g^*\alpha'' = \det(g)\alpha''$ (which makes sense because $p\alpha'' = 0$).

Our main result is as follows.

Theorem 2.8. *We have $\alpha = \alpha' = \pm\alpha'' \in E^d BV_w$, where $d = 2p^m(p^w - 1)/(p - 1)$. The annihilator of v_m on E^*BV_w is the same as the annihilator of I_{n+1}. It is a free module of rank one over $E^*BV_w/I_{n+1} = P(\mathbb{F}_p; w) = \mathbb{F}_p[\![x_0, \dots, x_{w-1}]\!]$ generated by α. It maps injectively to $H^*(BV_w; \mathbb{F}_p)$.*

This will be proved after Proposition 5.13. Note that α' and α'' appear to depend on the choice of generators v_k, but in fact they do not, because they are equal to α.

The basic structure of the proof is as follows. It is quite easy to see that $I_{n+1}\alpha' = I_{n+1}\alpha'' = 0$ and that α, α', and α'' all have the same degree. We shall show in Section 3 that q_k is compatible up to sign with the Milnor Bockstein operation Q_k in mod p cohomology. Given this, classical arguments about Dickson invariants show that $\alpha = \alpha' = \pm\alpha'' \pmod{I_{n+1}}$. We will show using an intricate argument inspired by the theory of multiple level structures [14, 4] that $\mathrm{ann}(v_m) = \mathrm{ann}(I_{n+1}) = (\alpha)$. It follows for degree reasons that $\alpha' = \lambda\alpha$ for some $u \in \mathbb{F}_p$, and we find that $\lambda = 1$ by reducing everything modulo I_{n+1}. The same argument shows that $\alpha'' = \pm\alpha$.

3. The spectra $BP\langle m, n\rangle$

For brevity, we will write MU for the p-local spectrum $MU_{(p)}$. Recall that MU can be made into a strictly commutative ring spectrum (or "commutative S-algebra") in the foundational setting of [3], so we can construct a derived category $\mathcal{D}_{MU}$ of strict MU-modules, and a category $\mathcal{R}_{MU}$ of ring objects in $\mathcal{D}_{MU}$ (referred to in [3] as MU-ring spectra). As usual, we let BP^* be the largest quotient ring of MU^* over which the standard formal group law becomes p-typical. We know from [13] that (even when $p = 2$) there is a commutative ring BP in $\mathcal{D}_{MU}$ with homotopy ring BP^*, and that this object is unique up to canonical isomorphism.

If $p > 2$ we choose once and for all a sequence of elements $v_k \in BP_{2(p^k-1)} = BP^{-2(p^k-1)}$ such that $[p](t) = v_k t^{p^k} \pmod{v_0, \dots, v_{k-1}, t^{p^k+1}}$. In particular, this forces $v_0 = p$. Two popular choices would be to take v_k to be the k'th Hazewinkel generator (so that $[p](t) = \exp_F(pt) +_F \sum_{k>0}^F v_k t^{p^k}$) or the k'th Araki generator (so that $[p](t) = \sum_{k\ge0}^F v_k t^{p^k}$). In any case, we define $BP\langle m, n\rangle^* = BP^*/(v_i \mid i < m \text{ or } i > n)$. We know from [13, Theorem 2.6] that there is a commutative ring $BP\langle m, n\rangle$ in $\mathcal{D}_{MU}$ with homotopy ring $BP\langle m, n\rangle^*$, and that this object is unique up to canonical isomorphism.

If $p = 2$ we do not have so much choice about the sequence of v's. Nonetheless, we know from [13, Proposition 2.10] that there exist sequences of v's for which the rings $BP\langle n\rangle^* = BP^*/(v_i \mid i > n)$ can be realised as the homotopy ring of a commutative ring object $BP\langle n\rangle$, which is unique up to canonical isomorphism (here we are writing $BP\langle n\rangle$ for the spectrum called $BP\langle n\rangle'$ in [13]). Unfortunately, neither of the popular choices listed above work in this context. We fix such a sequence once and for all. By the proof of [13, Theorem 2.13], we know that there is a central $BP\langle n\rangle$-algebra $BP\langle m,n\rangle$ with homotopy ring $BP\langle m,n\rangle^* = BP\langle n\rangle^*/I_m$ and a derivation $Q_{m-1}\colon BP\langle m,n\rangle \to \Sigma^{2^m-1}BP\langle m,n\rangle$ such that $ab - ba = v_m Q_{m-1}(a)Q_{m-1}(b)$. It follows that when X is a space such that $BP\langle m,n\rangle^*X$ is concentrated in even degrees, the operation Q_{n-1} is trivial on the ring $BP\langle n,m\rangle^*X$, and thus this ring is commutative. If $BP\langle m,n\rangle$ and $BP'\langle m,n\rangle$ are two such central $BP\langle n\rangle$-algebras then either there is a unique isomorphism $BP\langle m,n\rangle \simeq BP'\langle m,n\rangle$, or there is a unique isomorphism $BP\langle m,n\rangle^{\mathrm{op}} \simeq BP'\langle m,n\rangle$.

Next, recall from [13] that EKMM theory gives a map $Q_k\colon MU/v_k \to \Sigma^{2p^k-1}MU/v_k$ in $\mathcal{D}_{MU}$ that is a derivation for any product on MU/v_k. By smashing this over MU with $BP/(v_i \mid i < m, i \neq k)$ we get a derivation $Q_k\colon P(m) \to \Sigma^{2p^k-1}P(m)$. We can then smash this over MU with $MU/(v_i \mid i > n)$ and then with $MU/(v_m, \dots, v_n)$ to get compatible derivations on $BP\langle m,n\rangle$ and $H\mathbb{F}_p$. It is also easy to see that there is a canonical map $q_n\colon BP\langle m,n-1\rangle \to \Sigma^{2p^n-1}BP\langle m,n\rangle$ and a ring map $\rho_n\colon BP\langle m,n\rangle \to BP\langle m,n-1\rangle$ fitting into a cofibre sequence

$$\Sigma^{2p^n-2}BP\langle m,n\rangle \xrightarrow{v_n} BP\langle m,n\rangle \xrightarrow{\rho_n} BP\langle m,n-1\rangle \xrightarrow{q_n} \Sigma^{2p^n-1}BP\langle m,n\rangle$$

such that $\rho_n q_n = Q_n$.

We claim that our operation Q_n on $H\mathbb{F}_p$ is the same as the Milnor Bockstein operation Q_n^M up to sign. It is well-known that derivations in mod p cohomology are the same as primitive elements in the Steenrod algebra $H\mathbb{F}_p^*H\mathbb{F}_p$, and that the space of primitives in dimension $2p^n - 1$ is spanned by Q_n^M, so we have $Q_n = \lambda Q_n^M$ for some $\lambda \in \mathbb{F}_p$. We need to show that $\lambda = \pm 1$, so we need only compute one nontrivial instance of Q_n. We shall do this in the case $p > 2$. The case where $p = 2$ and $n > 0$ requires essentially only notational changes, and the case where $p = 2$ and $n = 0$ can be done with simple *ad hoc* arguments. Recall that $H\mathbb{F}_p^*BV_1 = \mathbb{F}_p[x] \otimes E[a]$, where a is the usual generator of $H\mathbb{F}_p^1BV_1$ and $x = \beta a$, which is the image of the usual generator of $MU^2\mathbb{C}P^\infty$. It is well-known that $Q_n^M(a) = x^{p^n}$. Given this, our claim follows easily from Proposition 3.1 below.

Proposition 3.1. *Let p be an odd prime, and let C_p be the cyclic group of p'th roots of unity in $\mathbb{C}$, which we can identify with V_1. Write*

$$X = \mathit{cofibre}(S^{2p^n-1} \to S^{2p^n-1}/C_p),$$

which is the $2p^n$-skeleton of BC_p. Let q_n be the Bockstein operation in the cofibration

$$\Sigma^{2p^n-2}P(n) \xrightarrow{v_n} P(n) \xrightarrow{\rho} P(n+1) \xrightarrow{q_n} \Sigma^{2p^n-1}P(n).$$

Then there is a unique element $b \in P(n+1)^1X$ that hits the usual generator $a \in H\mathbb{F}_p^1X = H\mathbb{F}_p^1BC_p$. Moreover, $q_nb = \pm x^{p^n}$, where x is the image in $P(n)^2BC_p$ of the usual generator $x \in MU^2\mathbb{C}P^\infty$.

The sign ambiguity could be resolved by a careful analysis of conventions, which we do not have the patience to do.

Proof. For brevity, write $m = p^n - 1$ and $R = P(n)$ and $\overline{R} = P(n+1)$, so we have a cofibration

$$\Sigma^{2m}R \xrightarrow{v_n} R \xrightarrow{\rho} \overline{R} \xrightarrow{q_n} \Sigma^{2m+1}R.$$

We also write

$$\begin{aligned}
P &= \mathbb{C}P^m = S^{2m+1}/S^1 \\
L &= \text{ tautological bundle over } P \\
P^L &= \text{ Thom space of } L \simeq \mathbb{C}P^{m+1} \\
Y &= S^{2m+1}/C_p \\
H &= H\mathbb{F}_p.
\end{aligned}$$

Note that Y can also be thought of as the sphere bundle $S(L^p)$ in the p'th tensor power of L, or as the $(2m+1)$-skeleton of $BV_1 = S^\infty/C_p$. Note also that

$$H^*X = P[x]\otimes E[u]/(x^{m+2}, ax^{m+1}) = \mathbb{F}_p\{1, a, x, \dots, ax^m, x^{m+1}\}.$$

An easy connectivity argument shows that there is a unique element $b \in \overline{R}^1X$ that hits $v \in H^1X$. Indeed, let F be the fibre of the map $\overline{R} \to H$, so that π_*F starts in dimension $2p^{n+1}-2$. The bottom cell of DX is in dimension $-2p^n$, so the bottom cell of $F\wedge DX$ is in dimension $d = 2(p-1)p^n - 2$. This is strictly greater than 1 because $p > 2$ and $n \geq 0$. It follows easily that $\pi_1(DX\wedge\overline{R}) = \pi_1(DX\wedge H)$, or in other words $\overline{R}^1X = H^1X$, so there is a unique element b as described.

We next consider the diagram

$$\begin{tikzcd}
 & P \arrow[dl, "z_L"'] \arrow[dr, "z_{L^p}"] & \\
P^L \arrow[rr, "\phi"'] & & P^{L^p}
\end{tikzcd}$$

Here z_L and z_{L^p} are the zero-section inclusions, and ϕ is obtained in the obvious way from the p'th power map of total spaces $E(L) \to E(L^p)$. There is also a corresponding diagram with P replaced by $\mathbb{C}P^\infty$. By applying R^* to this, we get a diagram

$$\begin{tikzcd}
 & R^*[\![x]\!] & \\
R^*[\![x]\!]u_L \arrow[ur, "z_L^*"] & & R^*[\![x]\!]u_{L^p} \arrow[ul, "z_{L^p}^*"'] \arrow[ll, "\phi^*"]
\end{tikzcd}$$

Here u_L and u_{L^p} are the Thom classes of L and L^p; the corresponding Euler classes are x and $[p](x)$. It is well-known that $z_L^*(f(x)u_L) = f(x)x$ and $z_{L^p}^*(f(x)u_{L^p}) = f(x)[p](x)$, and it follows easily (because $\langle p\rangle(x) = [p](x)/x$ is not a zero-divisor in $R^*[\![x]\!]$) that

$$\phi^*(f(x)u_{L^p}) = f(x)\langle p\rangle(x)u_L.$$

By mapping the $\mathbb{C}P^m$ diagram into the $\mathbb{C}P^\infty$ diagram, we see that the same fomula holds for the map

$$\phi^*\colon \tilde{R}^*P^{L^p} = (R^*[x]/x^{m+1}).u_{L^p} \to (R^*[x]/x^{m+1}).u_L = \tilde{R}^*P^L.$$

In this context, however, we simply have $\langle p\rangle(x) = v_nx^m$, so

$$\phi^*u_{L^p} = v_nx^mu_L.$$

We next apply the octahedral axiom to the maps $S^{2m+1} \to Y \to P$ (which comes down to replacing the maps by cofibrations and using the fact that $P/Y = (P/S^{2m+1})/(Y/S^{2m+1})$). As $S^{2m+1} = S(L)$ and $Y = S^{2m+1}/C_p = S(L^p)$, we see that the cofibres of $S^{2m+1} \to P$ and $Y \to P$ are homeomorphic to P^L and P^{L^p}. With these identifications, the map $S^{2m+1} \to Y$ is the p'th power map $S(L) \to S(L^p)$, and thus the induced map $P^L \to P^{L^p}$ is just ϕ. We therefore have an octahedral diagram

$$\begin{tikzcd}
 & & P^L \arrow[dl, "\circ" marking, "\tau_L"'] \arrow[ddrr, bend left, "\phi"] & & \\
 & S^{2m+1} \arrow[rr, "\pi_L"] \arrow[dr, "\phi"] & & P \arrow[ul, "z_L"'] \arrow[dr, "z_{L^p}"] & \\
X \arrow[ur, "\circ" marking, "\tau_{L^p}"] \arrow[uurr, bend left, "r"] & & Y \arrow[ll, "j"] \arrow[ur, "\pi_{L^p}"'] & & P^{L^p} \arrow[ll, "\circ" marking, "e"]
\end{tikzcd}$$

(A circled arrow $U \xrightarrow{\circ} V$ means a map $U \to \Sigma V$. The diagram can be made to look more like an octahedron by lifting up the outer three vertices and drawing in an extra arrow to represent the composite $je\colon P^{L^p} \xrightarrow{\circ} X$.) In particular, we see that the stable fibre of ϕ is just X.

We next claim that the map $e^*\colon \widetilde{H}^1 Y \to \widetilde{H}^2 P^{L^p}$ is an isomorphism, and sends a to the Thom class u_{L^p}. To see this, let M be the restriction of L to the basepoint in P, so M is just a one-dimensional complex vector space. The bottom cell of $Y = S(L^p)$ is just the circle $S(M^p)$, and the bottom cell of P^{L^p} is the one-point compactification of M^p, written S^{M^p}. The restriction of e to this bottom cell is the standard homeomorphism of S^{M^p} with the unreduced suspension of $S(M^p)$, followed by the projection to the reduced suspension. The claim follows easily from this.

We now consider the following diagram.

$$\begin{tikzcd}
\Sigma^{-2} P^L \arrow[r, "\phi"] \arrow[d, "x^m u_L"'] & \Sigma^{-2} P^{L^p} \arrow[r, "je"] \arrow[d, "u_{L^p}"'] & \Sigma^{-1} X \arrow[r, "r"] \arrow[d, dashed, "c"] & \Sigma^{-1} P^L \arrow[d, "x^m u_L"] \\
\Sigma^{2m} R \arrow[r, "v_n"'] & R \arrow[r, "\rho"'] & \overline{R} \arrow[r, "q_n"'] & \Sigma^{2m+1} R
\end{tikzcd}$$

We see from the octahedron that the top line is a cofibration, and the bottom line is a cofibration by construction. The left hand square commutes because $\phi^* u_{L^p} = v_n x^m u_L$. It follows that there exists a map $c\colon \Sigma^{-1} X \to \overline{R}$ (i.e. $c \in \overline{R}^1 X$) making the whole diagram commute.

From our discussion of e^* in cohomology and the commutativity of the middle square, we see that the image of c in $H^1 X$ is just a. By the uniqueness of b, we deduce that $c = b$. Thus, the commutativity of the right hand square tells us that $q_n b$ is the image of $x^m u_L \in R^{2m+2} P^L$ in $R^* X$. Under the usual identification of P^L with $\mathbb{C}P^{m+1}$, the element $x^m u_L$ becomes x^{m+1}, and the map $X \to \mathbb{C}P^{m+1}$ is a restriction of the usual map $BC_p \to \mathbb{C}P^\infty$. It follows that $q_n b = x^{m+1}$ as claimed. $\square$

4. Commutative algebra

In this section, we recall some basic ideas from commutative algebra.

We will need to be a little more careful than is usual about the relationship between graded and ungraded rings. For us, a graded ring R^* will mean a sequence of Abelian groups R^k (for $k \in \mathbb{Z}$) with product maps $R^i \otimes R^j \to R^{i+j}$ with the usual properties. We will assume that $R^k = 0$ when k is odd, and that the product is commutative. This will apply to all rings that we consider except for $H^*(BV_k; \mathbb{F}_p)$, and in that case we will not need to use the results of this section. It is common to identify a graded ring R^* with the ungraded ring $\bigoplus_k R^k$. We shall not do this, for the following reason. There is an obvious way to interpret the expression $R^* = \mathbb{Z}_{(p)}[v_1, \dots, v_n][\![x_0, \dots, x_{k-1}]\!]$ as a graded ring (with $|v_k| = -2(p^k - 1)$ and $|x_i| = 2$). Explicitly, R^k is the set of expressions $\sum_\alpha a_\alpha x^\alpha$, where $\alpha = (\alpha_0, \dots, \alpha_{k-1})$ is a multiindex, $|\alpha| = \sum_i \alpha_i$ and $a_\alpha \in BP\langle n\rangle^{-2|\alpha|}$. With this interpretation, the ring $\bigoplus_k R^k$ is not the same as the ungraded ring $R' = \mathbb{Z}_{(p)}[v_1, \dots, v_n][\![x_0, \dots, x_{k-1}]\!]$ (it does not contain $\sum_{i\geq 0} x_0^i$, for example). The set $\prod_k R_k$ is different again. It is not clear how many of the nice ring-theoretic properties of R' are shared by $\bigoplus_k R^k$. For this reason, we prefer to work with graded rings as defined above. Most theorems for ungraded rings have graded counterparts, which are proved by a straightforward adaptation of the ungraded proofs. We will outline the results that we need, leaving most of the task of adaptation to the reader.

We will only consider homogeneous elements of R^*, in other words elements of R^k for various k. The word "module" will always mean "graded module", and similarly for ideals.

Given an ideal I^* in R^*, let $(I^N)^k$ denote the part of the N'th power of I in degree k. The cosets $a + (I^N)^k$ (for $a \in R^k$) form a basis for a topology on R^k. We say that a topology of this form is a *linear topology* on R^*, and we say that I^* is an *ideal of definition.* If I^* is one ideal of definition, it is clear that another ideal J^* is an ideal of definition for the same topology if I^* contains a power of J^* and *vice versa.* We say that a linear topology is *complete* if $R^k = \varprojlim_N R^k/(I^N)^k$ for all k. We say that a homogeneous element is *topologically nilpotent* if some power of it lies in I^*. In the Noetherian case, it is not hard to see that the topologically nilpotent elements form an ideal of definition.

If R^* is a quotient of $E^*[\![x_0, \dots, x_{k-1}]\!]$ then we give R^* the complete linear topology defined by the ideal $(x_0, \dots, x_{k-1})$. In this context, we can consider the ungraded ring $\mathrm{Tot}(R^*) = \varprojlim_N \bigoplus_k (R/I^N)^k$ as a substitute for $\bigoplus_k R^k$. This avoids the difficulty mentioned above: if

$$R^* = \mathbb{F}_p[v_m, \dots, v_n][\![x_0, \dots, x_{k-1}]\!]$$

(in the usual graded sense) then

$$\mathrm{Tot}(R^*) = \mathbb{F}_p[v_m, \dots, v_n][\![x_0, \dots, x_{k-1}]\!]$$

(in the usual ungraded sense). However, the relationship between properties of R^* and those of $\mathrm{Tot}(R^*)$ is not as close as one might like, so we will not stress this point of view.

We will need the following version of the Weierstrass preparation theorem.

Proposition 4.1. *Let R^* be as above. Let $y = \sum_{k\geq 0} a_k x^k \in R^*[\![x]\!]$ be a homogeneous element such that a_i is topologically nilpotent for $i < d$, and a_d is a unit. Then $R^*[\![x]\!]$ is freely generated by $\{x^i \mid i < d\}$ as a module over $R^*[\![y]\!]$, and thus $R^*[\![x]\!]/y$ is freely generated by $\{x^i \mid i < d\}$ as a module over R^*.*

Proof. We may assume that $a_d = 1$. For any $m \geq 0$, we can write $m = ld + k$ with $l \geq 0$ and $0 \leq k < d$, and then write $w_m = x^k y^l$. Let I^* be the ideal of topological nilpotents, so that $w_m = x^m \pmod{I^*, x^{m+1}}$. For any R^*-module M^* that is complete with respect to I^*, we can define a map

$$\theta_M\colon \prod_{m\geq 0} M^* \to M^*[\![x]\!]$$

by $\theta_M(b) = \sum_m b_m w_m$. Suppose that $I^*M^* = 0$, and that $c = \sum_{m\geq 0} c_m x^m \in M^*[\![x]\!]$. Given the form of w_m mod I^*, we see easily by induction that there is a unique sequence of b_m's such that $c = \sum_{k=0}^m b_m w_m \pmod{x^{m+1}}$ and thus that θ_M is an isomorphism. Moreover, if we have a pair of modules $N^* \leq M^*$ such that θ_N and $\theta_{M/N}$ are isomorphisms, then so is θ_M (by a five-lemma argument). It follows by induction that θ_{M/I^k} is iso for all k, and thus by taking inverse limits that θ_M is iso for all M^*. In particular, θ_R is an isomorphism. This means that for any series $c(x) \in R^*[\![x]\!]$, there are unique series $b_0(y), \dots, b_{d-1}(y) \in R^*[\![y]\!]$ such that $c(x) = \sum_i b_i(y) x^i$, which proves the proposition. $\square$

We say that a graded ring R^* is *local* if it has only one maximal ideal, or equivalently if it has an ideal such that every homogeneous element in the complement is invertible. It is clear that E^* is local in this sense, with maximal ideal $(v_m, \dots, v_n)$. Similarly, any quotient of the ring $E^*[\![x_0, \dots, x_{k-1}]\!]$ is local, with maximal ideal $(v_m, \dots, v_n, x_0, \dots, x_{k-1})$.

We say that a graded ring R^* is Noetherian if every ideal is generated by a finite set of homogeneous elements. Simple adaptations of the usual arguments in an ungraded context [9, Theorem 3.3] show that any quotient of $E^*[\![x_0,\dots,x_{k-1}]\!]$ is Noetherian.

We say that a graded ring R^* is a *domain* if the product of two nonzero homogeneous elements of R^* is nonzero. This will hold if $\mathrm{Tot}(R^*)$ is an ungraded domain, but unfortunately we have not been able to prove the converse. We say that an ideal P^* in R^* is *prime* if R^*/P^* is a domain, and define the *Krull dimension* of R^* to be the largest integer d such that there exists a chain $P_0^* < \dots < P_d^*$ of prime ideals in R^*.

We say that a Noetherian graded local ring R^* of Krull dimension d is *regular* if there is a sequence of d homogeneous elements that generates the maximal ideal. Such a sequence is called a *regular system of parameters*; it is necessarily a regular sequence. Simple adaptations of the usual arguments in an ungraded context [9, Theorem 15.4] show that $R^* = E^*[\![x_0,\dots,x_{k-1}]\!]$ has dimension $w+k$, so the sequence $\{v_m,\dots,v_n,x_0,\dots,x_{k-1}\}$ is a regular sequence of parameters and R^* is regular. More generally, if R^* is any graded regular local ring and x is a homogeneous indeterminate then $R^*[\![x]\!]$ is again a graded regular local ring.

By graded versions of [9, Theorems 14.3 and 20.3], we see that a graded regular local ring is an integral domain, with unique factorisation for homogeneous elements.

The following result is a graded version of [9, Theorem 14.2], and can be proved in the same way.

Theorem 4.2. *Let R^* be a graded regular local ring of dimension d, and $\{x_0,\dots,x_{k-1}\}$ a sequence of homogeneous elements of R^*. Then the following are equivalent:*

(a) *$\{x_0,\dots,x_{k-1}\}$ is a subset of a regular system of parameters.*
(b) *The images of $\{x_0,\dots,x_{k-1}\}$ in $\mathfrak{m}/\mathfrak{m}^2$ (where $\mathfrak{m}$ is the unique maximal ideal of R^*) are linearly independent over the graded field $R^*/\mathfrak{m}$.*
(c) *$R^*/(x_0,\dots,x_{k-1})$ is a graded regular local ring of dimension $d-k$.*

We can also show that two weaker conditions are equivalent to each other.

Theorem 4.3. *Let R^* be a graded regular local ring of dimension d, and $\{x_0,\dots,x_{k-1}\}$ a sequence of homogeneous elements of R^*. Then the following are equivalent:*

(a) *$\{x_0,\dots,x_{k-1}\}$ is a regular sequence.*
(b) *$R^*/(x_0,\dots,x_{k-1})$ has Krull dimension $d-k$.*

Proof. This is a graded version of the equivalence (1)$\Leftrightarrow$(3) in part (iii) of [9, Theorem 17.4], taking into account the equation $\mathrm{ht}(I^*) = \dim(R^*) - \dim(R^*/I^*)$ from part (i) of that theorem, and the fact that regular local rings are Cohen-Macaulay. $\square$

5. The structure of E^*BV_k

Proposition 5.1. *Let R^* be an algebra over E^* of chromatic depth $d > 0$. Then $[p](x)$ is not a zero-divisor in $R^*[\![x]\!]$, and the ring $R^*[\![x]\!]/[p](x)$ has chromatic depth at least $d-1$.*

Proof. Suppose that $0 \neq f(x) \in R^*[\![x]\!]$, say $f(x) = ax^k \pmod{x^{k+1}}$ with $0 \neq a \in R^*$. As R^* has nonzero chromatic depth, we see that $v_m a \neq 0$ and $[p](x)f(x) = v_m a x^{k+p^m} \pmod{x^{1+k+p^m}}$ so $[p](x)f(x) \neq 0$. This shows that $[p](x)$ is not a zero-divisor in $R^*[\![x]\!]$.

Now suppose that $d > 1$, so that $\{v_m, v_{m+1}\}$ is regular on R^*. Consider a series $f(x) \in R^*[\![x]\!]$ such that $v_m f(x)$ is divisible by $[p](x)$. We claim that $f(x)$ is divisible by $[p](x)$. Indeed, suppose that $v_m f(x) = [p](x)g(x)$. We then have $[p](x)g(x) = 0$ in $(R^*/v_m)[\![x]\!]$. However, v_{m+1} is not a zero divisor in R^*/v_m, so $[p](x)$ is not a zero divisor in $(R^*/v_m)[\![x]\!]$ by the previous paragraph. Thus $g(x) = 0$ in $R^*/v_m[\![x]\!]$, or $g(x) = v_m h(x)$ in $R^*[\![x]\!]$ say. This means that $v_m(f(x) - [p](x)h(x)) = 0$ in $R^*[\![x]\!]$, but v_m is not a zero-divisor so $f(x) = [p](x)h(x)$ as claimed. The conclusion is that v_m is regular on $R^*[\![x]\!]/[p](x)$. If $d > 2$ then we can replace R^* by R^*/v_m and m by $m+1$ and d by $d-1$ and run the same argument again. It follows by induction that $R^*[\![x]\!]/[p](x)$ has chromatic depth at least $d-1$. $\square$

Corollary 5.2. *Let R^* be an algebra over E^* of chromatic depth $d>0$. Then for $j\leq d$ and $k\geq j$ the sequence $[p](x_0),\dots,[p](x_{j-1})$ is regular in $R^*[\![x_0,\dots,x_{k-1}]\!]$ and the quotient $P(R^*;j;k)$ has chromatic depth at least $d-j$.*

Proof. This follows easily from the proposition, using the obvious fact that $S^*[\![x]\!]$ has the same chromatic depth as S^* for any algebra S^* over $BP\langle m,n\rangle^*$. ☐

Corollary 5.3. *For $k\leq w$ we have $E^*BV_k=P(E^*;k)$, and this has chromatic depth at least $w-k$.*

Proof. Write $B=B\mathbb{Z}/p$ and $Z=\mathbb{C}P^\infty$, so $E^*Z=E^*[\![x]\!]$ and B is a circle bundle over Z with Chern class $[p](x)$. Thus $B^i\times Z^{k-i}$ is a circle bundle over $B^{i-1}\times Z^{k+1-i}$ with Chern class $[p](x_i)$. This gives a long exact Gysin sequence

$$\dots\leftarrow E^*(B^i\times Z^{k-i})\leftarrow E^*(B^{i-1}\times Z^{k+1-i})\xleftarrow{[p](x_i)}E^{*-2}(B^{i-1}\times Z^{k+1-i})\leftarrow\cdots.$$

It follows easily by induction from Corollary 5.2 that these sequences are short exact, and that $E^*(B^i\times Z^{k-i})=P(E^*;i;k)$. The case $i=k$ gives the corollary. ☐

Definition 5.4. If $k\leq w$, we define $A(k)^*=A(m,n,k)^*$ to be the largest quotient ring of $R^*=P(E^*;k)$ over which the series $\phi_k(t)=\prod_{\underline{\lambda}\in\mathbb{F}_p^k}(t-_F[\underline{\lambda}](\underline{x}))^{p^m}$ divides $[p](t)$. We also write $\psi_k(t)=[p](t)/\phi_k(t)$.

In more detail, we note that $t-_F[\underline{\lambda}](\underline{x})$ is a unit multiple of $t-[\underline{\lambda}](\underline{x})$, so $\phi_k(t)$ is a unit multiple of a monic polynomial of degree p^{m+k}, whose lower coefficients are topologically nilpotent. It follows from Proposition 4.1 that $R^*[\![t]\!]/\phi_k(t)$ is a free module over R^* on generators $1,t,\dots,t^{p^{m+k}-1}$. In particular, we can write $[p](t)=\sum_{i=0}^{p^{m+k}-1}c_it^i \pmod{[p](t)}$, for uniquely defined coefficients $c_i\in R^*$. We define $A(k)^*$ to be the quotient ring $R^*/(c_0,\dots,c_{p^{m+k}-1})$. It also follows from Proposition 4.1 that $\phi_k(t)$ is not a zero-divisor in $A(k)^*[\![t]\!]$, so there is a unique series $\psi_k(t)\in A(k)^*[\![t]\!]$ such that $[p](t)=\phi_k(t)\psi_k(t)$.

Theorem 5.5. *For $k\leq w$ we have*
(a) *$A(k)^*$ is a regular local ring in the graded sense.*
(b) *There is a unique formal group law F_k over $A(k)^*$ for which $\phi_k(t)$ is a homomorphism $F\to F_k$.*
(c) *There is a unique series $\theta_k(t)=t\overline{\theta}_k(t)$ over $A(k)^*$ such that $[p]_F(t)=\theta_k(\phi_k(t))$.*
(d) *If $k>0$, we have $A(k)^*=A(k-1)^*[\![x_{k-1}]\!]/\psi_{k-1}(x_{k-1})$.*

Moreover, when $k=w$ we have $A(w)^*=P(\mathbb{F}_p;w)=\mathbb{F}_p[\![x_0,\dots,x_{w-1}]\!]$ and $I_{n+1}=0$ in $A(w)^*$.

Proof. Suppose that $k<w$ and that (a)…(d) hold up to stage k (which is trivial for $k=0$). Define $B^*=A(k)^*[\![x_k]\!]/\psi_k(x_k)$.

We first claim that $\psi_k(x_k)=0$ in $A(k+1)^*$. Indeed, we have

$$\phi_{k+1}(t)=\phi_k(t)\prod_{i=1}^{p-1}\phi_k(t-_F[i](x_k)),$$

and this divides $[p](t)=\phi_k(t)\psi_k(t)$ over $A(k+1)^*$. As $\phi_k(t)$ is a unit multiple of a monic polynomial, it is not a zero divisor in $A(k+1)^*[\![t]\!]$, so we conclude that $\prod_{i=1}^{p-1}\phi_k(t-_F[i](x_k))$ divides $\psi_k(t)$ in $A(k+1)^*[\![t]\!]$. We now set $t=x_k$ to conclude that $\psi_k(x_k)=0$ in $A(k+1)^*$ as claimed.

It is also clear that $\phi_k(t)$ divides $\phi_{k+1}(t)$, so it divides $[p](t)$ over $A(k+1)^*$, so $A(k+1)^*$ is an algebra over $A(k)^*$. This means that $A(k+1)^*$ is a quotient of B^* in a natural way.

Next, we claim that B^* is a regular local ring in the graded sense. For this, we write $C^*=E^*[\![x_0,\dots,x_k]\!]$. This is clearly a regular local ring, and $B^*=C^*/(\psi_0(x_0),\dots,\psi_k(x_k))$. By Theorem 4.2 it suffices to check that the list

$$L=\{\psi_0(x_0),\dots,\psi_k(x_k),x_0,\dots,x_k,v_{m+k+1},\dots,v_n\}$$

is a regular system of parameters for C^*. The length of L is $n+k-m+2$ which is the same as the Krull dimension of C^*, so it is enough to check that L generates the maximal ideal of

C^*, or equivalently that $\overline{C}^* = C^*/(L) = \mathbb{F}_p$, or equivalently that $v_m, \dots, v_{m+k}$ vanish in $\overline{C}^*$, or equivalently that $[p](t)$ is divisible by $t^{p^{m+k+1}}$ over $\overline{C}^*$. However, we know that C^* is an algebra over $A(k)^*$ so $[p](t) = \phi_k(t)\psi_k(t)$ over $\overline{C}^*$. Moreover, as $x_0, \dots, x_{k-1}$ vanish in $\overline{C}^*$ we see from the definition that $\phi_k(t) = t^{p^{k+m}}$ in $\overline{C}^*[\![t]\!]$. This means that $[p](t) = t^{p^{k+m}}\psi_k(t)$ over $\overline{C}^*$, so $v_0 = \dots = v_{k+m-1} = 0$ in $\overline{C}^*$ and $v_{k+m} = \psi_k(0)$. On the other hand, we have $x_k = 0$ and $\psi_k(x_k) = 0$ so $\psi_k(0) = 0$ so $v_{k+m} = 0$ as required. This completes the proof that B^* is a regular local ring.

It is clear from the above that $\{x_0, \dots, x_k, v_{m+k+1}, \dots, v_n\}$ is a regular system of parameters for B^*, and thus that the Krull dimension of B^* is w. As B^* is an integral domain, it follows that any proper quotient of B^* has dimension strictly less than w. Thus, if we can show that $A(k+1)^*$ has dimension w, we can deduce that the quotient map $B^* \twoheadrightarrow A(k+1)^*$ is an isomorphism.

For the rest of the argument we need to separate the cases $k+1 = w$ and $k+1 < w$. If $k+1 = w$ we argue as follows. It follows from the definition of $A(w)^*$ that $A(w)^*/I_{n+1}$ is the largest quotient of $E^*[\![x_0, \dots, x_{w-1}]\!]/I_{n+1} = P(\mathbb{F}_p; w)$ over which $\phi_w(t)$ divides $[p](t)$. However, in this context $[p](t) = 0$ which is automatically divisible by $\phi_w(t)$, so $A(w)^*/I_{n+1} = P(\mathbb{F}_p; w)$. This has the same Krull dimension as B^*, so we must have $B^* = A(w)^* = A(w)^*/I_{n+1}$. In particular, we see that $I_{n+1} = 0$ in $A(w)^*$, and thus that the formal group law F becomes $F(s,t) = s+t$ in $A(w)^*[\![s,t]\!]$. It follows that parts (b) and (c) of the theorem hold with $F_w = F$ and $\theta_w = 0$.

We now consider the case $k+1 < w$, so that $k < w-1$. For this, we use the theory of multiple level structures developed in [4, Section 5]; we will assume that the reader is familiar with this.

Let $\widehat{E}^*$ be the graded ring $(E^*[u, u^{-1}]/(u^{p^n-1} - v_n))^\wedge_{I_n}$. Write $u_k = v_k u^{1-p^k} \in \widehat{E}^0$. It is easy to see that $\widehat{E}^0 = \mathbb{F}_p[\![u_m, \dots, u_{n-1}]\!]$, or $\mathbb{Z}_p[\![u_1, \dots, u_{n-1}]\!]$ if $m = 0$. Moreover, we have $\widehat{E}^* = \widehat{E}^0[u, u^{-1}]$.

Let F be the usual formal group law over E^*, and define $\widehat{F}(s,t) = uF(s/u, t/u)$, so that $\widehat{F}$ is a formal group law over $\widehat{E}^0$. Let $\mathbb{G}$ be the associated formal group over $X = \operatorname{spf}(\widehat{E}^0)$, which has height n and strict height m. This puts us in the context studied in [4].

Now write $J = I_n + (x_0, \dots, x_k) \leq A(k+1)^*$, and $\widehat{A}(k+1)^* = (A(k+1)^*[u, u^{-1}]/(u^{p^n-1} - v_n))^\wedge_J$. It is clear from the definitions that $\widehat{A}(k+1)^0$ is just the ring $\mathcal{O}_{\mathrm{Level}_m(V^*_{k+1}, \mathbb{G})}$ which classifies p^m-fold level-V^*_{k+1} structures on $\mathbb{G}$, and $\widehat{A}(k+1)^* = \widehat{A}(k+1)^0[u, u^{-1}]$. We know from [4, Theorem 5.6] that $\widehat{A}(k+1)^0$ is an integral domain and a finitely generated free module over $\widehat{E}^0$. It is nonzero because $k+1 \leq n-m$. It follows that $\widehat{A}(k+1)^*$ is a graded domain and a finitely generated free module on homogeneous generators over $\widehat{E}^*$.

We now have a diagram as follows.

$$\begin{tikzcd} E^* \arrow[r, "i"] \arrow[d, "c"'] & A(k+1)^* \arrow[d, "d"] \\ \widehat{E}^* \arrow[r, "\hat{\imath}"'] & \widehat{A}(k+1)^*. \end{tikzcd}$$

For $m \leq t \leq n$ we let $\widehat{I}_t$ be the ideal in $\widehat{E}^*$ generated by $\{v_m, \dots, v_{t-1}\}$. This is clearly prime, with $\widehat{I}_t \cap E^* = I_t$ and $0 = I_m < \dots < I_n$. As $\widehat{A}(k+1)^*$ is integral over $\widehat{E}^*$, the going-up theorem [9, Theorem 9.3] gives us a chain $0 = \widehat{J}_m < \dots < \widehat{J}_n$ of primes in $\widehat{A}(k+1)^*$ such that $\widehat{J}_t \cap \widehat{E}^* = \widehat{I}_t$. Let J_t be the preimage of $\widehat{J}_t$ under the map $d\colon A(k+1)^* \to \widehat{A}(k+1)^*$. It is clear that the preimage of this under the map $i\colon E^* \to A(k+1)^*$ is just I_t, so we have a chain of strict inclusions of prime ideals $0 = J_m < \dots < J_n$. Let J_{n+1} be the maximal ideal in $A(k+1)^*$. As v_n is a unit in $\widehat{A}(k+1)^*$ it is clear that $v_n \notin J_n$, but $v_n \in J_{n+1}$ so we have a chain of strict inclusions $0 = J_m < \dots < J_{n+1}$. This shows that $A(k+1)^*$ has Krull dimension w, so $B^* = A(k+1)^*$ as explained earlier.

We now define $\hat{x}_i = ux_i \in \widehat{A}(k+1)^0$, and for $\underline{\lambda} \in \mathbb{F}_p^{k+1}$ we define

$$[\underline{\lambda}](\hat{x}) = [\lambda_0]_{\widehat{F}}(\hat{x}_0) +_{\widehat{F}} \dots +_{\widehat{F}} [\lambda_k]_{\widehat{F}}(\hat{x}_k).$$

We also write

$$\hat{\phi}_{k+1}(t) = \prod_{\underline{\lambda}} (t -_{\widehat{F}} [\underline{\lambda}](\hat{x}))^{p^m} = u^{p^{m+k}} \phi_{k+1}(t/u) \in \widehat{A}(k+1)^0[\![t]\!].$$

Clearly, the map $\underline{\lambda} \mapsto [\underline{\lambda}](\underline{x})$ defines a p^m-fold level-V_{k+1}^* structure on $\mathbb{G}$ over $\mathrm{spf}(\widehat{A}(k+1)^0)$. If we compose this with the projection $V_m^* \times V_{k+1}^* \to V_{k+1}^*$ we get an ordinary (1-fold) level-$(V_m \times V_{k+1})^*$ structure. It follows from [14, Proposition 32 and Corollary 33] that $\hat{\phi}_{k+1}(t)$ is a coordinate on a quotient formal group of $\mathbb{G}$, and that the kernel of the quotient map is contained in the kernel of $p\colon \mathbb{G} \to \mathbb{G}$. This means that there is a unique formal group law $\widehat{F}_{k+1}$ defined over $\widehat{A}(k+1)^0$ such that $\hat{\phi}_{k+1}$ is a homomorphism $\widehat{F} \to \widehat{F}_{k+1}$, and that there is a unique power series $\hat{\theta}_{k+1}(t)$ defined over $\widehat{A}(k+1)^0$ such that $[p]_{\widehat{F}}(t) = \hat{\theta}_{k+1}(\hat{\phi}_{k+1}(t))$. We define $F_{k+1}(s,t) = u^{-p^k}\widehat{F}_{k+1}(u^{p^k}s, u^{p^k}t)$ and $\theta_k(t) = u^{-1}\hat{\theta}_{k+1}(u^{p^k}t)$. It is easy to deduce that F_{k+1} is the unique formal group law over $\widehat{A}(k+1)^*$ such that ϕ_{k+1} is a homomorphism $F \to F_{k+1}$, and that θ_{k+1} is the unique series such that $[p]_F(t) = \theta_{k+1}(\phi_{k+1}(t))$.

We next remark that $A(k+1)^*$ is a Noetherian domain, so it is not hard to see that the maps

$$A(k+1)^* \to v_n^{-1}A(k+1)^* \to (v_n^{-1}A(k+1)^*)^{\wedge}_J = \widehat{A}(k+1)^*$$

are injective.

Now write $s' = \phi_{k+1}(s)$ and $t' = \phi_{k+1}(t)$. We know from Proposition 4.1 that $A(k+1)^*[\![s,t]\!]$ is a free module over the subring $A(k+1)^*[\![s',t']\!]$ on generators s^it^j for $0 \le i,j < p^{k+1}$. We can thus write $\phi_{k+1}(F(s,t)) = \sum_{i,j} F_{i,j}(s',t')s^it^j$ for uniquely determined series $F_{i,j}$. Similarly, $\widehat{A}(k+1)^*[\![s,t]\!]$ is a free module over $\widehat{A}(k+1)^*[\![s',t']\!]$ on $\{s^it^j\}$, so the equation $\phi_{k+1}(F(s,t)) = \sum_{i,j} F_{i,j}(s',t')s^it^j$ is the *unique* way to write $\phi_{k+1}(F(s,t))$ in terms of the generators s^it^j. On the other hand, we also have $\phi_{k+1}(F(s,t)) = F_{k+1}(s',t') \in \widehat{A}(k+1)^*[\![s,t]\!]$. It follows that $F_{i,j} = 0$ for $(i,j) \neq (0,0)$ and that $F_k = F_{0,0}$, so the series F_k is actually defined over $A(k+1)^*$ rather than $\widehat{A}(k+1)^*$. It is clearly the unique formal group law over $A(k+1)^*$ for which the series ϕ_{k+1} is a homomorphism $F \to F_k$. Similarly, we see that $\theta_{k+1}(t)$ is defined over $A(k+1)^*$, and it is the unique series over $A(k+1)^*$ for which $[p](t) = \theta_{k+1}(\phi_{k+1}(t))$. By putting $t = 0$ we see that $\theta_{k+1}(t)$ is divisible by t, so it can be written as $t\overline{\theta}_{k+1}(t)$. This completes our induction step. ☐

Proposition 5.6. *The ring $A(k)^*$ has chromatic depth at least $w-k$.*

Proof. The claim is that the sequence $\{v_m, \dots, v_{n-k}\}$ (of length $w-k$) is regular on $A(k)^*$. By Theorem 4.3, it is enough to check that the quotient $A(k)^*/(v_m, \dots, v_{n-k})$ has dimension at most $\dim(A(k)^*) - (w-k) = k$. Let B^* be a graded integral domain which is a quotient of $A(k)^*/(v_m, \dots, v_{n-k})$. It is enough to show that every such B^* has dimension at most k, and thus enough to show that the maximal ideal $\mathfrak{m}$ of B^* needs at most k generators. From now on we work in B^*.

Write $W = \{\underline{\lambda} \in \mathbb{F}_p^k \mid [\underline{\lambda}](\underline{x}) = 0\}$. This is a subgroup of V_k^*, of dimension d say. Note that $\mathrm{Aut}(\mathbb{F}_p^k) = GL_k(\mathbb{F}_p)$ acts on $A(k)^*$ in a natural way. After applying a suitable element of this group, we may assume that W is the evident copy of $\mathbb{F}_p^d$ in $\mathbb{F}_p^k$, spanned by the first d standard basis vectors, so that $x_0 = \ldots = x_{d-1} = 0$ in B. We write U for the space spanned by the remaining standard basis vectors, so that $\mathbb{F}_p^k = W \oplus U$. Define $\overline{\phi}(t) = \prod_{\underline{\lambda}\in U}(t -_F [\underline{\lambda}](\underline{x}))$, so that $\phi_k(t) = \overline{\phi}(t)^{p^{m+d}}$. As B^* is an integral domain, we see that

$$\overline{\phi}'(0) = \prod_{\underline{\lambda}\in U\setminus 0} [\underline{\lambda}](\underline{x}) \neq 0$$

and thus that $\mathrm{ord}_t\, \phi_k(t) = p^{m+d}$, where $\mathrm{ord}_t\, f(t)$ means the largest integer N such that t^N divides $f(t)$.

As $v_0 = \ldots = v_{n-k} = 0$ in B^* we see that $\mathrm{ord}_t[p](t) \ge p^{n+1-k}$. On the other hand, we have $\mathrm{ord}_t[p](t) = \mathrm{ord}_t\, \theta_k(\overline{\phi}(t)^{p^{m+d}}) = p^{m+d}\, \mathrm{ord}_s\, \theta_k(s)$. Thus $\mathrm{ord}_s\, \theta_k(s) \ge p^{w-d-k}$.

Now consider the list

$$L = \{x_d, \dots, x_{k-1}, v_{n+1-d}, \dots, v_n\},$$

so that L has length k. It will be enough to show that $\overline{B} = B/(L) = \mathbb{F}_p$, or equivalently that $x_0 = \ldots = x_k = 0$ in $\overline{B}$ and $v_0 = \ldots = v_n = 0$ in $\overline{B}$. We already have $x_0 = \ldots = x_{d-1} = 0$ in B and the remaining x's are in L so all x's vanish in $\overline{B}$, as required. We next note that $\phi_k(t)$ becomes $t^{p^{m+k}}$ over $\overline{B}$, and $[p](t) = \theta_k(\phi_k(t))$, and $\mathrm{ord}_s(\theta_k(s)) \geq p^{w-d-k}$ so $[p](t)$ has height at least $m+k+w-d-k = n+1-d$ over $\overline{B}$. This means that $v_0 = \ldots = v_{n-d} = 0$ over $\overline{B}$ and the remaining v's are in L so they vanish in $\overline{B}$ also. Thus $(L) = \mathfrak{m}$, and $\mathfrak{m}$ needs only k generators, as required. □

We leave the proof of the following simple lemmas to the reader.

Lemma 5.7. *Let R^* be a ring, and let ϕ, ψ be elements of R^* such that ϕ is not a zero-divisor. Then the annihilator of ϕ in $R^*/\phi\psi$ is generated by ψ, and thus the ideal generated by ϕ in $R^*/\phi\psi$ is a free module of rank one over R^*/ψ.* □

Lemma 5.8. *For any elements ϕ and χ in any ring R^*, the annihilator $\mathrm{ann}(\phi\chi, R^*)$ in R^* is the preimage of $\mathrm{ann}(\phi, R^*/\mathrm{ann}(\chi))$ under the quotient map $R^* \to R^*/\mathrm{ann}(\chi)$.* □

Definition 5.9. We write ϕ_i for $\phi_i(x_i)$ and $\chi_j = \prod_{i<j} \phi_i \in E^*BV_j \leq E^*BV_w$. We also write ψ_i for $\psi_i(x_i) = \overline{\theta}_i(\phi_i(x_i))$, so that $\phi_i\psi_i = [p](x_i)$. We also have $E^*BV_k/(\psi_i \mid i < j) = P(A(j)^*; j, k)$ by Theorem 5.5(d).

Proposition 5.10. *If $j \leq k \leq w$ then the annihilator of χ_j in E^*BV_k is precisely the ideal $(\psi_0, \ldots, \psi_{j-1})$. Thus, the ideal generated by χ_j is a free module of rank one over $P(A(j)^*; j, k)$.*

Proof. The proposition is trivial when $j = 0$ (so $\chi_j = 1$). We may thus assume the statement for χ_j and prove the one for χ_{j+1}. As $\chi_{j+1} = \phi_j\chi_j$, we see that the annihilator of χ_{j+1} is the preimage of the annihilator of ϕ_j in $S^* = E^*BV_k/\mathrm{ann}(\chi_j) = P(A(j)^*; j, k)$. We write $R^* = P(A(j)^*; j+1, k)$, so that $S^* = R^*[\![x_j]\!]/\phi_j\psi_j$. It follows from Lemma 5.7 that the annihilator of ϕ_j in S^* is just the ideal generated by ψ_j. Thus, we have

$$\begin{aligned} \mathrm{ann}(\chi_{j+1}, E^*BV_k) &= \mathrm{ann}(\chi_j, E^*BV_k) + (\psi_j) \\ &= (\psi_0, \ldots, \psi_{j-1}) + (\psi_j) \\ &= (\psi_0, \ldots, \psi_j), \end{aligned}$$

as required. □

The following corollary is immediate.

Corollary 5.11. *If $0 \leq j < k \leq w$ then the quotient $(\chi_j)/(\chi_{j+1})$ is isomorphic as a module over E^*BV_k to $P(A(j)^*; j+1, k)[\![x_j]\!]/\phi_j$, which is (by Proposition 4.1) a free module of rank p^{mj} over $P(A(j)^*; j+1, k)$.* □

Proposition 5.12. *The ideal of v_m-torsion elements in E^*BV_w is a free module over the ring $E^*BV_w/I_{n+1} = \mathbb{F}_p[\![x_0, \ldots, x_{w-1}]\!]$, generated by $\alpha = \chi_w$. The map $\mathrm{ann}(v_m) \rightarrowtail E^*BV_w \twoheadrightarrow E^*BV_w/I_{n+1}$ is injective.*

Proof. Take $k = w$ in Corollary 5.11. We know from Proposition 5.6 that $A(j)^*$ has chromatic depth at least $w-j$, and it follows from Corollary 5.2 that $P(A(j)^*; j+1, w)$ has chromatic depth at least $(w-j)-(w-(j+1)) = 1$, so v_m is regular on $(\chi_j)/(\chi_{j+1})$. It follows easily that v_m is regular on $E^*BV_w/(\chi_w)$, and thus that the v_m-torsion is contained in the ideal (χ_w). We know from Proposition 5.10 that the annihilator of χ_w is the same as the kernel of the map $E^*BV_w \twoheadrightarrow A(w)^*$. We also know from Theorem 5.5 that $A(w)^* = E^*BV_w/I_{n+1} = \mathbb{F}_p[\![x_0, \ldots, x_{w-1}]\!]$. It follows immediately that the v_m-torsion is precisely the ideal generated by χ_w and that it is a free module over E^*BV_w/I_{n+1}. It is easy to see from the definitions that $\chi_w = \alpha$. Moreover, modulo I_{n+1} we see that χ_w is just the product of all linear polynomials of the form $\lambda_0 x_0 + \ldots + \lambda_{j-1}x_{j-1} + x_j$ for some $j < w$. In particular, it is nonzero. As E^*BV_w/I_{n+1} is an integral domain, it follows easily that the map $(E^*BV_w/I_{n+1}).\chi_w \rightarrowtail E^*BV_w \twoheadrightarrow E^*BV_w/I_{n+1}$ is injective. □

Our proof of Theorem 2.8 relies on some facts from the classical theory of Dickson invariants [16]. For ease of reference, we give a swift proof of what we need.

Proposition 5.13. *In $H^*(BV_k;\mathbb{F}_p)=E[a_0,\dots,a_{k-1}]\otimes\mathbb{F}_p[x_0,\dots,x_{k-1}]$ we define*

$$\beta=\prod_u\sum_i\lambda_ix_i,$$

where the product runs over nonzero sequences $(\lambda_0,\dots,\lambda_{k-1})\in\mathbb{F}_p^k$ whose last nonzero entry is one. We also define

$$\beta'=\det_{i,j}(x_i^{p^j})$$
$$\beta''_m=Q_{m+k-1}\dots Q_m(a_0a_1\dots a_{k-1})$$

Then $\beta''_m=(\beta')^{p^m}=\beta^{p^m}$.

Proof. Using column operations and the fact that $(u+v)^{p^j}=u^{p^j}+v^{p^j}\pmod p$, we see easily that $g^*\beta'=\det(g)\beta'$ for all $g\in\mathrm{Aut}(V_k)$. It is also clear that the element $a_0a_1\dots a_{k-1}$ transforms in the same way, and thus that $g^*\beta''_m=\det(g)\beta''_m$ for all g and m. It is immediate from the definition that β' lies in $\mathbb{F}_p[x_0,\dots,x_{k-1}]$ and that it is divisible by x_0. It is well-known that Q_i is a derivation with $Q_i(a_j)=x_j^{p^i}$ and $Q_i(x_j)=0$, and from this it follows that β''_0 is a sum of terms of the form $\pm\prod_{i<k}x_i^{p^{\sigma(i)}}$ for various permutations σ. In particular, we see that β''_0 lies in $\mathbb{F}_p[x_0,\dots,x_{k-1}]$ and that it is divisible by x_0. As β' and β''_0 are divisible by x_0 and invariant under $SL_k(\mathbb{F}_p)$, we see that they are both divisible by each of the terms $\sum_i\lambda_ix_i$ in the product formula for β. As these terms are inequivalent irreducibles and $\mathbb{F}_p[x_0,\dots,x_{k-1}]$ has unique factorisation, we see that β''_0 and β' are divisible by β. It is easy to check that β''_0/β and β'/β have degree zero, so they lie in $\mathbb{F}_p$. By comparing coefficients for the monomial $\prod_{i<k}x_i^{p^i}$, we see that $\beta=\beta'=\beta''_0$.

Finally, we can define an (ungraded) ring map $F\colon H^*(BV_k;\mathbb{F}_p)\to H^*(BV_k;\mathbb{F}_p)$ by $F(a_i)=a_i$ and $F(x_i)=x_i^p$. We then have $F\circ Q_i=Q_{i+1}\circ F$, from which it follows easily that $(\beta''_0)^{p^m}=F^m\beta''_0=\beta''_m$, which completes the proof of the proposition. ∎

Proof of Theorem 2.8. Given Proposition 5.12, all that is left is to prove that $\alpha=\alpha'=\pm\alpha''$.

We first show that $v_i\alpha'=v_i\alpha''=0$ for $i=m,\dots,n$.

Recall that α' is the determinant of the matrix with entries $\pi_i(x_j)$ for $m\le i\le n$ and $0\le j<w$. As we work modulo I_m we have $[p](t)=\sum_{i=m}^n v_i\pi_i(t)$, so $v_i\pi_i(x_j)=-\sum_{k\neq i}^n v_k\pi_k(x_j)$ for all j. This shows that when we multiply the i'th row of our matrix by v_i, it becomes a linear combination of the other rows, so the determinant becomes zero. It follows that v_i annihilates the determinant of the original matrix, in other words $v_i\alpha'=0$.

Next, choose elements $v_i\in\pi_*MU$ lifting our elements $v_i\in\pi_*BP$. We then have natural maps $v_i\colon\Sigma^{2p^i-2}M\to M$ for all $M\in\mathcal{D}_{MU}$. Working in $\mathcal{D}_{MU}$ we see that $v_iq_i=0$ and thus $v_iq_n\dots q_{m+1}q_m=0$. After applying the forgetful functor to the category of spectra, we conclude that $v_i\alpha''=0$.

In view of Proposition 5.12, it is now enough to check that α, α' and $\pm\alpha''$ have the same image modulo I_{n+1}, or equivalently the same image in $H^*(BV_w;\mathbb{F}_p)$. One can check from the definitions that these are the same as the elements β^{p^m}, $(\beta')^{p^m}$ and $\pm\beta''_m$ of Proposition 5.13, so they are all the same, as required. ∎

We conclude with a finer filtration of E^*BV_w.

Proposition 5.14. *The ring E^*BV_w admits a finite filtration by ideals, whose quotients are finitely generated free modules over regular local rings of dimension w.*

This could be made more explicit but the bookkeeping would be tedious.

Proof. We actually prove that the ring $P(A(i)^*;j,k)$ admits such a filtration whenever $i\le j\le k\le w$. The case when $i=j=0$ and $k=w$ gives the proposition. When $j=k$ we have $P(A(i)^*;j,k)=A(i)^*$ so the claim follows from Theorem 5.5, so we work by induction on $k-j$. We have $P(A(i)^*;j,k)\simeq P(A(i)^*;i,k+i-j)$, so we may assume that $j=i$. Note that $P(A(i)^*;i,k)=P(A(i)^*;i+1,k)[\![x_i]\!]/\phi_i(x_i)\psi_i(x_i)$, so we can apply Lemma 5.7. This gives us a two stage filtration of $P(A(i)^*;i,k)$ in which one quotient is $P(A(i+1)^*;i+1,k)$ and the other is a finitely generated free module over $P(A(i)^*;i+1,k)$. The proposition follows by induction. $\square$

In the case where $n=1$ and $m=0$, the reduced E-cohomology of BV_1 is a free module of rank one over $A(1)^* = E^*[\![x]\!]/\langle p\rangle(x)$, and we have $A(2)^* = \mathbb{F}_p[\![x_0,x_1]\!] = H\mathbb{F}_p^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Ossa has shown that $E \wedge BV_{2+}$ splits as an E-module as a wedge of copies of E, $E \wedge BV_1$ and $H\mathbb{F}_p \wedge (\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+$. (He actually works with the connective K-theory spectrum kU, but it is well-known that this splits p-locally as a wedge of copies of E. He also works with $BV_1 \wedge BV_1$ rather than BV_{2+} but again the translation is trivial.) One can check that the induced splitting of E^*BV_2 splits our filtration of E^*BV_2.

In more general cases, it is unclear what happens. The most plausible idea seems to be that there should be an $BP\langle m,n\rangle$-algebra spectrum $A(m,n,k)$ with homotopy ring $A(m,n,k)^*$ and that $BP\langle m,n\rangle \wedge BV_w$ should be a finite wedge of spectra of the form $A(m',n,k')$ for various m' and k'. However, much work remains to be done in this direction.

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