PRODUCTS ON MU-MODULES

N. P. STRICKLAND

Abstract. Elmendorf, Kriz, Mandell and May have used their technology of modules over highly structured ring spectra to give new constructions of MU-modules such as BP, $K(n)$ and so on, which makes it much easier to analyse product structures on these spectra. Unfortunately, their construction only works in its simplest form for modules over $MU[\frac{1}{2}]_*$ that are concentrated in degrees divisible by 4; this guarantees that various obstruction groups are trivial. We extend these results to the cases where $2 = 0$ or the homotopy groups are allowed to be nonzero in all even degrees; in this context the obstruction groups are nontrivial. We shall show that there are never any obstructions to associativity, and that the obstructions to commutativity are given by a certain power operation; this was inspired by parallel results of Mironov in Baas-Sullivan theory. We use formal group theory to derive various formulae for this power operation, and deduce a number of results about realising 2-local MU_*-modules as MU-modules.

1. Introduction

A great deal of work in algebraic topology has exploited the generalised cohomology theory $MU^*(X)$ (for spaces X), which is known as complex cobordism; good entry points to the literature include [2, 26, 25, 29]. This theory is interesting because of its connection with the theory of formal group laws (FGL's), starting with Quillen's fundamental theorem [23, 24] that MU^* is actually the universal example of a ring equipped with an FGL.

Suppose that we have a graded ring A^* equipped with an FGL. In the cases discussed below, the FGL involved will generally be the universal example of an FGL with some interesting property. Examples include the rings known to topologists as BP^*, $P(n)^*$, $K(n)^*$ and $E(n)^*$; see Section 2 for the definitions. It is natural to ask whether there is a generalised cohomology theory $A^*(X)$ whose value on a point is the ring A^*, and a natural transformation $MU^*(X) \to A^*(X)$, such that the resulting map $MU^* \to A^*$ carries the universal FGL over MU^* to the given FGL over A^*. This question has a long history, and has been addressed by a number of different methods for different rings A^*. The simplest case is when A^* is obtained from MU^* by inverting some set S of nonzero homogeneous elements, in other words $A^* = S^{-1}MU^*$. In that case the functor $A^*(X) = A^* \otimes_{MU^*} MU^*(X)$ is a generalised cohomology theory on finite complexes, which can be extended to infinite complexes or spectra by standard methods. For example, given a prime p one can invert all other primes to get a cohomology theory $MU^*_{(p)}(X)$. Cartier had previously introduced the notion of a p-typical FGL and constructed the universal example of such a thing over BP^*, which is a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_k in degree $-2(p^k-1)$ for $k > 0$. It was thus natural to ask our "realisation question" for $A^* = BP^*$. Quillen [23] constructed an idempotent self map $\epsilon\colon MU^*_{(p)}(X) \to MU^*_{(p)}(X)$, whose image is a subring, which we call $BP^*(X)$. He showed that this is a cohomology theory whose value on a point is the ring BP^*, and that the FGL's are compatible in the required manner. This cohomology theory was actually defined earlier by Brown and Peterson [6] (hence the name), but in a less structured and precise way. It is not hard to check that we again have $BP^*(X) = BP^* \otimes_{MU^*} MU^*(X)$ when X is finite. This might tempt us to just define $A^*(X) = A^* \otimes_{MU^*} MU^*(X)$ for any A^*, but unfortunately this does not usually have the exactness properties required of a generalised cohomology theory. Another major advance was Landweber's determination [13] of the precise conditions under which $A^* \otimes_{MU^*} MU^*(X)$ does have the required exactness properties, which turned out to be natural ones from the point of view of formal groups. However, there are many cases of interest in which Landweber's exactness conditions are not satisfied, and for these different methods are required. Many of them are of the form $A^* = (S^{-1}MU^*)/I$ for set S of homogeneous elements and some homogeneous ideal $I \leq S^{-1}MU^*$. For technical reasons things are easier if we assume that I is generated by a regular sequence, in other words $I = (x_1, x_2, \dots)$ and x_k is not a zero-divisor in $(S^{-1}MU^*)/(x_j \mid j < k)$. If A^* arises in this way, we say that it is a *localised regular quotient (LRQ)* of MU^*. If $S = \emptyset$ we say that A^* is a *regular quotient* of MU^*. The first advance in this context was the Baas-Sullivan theory of cobordism of manifolds with singularities [4]. Given a regular quotient A^* of MU^*, this theory constructed a cohomology theory $A^*(X)$, landing in the category of MU^*-modules, and a map $MU^*(X) \to A^*(X)$. Unfortunately, the details were technically unwieldy, and it was not clear whether $A^*(X)$ was unique or whether it had a natural product structure, and if so whether it was commutative or associative. Some of these questions were addressed by Shimada and Yagita [27], Mironov [18] and Morava [20], largely using the geometry of cobordisms. Another idea was (in special cases, modulo some technical details) to calculate the group of all natural transformations $A^*(X) \otimes A^*(X) \to A^*(X)$ and then see which of them are commutative, associative and unital. This was the approach of Würgler [33, 31, 32]; much more recently, Nassau has corrected some inaccuracies and extended these results [22, 21].

Baas-Sullivan theory eventually yielded satisfactory answers for rings of the form MU^*/x, but the work involved in handling ideals with more than one generator remained rather hard. The picture changed dramatically with the publication of [9] by Elmendorf, Kriz, Mandell and May (hereafter referred to as EKMM), which we now explain. Firstly, the natural home for our investigation is not really the category of generalised cohomology theories, but rather Boardman's homotopy category of spectra [2, 16], which we call $\mathcal{B}$. There is a functor Σ^∞ from finite complexes to $\mathcal{B}$, and any cohomology theory $A^*(X)$ on finite complexes is represented by a spectrum $A \in \mathcal{B}$ in the sense that $A^n(X) = [\Sigma^\infty X, \Sigma^n A]$ for all n and X. The representing spectrum A is unique up to isomorphism [5, 1], and the isomorphism is often unique. There have been many different constructions of categories equivalent to $\mathcal{B}$. The starting point of [9] was EKMM's construction of a topological model category $\mathcal{M}$ with a symmetric monoidal smash product, whose homotopy category is equivalent to $\mathcal{B}$. This was previously feared to be impossible, for subtle technical reasons [14]. EKMM were also able to construct a version of MU which was a strictly commutative monoid in $\mathcal{M}$, which allowed them to define the category $\mathcal{M}_{MU}$ of MU-modules. They showed how to make this into topological model category, and thus defined an associated homotopy category $\mathcal{D}_{MU}$. This again has a symmetric monoidal smash product, which should be thought of as a sort of tensor product over MU. They showed that the problem of realising LRQ's of MU^* becomes very much easier if we work in $\mathcal{D}_{MU}$ (and then apply a forgetful functor to $\mathcal{B}$ if required). In fact their methods work when MU is replaced by any strictly commutative monoid R in $\mathcal{M}$ such that R^* is concentrated in even degrees. They show that if A^* is an LRQ of R^* and 2 is invertible in A^* and A^* is concentrated in degrees divisible by 4, then A can be realised as a commutative and associative ring object in $\mathcal{D}_R$.

In the present work, we will start by sharpening this slightly. The main point here is that EKMM notice an obstruction to associativity in A^{4k+2}, so they assume that these groups are zero. Motivated by a parallel result in Baas-Sullivan theory [19], we show that the associativity obstructions are zero even if the groups are not (see Remark 3.10). We deduce that if A^* is an LRQ of R^* and 2 is invertible in A^* then A can be realised as a commutative and associative ring in $\mathcal{D}_R$, in a way which is unique up to unique isomorphism (Theorem 2.6). We also prove a number of subsidiary results about the resulting ring objects.

The more substantial part of our work is the attempt to remove the condition that 2 be invertible in A^*, without which the results become somewhat more technical. We show that the obstruction to defining a commutative product on R/x is given by $\widetilde{P}(x)$ for a certain power operation $\widetilde{P}\colon R^d \to R^{2d+2}/2$. This was again inspired by a parallel result of Mironov [19]. We deduce that if $A^* = S^{-1}R^*/I$ is an LRQ of R^* without 2-torsion and $\widetilde{P}(I) \leq I \pmod 2$ then A^* is again uniquely realisable (Theorem 2.7). When A^* has 2-torsion we have no such general result and must proceed case by case. Again following Mironov, we show that when $R = MU$, the operation $\widetilde{P}$ can be computed using formal group theory. We considerably extend and sharpen Mironov's calculations, using techniques which I hope will be useful in more general work on power operations. Using these results, we show that many popular LRQ's of $MU^*_{(2)}$ have almost unique realisations as associative, almost commutative rings in $\mathcal{D}_{MU}$. See Theorems 2.12 and 2.13 for precise statements. The major exceptions are the rings $BP\langle n\rangle^*$ and $E(n)^*$, but we show that even these become uniquely realisable as commutative rings in $\mathcal{D}_{MU}$ if we allow ourselves to modify the usual definition slightly. We call the resulting spectra $BP\langle n\rangle'$ and $E(n)'$; they are acceptable substitutes for $BP\langle n\rangle$ and $E(n)$ in almost all situations.

2. Statement of Results

We use the category $\mathcal{M}$ of S-modules as constructed in [9]; we recall some details in Section 8. The main point is that $\mathcal{M}$ is a symmetric monoidal category with a closed model structure whose homotopy category is Boardman's homotopy category of spectra. We shall refer to the objects of $\mathcal{M}$ simply as spectra.

Because $\mathcal{M}$ is a symmetric monoidal category, it makes sense to talk about strictly commutative ring spectra; these are essentially equivalent to E_∞ ring spectra in earlier foundational settings. Let R be such an object, such that $R_* = \pi_* R$ is even (by which we mean, concentrated in even degrees). We also assume that R is q-cofibrant in the sense of [9, Chapter VII] (if not, we replace R by a weakly equivalent cofibrant model). The main example of interest to us is $R = MU$. There are well-known constructions of MU as a spectrum in the earlier sense of Lewis and May [15], with an action of the E_∞ operad of complex linear isometries. Thus, the results of [9, Chapter II] allow us to construct MU as a strictly commutative ring spectrum.

One can define a category $\mathcal{M}_R$ of R-modules in the evident way, with all diagrams commuting at the geometric level. After inverting weak equivalences, we obtain a homotopy category $\mathcal{D} = \mathcal{D}_R$, referred to as the derived category of $\mathcal{M}_R$. We shall mainly work in this derived category, and the category $\mathcal{R} = \mathcal{R}_R$ of ring objects in $\mathcal{D}$ (referred to in [9] as R-ring spectra). All our ring objects are assumed to be associative and to have a two-sided unit. Thus, an object $A \in \mathcal{R}$ has an action $R \wedge_S A \to A$ which makes various diagrams commute at the geometric level, and a product $A \wedge_S A \to A$ that is geometrically compatible with the R-module structures, and is homotopically associative and unital. We also write $\mathcal{R}_*$ for the category of algebras over the discrete ring R_*. We write $\mathcal{R}_*^e$ for the category of even R_*-algebras, and $\mathcal{R}_*^c$ for the commutative ones, and similarly $\mathcal{R}_*^{ec}$, $\mathcal{R}^e$, $\mathcal{R}^c$ and $\mathcal{R}^{ec}$.

Definition 2.1. Let A_* be an even commutative R_*-algebra without 2-torsion. A *strong realisation* of A_* is a commutative ring object $A \in \mathcal{R}^{ec}$ with a given isomorphism $\pi_*(A) \simeq A_*$, such that the resulting map

$$\mathcal{R}(A, B) \to \mathcal{R}(A_*, \pi_*(B))$$

is an isomorphism whenever $B \in \mathcal{R}^{ec}$ and B_* has no 2-torsion. We say that A_* is *strongly realisable* if such a realisation exists.

Remark 2.2. It is easy to see that the category of strongly realisable R_*-algebras is equivalent to the category of those $A \in \mathcal{R}^{ec}$ for which $\pi_*(A)$ is strongly realisable. In particular, any two strong realisations of A_* are canonically isomorphic.

Our main aim is to prove that certain R_*-algebras are strongly realisable, and to prove some more *ad hoc* results for certain algebras over $MU_*/2$.

Definition 2.3. A *localised regular quotient (LRQ)* of R_* is an algebra A_* over R_* that can be written in the form $A_* = (S^{-1}R_*)/I$, where S is any set of (homogeneous) elements in R_* and I is an ideal which can be generated by a regular sequence. We say that A_* is a *positive localised regular quotient (PLRQ)* if it can be written in the form $(S^{-1}R_*)/I$ as above, where I can be generated by a regular sequence of elements of nonnegative degree.

Remark 2.4. If A_* is an LRQ of R_* and B_* is an arbitrary R_*-algebra then $\mathcal{R}_*(A_*, B_*)$ has at most one element. Suppose that A is a commutative ring object in $A \in \mathcal{R}^{ec}$ with a given isomorphism $\pi_*(A) \simeq A_*$. It follows that A is a strong realisation of A_* if and only if: whenever there is a map $A_* \to \pi_*(B)$ of R_*-algebras, there is a unique map $A \to B$ in $\mathcal{R}^{ec}$.

Remark 2.5. Let S be a set of homogeneous elements in R_*. Using the results of [9, Section VIII.2] one can construct a strictly commutative ring spectrum $S^{-1}R$ and a map $R \to S^{-1}R$ inducing an isomorphism $S^{-1}\pi_*(R) \to \pi_*(S^{-1}R)$. Results of Wolbert show that $\mathcal{D}_{S^{-1}R}$ is equivalent to the subcategory of $\mathcal{D}_R$ consisting of objects M such that each element of S acts invertibly on $\pi_*(M)$. Using this it is easy to check that any algebra over $S^{-1}R_*$ is strongly realisable over R if and only if it is strongly realisable over $S^{-1}R$. For more discussion of this, see Section 4.

We start by stating a result for odd primes, which is relatively easy.

Theorem 2.6. *If A_* is an LRQ of R_* and 2 is a unit in A_* then A_* is strongly realisable.*

This will be proved as Theorem 4.11.

Our main contribution is the extension to the case where 2 is not inverted. Our results involve a certain "commutativity obstruction" $\overline{c}(x) \in \pi_{2|x|+2}(R)/(2,x)$, which is defined in Section 3. In Section 10, we show that when $d \geq 0$ this arises from a power operation $\widetilde{P} \colon \pi_d(R) \to \pi_{2d+2}(R)/2$. This result was inspired by a parallel result of Mironov in Baas-Sullivan theory [19]. The restriction $d \geq 0$ is actually unneccessary but the argument for the case $d < 0$ is intricate and we have no applications so we have omitted it. In Section 5 we show how to compute this power operation using formal group theory, at least in the case $R = MU$. The first steps in this direction were also taken by Mironov [19], but our results are much more precise.

By Remark 2.5 we also have a power operation $\widetilde{P} \colon \pi_d(S^{-1}R) \to \pi_{2d+2}(S^{-1}R)/2$. This is in fact determined algebraically by the power operation on π_*R, as we will see in Section 5.

Our result for the case where A_* has no 2-torsion is quite simple and similar to the case where 2 is inverted.

Theorem 2.7. *Let $A_* = (S^{-1}R_*)/I$ be a PLRQ of R_* which has no 2-torsion. Suppose also that $\widetilde{P}(I) \subseteq S^{-1}R_*$ maps to 0 in $A_*/2$. Then A_* is strongly realisable.*

This will be proved as Theorem 4.12.

We next recall the definitions of some algebras over MU_* which one might hope to realise as spectra using the above results. First, we have the rings

$$\begin{aligned} kU_* &:= \mathbb{Z}[u] && |u| = 2 \\ KU_* &:= \mathbb{Z}[u^{\pm 1}] \\ H_* &:= \mathbb{Z} && \text{(in degree zero)} \\ H\mathbb{F}_* &:= \mathbb{F}_p && \text{(in degree zero).} \end{aligned}$$

These are PLRQ's of MU_* in well-known ways. Next, we consider the Brown-Peterson ring

$$BP_* := \mathbb{Z}_{(p)}[v_k \mid k > 0] \qquad |v_k| = 2(p^k - 1).$$

We take $v_0 = p$ as usual. There is a unique p-typical formal group law F over this ring such that

$$[p]_F(x) = \exp_F(px) +_F \sum_{k>0}^{F} v_k x^{p^k}.$$

(Thus, our v_k's are Hazewinkel's generators rather than Araki's.) We use this FGL to make BP_* into an algebra over MU_* in the usual way. We define

$$\begin{aligned} P(n)_* &:= BP_*/(v_i \mid i < n) = \mathbb{F}_p[v_j \mid j \geq n] \\ B(n)_* &:= v_n^{-1} BP_*/(v_i \mid i < n) = v_n^{-1}\mathbb{F}_p[v_j \mid j \geq n] \\ k(n)_* &:= BP_*/(v_i \mid i \neq n) = \mathbb{F}_p[v_n] \\ K(n)_* &:= v_n^{-1} BP_*/(v_i \mid i \neq n) = \mathbb{F}_p[v_n^{\pm 1}] \\ BP\langle n\rangle_* &:= BP_*/(v_i \mid i > n) = \mathbb{Z}_{(p)}[v_1, \dots, v_n] \\ E(n)_* &:= v_n^{-1} BP_*/(v_i \mid i > n) = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}] \end{aligned}$$

These are all PLRQ's of BP_*, and it is not hard to check that BP_* is a PLRQ of $MU_{(p)*}$, and thus that all the above rings are PLRQ's of $MU_{(p)*}$.

We also let $w_k \in \pi_{2(p^k-1)}MU$ denote the bordism class of a smooth hypersurface W_{p^k} of degree p in $\mathbb{C}P^{p^k}$. It is well-known that $I_n = (w_i \mid i < n)$ is the smallest ideal modulo which the universal formal group law over MU_* has height n, and that the image of I_n in BP_* is the ideal $(v_i \mid i < n)$. In fact, we have

$$\sum_{m>0} [W_m] x^m dx = [p]_F(x) d\log_F(x) = [p]_F(x) \sum_{m\geq 0} [\mathbb{C}P^m] x^m dx.$$

Moreover, the sequence of w_i's is regular, so that MU_*/I_n is a PLRQ of MU_*.

One can also define PLRQ's of $MU[\frac{1}{6}]_*$ giving rise to various versions of elliptic homology, but we refrain from giving details here. If we do not invert 6 then the relevant rings seem not to be LRQ's of MU_*. If we take $R = MU_p^\wedge$ then we can make $\mathbb{Z}_p[v_n]$ into an LRQ of R_* in such a way that the resulting formal group law is of the (non-p-typical) type considered by Lubin and Tate in algebraic number theory. We can also take $R = L_{K(n)}MU$ and consider $\widehat{E(n)}_*$ as an LRQ of R_* via the Ando orientation [3] rather than the more usual p-typical one. We leave the details of these applications to the reader.

The following proposition is immediate from Theorem 2.6.

Proposition 2.8. *If $p > 2$ and $R = MU$ or $R = MU_{(p)}$ then $kU_{(p)*}$, $KU_{(p)*}$, $H_{(p)*}$, $H\mathbb{F}_{p*}$, BP_*, $P(n)_*$, $B(n)_*$, $k(n)_*$, $BP\langle n\rangle_*$, $E(n)_*$ and MU_*/I_n are all strongly realisable.*

After doing some computations with the power operation $\widetilde{P}$, we will also prove the following.

Proposition 2.9. *If $R = MU$ then kU_*, KU_*, H_* and $H\mathbb{F}_*$ are strongly realisable. If $R = MU_{(2)}$ then $kU_{(2)*}$, $KU_{(2)*}$, $H_{(2)*}$ and BP_* are strongly realisable.*

The situation is less satisfactory for the rings $BP\langle n\rangle_*$ and $E(n)_*$ at $p = 2$. For $n > 1$, they cannot be realised as the homotopy rings of commutative ring objects in $\mathcal{D}$. However, if we kill off a slightly different sequence of elements instead of the sequence $(v_{n+1}, v_{n+2}, \dots)$, we get a quotient ring that is realisable. The resulting spectrum serves as a good substitute for $BP\langle n\rangle$ in almost all arguments.

Proposition 2.10. *If $R = MU_{(2)}$ and $n > 0$, there is a quotient ring $BP\langle n\rangle'_*$ of BP_* such that*

1. *The evident map*

$$\mathbb{Z}_{(2)}[v_1, \dots, v_n] \to BP_* \to BP\langle n\rangle'_*$$

is an isomorphism.

2. *$BP\langle n\rangle'_*$ is strongly realisable.*
3. *We have $BP\langle n\rangle'_*/I_n = k(n)_* = BP_*/(v_i \mid i \neq n)$ as MU_*-algebras.*

Moreover, the ring $E(n)'_* = v_n^{-1}BP\langle n\rangle'_*$ is also strongly realisable. If $n = 1$ then we can take $BP\langle 1\rangle'_* = BP\langle 1\rangle_*$.

This is proved in Section 7.

The situation for $MU_*/2$ and algebras over it is also more complicated than for odd primes.

Definition 2.11. Throughout this paper, we write τ for the twist map $X \wedge X \to X \wedge X$, for any object X for which this makes sense. We say that a ring map $f\colon A \to B$ in $\mathcal{R}$ is *central* if

$$\phi \circ \tau \circ (f \wedge 1) = \phi \circ (f \wedge 1)\colon A \wedge B \to B,$$

where $\phi\colon B \wedge B \to B$ is the product. We say that B is a *central A-algebra* if there is a given central map $A \to B$.

Theorem 2.12. *When $R = MU_{(2)}$, there is a ring $MU/I_n \in \mathcal{R}$ with $\pi_*(MU/I_n) = MU_*/I_n$, and derivations $Q_i\colon MU/I_n \to \Sigma^{2^{i+1}-1}MU/I_n$ for $0 \leq i < n$. If ϕ is the product on MU/I_n we have*

$$\phi \circ \tau - \phi = w_n \phi \circ (Q_{n-1} \wedge Q_{n-1}).$$

This is proved in Section 7. There are actually many non-isomorphic rings with these properties. We will outline an argument that specifies one of them unambiguously.

We get a sharper statement for algebras over $P(n)_*$.

Theorem 2.13. *When $R = MU_{(2)}$, there is a central BP-algebra $P(n) = BP \wedge MU/I_n \in \mathcal{R}$ and an isomorphism $\pi_* P(n) = P(n)_*$. This has derivations $Q_i\colon P(n) \to \Sigma^{2^{i+1}-1} P(n)$ for $0 \leq i < n$. If ϕ is the product on $P(n)$ we have*

$$\phi \circ \tau - \phi = v_n \phi \circ (Q_{n-1} \wedge Q_{n-1}).$$

If B is another central BP-algebra such that

$$\pi_k B = \begin{cases} \{0,1\} & \text{if } k = 0 \\ 0 & \text{if } 0 < k < |v_n| \\ \{0, v_n\} & \text{if } k = |v_n| \end{cases}$$

then either there is a unique map $P(n) \to B$ of BP-algebras, or there is a unique map $P(n) \to B^{op}$. Analogous statements hold for $B(n)$, $k(n)$ and $K(n)$ with BP replaced by $v_n^{-1}BP$, $BP\langle n\rangle'$ and $E(n)'$ respectively.

This is also proved in Section 7. Related results were announced by Würgler in [33], but there appear to be some problems with the line of argument used there. A correct proof on similar lines has recently been given by Nassau [21, 22].

3. Products on R/x

Suppose that $x \in R_d$ is not a zero-divisor (so d is even). We then have a cofibre sequence in the triangulated category $\mathcal{D}$:

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho} R/x \xrightarrow{\beta} \Sigma^{d+1} R.$$

Because x is not a zero divisor, we have $\pi_*(R/x) = R_*/x$. In particular, $\pi_{d+1} R/x = 0$ (because $d+1$ is odd), and thus $\rho^*\colon [R/x, R/x] \simeq [R, R/x]$. It follows that R/x is unique up to unique isomorphism as an object under R.

We next set up a theory of products on objects of the form R/x. Apart from the fact that all such products are associative, our results are at most minor sharpenings of the those in [9, Chapter V].

Observe that $(R/x)^{(2)}$ is a cell R-module with one 0-cell, two $(d+1)$-cells and one $(2d+2)$-cell. We say that a map $\phi\colon (R/x)^{(2)} \to R/x$ is a *product* if it agrees with ρ on the bottom cell, in other words $\phi \circ (\rho \wedge \rho) = \rho\colon R \to R/x$.

The main result is as follows.

Proposition 3.1.
1. *All products are associative, and have ρ as a two-sided unit.*
2. *The set of products on R/x has a free transitive action of the group R_{2d+2}/x (in particular, it is nonempty).*
3. *There is a naturally defined element $\overline{c}(x) \in \pi_{2d+2}(R)/(2, x)$ such that R/x admits a commutative product if and only if $\overline{c}(x) = 0$.*
4. *If so, the set of commutative products has a free transitive action of* $\mathrm{ann}(2, R_{2d+2}/x) = \{y \in R_{2d+2}/x \mid 2y = 0\}$.
5. *If $d \geq 0$ there is a power operation $\widetilde{P}\colon R_d \to R_{2d+2}/2$ such that $\overline{c}(x) = \widetilde{P}(x) \pmod{2, x}$ for all x.*

Proof. Part (1) is proved as Lemma 3.4 and Proposition 3.8. In part (2), the fact that products exist is [9, Theorem V.2.6]; we also give a proof in Corollary 3.3, which is slightly closer in spirit with our other proofs. Parts (3) and (4) form Corollary 3.12. Part (5) is explained in more detail and proved in Section 10. □

From now on we will generally state our results in terms of $\widetilde{P}(x)$ instead of $\overline{c}(x)$, as that is the form in which the results are actually applied.

Lemma 3.2. *The map $x\colon \Sigma^d R/x \to R/x$ is zero.*

Proof. Using the cofibration

$$\Sigma^d R \xrightarrow{\rho} \Sigma^d R/x \xrightarrow{\beta} \Sigma^{d+1} R$$

and the fact that $\pi_{d+1}(R/x) = \pi_{d+1}(R)/x = 0$, we find that $\rho^*\colon [R/x, R/x]_d \to [R, R/x]_d = \pi_d(R/x)$ is injective. It is clear that x gives zero on the right hand side, so it is zero on the left hand side as claimed. $\square$

Corollary 3.3. *There exist products on R/x.*

Proof. There is a cofibration $\Sigma^d R/x \xrightarrow{x} R/x \xrightarrow{1\wedge\rho} (R/x)^{(2)}$. The lemma tells us that the first map is zero, so $1 \wedge \rho$ is a split monomorphism, and any splitting is clearly a product. $\square$

Lemma 3.4. *If $\phi\colon (R/x)^{(2)} \to R/x$ is a product then ρ is a two-sided unit for ϕ, in the sense that*

$$\phi \circ (\rho \wedge 1) = \phi \circ (1 \wedge \rho) = 1\colon R/x \to R/x.$$

Proof. By hypothesis, $\phi \circ (\rho \wedge 1)\colon R/x \to R/x$ is the identity on the bottom cell of R/x. We observed earlier that $[R/x, R/x] \simeq [R, R/x]$, and it follows that $\phi\circ(\rho\wedge 1) = 1$. Similarly $\phi\circ(1\wedge\rho) = 1$. $\square$

Remark 3.5. EKMM study products for which ρ is a one-sided unit, and our definition of products is *a priori* even weaker. It follows from the lemma that EKMM's products are the same as ours and have ρ as a two-sided unit.

Lemma 3.6. *Let $A \in \mathcal{D}$ be such that $x\colon \Sigma^d A \to A$ is zero. Then the diagram*

$$R/x \vee R/x \xrightarrow{(\rho\wedge 1, 1\wedge\rho)} (R/x)^{(2)} \xrightarrow{\beta\wedge\beta} \Sigma^{2d+2} R$$

induces a left-exact sequence

$$[\Sigma^{2d+2} R, A] \to [(R/x)^{(2)}, A] \to [R/x \vee R/x, A].$$

Similarly, the diagram

$$(R/x)^{(2)} \vee (R/x)^{(2)} \vee (R/x)^{(2)} \xrightarrow{(\rho\wedge 1\wedge 1, 1\wedge\rho\wedge 1, 1\wedge 1\wedge\rho)} (R/x)^{(3)} \xrightarrow{\beta^{(3)}} \Sigma^{3d+3} R$$

gives a left-exact sequence

$$[\Sigma^{3d+3} R, A] \to [(R/x)^{(3)}, A] \to [(R/x)^{(2)} \vee (R/x)^{(2)} \vee (R/x)^{(2)}, A].$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
& R/x \arrow[d, "\rho\wedge 1"] & \\
R/x \arrow[r, "1\wedge\rho"] \arrow[d, "\beta"'] & (R/x)^{(2)} \arrow[d, "\beta\wedge 1"] \arrow[dr, "\beta\wedge\beta"] & \\
\Sigma^{d+1} R \arrow[r, "\rho"'] & \Sigma^{d+1} R/x \arrow[r, "\beta"'] & \Sigma^{2d+2} R
\end{tikzcd}$$

We now apply the functor $[-, A]$ and make repeated use of the cofibration

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho} R/x \xrightarrow{\beta} \Sigma^{d+1} R.$$

The conclusion is that all maps involving β become monomorphisms, all maps involving ρ become epimorphisms, and the bottom row and the middle column become short exact. The first claim follows by diagram chasing. For the second claim, consider the diagram

$$\begin{tikzcd}
 & (R/x)^{(2)} \vee (R/x)^{(2)} \arrow[d, "(\rho\wedge 1\wedge 1, 1\wedge\rho\wedge 1)"] & \\
(R/x)^{(2)} \arrow[r, "1\wedge 1\wedge\rho"] \arrow[d, "\beta\wedge\beta"'] & (R/x)^{(3)} \arrow[d, "\beta\wedge\beta\wedge 1"] \arrow[dr, "\beta\wedge\beta\wedge\beta"] & \\
\Sigma^{2d+2}R \arrow[r, "\rho"'] & \Sigma^{2d+2}R/x \arrow[r, "\beta"'] & \Sigma^{3d+3}R
\end{tikzcd}$$

We apply the same logic as before, using the first claim (with A replaced by $F(R/x, A)$) to see that the middle column becomes left exact. □

We next determine how many different products there are on R/x.

Lemma 3.7. *If ϕ is a product on R/x and $u \in \pi_{2d+2}(R)/x = [\Sigma^{2d+2}R, R/x]$ then $\phi' = \phi + u \circ (\beta\wedge\beta)$ is another product. Moreover, this construction gives a free transitive action of $\pi_{2d+2}(R)/x$ on the set of all products.*

Proof. Let P be the set of products. As $(\beta\wedge\beta)\circ(\rho\wedge\rho) = 0$, it is clear that the above construction gives an action of $\pi_{2d+2}(R)/x$ on P. Now suppose that $\phi, \phi' \in P$. We need to show that there is a unique $u\colon \Sigma^{2d+2}R \to R/x$ such that $\phi' = \phi + u \circ (\beta \wedge \beta)$. Using the unital properties of ϕ and ϕ' given by Lemma 3.4, we see that

$$(\phi' - \phi) \circ (\rho \wedge 1) = (\phi' - \phi) \circ (1 \wedge \rho) = 0.$$

Because of Lemma 3.2, we can apply Lemma 3.6 to see that $\phi' - \phi = u \circ (\beta \wedge \beta)$ for a unique element u, as claimed. □

Proposition 3.8. *Any product on R/x is associative.*

Proof. Let ϕ be a product, and write

$$\delta := \phi \circ (\phi \wedge 1 - 1 \wedge \phi)\colon (R/x)^{(3)} \to R/x,$$

so the claim is that δ is nullhomotopic. Using the unital properties of ϕ we see that

$$\delta \circ (\rho \wedge 1 \wedge 1) = \delta \circ (1 \wedge \rho \wedge 1) = \delta \circ (1 \wedge 1 \wedge \rho) = 0.$$

Using Lemma 3.6, we conclude that $\delta = u \circ (\beta \wedge \beta \wedge \beta)$ for a unique element $u \in [\Sigma^{3d+3}R, R/x] = \pi_{3d+3}(R)/x = 0$ (because $3d + 3$ is odd). Thus $\delta = 0$ as claimed. □

Remark 3.9. The corresponding result in Baas-Sullivan theory was already known (this is proved in [18] in a form which is valid when R_* need not be concentrated in even degrees, for example for $R = MSp$).

Remark 3.10. The EKMM approach to associativity is essentially as follows. They note that R/x has cells of dimension 0 and $d{+}1$, so $(R/x)^{(3)}$ has cells in dimensions 0, $d{+}1$, $2d{+}2$ and $3d{+}3$. The map δ vanishes on the zero-cell and $\pi_{d+1}(R/x) = \pi_{3d+3}(R/x) = 0$ so the only obstruction to concluding that $\delta = 0$ lies in $\pi_{2d+2}(R/x)$. EKMM work only with LRQ's that are concentrated in degrees divisible by 4, so the obstruction goes away. We instead use Lemma 3.6 to analyse the attaching maps in $(R/x)^{(3)}$; implicitly, we show that the obstruction is divisible by x and thus is zero.

We now discuss commutativity.

Lemma 3.11. *There is a natural map c from the set of products to $\pi_{2d+2}R/x$ such that $c(\phi) = 0$ if and only if ϕ is commutative. Moreover,*

$$c(\phi + u \circ (\beta \wedge \beta)) = c(\phi) - 2u.$$

Proof. Let $\tau\colon (R/x)^{(2)} \to (R/x)^{(2)}$ be the twist map. Clearly, if ϕ is a product then so is $\phi \circ \tau$. Thus, there is a unique element $v \in \pi_{2d+2}R/x$ such that

$$\phi \circ \tau = \phi + v \circ (\beta \wedge \beta).$$

We define $c(\phi) := v$. Next, recall that the twist map on $\Sigma^{2d+2}R = \Sigma^{d+1}R \wedge \Sigma^{d+1}R$ is homotopic to (-1), because $d+1$ is odd. It follows by naturality that $(\beta \wedge \beta) \circ \tau = \tau \circ (\beta \wedge \beta) = -\beta \wedge \beta$. Consider a second product $\phi' = \phi + u \circ (\beta \wedge \beta)$. We now see that

$$\phi' \circ \tau = \phi + v \circ (\beta \wedge \beta) - u \circ (\beta \wedge \beta) = \phi' + (v - 2u) \circ (\beta \wedge \beta).$$

Thus $c(\phi') = c(\phi) - 2u$ as claimed. □

Corollary 3.12. *There is a naturally defined element $\overline{c}(x) \in \pi_{2d+2}(R)/(2,x)$ such that R/x admits a commutative product if and only if $\overline{c}(x) = 0$. If so, the set of commutative products has a free transitive action of the group $\mathrm{ann}(2, \pi_{2d+2}(R)/x) := \{y \in \pi_{2d+2}(R)/x \mid 2y = 0\}$. In particular, if $\pi_*(R)/x$ has no 2-torsion then there is a unique commutative product.*

Proof. We choose a product ϕ on R/x and define $\overline{c}(x) := c(\phi) \pmod 2$. This is well-defined, by the lemma. If $\overline{c}(x) \neq 0$ then $c(\phi') \neq 0$ for all ϕ', so there is no commutative product. If $\overline{c}(x) = 0$ then $c(\phi) = 2w$, say, so that $\phi' = \phi + w \circ (\beta \wedge \beta)$ is a commutative product. In this case, the commutative products are precisely the products of the form $\phi' + z \circ (\beta \wedge \beta)$ where $2z = 0$, so they have a free transitive action of $\mathrm{ann}(2, \pi_{2d+2}(R)/x)$. □

Next, we consider the Bockstein operation:

$$\overline{\beta} := \rho\beta\colon R/x \to \Sigma^{d+1}R/x.$$

Definition 3.13. Let $A \in \mathcal{R}$ be a ring, with product $\phi\colon A \wedge A \to A$. We say that a map $Q\colon A \to \Sigma^k A$ is a *derivation* if we have

$$Q \circ \phi = \phi \circ (Q \wedge 1 + 1 \wedge Q)\colon A^{(2)} \to A.$$

Proposition 3.14. *The map $\overline{\beta}$ is a derivation with respect to any product ϕ on R/x.*

Proof. Write $\delta := \overline{\beta} \circ \phi - \phi \circ (\overline{\beta} \wedge 1 + 1 \wedge \overline{\beta})$, so the claim is that $\delta = 0$. It is easy to see that $\delta \circ (\rho \wedge 1) = \delta \circ (1 \wedge \rho) = 0$, so by Lemma 3.6 we see that δ factors through a unique map $\Sigma^{2d+2}R \to \Sigma^{d+1}R/x$. This is an element of $\pi_{d+1}(R)/x$, which is zero because $d+1$ is odd. □

We end this section by analysing maps out of the rings R/x.

Proposition 3.15. *Let $A \in \mathcal{R}^e$ be an even ring. If x maps to zero in π_*A then there is precisely one unital map $f\colon R/x \to A$, and otherwise there are no such maps. If f exists and ϕ is a product on R/x, then there is a naturally defined element $d_A(\phi) \in \pi_{2d+2}(A)$ such that*

(a) *$d_A(\phi) = 0$ if and only if f is a ring map with respect to ϕ.*
(b) *$d_A(\phi + u \circ (\beta \wedge \beta)) = d_A(\phi) + u$.*
(c) *If A is commutative then $2d_A(\phi) = c(\phi) \in \pi_{2d+2}A$.*

Proof. The statement about the existence and uniqueness of f follows immediately from the cofibration $\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho} R/x \xrightarrow{\beta} \Sigma^{d+1}R$, and the fact that $\pi_{d+1}A = 0$. Suppose that f exists; it follows easily using the product structure on A that $x\colon \Sigma^d A \to A$ is zero. Now let ψ be the given product on A, and let ϕ be a product on R/x. Consider the map

$$\delta := \psi \circ (f \wedge f) - f \circ \phi\colon (R/x)^{(2)} \to A.$$

By the usual argument, we have $\delta = v \circ (\beta \wedge \beta)$ for a unique map $v\colon \Sigma^{2d+2}R \to A$. We define $d_A(\phi) := v \in \pi_{2d+2}A$. It is obvious that this vanishes if and only if f is a ring map, and that $d_A(\phi + u \circ (\beta \wedge \beta)) = d_A(\phi) + u$.

Now suppose that A is commutative, so $\psi = \psi \circ \tau$. On the one hand, using the fact that $(\beta \wedge \beta) \circ \tau = -\beta \wedge \beta$ we see that $\delta - \delta \circ \tau = 2d_A(\phi) \circ (\beta \wedge \beta)$. On the other hand, from the definition of δ and the fact that $\psi \circ \tau = \psi$, we see that

$$\delta - \delta \circ \tau = f \circ (\phi - \phi \circ \tau) = c(\phi) \circ (\beta \wedge \beta).$$

Because $(\beta \wedge \beta)^*\colon \pi_{2d+2}A \to [(R/x)^{(2)}, A]$ is a split monomorphism, we conclude that $2d_A(\phi) = c(\phi)$ in $\pi_{2d+2}A$. □

4. Strong realisations

In this section we assemble the products which we have constructed on the R-modules R/x to get products on more general R_*-algebras. We will work entirely in the derived category $\mathcal{D}$, rather than the underlying geometric category. All the main ideas in this section come from [9, Chapter V].

We start with some generally nonsensical preliminaries.

Definition 4.1. Given a diagram $A \xrightarrow{f} C \xleftarrow{g} B$ in $\mathcal{R}$, we say that f commutes with g if and only if we have

$$\phi_C \circ (f \wedge g) = \phi_C \circ \tau \circ (f \wedge g)\colon A \wedge B \to C.$$

Note that this can be false when $f = g$; in particular A is commutative if and only if 1_A commutes with itself.

The next three lemmas become trivial if we replace $\mathcal{D}$ by the category of modules over a commutative ring, and the smash product by the tensor product. The proofs in that context can easily be made diagrammatic and thus carried over to $\mathcal{D}$.

Lemma 4.2. *If A and B are rings in $\mathcal{R}$, then there is a unique ring structure on $A \wedge B$ such that the evident maps $A \xrightarrow{i} A \wedge B \xleftarrow{j} B$ are commuting ring maps. Moreover, with this product, (i, j) is the universal example of a commuting pair of maps out of A and B.* □

Lemma 4.3. *A map $f\colon A \wedge B \to C$ commutes with itself if and only if $f \circ i$ commutes with itself and $f \circ j$ commutes with itself. In particular, $A \wedge B$ is commutative if and only if i and j commute with themselves.* □

Lemma 4.4. *If A and B are commutative, then so is $A \wedge B$, and it is the coproduct of A and B in $\mathcal{R}^c$.* □

Corollary 4.5. *Suppose that*

- *A and B are strong realisations of A_* and B_*.*
- *The ring $A_* \otimes_{R_*} B_*$ has no 2-torsion.*
- *The natural map $A_* \otimes_{R_*} B_* \to \pi_*(A \wedge B)$ is an isomorphism.*

Then $A \wedge B$ is a strong realisation of $A_* \otimes_{R_*} B_*$. □

We next consider the problem of realising $S^{-1}R_*$, where S is a set of homogeneous elements of R_*. If S is countable then we can construct an object $S^{-1}R \in \mathcal{D}$ by the method of [9, Section V.2]; this has $\pi_*(S^{-1}R) = S^{-1}\pi_*(R)$. If we want to allow S to be uncountable then it seems easiest to construct $S^{-1}R$ as the finite localisation of R away from the R-modules $\{R/x \mid s \in S\}$; see [17] or [11, Theorem 3.3.7]. In either case, we note that $S^{-1}R$ is the Bousfield localisation of R in $\mathcal{D}$ with respect to $S^{-1}R$. We may thus use [9, Section VIII.2] to construct a model of $S^{-1}R$ which is a strictly commutative algebra over R in the underlying topological category of spectra. The localisation functor involved here is smashing, so results of Wolbert [30] [9, Section VIII.3] imply that $\mathcal{D}_{S^{-1}R}$ is equivalent to the full subcategory of $\mathcal{D}_R$ consisting of R-modules M for which $\pi_*(M)$ is a module over $S^{-1}R_*$. This makes the following result immediate.

Proposition 4.6. *Let S be a set of homogeneous elements of R_*, and let A_* be an algebra over $S^{-1}R_*$. Then A_* is strongly realisable over R if and only if it is strongly realisable over $S^{-1}R$.* □

This allows us to reduce everything to the case $S = \emptyset$.

Now consider a sequence (x_i) in R_*, with products ϕ_i on R/x_i. Write $A_i = R/x_1 \wedge \ldots \wedge R/x_i$, and make this into a ring as in Lemma 4.2. There are evident maps $A_i \to A_{i+1}$, so we can form the telescope $A = \underrightarrow{\operatorname{holim}}_i A_i$.

Lemma 4.7. *If $M \in \mathcal{D}$ and $I = (x_1, x_2, \dots) \leq R_*$ acts trivially on M and $r \geq 0$ then $[A^{(r)}, M] = \varprojlim_i [A_i^{(r)}, M]$.*

Proof. This will follow immediately from the Milnor sequence if we can show that $\varprojlim_i^1 [A_i^{(r)}, M]_* = 0$. For this, it suffices to show that the map $\rho^* \colon [B \wedge R/x_i, M] \to [B, M]$ is surjective for all B. This follows from the cofibration $\Sigma^{|x_i|} B \xrightarrow{x_i} B \to B \wedge R/x_i$ and the fact that x_i acts trivially on M. □

Proposition 4.8. *Let (x_i) be a sequence in R_*, and ϕ_i a product on R/x_i for each i. Let A be the homotopy colimit of the rings $A_i = R/x_1 \wedge \dots \wedge R/x_i$, and let $f_i \colon R/x_i \to A$ be the evident map. Then there is a unique associative and unital product on A such that maps f_i are ring maps, and f_i commutes with f_j when $i \neq j$. This product is commutative if and only if each f_i commutes with itself. Ring maps from A to any ring B biject with systems of ring maps $g_i \colon R/x_i \to B$ such that g_i commutes with g_j for all $i \neq j$.*

Proof. Because R/x_i admits a product, we know that x_i acts trivially on R/x_i. Because A has the form $R/x_i \wedge B$, we see that x_i acts trivially on A. Thus I acts trivially on A, and Lemma 4.7 assures us that $[A^{(r)}, A] = \varprojlim_i [A_i^{(r)}, A]$.

Let ψ_i be the product on A_i. By the above, there is a unique map $\psi \colon A \wedge A \to A$ which is compatible with the maps ψ_i. It is easy to check that this is an associative and unital product, and that it is the only one for which the f_i are commuting ring maps. It is also easy to check that ψ is commutative if and only if each of the maps $A_i \to A$ commutes with itself, if and only if each f_i commutes with itself.

Now let B be any ring in $\mathcal{R}$. We may assume that each x_i maps to zero in $\pi_*(B)$, for otherwise the claimed bijection is between empty sets. As B is a ring, this means that each x_i acts trivially on B, so that $[A^{(r)}, B] = \varprojlim_i [A_i^{(r)}, B]$. We see from Lemma 4.2 that ring maps from A_i to B biject with systems of ring maps $g_j \colon R/x_j \to B$ for $j < i$ such that g_j commutes with g_k for $j \neq k$. The claimed description of ring maps $A \to B$ follows easily. □

Corollary 4.9. *If each R/x_i is commutative, then A is the coproduct of the R/x_i in $\mathcal{R}^c$.* □

Remark 4.10. If the sequence (x_i) is regular, then it is easy to see that $\pi_*(A) = R_*/(x_1, x_2, \dots)$. Note also that ring maps out of R/x were analysed in Proposition 3.15.

We now restate and prove Theorems 2.6 and 2.7. Of course, the former is a special case of the latter, but it seems clearest to prove Theorem 2.6 first and then explain the improvements necessary for Theorem 2.7.

Theorem 4.11. *If A_* is an LRQ of R_* and 2 is a unit in A_* then A_* is strongly realisable.*

Proof. We can use Proposition 4.6 to reduce to the case where $A_* = R_*/I$ where 2 is invertible in R_* and I is generated by a regular sequence $(x_1, x_2, \dots)$. We know from Proposition 3.1 that there is a unique commutative product ϕ_i on R/x_i. If $C \in \mathcal{R}^{ec}$ and $x_i = 0$ in $\pi_*(C)$ then in the notation of Proposition 3.15 we have $2d_C(\phi_i) = 0$ and thus $d_C(\phi_i) = 0$, so the unique unital map $R/x_i \to C$ is a ring map. It follows that R/x_i is a strong realisation of R_*/x_i, and thus that $A_i = R/x_1 \wedge \dots \wedge R/x_i$ is a strong realisation of $R_*/(x_1, \dots, x_i)$. Using Proposition 4.8, we get a ring A which is a strong realisation of R_*/I. □

We next address the case where 2 is not a zero-divisor, but is not invertible either.

Theorem 4.12. *Let $A_* = (S^{-1}R_*)/I$ be a PLRQ of R_* which has no 2-torsion. Suppose also that $\widetilde{P}(I) \subseteq S^{-1}R_*$ maps to 0 in $A_*/2$, where $\widetilde{P} \colon R_d \to R_{2d+2}/2$ is the power operation defined in Section 10. Then A_* is strongly realisable.*

Proof. After using Proposition 4.6, we may assume that $S = \emptyset$. Choose a regular sequence (x_i) generating I. As $\overline{c}(x_i) = \widetilde{P}(x_i) \in I \pmod 2$, we can choose a product ϕ_i on R/x_i such that $c(\phi_i) \in I$. We let A be the "infinite smash product" of the R/x_i, as in Proposition 4.8, so that

$\pi_*(A) = A_*$. Because $c(\phi_i)$ maps to zero in $\pi_*(A)$, we see easily that the map $R/x_i \to A$ commutes with itself. By Proposition 4.8, we conclude that A is commutative.

Let $B \in \mathcal{R}^{ec}$ be an even commutative ring, and that $\pi_*(B)$ has no 2-torsion. The claim is that $\mathcal{R}(A, B) = \mathcal{R}_*(A_*, \pi_*(B))$. The right hand side has at most one element, and if it is empty, then the left hand side is also. Thus, we may assume that there is a map $A_* \to \pi_*(B)$ of R_*-algebras, and we need to show that there is a unique ring map $A \to B$.

By Proposition 4.8, we know that ring maps $A \to B$ biject with systems of ring maps $R/x_i \to B$ (which automatically commute as B is commutative). There is a unique unital map $f\colon R/x_i \to B$, and Proposition 3.15 tells us that the obstruction to f being a homomorphism satisfies $2d_B(\phi_i) = c(\phi_i) = 0 \in \pi_*(B)$. Because $\pi_*(B)$ has no 2-torsion, we have $d_B(\phi_i) = 0$, so there is a unique ring map $R/x_i \to B$, and thus a unique ring map $A \to B$ as required. □

The following result is also useful.

Proposition 4.13. *Let A_* be a strongly realisable R_*-algebra, and let $A_* \to B_*$ be a map of R_*-algebras that makes B_* into a free module over A_*. Then B_* is strongly realisable.*

Proof. First, observe that if F and M are A-modules, there is a natural map

$$\mathrm{Hom}_A(F, M) \to \mathrm{Hom}_{A_*}(F_*, M_*),$$

which is an isomorphism if F is a wedge of suspensions of A (in other words, a free A-module).

Choose a homogeneous basis $\{e_i\}$ for B_* over A_*, where e_i has degree d_i. Define $B := \bigvee_i \Sigma^{d_i} A$, so that B is a free A-module with a given isomorphism $\pi_* B \simeq B_*$ of A_*-modules. Define $B_0 := A$ and $B_1 := B$ and

$$B_2 := \bigvee_{i,j} \Sigma^{d_i+d_j} A$$
$$B_3 := \bigvee_{i,j,k} \Sigma^{d_i+d_j+d_k} A.$$

The product map $\mu\colon A \wedge A \to A$ gives rise to evident maps $\phi_k\colon B^{(k)} \to B_k$ which in turn give isomorphisms $B_*^{\otimes_{A_*} k} = \pi_* B_k$ of A_*-modules. The multiplication map $B_* \otimes_{A_*} B_* \to B_*$ corresponds under the isomorphism $\mathrm{Hom}_A(B_2, B) = \mathrm{Hom}_{A_*}(\pi_* B_2, B_*)$ to a map $B_2 \to B$. After composing this with ϕ_2, we get a product map $\mu_B\colon B \wedge B \to B$. A similar procedure gives a unit map $A \to B$.

We next prove that this product is associative. Each of the two associated products $B^{(3)} \to B$ factors as ϕ_3 followed by a map $B_3 \to B$, corresponding to a map $B_*^{\otimes_{A_*} 3} \to A_*$. The two maps $B_*^{\otimes_{A_*} 3} \to A_*$ in question are just the two possible associated products, which are the same because B_* is associative. It follows that B is associative. Similar arguments show that B is commutative and unital.

Now consider an object $C \in \mathcal{R}$ equipped with a map $B_* \to C_*$ (and thus a map $A_* \to C_*$). As A is a strong realisation of A_*, there is a unique map $A \to C$ compatible with the map $A_* \to C_*$. This makes C into an A-module, and thus gives an isomorphism $\mathrm{Hom}_A(B, C) = \mathrm{Hom}_{A_*}(B_*, C_*)$. There is thus a unique A-module map $B \to C$ inducing the given map $B_* \to C_*$. It follows easily that B_* is a strong realisation of B_*. □

We will need to consider certain R_*-algebras that are not strongly realisable. The following result assures us that weaker kinds of realisation are not completely uncontrolled.

Proposition 4.14. *Let A_* be an LRQ of R_*, and let $B, C \in \mathcal{R}^e$ be rings (not necessarily commutative) such that $\pi_*(B) = A_* = \pi_*(C)$. Then there is an isomorphism $f\colon B \to C$ (not necessarily a ring map) that is compatible with the unit maps $B \leftarrow R \to C$.*

Proof. We may as usual assume that $S = \emptyset$, and write $I = (x_1, x_2, \dots)$. Let A be the infinite smash product of the R/x_i's, so that $\pi_*(A) = A_*$. It will be enough to show that there is a unital isomorphism $A \to B$. Moreover, any unital map $A \to B$ is automatically an isomorphism, just by looking at the homotopy groups.

There is a unique unital map $f_i\colon R/x_i \to B$. Write $A_i = R/x_1 \wedge \ldots \wedge R/x_i$, and let g_i be the map

$$A_i \xrightarrow{f_1\wedge\ldots\wedge f_i} B^{(i)} \to B,$$

where the second maps is the product. Because B is a ring and each x_i goes to zero in $\pi_*(B)$, we can apply Lemma 4.7 to get a unital map $g\colon A \to B$ as required. $\square$

We conclude this section by investigating R-module maps $A \to A$ for various R-algebras $A \in \mathcal{R}$.

Proposition 4.15. *Let $\{x_1, x_2, \ldots\}$ be a regular sequence in R_*, let ϕ_i be a product on R/x_i, and let A be the infinite smash product of the rings R/x_i. Let $Q_i\colon A \to \Sigma^{|x_i|+1}A$ be obtained by smashing the Bockstein map $\overline{\beta}_{x_i}\colon R/x_i \to \Sigma^{|x_i|+1}R/x_i$ with the identity map on all the other R/x_j's. Then $\mathcal{D}(A, A)_*$ is isomorphic as an algebra over A_* to the completed exterior algebra on the elements Q_i.*

Proof. It is not hard to see that $Q_iQ_j = -Q_jQ_i$, with a sign coming from an implicit permutation of suspension coordinates. We also have $\overline{\beta}_i^2 = 0$ and thus $Q_i^2 = 0$. Given any finite subset $S = \{i_1 < \ldots < i_n\}$ of the positive integers, we define

$$Q_S := Q_{i_1}Q_{i_2}\ldots Q_{i_n}\colon A \to \Sigma^{d_S}A,$$

where $d_S = \sum_j(|x_{i_j}| + 1)$. The claim is that one can make sense of homogeneous infinite sums of the form $\sum_S a_S Q_S$ with $a_S \in A_*$, and that any graded map $A \to A$ of R-modules is uniquely of that form.

Write $A_n = R/x_1 \wedge \ldots \wedge R/x_n$, and let $i_n\colon A_n \to A$ be the evident map. It is easy to check that $Q_S \circ i_n = 0$ if $\max(S) > n$, and a simple induction shows that $\mathcal{D}(A_n, A)_*$ is a free module over A_* generated by the maps $Q_S \circ i_n$ for which $\max(S) \le n$. Moreover, Lemma 4.7 implies that $\mathcal{D}(A, A)_* = \varprojlim_n \mathcal{D}(A_n, A)_*$. The claim follows easily. $\square$

The above result relies more heavily than one would like on the choice of a regular sequence generating the ideal $\ker(R_* \to A_*)$. We will use the following construction to make things more canonical.

Construction 4.16. Let $A \in \mathcal{R}^e$ be an even ring, with unit $\eta\colon R \to A$, and let I be the kernel of $\eta_*\colon R_* \to A_*$. Given a derivation $Q\colon A \to \Sigma^k A$, we define a function $d(Q)\colon I \to A_*$ as follows. Given $x \in I$, we have a cofibration

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho_x} R/x \xrightarrow{\beta_x} \Sigma^{d+1}R$$

as usual. Here x may be a zero-divisor in R_*, so we need not have $\pi_*(R/x) = \pi_*(R)/x$. Nonetheless, we see easily that there is a unique map $f_x\colon R/x \to A$ such that $f_x \circ \rho_x = \eta$. As Q is a derivation, one checks easily that $Q \circ \eta = 0$, so $(Q \circ f_x) \circ \rho_x = 0$, so $Q \circ f_x = y \circ \beta_x$ for some $y\colon \Sigma^{d+1}R \to \Sigma^k A$. Because x acts as zero on A, we see that y is unique. We can thus define $d(Q)(x) := y \in \pi_{d+1-k}A$.

Proposition 4.17. *Let $A \in \mathcal{R}^e$ be such that $\pi_*(A) = R_*/I$, where I can be generated by a regular sequence. Let $\mathrm{Der}(A)$ be the set of derivations $A \to A$. Then Construction 4.16 gives rise to a natural monomorphism $d\colon \mathrm{Der}(A) \to \mathrm{Hom}_{R_*}(I/I^2, A_*)$ (with degrees shifted by one).*

Proof. Choose a regular sequence $\{x_1, x_2, \ldots\}$ generating I. Write $A_n = R/x_1 \wedge \ldots \wedge R/x_n$, and let j_n be the map

$$R/x_1 \wedge \ldots \wedge R/x_n \xrightarrow{f_{x_1}\wedge\ldots\wedge f_{x_n}} A^{(n)} \xrightarrow{\text{product}} A.$$

It is easy to see that A is the homotopy colimit of the objects A_n (although there may not be a ring structure on A_n for which j_n is a homomorphism). We also write $A_{n,i}$ for the smash product of the R/x_j for which $j \le n$ and $j \ne i$, and $j_{n,i}$ for the evident map $A_{n,i} \to A_n \xrightarrow{j_n} A$.

Consider a derivation $Q\colon A \to \Sigma^k A$, and write $b_i = d(Q)(x_i)$. Because Q is a derivation, we see that $Q \circ j_n$ is a sum of n terms, of which the i'th is b_i times the composite

$$A_n = A_{n,i} \wedge R/x_i \xrightarrow{1\wedge\beta_{x_i}} \Sigma^{|x_i|+1} A_{n,i} \xrightarrow{j_{n,i}} \Sigma^{|x_i|+1} A.$$

Now consider an element $x = \sum_{i=1}^n a_i x_i$ of I. It is easy to see that there is a unique unital map $f'_x\colon R/x \to A_n$, and that $j_n \circ f'_x = f_x$. Now consider the following diagram.

$$\begin{tikzcd}
\Sigma^d R \arrow[r, "x"] \arrow[d, "a_i"'] & R \arrow[r, "\rho_x"] \arrow[d, "1"'] & R/x \arrow[r, "\beta_x"] \arrow[d, "f'_x"] & \Sigma^{d+1} R \arrow[d, "a_i"] \\
\Sigma^{|x_i|} A_{n,i} \arrow[r, "x_i"'] & A_{n,i} \arrow[r, "1\wedge\rho_{x_i}"'] & A_n \arrow[r, "1\wedge\beta_{x_i}"'] & \Sigma^{|x_i|+1} A_{n,i}
\end{tikzcd}$$

The left hand square commutes because the terms $a_j x_j$ for $j \neq i$ become zero in $\pi_*(A_{n,i})$. It follows that there exists a map $R/x \to A_n$ making the whole diagram commute. However, f'_x is the *unique* map making the middle square commute, so the whole diagram commutes as drawn. Thus $j_{n,i} \circ (\beta_{x_i} \wedge 1) \circ f'_x = a_i \circ \beta_x$ (thinking of a_i as an element of $\pi_*(A)$). As $Q \circ j_n = \sum_i b_i.(j_{n,i} \circ (\beta_{x_i} \wedge 1))$, we conclude that $Q \circ f_x = Q \circ j_n \circ f'_x = (\sum_i a_i b_i) \circ \beta_x$. Thus $d(Q)(x) = \sum_i a_i b_i$.

This shows that $d(Q)$ is actually a homomorphism $I/I^2 \to A_*$. It is easy to check that the whole construction gives a homomorphism $d\colon \mathrm{Der}(A) \to \mathrm{Hom}_{A_*}(I/I^2, A_*)$. If $d(Q) = 0$ then all the elements b_i are zero, so $Q \circ j_n = 0$. As A is the homotopy colimit of the objects A_n, we conclude from Lemma 4.7 that $Q = 0$. Thus, d is a monomorphism. □

The meaning of the proposition is elucidated by the following elementary lemma.

Lemma 4.18. *If $\{x_1, x_2, \dots\}$ is a regular sequence in R_*, and I is the ideal that it generates, then I/I^2 is freely generated over R_*/I by the elements x_i.*

Proof. It is clear that I/I^2 is generated by the elements x_i. Suppose that we have a relation $\sum_{i=1}^n a_i x_i = 0$ in I (not I/I^2). We claim that $a_i \in I$ for all i. Indeed, it is clear that $a_n x_n \in (x_1, \dots, x_{n-1})$ so by regularity we have $a_n = \sum_{i=1}^{n-1} b_i x_i$ say; in particular, $a_n \in I$. Moreover, $\sum_{i=1}^{n-1} (a_i + b_i x_n) x_i = 0$, so by induction we have $a_i + b_i x_n \in I$ for $i < n$, and thus $a_i \in I$ as required.

Now suppose that we have a relation $\sum_i a_i x_i \in I^2$, say $\sum_i a_i x_i = \sum_{j \leq i} b_{ij} x_i x_j$. We then have $\sum_i (a_i - \sum_{j \leq i} b_{ij} x_j) x_i = 0$, so by the previous claim we have $a_i - \sum_{j \leq i} b_{ij} x_j \in I$, so $a_i \in I$. This shows that the elements x_i generate I/I^2 freely. □

Corollary 4.19. *In the situation of Proposition 4.15 the map $d\colon \mathrm{Der}(A) \to \mathrm{Hom}(I/I^2, A_*)$ is an isomorphism, and $\mathcal{D}(A,A)_*$ is the completed exterior algebra generated by $\mathrm{Der}(A)$.*

Proof. It is easy to see that Q_i is a derivation and that $d(Q_i)(x_j) = \delta_{ij}$ (Kronecker's delta). This shows that d is surjective, and the rest follows. □

5. Formal group theory

In this section, we take $R = MU$, and let F be the usual formal group law over MU_*. In places it will be convenient to use cohomological gradings; we recall the convention $A^* = A_{-*}$. We will write q for the usual map $MU^* \to BP^*$, and note that $q(w_1) = v_1 \pmod 2$.

A well-known construction gives a power operation

$$P\colon R^d X \to R^{2d}(\mathbb{R}P^2 \times X),$$

which is natural for spaces X and strictly commutative ring spectra R. A good reference for such operations is [7]; in the case of MU, the earliest source is probably [28].

In the case $R = S^{-1}MU$ there is an element $\epsilon \in R^2\mathbb{R}P^2$ such that $R^*\mathbb{R}P^2 = R^*[\epsilon]/(2\epsilon, \epsilon^2)$. More generally, the even-dimensional part of $R^*(\mathbb{R}P^2 \times X)$ is $R^*(X)[\epsilon]/(2\epsilon, \epsilon^2)$, and $P(x) = x^2 + \epsilon \widetilde{P}(x)$

for a uniquely determined operation $R^d(X) \to R^{2d-2}(X)/2$. We also have the following properties:

$$\begin{aligned} P(1) &= 1 \\ P(xy) &= P(x)P(y) \\ P(x+y) &= P(x) + P(y) + (2 + \epsilon w_1)xy \\ P(x) &= x(x +_F \epsilon) \qquad \text{if } x \text{ is the Euler class of a complex line bundle.} \end{aligned}$$

To handle the nonadditivity of P, we make the following construction. For any MU^*-algebra A^*, we define

$$T(A^*) := \{(r,s) \in A^*/2 \times A^*[\epsilon]/(2,\epsilon^2) \mid s = r^2 \pmod{\epsilon}\}.$$

Given $a, b \in A^*$ (with $|b| = 2|a| - 2$) we define $[a,b] := (a, a^2 + \epsilon b) \in T(A^*)$. We make $T(A^*)$ into a ring by defining

$$\begin{aligned} (r,s) + (t,u) &:= (r+t, s+u+\epsilon w_1 rt) \\ (r,s).(t,u) &:= (rt, su) \end{aligned}$$

or equivalently

$$\begin{aligned} [a,b] + [c,d] &:= [a+c, b+d+w_1 ac] \\ [a,b].[c,d] &:= [ac, a^2 d + bc^2]. \end{aligned}$$

Note that $2[a,b] = [0, w_1 a^2]$ and $4[a,b] = 0$, so $4T(A^*) = 0$. If we define $Q(a) := (a, P(a)) = [a, \widetilde{P}(a)]$, then Q gives a ring map $MU^*X \to T(MU^*X)$.

Definition 5.1. Suppose that A^* is a PLRQ of MU^*, and let $f\colon MU^* \to A^*$ be the unit map. We say that A^* has an induced power operation (IPO) if there is a ring map $\overline{Q}\colon A^* \to T(A^*)$ making the following diagram commute:

$$\begin{CD} MU^* @>{Q}>> T(MU^*) \\ @V{f}VV @VV{T(f)}V \\ A^* @>>{\overline{Q}}> T(A^*). \end{CD}$$

Because A^* is an LRQ, we know that such a map is unique if it exists.

If $A^* = S^{-1}MU^*$ then we know that $S^{-1}MU$ can be constructed as a strictly commutative MU-algebra and thus an E_∞ ring spectrum, and the power operation coming from this E_∞ structure clearly gives an IPO on $S^{-1}MU_*$. For a more elementary proof, it suffices to show that when $x \in S$ the image of $Q(x)$ in $T(S^{-1}MU_*)$ is invertible. However, the element (x, x^2) is trivially invertible in $T(S^{-1}MU)$ and $Q(x)$ differs from this by a nilpotent element, so it too is invertible.

It is now easy to reduce the following result to Theorem 2.7.

Proposition 5.2. *Let A_* be a PLRQ of MU_* which has no 2-torsion and admits an IPO. Then A_* is strongly realisable.* $\square$

We now give a formal group theoretic criterion for the existence of an IPO.

Definition 5.3. Let F be a formal group law over a ring A^*. Given an algebra B^* over A^* and an element $x \in B^2$, we define $Z(x) := (x, x(x +_F \epsilon)) \in T(B^*)$. (We need x to be topologically nilpotent in a suitable sense to interpret this, but we leave the details to the reader.) Thus, if X is a space, $A^* = MU^*$, $B^* = MU^*X$ and x is the Euler class of a complex line bundle over X then $Z(x) = Q(x)$.

Proposition 5.4. *Let A^* be a LRQ of MU^*, and let F be the obvious formal group law over A^*. Then a ring map $\overline{Q}\colon A^* \to T(A^*)$ is an IPO if and only if we have*

$$Z(x) +_{\overline{Q}_* F} Z(y) = Z(x +_F y) \in T(A^*[\![x,y]\!]).$$

Proof. Let F' be the universal FGL over MU^* and put $Z'(x) = (x, x(x +_{F'} \epsilon))$. Let $f\colon MU^* \to A^*$ be the unit map, so that $F = f_*F'$. Using the universality of F', we see that $\overline{Q}$ is an IPO if and only if we have

$$X +_{\overline{Q}_* f_* F'} Y = X +_{T(f)_* Q_* F'} Y \in T(A^*)[\![X, Y]\!].$$

The left hand side is of course $X +_{\overline{Q}_* F} Y$. There is an evident map

$$T(A^*)[\![X, Y]\!] \to T(A^*[\![x, y]\!]),$$

sending X to $Z(x)$ and Y to $Z(y)$, and one can check that this is injective. Thus, $\overline{Q}$ is an IPO if and only if

$$Z(x) +_{\overline{Q}_* F} Z(y) = Z(x) +_{T(f)_* Q_* F'} Z(y) \in T(A^*[\![x, y]\!]).$$

The right hand side here is $T(f)(Z'(x) +_{Q_* F'} Z'(y))$ and $Z(x +_F y) = T(f)(Z'(x +_{F'} y))$ so the proposition will follow once we prove that $Z'(x +_{F'} y) = Z'(x) +_{Q_* F'} Z'(y) \in T(MU^*[\![x, y]\!])$. To do this, we use the usual isomorphism $MU^*[\![x, y]\!] = MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, so that x, y and $x +_{F'} y$ are Euler classes, so $Q(x) = Z'(x)$ and $Q(y) = Z'(y)$ and $Q(x +_{F'} y) = Z'(x +_{F'} y)$. As Q is a natural multiplicative operation we also have $Q(x +_{F'} y) = Q(x) +_{Q_* F'} Q(y) = Z(x) +_{Q_* F'} Z(y)$, which gives the desired equation. □

We now use this to show that there is an IPO on kU^*. In this case the real reason for the IPO is that the Todd genus gives an H_∞ map $MU \to kU$, but we give an independent proof as a warm-up for the case of BP^*.

Proposition 5.5. *Let $f\colon MU^* \to kU^* := \mathbb{Z}[u]$ be the Todd genus. Then there is an induced power operation on kU^*, given by $\overline{Q}(u) = [u, u^3]$. Thus, kU^* and KU^* are strongly realisable.*

Proof. The FGL over kU^* coming from f is just the multiplicative FGL $x +_F y = x + y + uxy$, so $Z(x) = (x, x^2 + x\epsilon + ux^2\epsilon) = [x, x + ux^2]$. If we put $U = [u, u^3]$ then $X +_{\overline{Q}_* F} Y = X + Y + UXY$. We thus need only verify that $Z(x) + Z(y) + U\,Z(x)Z(y) = Z(x + y + uxy)$. This is a straightforward calculation; some steps are as follows.

$$\begin{aligned}
Z(x) + Z(y) &= [x + y, x + y + u(x^2 + xy + y^2)] \\
Z(x)Z(y) &= [xy, xy(x + y)] \\
U\,Z(x)Z(y) &= [uxy, u^2xy(x + y) + u^3x^2y^2] \\
Z(x + y + uxy) &= [x + y + uxy, x + y + u(x^2 + xy + y^2) + u^3x^2y^2].
\end{aligned}$$

□

We now turn to the case of BP^*. For the moment we prove only that an IPO exists; in the next section we will calculate it.

Proposition 5.6. *There is an IPO on BP^*, so BP^* is strongly realisable.*

This is proved after Lemma 5.9.

Definition 5.7. For the rest of this section, we will write

$$z := z(x) := \sum_{k \geq 0} v_1^{2^k} x^{2^k} \in \mathbb{F}_2[v_1][\![x]\!].$$

Note that

$$z^2 = z + v_1 x$$

so

$$z/(v_1 x) = 1/(1 + z).$$

Lemma 5.8. *We have*

$$x +_F \epsilon = x + (1 + z)\epsilon \qquad \textit{in} \qquad BP^*[\epsilon][\![x]\!]/(2\epsilon, \epsilon^2).$$

Proof. Working rationally and modulo ϵ^2, we have

$$\log_F(\epsilon) = \epsilon$$

so

$$x +_F \epsilon = \exp_F(\log_F(x) + \epsilon) = x + \exp_F'(\log_F(x))\epsilon = x + \epsilon/\log_F'(x).$$

Note that $\log_F'(x)$ is integral and its constant term is 1, so the above equation is between integral terms and we can sensibly reduce it modulo 2.

We next recall the formula for $\log_F(x)$ given in [25, Section 4.3]. We consider sequences $I = (i_1, \dots, i_r)$ with $l \geq 0$ and $i_j > 0$ for each j. We write $|I| := r$ and $\|I\| := i_1 + \dots + i_r$. We also write

$$v_I := v_{i_1}^{m_1} \dots v_{i_r}^{m_r}$$

where

$$m_j := 2^{\sum_{k<j} i_k}.$$

The formula is

$$\log_F(x) = \sum_I v_I x^{2^{\|I\|}}/2^{|I|}.$$

The only terms which contribute to $\log_F'(x)$ modulo 2 are those for which $\|I\| = |I|$, so $i_j = 1$ for all j. If I has this form and $|I| = k$ then $v_I = v_1^{2^k-1}$. Thus

$$\log_F'(x) = \sum_k v_1^{2^k-1} x^{2^k-1} = z/(v_1 x) \pmod 2.$$

As remarked in Definition 5.7, we have $z/(v_1x) = 1/(1+z)$, so

$$x +_F \epsilon = x + \epsilon/\log_F'(x) = x + (1+z)\epsilon$$

as claimed. □

Lemma 5.9. *In $T(BP^*[\![x, y]\!])$ we have*

$$[0, x] +_{QF} [0, y] = [0, x + y]$$

and

$$[x, y] = [x, 0] +_{QF} \left[0, \sum_{k\geq 0} (v_1 x)^{2(2^k-1)} y\right].$$

In particular, we have

$$Z(x) = [x, 0] +_{QF} [0, z/v_1] = [x, 0] +_{QF} \left[0, \sum_{k\geq 0} v_1^{2^k-1} x^{2^k}\right].$$

Proof. The first statement is clear, just because $[0, x][0, y] = 0$. For the second statement, write $X := [x, 0]$ and

$$w := \sum_k (v_1 x)^{2(2^k-1)} y = y(z/v_1 x)^2 = y/(1+z)^2,$$

and $W := [0, w]$. Let $a_{ij} \in MU_{2(i+j-1)}$ be the coefficient of $x^i y^j$ in $x +_F y$. Because $W^2 = 0$ we have $X +_{QF} W = X + W + \sum_{j>0} Q(a_{1j}) X^j W$, and

$$Q(a_{1j}) X^j W = [a_{1j}, \widetilde{P}(a_{1j})][x^j, 0][0, w] = [0, a_{1j}^2 x^{2j} w].$$

This expression is to be interpreted in $T(BP^*[\![x,y]\!])$, so we need to interpret a_{1j} in $BP^*/2$. Thus Lemma 5.8 tells us that $a_{10}=1$ and $a_{1,2^k}=v_1^{2^k}$ and all other a_{1j}'s are zero. Thus

$$X+_{QF}W=X+W+\sum_{k\geq 0}[0,(v_1x)^{2^{k+1}}w]=$$
$$\left[x,w\left(1+\sum_{k\geq 0}(v_1x)^{2^{k+1}}\right)\right]=[x,w(1+z^2)]=[x,y]$$

as claimed.

For the last statement, Lemma 5.8 gives

$$Z(x)=(x,x(x+_F\epsilon))=(x,x^2+x(1+z)\epsilon)=[x,x(1+z)].$$

By the previous paragraph, this can be written as $[x,0]+_{QF}[0,x(1+z)/(1+z)^2]=[x,0]+_{QF}[0,z/v_1]$. □

Proof of Proposition 5.6. To show that $\overline{Q}$ exists, it is enough to show that the formal group law on $T(BP^*)$ obtained from the map $MU^*\xrightarrow{Q}T(MU^*)\xrightarrow{T(q)}T(BP^*)$ is 2-typical. Let p be an odd prime, so the associated cyclotomic polynomial is $\Phi_p(t)=1+t+\cdots+t^{p-1}$. We need to show that

$$X+_{QF}\Omega X+_{QF}\cdots+_{QF}\Omega^{p-1}X=0\qquad\text{in}\qquad B^*:=T(BP^*)[\Omega][\![X]\!]/\Phi_p(\Omega).$$

(This is just the definition of 2-typicality for formal groups over rings which may have torsion.) Consider the ring $C^*:=T(BP^*[\omega][\![x]\!]/\Phi_p(\omega))$. As $\Phi_p(\omega)=0$ we have $t^p-1=\prod_{j=0}^{p-1}(t-\omega^j)$, and by looking at the coefficient of t^{p-2} we find that $\sum_{0\leq i<j<p}\omega^{i+j}=0$. Now write $\Omega:=[\omega,0]$ and $X:=[x,0]$, so that $\Omega,X\in C^*$. We find that

$$\Phi_p(\Omega)=\sum_{i=0}^{p-1}[\omega^i,0]=\left[\Phi_p(\omega),v_1\sum_{0\leq i<j<p}\omega^{i+j}\right]=0.$$

This gives us a ring map $B^*\to C^*$; we claim that this is injective. Indeed, it is easy to see that $\{\omega,\omega^2,\dots,\omega^{p-1}\}$ is a basis for $\mathbb{Z}[\omega]/\Phi_p(\omega)$, and that $\alpha\mapsto\alpha^2$ is a permutation of this basis. Suppose that we have

$$\sum_{i=1}^{p-1}\sum_{j\geq 0}[a_{ij},b_{ij}]\Omega^iX^j=0\text{ in }C^*.$$

Using the evident map $C^*\to BP^*[\omega][\![x]\!]/(2,\Phi_p(\omega))$, we see that $a_{ij}=0$ for all i,j. As $[0,b]\Omega^iX^j=[0,\omega^{2i}X^{2j}b]$, we see that

$$\sum_{i=1}^{p-1}\sum_{j\geq 0}b_{ij}\omega^{2i}X^{2j}=0.$$

As the elements ω^{2i} are a permutation of the elements ω^i, we see that $b_{ij}=0$ for all i,j. We may thus regard B^* as a subring of C^*.

Next, we know that

$$Z(x)+_{QF}Z(\omega x)+_{QF}\cdots+_{QF}Z(\omega^{p-1}x)=Z(x+_F\omega x+_F\cdots+_F\omega^{p-1}x)=0,\tag{1}$$

because F is 2-typical over BP_*. By Lemma 5.9, we also know that

$$Z(\omega^ix)=\Omega^iX+_{QF}[0,w_i],\tag{2}$$

where $w_i=\sum_k v_1^{2^k-1}\omega^{2^ki}x^{2^k}$. It is easy to see that $[0,w_i][0,w_j]=0$, so that $[0,w_i]+_{QF}[0,w_j]=[0,w_i+w_j]$. We also have $\sum_{i=0}^{p-1}\omega^{2^ki}=0$ for all k. This means that

$$\sum_i^{QF}[0,w_i]=[0,\sum_k v_1^{2^k-1}x^{2^k}\sum_{i=0}^{p-1}\omega^{2^ki}]=0.\tag{3}$$

By combining equations (1) to (3), we see that

$$\sum_i^{QF} \Omega^i X = 0$$

as required. □

6. The power operation on BP^*

We now give explicit formulae for the IPO on BP^*.

Definition 6.1. Given a subset $J = \{j_1 < \ldots < j_r = n\} \subseteq \{1, \ldots, n\}$, we define

$$u_J := v_{j_1+1} \prod_{k=1}^{r-1} (v_1 v_{j_k})^{2(2^{j_{k+1}-j_k}-1)} \in \pi_{|v_{n+1}|} BP$$

and $u_n = \sum_J u_J$, where J runs over subsets of $\{1, \ldots, n\}$ that contain n. By separating out the case $r = 1$ and putting $j = n - j_{r-1}$ in the remaining cases we obtain a recurrence relation

$$u_n = v_{n+1} + \sum_{j=1}^{n-1} (v_1 v_{n-j})^{2(2^j-1)} u_{n-j}.$$

Proposition 6.2. *The induced power operation on BP^* is given by*

$$\overline{Q}(v_n) = \begin{cases} [0, v_1] & \text{if } n = 0 \\ [v_1, v_2] & \text{if } n = 1 \\ [v_n, v_1 v_n^2 + u_n] & \text{if } n > 1 \end{cases}$$

Moreover, we have $u_n = v_{n+1} \pmod{v_1^2}$.

This is proved after Corollary 6.4. We will reuse the notation of Definition 5.7.

Lemma 6.3. *We have $\exp_F(2x) = 2z/v_1$ in $BP^*[\![x]\!]/4$.*

Proof. Using Ravenel's formulae as in the proof of Lemma 5.8, we have

$$\log_F(2x)/2 = \sum_I 2^{2^{\|I\|}-|I|-1} v_I x^{2^{\|I\|}}.$$

When $k \geq 0$ we have $2^k \geq k + 1$, with equality only when $k = 0$ or $k = 1$. It follows easily that

$$\log_F(2x)/2 = x + v_1 x^2 \pmod 2.$$

By inverting this, we find that

$$\exp_F(2x)/2 = \sum_{k \geq 0} v_1^{2^k-1} x^{2^k} = z/v_1 \pmod 2,$$

and thus that $\exp_F(2x) = 2z/v_1 \pmod 4$. □

Because $T(BP^*)$ is a torsion ring, the formal group law QF has no exp series. Nonetheless, $\exp_F(2X)$ is a power series over BP^*, so we can apply Q to the coefficients to get a power series over $T(BP^*)$ which we call $\exp_{QF}(2X)$. This makes perfect sense even though $\exp_{QF}(X)$ does not.

Corollary 6.4. *In $T(BP^*)[\![X]\!]$, we have*

$$\exp_{QF}(2X) = \sum_{k \geq 0} [0, v_1^{2^{k+1}-1}] X^{2^k}.$$

By taking $X = Z(x) \in T(BP^*[\![x]\!])$, we get

$$\exp_{QF}(2Z(x)) = \left[0, \sum_{j>0} v_1^{2^j-1} x^{2^j} \right] = [0, z/v_1 + x].$$

Proof. Because $4 = 0$ in $T(BP^*)$, it follows immediately from the lemma that $\exp_{QF}(2X) = \sum_k 2\overline{Q}(v_1)^{2^k-1}X^{2^k}$. Using $\overline{Q}(v_1) = [v_1, \widetilde{P}(v_1)]$, we see that $2\overline{Q}(v_1)^{2^k-1} = [0, v_1^{2^{k+1}-1}]$, and the first claim follows. If we now put $X = Z(x) = [x, x(1+z)]$ then $[0, v_1^{2^{k+1}-1}]X^{2^k} = [0, v_1^{2^{k+1}-1}x^{2^{k+1}}]$, and the second claim follows. □

Proof of Proposition 6.2. Let p_k denote the image of $\widetilde{P}(v_k) \in MU^{2^{k+2}-2}/2$ in $BP^*/2$ and write $V_k = \overline{Q}(v_k) = [v_k, p_k] \in T(BP^*)$. Recall that the Hazewinkel generators v_k are characterised by the formula

$$[2]_F(x) = \exp_F(2x) +_F \sum_{k>0}^{F} v_k x^{2^k} \in BP^*[\![x]\!].$$

By applying the ring map $\overline{Q}$ and putting $X = Z(x)$ we obtain

$$[2]_{QF}(Z(x)) = \exp_{QF}(2Z(x)) +_{QF} \sum_{k>0}^{QF} V_k Z(x)^{2^k} \in T(BP^*[\![x]\!]).$$

The first term can be evaluated using Corollary 6.4. For the remaining terms, we have

$$V_k Z(x)^{2^k} = [v_k, p_k][x^{2^k}, 0] = [v_k x^{2^k}, p_k x^{2^{k+1}}].$$

We can use Lemma 5.9 to rewrite this as

$$\begin{aligned} V_k Z(x)^{2^k} &= [v_k x^{2^k}, 0] +_{QF} \left[0, \sum_{l>0} (v_1 v_k x^{2^k})^{2^l-2} p_k x^{2^{k+1}}\right] \\ &= [v_k x^{2^k}, 0] +_{QF} \left[0, \sum_{l>0} (v_1 v_k)^{2^l-2} p_k x^{2^{k+l}}\right]. \end{aligned}$$

After using the formula $[0, b] +_{QF} [0, c] = [0, b + c]$ to collect terms, we find that

$$[2]_{QF}(Z(x)) = \left[0, \sum_{l>0} v_1^{2^l-1} x^{2^l} + \sum_{k,l>0} (v_1 v_k)^{2^l-2} p_k x^{2^{k+l}}\right] +_{QF} \sum_{k>0}^{QF} [v_k x^{2^k}, 0]. \tag{4}$$

On the other hand, we know that

$$\begin{aligned} [2]_{QF}(Z(x)) &= Z([2]_F(x)) \\ &= Z\left(\exp_F(2x) +_F \sum_{k>0}^{F} v_k x^{2^k}\right) \\ &= Z(\exp_F(2x)) +_{QF} \sum_{k>0}^{QF} Z(v_k x^{2^k}). \end{aligned}$$

The first term is zero because $\exp_F(2x)$ is divisible by 2. For the remaining terms, Lemma 5.9 gives

$$Z(v_k x^{2^k}) = [v_k x^{2^k}, 0] +_{QF} [0, \sum_{j\geq 0} v_1^{2^j-1} v_k^{2^j} x^{2^{k+j}}].$$

Thus, we have

$$[2]_{QF}(Z(x)) = \left[0, \sum_{k>0} \sum_{l\geq 0} v_1^{2^l-1} v_k^{2^l} x^{2^{k+l}}\right] +_{QF} \sum_{k>0}^{QF} [v_k x^{2^k}, 0]. \tag{5}$$

By comparing this with equation (4) and equating coefficients of $x^{2^{n+1}}$, we find that

$$v_1^{2^{n+1}-1} + \sum_{j=1}^{n} (v_1 v_{n+1-j})^{2^j-2} p_{n+1-j} = \sum_{j=0}^{n} v_1^{2^j-1} v_{n+1-j}^{2^j}.$$

After some rearrangement and reindexing, this becomes

$$p_n + v_1 v_n^2 = v_1^{2^{n+1}-1} + v_{n+1} + \sum_{j=1}^{n-1} (v_1 v_{n-j})^{2(2^j-1)} (p_{n-j} + v_1 v_{n-j}^2).$$

In particular, we have $p_1 = v_2$. We now define

$$p'_n = \begin{cases} v_1 & \text{if } n = 0 \\ v_2 & \text{if } n = 1 \\ v_1 v_n^2 + u_n & \text{if } n > 1 \end{cases}$$

The claim of the proposition is just that $p_n = p'_n$ for all $n \geq 0$. Using the recurrence relation given in definition 6.1, one can check that for all $n > 0$ we have

$$p'_n + v_1 v_n^2 = v_1^{2^{n+1}-1} + v_{n+1} + \sum_{j=1}^{n-1} (v_1 v_{n-j})^{2(2^j-1)} (p'_{n-j} + v_1 v_{n-j}^2).$$

In particular, we have $p'_1 = v_2 = p_1$, and it follows inductively that $p_n = p'_n$ for all $n > 0$. We also have

$$Q(v_0) = Q(1) + Q(1) = [1, 0] + [1, 0] = [0, v_1]$$

so $p_0 = v_1 = p'_0$. □

Remark 6.5. The first few cases are

$$\begin{aligned} p_0 &= v_1 \\ p_1 &= v_2 \\ p_2 &= v_1^4 v_2 + v_1 v_2^2 + v_3 \\ p_3 &= v_1^{12} v_2 + v_1^6 v_2^3 + v_1^2 v_2^2 v_3 + v_1 v_3^2 + v_4. \end{aligned}$$

In particular, we find that $p_3 \notin (v_k \mid k \geq 3)$, which shows that there is no commutative product on $BP\langle 2 \rangle$, considered as an object of $\mathcal{D}$. This problem does not go away if we replace the Hazewinkel generator v_k by the corresponding Araki generator, or the bordism class w_k of a smooth quadric hypersurface in $\mathbb{C}P^{2^k}$. However, it is possible to choose a more exotic sequence of generators for which the problem does go away, as indicated by the next result.

Proposition 6.6. *Fix an integer $n > 0$. There is an ideal $J \leq BP^*$ such that*

1. *The evident map*

$$\mathbb{Z}_{(2)}[v_1, \ldots, v_n] \to BP^* \to BP^*/J$$

   is an isomorphism.
2. $\widetilde{P}(J) \leq J \pmod 2$.
3. $I_n + J = I_n + (v_k \mid k > n) = (v_k \mid k \neq n)$.

The proof will construct an ideal explicitly, but it is not the only one with the stated properties. If $n = 1$ we can take $J = (v_k \mid k > n)$, but for $n > 1$ this violates condition (2).

Remark 6.7. The subring $\mathbb{Z}_{(2)}[v_1, \ldots, v_n]$ of BP^* is the same as the subring generated by all elements of degree at most $2^{n+1} - 2$; it is thus defined independently of the choice of generators for BP_*.

Proof. First consider the case $n = 1$, and write $J = (v_k \mid k > 1)$. By inspecting definition 6.1, we see that $u_n \in J$ for all $n > 1$, and thus Proposition 6.2 tells us that $\widetilde{P}(J) \leq J \pmod 2$. We may thus assume that $n > 1$. Write $B^* = \mathbb{Z}_{(2)}[v_1, \ldots, v_n]$, thought of as a subring of BP^*. We will recursively define a sequence of elements $x_k \in BP^*$ for $k > n$ such that

(a) $x_k \in v_k + v_1^2 B^*$
(b) $\widetilde{P}(x_{k-1}) \in (x_{n+1}, \ldots, x_k) \pmod 2$ if $k > n + 1$.

It is clear that we can then take $J = (x_k \mid k > n)$. We start by putting $x_{n+1} = v_{n+1}$. Suppose that we have defined $x_{n+1}, \dots, x_k$ with the stated properties. There is an evident map

$$\mathbb{F}[v_1, \dots, v_n, v_{k+1}] \xrightarrow{f} BP_*/(2, x_{n+1}, \dots, x_k),$$

which is an isomorphism in degree $2(2^{k+1} - 1) = |v_{k+1}|$. Let $\overline{p}_k$ be the image of $\widetilde{P}(x_k)$ in $BP_*/(2, x_{n+1}, \dots, x_k)$, and write $\overline{x}_{k+1} = f^{-1}(\overline{p}_k)$. We can lift this to get an element x_{k+1} of $\mathbb{Z}_{(2)}[v_1, \dots, v_n, v_{k+1}]$ such that $\overline{x}_{k+1} = x_{k+1} \pmod 2$ and every coefficient in x_{k+1} is 0 or 1. It is easy to see that condition (b) is satisfied, and that $x_{k+1} \in v_{k+1} + B^*$. However, we still need to show that $x_{k+1} - v_{k+1}$ is divisible by v_1^2. By assumption we have $x_k = v_k + v_1^2 b$ for some $b \in B^*$. Recall from Proposition 6.2 that $\widetilde{P}(v_k) = v_{k+1} + v_1 v_k^2 \pmod{2, v_1^2}$. It follows after a small calculation that $\widetilde{P}(x_k) = v_{k+1} + v_1 v_k^2 \pmod{2, v_1^2}$ also. Moreover, we have $v_k^2 = v_1^4 b^2 \pmod{2, x_k}$, so $\overline{p}_k = v_{k+1} \pmod{2, v_1^2}$. It follows easily that $x_{k+1} = v_{k+1} \pmod{v_1^2}$, as required. □

We give one further calculation, closely related to Proposition 6.2.

Proposition 6.8. *Recall that $I_k := (w_0, \dots, w_{k-1}) < MU^*$, where w_i is the bordism class of a smooth quadric hypersurface in $\mathbb{C}P^{2^i}$. We have $\widetilde{P}(I_{k-1}) \leq I_k$, and $\widetilde{P}(w_{k-1}) = w_k \pmod{I_k}$.*

Proof. If $k = 1$ we have $I_0 = 0$ and $w_0 = 2$, so $P(w_0) = P(1) + P(1) + (2 + w_1\epsilon) = w_1\epsilon \pmod 2$, as required. Thus, we may assume that $k > 1$, and it follows easily from the formulae for $P(x+y)$ and $P(xy)$ that P induces a ring map $MU^* \to B^* = (MU^*/I_k)[\epsilon]/\epsilon^2$. Note that $[2]_F(x) = w_k x^{2^k} + O(x^{2^k+1})$ over B^*. Write $X = x(x +_F \epsilon) \in MU^*[\epsilon][\![x]\!]/(I_k, \epsilon^2)$. Arguing in the usual way, we see that

$$[2]_{P_*F}(X) = [2]_F(x)([2]_F(x) +_F \epsilon) = \epsilon w_k x^{2^k} + O(x^{2^k+1}).$$

It follows easily that we must have

$$[2]_{P_*F}(X) = \epsilon w_k X^{2^{k-1}} + O(X^{2^{k-1}+1}).$$

It follows that $P(w_i) = 0 \in B^*$ for $i < k-1$, and that $P(w_{k-1}) = \epsilon w_k \in B^*$, as required. □

7. Applications to MU

Proof of Proposition 2.9. The claims involving kU and KU follow from Proposition 5.5, and those for BP follow from Proposition 5.6. The claim H follows from Theorem 2.7, as the condition $\widetilde{P}(I) \subseteq I \pmod 2$ is trivially satisfied for dimensional reasons. The claim for $H\mathbb{F}$ can be proved in the same way as Theorem 4.12 after noting that all the obstruction groups are trivial. □

Proof of Proposition 2.10. Choose an ideal J as in Proposition 6.6 and set $BP\langle n\rangle'_* = BP_*/J$. Everything then follows from Theorem 2.7. □

We now take $R = MU_{(2)}$ and turn to the proof of Theorem 2.13. As previously, we let $w_k \in \pi_{2^{k+1}-2}R$ denote the bordism class of the quadric hypersurface W_{2^k} in $\mathbb{C}P^{2^k}$. Recall that the image of w_k in BP_* is v_k modulo $I_k = (v_0, \dots, v_{k-1})$, and thus $P(n)_* = BP_*/(w_0, \dots, w_{n-1})$.

We next choose a product ϕ_k on MU/w_k for each k. For $k = 0$ we choose there are two possible products, and we choose one of them randomly. (It is possible to specify one of them precisely using Baas-Sullivan theory, but that would lead us too far afield.) For $k > 0$, we recall from Proposition 6.8 that $\widetilde{P}(w_k) = w_{k+1} \pmod{I_{k+1}}$. It follows easily that there is a product ϕ_k such that $c(\phi_k) = w_{k+1} \pmod{w_1, \dots, w_k}$, and that this is unique up to a term $u \circ (\beta \wedge \beta)$ with $u \in (w_1, \dots, w_k)$. From now on, we take ϕ_k to be a product with this property. It is easy to see that the resulting product $MU/w_0 \wedge \dots \wedge MU/w_{n-1}$ is independent of the choice of ϕ_k's (except for ϕ_0).

Definition 7.1. We write

$$MU/I_n = MU/w_0 \wedge \dots \wedge MU/w_{n-1},$$

made into a ring as discussed above. For $i < n$, we define

$$Q_i \colon MU/I_n \to \Sigma^{2^{i+1}-1} MU/I_n$$

by smashing the Bockstein map $\overline{\beta}\colon MU/w_i \to \Sigma^{2^{i+1}-1} MU/w_i$ with the identity on the other factors. We also define

$$\begin{aligned} P(n) &:= BP \wedge MU/I_n \\ B(n) &:= w_n^{-1} P(n) \\ k(n) &:= BP\langle n\rangle' \wedge MU/I_n \\ K(n) &:= w_n^{-1} k(n). \end{aligned}$$

It is clear that $\pi_*(MU/I_n) = MU_*/I_n$ and $\pi_*(P(n)) = P(n)_*$ and $\pi_*(B(n)) = B(n)_*$. Condition (2) in Proposition 2.10 assures us that $\pi_* k(n) = k(n)_*$ and $\pi_* K(n) = K(n)_*$ as well. As BP and $BP\langle n\rangle'$ are commutative, it is easy to see that $P(n)$, $B(n)$, $k(n)$ and $K(n)$ are central algebras over BP, $v_n^{-1}BP$, $BP\langle n\rangle'$ and $E(n)'$ respectively. The derivations Q_i on MU/I_n clearly induce compatible derivations on $P(n)$, $B(n)$, $k(n)$ and $K(n)$.

Proposition 7.2. *The product ϕ on MU/I_n satisfies*

$$\phi - \phi \circ \tau = w_n \phi \circ (Q_{n-1} \wedge Q_{n-1}).$$

Similarly for $P(n)$, $B(n)$, $k(n)$ and $K(n)$.

Proof. This follows easily from the fact that $c(\phi_{k-1}) = w_k \pmod{I_k}$, given by Proposition 6.8. □

Proposition 7.3. *Let A be a central BP-algebra such that $\pi_0(A) = \{0, 1\}$, $\pi_{2^{n+1}-2}(A) = \{0, v_n\}$ and $\pi_k(A) = 0$ for $0 < k < 2^{n+1}-2$. Then either there is a unique map $P(n) \to A$ of BP-algebras, or there is a unique map $P(n) \to A^{op}$ (but not both). Analogous statements hold for $B(n)$, $k(n)$ and $K(n)$ with BP replaced by $v_n^{-1}BP$, $BP\langle n\rangle'$ and $E(n)'$ respectively.*

Proof. We treat only the case of $P(n)$; the other cases are essentially identical. Any ring map $MU/I_n \to A$ commutes with the given map $BP \to A$, because the latter is central. It follows that maps $P(n) \to A$ of BP-algebras biject with maps $MU/I_n \to A$ of rings, which biject with systems of commuting ring maps $MU/w_i \to A$ for $0 \le i < n$. For $i < n-1$ we have $\pi_{2|w_i|+2}(A) = 0$, so Proposition 3.15 tells us that the unique unital map $f_i\colon MU/w_i \to A$ is a ring map. This remains the case if we replace the product ψ on A by $\psi \circ \tau$, or in other words replace A by A^{op}. There is an obstruction $d_A(\phi_{n-1}) \in \pi_{2^{n+1}-2}(A) = \{0, v_n\}$ which may prevent f_{n-1} from being a ring map. If it is nonzero, we have

$$d_A(\phi_{n-1} \circ \tau) = d_A(\phi_{n-1} + \widetilde{P}(w_{n-1}) \circ (\beta \wedge \beta)) = d_A(\phi_{n-1}) + v_n = 0$$

This shows that $f_{n-1}\colon MU/w_{n-1} \to A^{\mathrm{op}}$ is a ring homomorphism. After replacing A by A^{op} if necessary, we may thus assume that all the $f_i\colon MU/w_i \to A$ are ring maps.

The obstruction to f_i commuting with f_j lies in $\pi_{|w_i|+|w_j|+2}(A)$. If i and j are different then at least one is strictly less than $n-1$; it follows that $|w_i| + |w_j| + 2 < 2^{n+1} - 2$ and thus that the obstruction group is zero. Thus f_i commutes with f_j when $i \neq j$, and we get a unique induced map $MU/I_n \to A$, as required. □

8. Point-set level foundations

In order to analyse the commutativity obstruction $\overline{c}(x)$ more closely and relate them to power operations, we need to recall some internal details of the EKMM category.

EKMM use the word "spectrum" in the sense defined by Lewis and May [15], rather than the sense we use elsewhere in this paper. They construct a category $\mathbb{LS}$ of "$\mathbb{L}$-spectra". This depends on a universe $\mathcal{U}$, but the functor $\mathcal{L}(\mathcal{U}, \mathcal{V}) \ltimes_{\mathcal{L}(U)} (-)$ gives a canonical equivalence of categories from $\mathbb{L}$-spectra over $\mathcal{U}$ to $\mathbb{L}$-spectra over $\mathcal{V}$, so the dependence is only superficial. (Here $\mathcal{L}(\mathcal{U}, \mathcal{V})$ is the space of linear isometries from $\mathcal{U}$ to $\mathcal{V}$.) We therefore take $\mathcal{U} = \mathbb{R}^\infty$. EKMM show that $\mathbb{LS}$ has a commutative and associative smash product $\wedge_{\mathcal{L}}$, which is not unital. However, there is a sort of "pre-unit" object S, with a natural map $S \wedge_{\mathcal{L}} X \to X$. They then define the subcategory $\mathcal{M} := \mathcal{M}_S = \{X \mid S \wedge_{\mathcal{L}} X = X\}$ of "S-modules", and prove that $S \wedge_{\mathcal{L}} S = S$ so that $S \wedge_{\mathcal{L}} X$ is an S-module for any X. We write $\wedge$ for the restriction of $\wedge_{\mathcal{L}}$ to $\mathcal{M}$.

We next give a brief outline of the properties of $\mathcal{M}$. Let $\mathcal{T}$ be the category of based spaces (all spaces are assumed to be compactly generated and weakly Hausdorff). We write 0 for the one-point space, or for the basepoint in any based space, or for the trivial map between based spaces.

We give $\mathcal{T}$ the usual Quillen model structure for which the fibrations are Serre fibrations. We write $h\mathcal{T}$ for the category with Hom sets $\pi_0 F(A,B) = \mathcal{T}(A,B)/\text{homotopy}$, and $\overline{h}\mathcal{T}$ for the category obtained by inverting the weak equivalences. We refer to $h\mathcal{T}$ as the strong homotopy category of $\mathcal{T}$, and $\overline{h}\mathcal{T}$ as the weak homotopy category.

The category $\mathcal{M}$ is a topological category: the Hom sets $\mathcal{M}(X,Y)$ are based spaces, and there are continuous composition maps

$$\mathcal{M}(X,Y) \wedge \mathcal{M}(Y,Z) \to \mathcal{M}(X,Z).$$

We again have a strong homotopy category $h\mathcal{M}$, with $h\mathcal{M}(X,Y) = \pi_0\mathcal{M}(X,Y)$; when we have defined homotopy groups, we will also define a weak homotopy category $\overline{h}\mathcal{M}$ in the obvious way.

$\mathcal{M}$ is a closed symmetric monoidal category, with smash product and function objects again written as $X \wedge Y$ and $F(X,Y)$. Both of these constructions are continuous functors of both arguments. The unit of the smash product is S.

There is a functor $\Sigma^\infty\colon \mathcal{T} \to \mathcal{M}$, such that

$$\begin{aligned}
\Sigma^\infty S^0 &= S \\
\Sigma^\infty(A \wedge B) &= \Sigma^\infty A \wedge \Sigma^\infty B \\
\mathcal{M}(\Sigma^\infty A \wedge X, Y) &= \mathcal{T}(A, \mathcal{M}(X,Y)) \\
\mathcal{M}(\Sigma^\infty A, \Sigma^\infty B) &= \mathcal{T}(A,B).
\end{aligned}$$

(For the last of these, see [8].)

The last equation shows that Σ^∞ is a full and faithful embedding of $\mathcal{T}$ in $\mathcal{M}$, so that all of unstable homotopy theory is embedded in the strong homotopy category $h\mathcal{M}$. In particular, $h\mathcal{M}$ is very far from Boardman's stable homotopy category $\mathcal{B}$. However, it turns out that the weak homotopy category $\overline{h}\mathcal{M}$ is equivalent to $\mathcal{B}$.

The definition of this weak homotopy category involves certain "cofibrant sphere objects" which we now discuss. It will be convenient for us to give a slightly more flexible construction than that used in [9], so as to elucidate certain questions of naturality. Let $\mathcal{U}$ be a universe. There is a natural way to make the Lewis-May spectrum $\Sigma^\infty\mathcal{L}(\mathcal{U},\mathbb{R}^\infty)_+$ into a $\mathbb{L}$-spectrum, using the action of $\mathcal{L}(1) = \mathcal{L}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ on $\mathcal{L}(\mathcal{U},\mathbb{R}^\infty)$ as well as on the suspension coordinates. One way to see this is to observe that $\Sigma^\infty\mathcal{L}(\mathcal{U},\mathbb{R}^\infty)_+ = \mathcal{L}(\mathcal{U},\mathbb{R}^\infty) \ltimes S^0$, where the S^0 on the right hand side refers to the sphere spectrum indexed on the universe $\mathcal{U}$. We then define $S(\mathcal{U}) = S \wedge_{\mathcal{L}} \Sigma^\infty\mathcal{L}(\mathcal{U},\mathbb{R}^\infty)_+$. This gives a contravariant functor $S\colon \{\text{Universes}\} \to \mathcal{M}$, and it is not hard to check that $S(\mathcal{U}) \wedge S(\mathcal{V}) = S(\mathcal{U} \oplus \mathcal{V})$.

Moreover, for any finite-dimensional subspace $U < \mathcal{U}$, there is a natural subobject $S(\mathcal{U},U) \leq S(\mathcal{U})$ and a canonical isomorphisms

$$\begin{aligned}
S(\mathcal{U},U) \wedge S(\mathcal{U},V) &= S(\mathcal{U} \oplus \mathcal{V}, U \oplus V) \\
\Sigma^U S(\mathcal{U}, U \oplus V) &= S(\mathcal{U},V).
\end{aligned}$$

This indicates that the objects $S(\mathcal{U},U)$ are in some sense stable. They can be defined as follows: take the Lewis-May spectrum $\Sigma^\infty_U S^0$ indexed on $\mathcal{U}$, and then take the twisted half smash product with the space $\mathcal{L}(\mathcal{U},\mathbb{R}^\infty)$ to get a Lewis-May spectrum indexed on $\mathbb{R}^\infty$ which is easily seen to be an $\mathbb{L}$-spectrum in a natural way. We then apply $S \wedge_{\mathcal{L}} (-)$ to get $S(\mathcal{U},U)$.

For any $n > 0$ and $d \geq 0$ we write

$$\begin{aligned}
\mathcal{L}(n) &:= \mathcal{L}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty) \\
S(n) &:= S((\mathbb{R}^\infty)^n) = S(1)^{(n)} \\
S^d(1) &:= \Sigma^d S(\mathbb{R}^\infty) \\
S^{-d}(1) &:= S(\mathbb{R}^\infty, \mathbb{R}^d)
\end{aligned}$$

We will also allow ourselves to write $S^d(n)$ for $\Sigma^k S((\mathbb{R}^\infty)^n, V)$ where V is a subspace of $(\mathbb{R}^\infty)^n$ of dimension $k-d$ and k and V are clear from the context.

Any object of the form $S^U \wedge S(\mathcal{V}, V)$ is non-canonically isomorphic to $S^d(1)$, where $d = \dim(U) - \dim(V)$, but when one is interested in the naturality or otherwise of various constructions it is often a good idea to forget this fact. There are isomorphisms $S^n(1) \wedge S^m(1) \simeq S^{n+m}(1)$ that become canonical and coherent in the homotopy category. The homotopy groups of an object $X \in \mathcal{M}$ are defined by

$$\pi_n(X) := h\mathcal{M}(S^n(1), X).$$

We say that a map $f\colon X \to Y$ is a weak equivalence if it induces an isomorphism $\pi_*(X) \to \pi_*(Y)$, and we define the weak homotopy category $\overline{h}\mathcal{M}$ by inverting weak equivalences. We define a cell object to be an object of $\mathcal{M}$ that is built from the sphere objects $S^n(1)$ in the usual sort of way; the category $\overline{h}\mathcal{M}$ is then equivalent to the category of cell objects and homotopy classes of maps.

Remark 8.1. In subsequent sections we will consider various spaces of the form $\mathcal{M}(S(1), X) = \Omega^\infty F_{\mathcal{L}}(S, X)$. This is weakly equivalent to $\Omega^\infty X$ but not homeomorphic to it; the functor $\Omega^\infty\colon \mathcal{M}_S \to \mathcal{T}$ is not representable and has rather poor behaviour. For this and many related reasons it is preferable to replace X by $F_{\mathcal{L}}(S, X)$ and thus work with EKMM's "mirror image" category $\mathcal{M}^S = \{Y \mid F_{\mathcal{L}}(S, Y) = Y\}$ rather than the equivalent category $\mathcal{M}_S$. However, our account of these considerations is still in preparation so we have used $\mathcal{M}_S$ in the present work.

Now let R be a commutative ring object in $\mathcal{M}$, in other words an object equipped with maps $S \xrightarrow{\eta} R \xleftarrow{\mu} R \wedge R$ making the relevant diagrams geometrically (rather than homotopically) commutative. (The term "ring" is something of a misnomer, as there is no addition until we pass to homotopy.) We let $\mathcal{M}_R$ denote the category of module objects over R in the evident sense. This is again a topological model category with a closed symmetric monoidal structure. The basic cofibrant objects are the free modules $S^d(1) \wedge R$ for $d \in \mathbb{Z}$. The weak homotopy category $\overline{h}\mathcal{M}_R$ obtained by inverting weak equivalences is also known as the derived category of R, and written $\mathcal{D} = \mathcal{D}_R$; it is equivalent to the strong homotopy category of cell R-modules. It is not hard to see that $\mathcal{D}$ is a monogenic stable homotopy category in the sense of [11]; in particular, it is a triangulated category with a compatible closed symmetric monoidal structure.

9. Strictly unital products

In the previous sections we worked in the derived category $\mathcal{D}$ of (strict) R-modules. In this section we sharpen the picture slightly by working with modules with strict units. These are not cell R-modules, so we need to distinguish between $\overline{h}\mathcal{M}_R(X, Y) := \mathcal{D}(X, Y) = [X, Y]$ and $h\mathcal{M}_R(X, Y) := \pi_0 \mathcal{M}_R(X, Y) = \mathcal{M}_R(X, Y)/\text{homotopy}$. Note that the latter need not have a group structure (let alone an Abelian one). However, most of the usual tools of unstable homotopy theory are available in $h\mathcal{M}_R$, because $\mathcal{M}_R$ is a topological category enriched over pointed spaces. In particular, we will need to use Puppe sequences.

As previously, we let x be a regular element in $\pi_d(R)$, so d is even. We regard x as an R-module map $S^d(1) \wedge R \to R$, and we write R/x for the cofibre. There is thus a pushout diagram

$$\begin{tikzcd} S^d(1) \wedge R \arrow[r] \arrow[d, "x"'] & I \wedge S^d(1) \wedge R \arrow[d] \\ R \arrow[r, "\rho"'] & R/x. \end{tikzcd}$$

As R is not a cell R-module, the same is true of R/x. However, the map $\rho\colon R \to R/x$ is a q-cofibration. One can also see that $S^0(1) \wedge R/x$ is a cell R-module which is the cofibre in $\mathcal{D}$ of the map $x\colon \Sigma^d R \to R$, so it has the homotopy type referred to as R/x in the previous section. Moreover, the map $S^0(1) \wedge R/x \to R/x$ is a weak equivalence. It follows that our new R/x has the same weak homotopy type as in previous sections.

Let W be defined by the following pushout diagram:

$$\begin{tikzcd} R \arrow[r, "\rho"] \arrow[d, "\rho"'] & R/x \arrow[d, "i_0"] \\ R/x \arrow[r, "i_1"'] & W \end{tikzcd}$$

There is a unique map $\nabla\colon W \to R/x$ such that $\nabla i_0 = 1 = \nabla i_1$, and there is an evident cofibration

$$S^{2d+1}(2) \wedge R \to W \to (R/x)^{(2)}.$$

Here

$$S^{2d+1}(2) = \Sigma S^d(1) \wedge S^d(1) = \begin{cases} \Sigma^{2d+1} S(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) & \text{if } d \geq 0 \\ \Sigma S(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^{|d|} \oplus \mathbb{R}^{|d|}) & \text{if } d < 0. \end{cases}$$

We define a *strictly unital product* on R/x to be a map $\phi\colon (R/x)^{(2)} \to R/x$ of R-modules such that $\phi|_W = \nabla$. Let P be the space of strictly unital products, and let $\overline{P}$ be the set of products on R/x in the sense of section 3.

Proposition 9.1. *The evident map $\pi_0(P) \to \overline{P}$ is a bijection.*

Proof. The cofibration

$$S^{2d+1}(2) \wedge R \xrightarrow{i} W \xrightarrow{j} (R/x)^{(2)}$$

gives a fibration $\mathcal{M}_R((R/x)^{(2)}, R/x) \xrightarrow{j^*} \mathcal{M}_R(W, R/x)$ of spaces. The usual theory of Puppe sequences and fibrations tells us that the image of j^* is the union of those components in $\pi_0\mathcal{M}_R((R/x)^{(2)}, R/x)$ that map to zero in $\pi_0\mathcal{M}_R(S^{2d+1}(2) \wedge R, R/x) = \pi_{2d+1}(R/x) = 0$, so j^* is surjective. In particular, we find that $P = (j^*)^{-1}\{\nabla\}$ is nonempty. Similar considerations then show that the H-space $H = \mathcal{M}_R(S^{2d+2}(1) \wedge R, R/x)$ acts on P, and that for any $\phi \in P$ the action map $h \mapsto h.\phi$ gives a weak equivalence $H \simeq P$. This shows that $\pi_0(H) = \pi_{2d+2}(R/x)$ acts freely and transitively on $\pi_0(P)$. This is easily seen to be compatible with our free and transitive action of $\pi_{2d+2}(R/x)$ on $\overline{P}$ (Lemma 3.7), and the claim follows. $\square$

Remark 9.2. These ideas also give another proof of associativity. Let Y be the union of all cells except the top one in $(R/x)^{(3)}$, so there is a cofibration $S^{3d+2}(3) \wedge R \to Y \to (R/x)^{(3)}$. Let ϕ be a product on R/x; by the proposition, we may assume that it is strictly unital. It is easy to see that $\phi \circ (\phi \wedge 1)$ and $\phi \circ (1 \wedge \phi)$ have the same restriction to Y (on the nose). It follows using the Puppe sequence that they only differ (up to homotopy) by the action of the group $\pi_{3d+3}(R/x) = 0$. Thus, ϕ is automatically associative up to homotopy.

We end this section with a more explicit description of the element $\overline{c}(x) \in \pi_{2d+d}(R)/(2, x)$. Define $X := \mathcal{M}(S^{2d}(2), R/x)$; this is a space with $\pi_k X = \pi_{2d+k}(R)/x$. The twist map τ of $S^{2d}(2) = S^d(1) \wedge S^d(1)$ gives a self-map of X, which we also call τ. Let y be the map

$$S^d(1) \wedge S^d(1) \xrightarrow{x \wedge x} R \wedge R \xrightarrow{\text{mult}} R \xrightarrow{\rho} R/x,$$

considered as a point of X. As R is commutative, this is fixed by τ.

Next, let $\gamma\colon I \wedge S^d(1) \to R/x$ be the obvious nullhomotopy of x, and consider the map

$$I \wedge S^d(1) \wedge S^d(1) \xrightarrow{\gamma \wedge x} R/x \wedge R \xrightarrow{\text{mult}} R/x.$$

This is adjoint to a path $\delta\colon I \to X$ with $\delta(0) = 0$ and $\delta(1) = y$. We could do a similar thing using $x \wedge \gamma$ to get another map $\delta'\colon I \to X$, but it is easy to see that $\delta' = \tau \circ \delta$. We now define a map $\phi_0\colon \partial(I^2) \to X$ by

$$\begin{aligned} \phi_0(s, 0) &= 0 \\ \phi_0(0, t) &= 0 \\ \phi_0(s, 1) &= \delta(s) \\ \phi_0(1, t) &= \tau\delta(t). \end{aligned}$$

We can use the pushout description of R/x to get a pushout description of $(R/x)^{(2)}$. Using this, we find that strictly unital products are just the same as maps $\phi\colon I^2 \to X$ that extend ϕ_0.

Let ϕ be such an extension. Let $\chi\colon I^2 \to I^2$ be the twist map; we find that $\phi' := \tau \circ \phi \circ \chi$ also extends ϕ_0 and corresponds to the opposite product on R/x. Let U be the space $(\{\pm 1\} \times I^2)/\sim$, where $(1,s,t) \sim (-1,s,t)$ if $(s,t) \in \partial(I^2)$; clearly this is homeomorphic to S^2. Define $\psi\colon U \to X$ by $\psi(1,s,t) = \phi(s,t)$ and $\psi(-1,s,t) = \phi'(s,t) = \tau\phi(t,s)$. It is not hard to see that the class in $\pi_2(X) = \pi_{2d+2}(R)/x$ corresponding to ψ is just $c(\phi)$, and thus that the image in $\pi_{2d+2}(R)/(2,x)$ is $\overline{c}(x)$.

Another way to think about this is to define a map $\tau\colon U \to U$ by $\tau(r,s,t) = (-r,t,s)$, and to think of I^2 as the image of $1 \times I^2$ in U. We can then say that ψ is the unique τ-equivariant extension of ϕ.

10. Power operations

In this section, we identify the commutativity obstruction $\overline{c}(x)$ of Proposition 3.1 with a kind of power operation. This is parallel to a result of Mironov in Baas-Sullivan theory, although the proofs are independent. We assume for simplicity that $d := |x| \geq 0$.

10.1. **The definition of the power operation.** Because R^* is concentrated in even degrees, we know that the Atiyah-Hirzebruch spectral sequence converging to $R^*\mathbb{C}P^\infty$ collapses and thus that R is complex orientable. We choose a complex orientation once and for all, taking the obvious one if R is (a localisation of) MU. This gives Thom classes for all complex bundles.

We write $R^{\mathrm{ev}}(X)$ for the even-degree part of $R^*(X)$, so that $R^{\mathrm{ev}}(\mathbb{R}P^2) = R^*[\epsilon]/(2\epsilon, \epsilon^2)$. (In the interesting applications the ring R^* has no 2-torsion and so $R^*\mathbb{R}P^2$ has no odd-degree part.)

We will need notation for various twist maps. We write ω for the twist map of $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$, or for anything derived from that by an obvious functor. Similarly, we write s for the twist map of $(\mathbb{R}^\infty)^2$, and $\sigma = S(s)$ for that of $S(2) = S(1) \wedge S(1) = S((\mathbb{R}^\infty)^2)$. We can thus factor the twist map τ of $S^{2d}(2)$ as $\tau = \omega\sigma = \sigma\omega$.

We will need to consider the bundle $V^d = \mathbb{R}^{2d} \times_{C_2} S^2$ over $S^2/C_2 = \mathbb{R}P^2$. Here C_2 is acting on $\mathbb{R}^{2d}$ by ω, and antipodally on S^2; the Thom space is $S^2_+ \wedge_{C_2} S^{2d}$. As d is even, we can regard V^d as $\mathbb{C} \otimes_{\mathbb{R}} V^{d/2}$, so we have a Thom class in $\widetilde{R}^{2d}(S^2_+ \wedge_{C_2} S^{2d})$ which generates $\widetilde{R}^{\mathrm{ev}}(S^2_+ \wedge_{C_2} S^{2d})$ as a free module over $R^{\mathrm{ev}}\mathbb{R}P^2 = R^*[\epsilon]/(2\epsilon, \epsilon^2)$.

Suppose that $x \in \pi_d(R)$. Recall that x is represented by a map $x\colon S^d(1) = S^d \wedge S(1) \to R$. By smashing this with itself and using the product structure of R we obtain a map $y\colon S^{2d}(2) \to R$. As R is commutative we have $y\tau = y$.

Because $S(\mathcal{U})$ is a continuous contravariant functor of $\mathcal{U}$, we have a map $\mathcal{L}(2)_+ \wedge S(1) \xrightarrow{\theta} S(2)$ and thus a map $\mathcal{L}(2)_+ \wedge S^{2d} \wedge S(1) \xrightarrow{y\theta} R$. If we let $s\colon (\mathbb{R}^\infty)^2 \to (\mathbb{R}^\infty)^2$ be the twist map and let C_2 act on $\mathcal{L}(2)$ by $g \mapsto g \circ s$ then $\mathcal{L}(2)$ is a model for EC_2 and thus $\mathcal{L}(2)/C_2 \simeq \mathbb{R}P^\infty$. As $y\tau = y$ we see that our map factors through $(\mathcal{L}(2)_+ \wedge_{C_2} S^{2d}) \wedge S(1) \simeq (\mathbb{R}P^\infty)^{V^d} \wedge S(1)$. For any CW complex A, the spectrum $A \wedge S(1)$ is a cofibrant approximation to $\Sigma^\infty A$, so we can regard this map as an element of $R^0(\mathbb{R}P^\infty)^{V^d}$. By restricting to $\mathbb{R}P^2$ and using the Thom isomorphism, we get an element of $R^{-2d}\mathbb{R}P^2$; we define $P(x)$ to be this element. We also recall that $R^{\mathrm{ev}}(\mathbb{R}P^2) = R^*[\epsilon]/(2\epsilon, \epsilon^2)$ and define $\widetilde{P}(x)$ to be the coefficient of ϵ in $P(x)$, so $\widetilde{P}(x) \in R^{-2d-2}/2 = \pi_{2d+2}(R)/2$. If A is a CW complex with only even-dimensional cells then we can replace R by $F(A_+, R)$ to get power operations $P\colon R^{-d}A \to R^{-2d}(\mathbb{R}P^2 \times A)$ and $\widetilde{P}\colon R^{-d}A \to R^{-2d-2}(A)/2$. It is not hard to check that this is the same as the more classical definition given in [7] and thus to deduce the properties listed at the beginning of Section 5.

We also need a brief remark about the process of restriction to $\mathbb{R}P^2$. The space of maps $\mu\colon S^2 \to \mathcal{L}(2)$ such that $\mu(-u) = \mu(u) \circ s$ is easily seen to be contractible. Choose such a map μ. We then have $(\mathbb{R}P^2)^{V^d} = S^2_+ \wedge_{C_2} S^{2d}$, and $P(x)$ is represented by the composite

$$(S^2_+ \wedge_{C_2} S^{2d}) \wedge S(1) \xrightarrow{\mu\wedge 1\wedge 1} (\mathcal{L}(2)_+ \wedge_{C_2} S^{2d}) \wedge S(1) \xrightarrow{y\theta} R.$$

We call this map β_0.

10.2. **A small modification.** Let M be the monoid $\mathcal{L}((\mathbb{R}^\infty)^2,(\mathbb{R}^\infty)^2)$. This acts contravariantly on $S(2)$, giving a map

$$(M_+ \wedge_{C_2} S^{2d}) \wedge S(2) \to S^{2d}(2) \xrightarrow{y} R.$$

Here we use the action of C_2 on M given by $g \mapsto g \circ s$. There is also a homotopically unique map $\lambda\colon S^2 \to M$ such that $\lambda(-u) = \lambda(u) \circ s$ for all $u \in S^2$. By combining this with the above map, we get a map

$$(S^2_+ \wedge_{C_2} S^{2d}) \wedge S(2) \to S^{2d}(2) \xrightarrow{y} R.$$

We call this map β_1. Recall that in the homotopy category there is a canonical isomorphism $S(1) \simeq S(2)$, so β_1 again represents an element of $R^0(\mathbb{R}P^2)^{V^d} = R^{-2d}\mathbb{R}P^2$. We claim that this is the same as $P(x)$. To see this, choose an isomorphism $v\colon (\mathbb{R}^\infty)^2 \simeq \mathbb{R}^\infty$, giving a map $v_*\colon M \to \mathcal{L}(2)$ and a map $S(v)\colon S(1) \to S(2)$. Take $v_* \circ \lambda\colon S^2 \to \mathcal{L}(2)$ as our choice of μ, and use $S(v)$ as a representative of the canonical equivalence $S(1) \simeq S(2)$ in the homotopy category; under these identifications, β_1 becomes β_0. We leave the rest of the details to the reader.

10.3. **An adjunction.** Define $Y := \mathcal{M}(S^{2d}(2), R)$. The twist maps ω, σ and τ induce commuting involutions of Y with $\tau = \omega\sigma$. We can think of $y\colon S^{2d}(2) \to R$ as a point of Y, which is fixed under τ. The contravariant action of M on $S(2)$ gives a covariant action on Y, which commutes with ω. Using this and our map $\lambda\colon S^2 \to M$ we can define a map $\beta_2\colon S^2_+ \to Y$ by $\beta_2(u) = \lambda(u).y$. If we let C_2 act on Y by ω then one finds that this is equivariant. We can think of β_2 as an adjoint of β_1 and thus a representative of $P(x)$.

10.4. **Mapping into R/x.** We are really only interested in the image of $\widetilde{P}(x)$ in $\pi_{2d+2}(R)/(2,x)$. To understand this, we reintroduce the space $X := \mathcal{M}(S^{2d}(2), R/x)$ as in Section 9. The unit map $\rho\colon R \to R/x$ induces an equivariant map $\rho_*\colon Y \to X$. We define $\beta_3 = \rho_* \circ \beta_2\colon S^2_+ \to X$.

Note that $\pi_0 Y = \pi_{2d} R$, and β_2 lands in the component corresponding to x^2, and $\rho(x^2) = 0$, so β_3 lands in the base component. Moreover, we have $\pi_1 X = \pi_{2d+1}(R)/x = 0$ and C_2 acts freely on S^2_+ so by equivariant obstruction theory we can extend β_3 over the cofibre of the inclusion $S^1_+ \to S^2_+$ to get a map β_4 say. This cofibre is equivariantly equivalent to $C_{2+} \wedge S^2$ and

$$[C_{2+} \wedge S^2, X]^{C_2} \simeq \pi_2(X) = \pi_{2d+2}(R)/x.$$

It is not hard to see that the element of $\pi_{2d+2}(R)/(2,x)$ coming from β_4 is just the image of $\widetilde{P}(x)$.

10.5. **The abstract argument.** We now set up an abstract situation in which we have a space X and we can define two elements $\alpha, \beta \in \pi_2(X)/2$ and prove that they are equal; later we apply this to show that $\overline{c}(x) = \widetilde{P}(x) \in \pi_{2d+2}(R)/(2,x)$. While this involves some repetition of previous constructions, we believe that it makes the argument clearer.

Let M be a 2-connected topological monoid, containing an involution σ. Let $C = \{1, \omega\}$ be the group of order two, and define $\tau = \sigma\omega \in C \times M$. Let X be a space with basepoint 0 and another distinguished point y in the base component. Suppose that $C \times M$ acts on X, the whole group fixes 0, and τ fixes y. Suppose also that $\omega \simeq 1\colon X \to X$ and $\pi_1(X) = 0$.

Definition 10.1. Write $V = \langle \sigma, \omega \rangle \simeq C_2^2$. Given $j, k \geq 0$ we can let V act on $\mathbb{R}^{j+k} = \mathbb{R}^j \oplus \mathbb{R}^k$ by $\omega = (-1) \oplus (-1)$ and $\sigma = 1 \oplus (-1)$ so $\tau = (-1) \oplus 1$. We write $\mathbb{R}^{j,k}$ for this representation of V, and $S^{j,k}$ for the sphere in $\mathbb{R}^{j,k}$, so that $S^{j,k} = S^{j+k-1}$ nonequivariantly.

Definition 10.2. Define a τ-equivariant map $\alpha_0\colon S^{0,1} = \{-1, 1\} \to X$ by $\alpha_0(-1) = 0$ and $\alpha_0(1) = y$. Using the evident τ-equivariant CW structure on $S^{2,1}$ and the fact that $\pi_1(X) = 0$ we find that there is an equivariant extension of α over $S^{2,1}$, which is unique modulo $1 + \tau^*$. Nonequivariantly we have $S^{2,1} = S^2$ and $\tau \simeq 1\colon S^2 \to S^2$ so we get a homotopy class of maps $S^2 \to X$, which is unique modulo 2. We write α for the corresponding element of $\pi_2(X)/2$.

Definition 10.3. Define $\lambda_0\colon S^{0,1} = \{-1, 1\} \to M$ by $\lambda_0(-1) = 1$ and $\lambda_0(1) = \sigma$; this is equivariant with respect to the evident right action of ω on M. As σ acts freely on $S^{2,1}$ and M is 2-connected, we see that there is a ω-equivariant extension $\lambda\colon S^{2,1} \to M$, which is unique up to equivariant homotopy.

Definition 10.4. Define $\beta\colon S^{2,1} \to X$ by $\beta(u) = \lambda(u).y$. As λ is σ-equivariant and y is fixed by $\tau = \omega\sigma$ and ω commutes with M we find that $\beta\sigma = \omega\beta\colon S^{2,1} \to X$. We next claim that β can be extended over the cofibre of the inclusion of $S^{2,0}$ in $S^{2,1}$ in such a way that we still have $\beta\sigma = \omega\beta$. This follows easily from the fact that σ acts freely on $S^{2,0}$ and y lies in the base component of X and $\pi_1(X) = 0$. The cofibre in question can be identified σ-equivariantly with $S^2 \wedge \{1, \sigma\}_+$. By composing with the inclusion $S^2 \to S^2 \wedge \{1, \sigma\}_+$ we get an element of $\pi_2(X)$. This can be seen to be unique modulo $1 + \omega_*$ but by hypothesis $\omega \simeq 1\colon X \to X$ so we get a well-defined element of $\pi_2(X)/2$, which we also call β.

Proposition 10.5. *We have $\alpha = \beta \in \pi_2(X)/2$.*

Proof. Consider the following picture of $\mathbb{R}^{2,1}$.

The axes are set up so that $N = (0, 0, 1)$ and $S = (0, 0, -1)$, so

$$\begin{aligned} \sigma(x, y, z) &= (-x, -y, -z) \\ \omega(x, y, z) &= (x, y, -z) \\ \tau(x, y, z) &= (-x, -y, z). \end{aligned}$$

We write H_+ and H_- for the upper and lower hemispheres and D for the unit disc in the plane $z = 0$. Thus $H_+ \cup H_- = S^{2,1}$ and $H_+ \cap H_- = S^{2,0}$, so $H_+ \cup H_- \cup D$ can be identified with the cofibre of the inclusion $S^{2,0} \to S^{2,1}$. Note also that $H_+ \cup D$ is τ-equivariantly homeomorphic to $S^{2,1}$ (by radial projection from the τ-fixed point $(0, 0, 1/2)$, say).

Let D' be the closed disc of radius $1/2$ centred at O and let A be the closure of $D \setminus D'$. Define $\alpha'\colon S^{2,1} \cup D' \to X$ by $\alpha' = y$ on $S^{2,1}$ and $\alpha' = 0$ on D'. We see by obstruction theory that α' can be extended τ-equivariantly over the whole of $S^{2,1} \cup D$. Moreover, if we identify $H_+ \cup D$ with $S^{2,1}$ as before then the restriction of α' to $H_+ \cup D$ represents the same homotopy class α as considered in Definition 10.2, as one sees directly from the definition.

Next, note that $S^{2,1} \cup A$ retracts σ-equivariantly onto $S^{2,1}$, so we can extend our map $\lambda\colon S^{2,1} \to M$ over $S^{2,1} \cup A$ equivariantly. As M is 1-connected, we can extend it further over the whole of $S^{2,1} \cup D$, except that we have no equivariance on D'.

Now define $\beta'\colon S^{2,1}\cup D\to X$ by $\beta'(u)=\lambda(u).\alpha'(u)$. We claim that $\beta'\sigma=\omega\beta'$. Away from D' this follows easily from the equivariance of λ and α', and on D' it holds because both sides are zero. Using this and our identification of $S^{2,1}\cup D$ with the cofibre of $S^{2,0}\to S^{2,1}$ we see that the restriction of β' to $H_+\cup D$ represents the class β in Definition 10.4.

Now observe that $S^{2,1}\cup D$ is 2-dimensional and M is 2-connected, so our map $\lambda\colon S^{2,1}\cup D\to M$ is nonequivariantly homotopic to the constant map with value 1. This implies that α' is homotopic to β', so $\alpha=\beta\in\pi_2(X)/2$ as claimed. □

10.6. **The proof that $\overline{c}(x)=\widetilde{P}(x)$.** We now prove that $\overline{c}(x)=\widetilde{P}(x)$. We take $X:=\mathcal{M}(S^{2d}(2),R/x)$ and $M:=\mathcal{L}((\mathbb{R}^\infty)^2,(\mathbb{R}^\infty)^2)$ as before, and define involutions ω, σ and τ as in Section 10.1. We also define y as in Section 10.3. It is then clear that the map β_4 of Section 10.4 represents the class β of Definition 10.4, so that $\beta=\widetilde{P}(x)\in\pi_{2d+2}(R)/(2,x)$. Now consider the constructions at the end of Section 9. It is not hard to see that the space U defined there is τ-equivariantly homeomorphic to $S^{2,1}$, with the two fixed points being $(0,0,1)$ and $(1,1,1)$. As the map $\psi\colon U\to X$ is equivariant and $\psi(0,0,1)=0$ and $\psi(1,1,1)=y$, we see that ψ represents the class α of Definition 10.2, so $\overline{c}(x)=\beta\in\pi_{2d+2}(R)/(2,x)$. It now follows from Proposition 10.5 that $\widetilde{P}(x)=\overline{c}(x) \pmod{2,x}$, as claimed.

References

[1] J. F. Adams. A variant of E. H. Brown's representability theorem. *Topology*, 10:185–198, 1971.

[2] J. F. Adams. *Stable Homotopy and Generalised Homology.* University of Chicago Press, Chicago, 1974.

[3] M. Ando. Isogenies of formal group laws and power operations in the cohomology theories E_n. *Duke Mathematical Journal*, 79(2):423–485, 1995.

[4] N. Baas. On bordism theory of manifolds with singularities. *Math. Scand.*, 33:279–302, 1973.

[5] E. H. Brown. Cohomology theories. *Annals of Mathematics*, 75:467–484, 1962.

[6] E. H. Brown and F. P. Peterson. A spectrum whose $\mathbf{Z}_p$ cohomology is the algebra of reduced p-th powers. *Topology*, 5:149–154, 1966.

[7] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ Ring Spectra and their Applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer–Verlag, 1986.

[8] A. D. Elmendorf. Stabilisation as a CW approximation. Available from `http://hopf.math.purdue.edu`, 1997.

[9] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, Modules and Algebras in Stable Homotopy Theory*, volume 47 of *Amer. Math. Soc. Surveys and Monographs*. American Mathematical Society, 1996.

[10] J. Franke. On the construction of elliptic cohomology. *Math. Nachr.*, 158:43–65, 1992.

[11] M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.

[12] M. Hovey and N. P. Strickland. Morava K-theories and localisation. To appear in the Memoirs of the American Mathematical Society, 1995.

[13] P. S. Landweber. Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$. *American Journal of Mathematics*, 98:591–610, 1976.

[14] L. G. Lewis. Is there a convenient category of spectra? *Journal of Pure And Applied Algebra*, 73:233–246, 1991.

[15] L. G. Lewis, J. P. May, and M. S. (with contributions by Jim E. McClure). *Equivariant Stable Homotopy Theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer–Verlag, New York, 1986.

[16] H. R. Margolis. *Spectra and the Steenrod Algebra.* North-Holland, 1983.

[17] H. R. Miller. Finite localizations. *Boletin de la Sociedad Matematica Mexicana*, 37:383–390, 1992. This is a special volume in memory of José Adem, and is really a book. The editor is Enrique Ramírez de Arellano.

[18] O. K. Mironov. Existence of multiplicative structures in the theory of cobordism with singularities. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(5):1065–1092,1219, 1975.

[19] O. K. Mironov. Multiplicativity in the theory of cobordism with singularities and the Steenrod-Tom Dieck operations. *Math USSR Izvestiya*, 13(1):89–107, 1979.

[20] J. Morava. A product for the odd primary bordism of manifolds with singularities. *Topology*, 18:177–186, 1979.

[21] C. Nassau. On the structure of $P(n)_*(P(n))$ for $p=2$. preprint, 1996.

[22] C. Nassau. *Eine nichtgeometrische Konstruktion der Spektren $P(n)$, Multiplikativen und antimultiplikativen Automorphismen von $K(n)$*. PhD thesis, Johann Wolfgang Goethe-Universität Frankfurt, October 1995.

[23] D. G. Quillen. On the formal group laws of unoriented and complex cobordism. *Bulletin of the American Mathematical Society*, 75:1293–1298, 1969.

[24] D. G. Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Mathematics*, 7:29–56, 1971.

[25] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, 1986.

[26] D. C. Ravenel. *Nilpotence and Periodicity in Stable Homotopy Theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, 1992.

[27] N. Shimada and N. Yagita. Multiplications in the complex bordism theory with singularities. *Publications of Research Institute of Mathematical Sciences, Kyoto University*, 12:259–293, 1976.

[28] T. tom Dieck. Steenrod-Operationen in Kobordismen-Theorien. *Math. Z.*, 107:380–401, 1968.

[29] W. S. Wilson. *Brown-Peterson Homology: An Introduction and Sampler*, volume 48 of *Regional Conference Series in Mathematics*. American Mathematical Society, 1982.

[30] J. J. Wolbert. Classifying modules over K-theory spectra. *J. Pure Appl. Algebra*, 124(1-3):289–323, 1998.

[31] U. Würgler. Cohomology theory of unitary manifolds with singularities and formal group laws. *Mathematische Zeitschrift*, 150:239–60, 1976.

[32] U. Würgler. On products in a family of cohomology theories associated to the invariant prime ideals of $\pi_*(BP)$. *Commentarii Mathematici Helvetici*, 52:457–81, 1977.

[33] U. Würgler. Commutative ring–spectra in characteristic 2. *Commentarii Mathematici Helvetici*, 61:33–45, 1986.

[34] Z.-i. Yosimura. Universal coefficient sequences for cohomology theories of cw-spectra. *Osaka J. Math.*, 12(2):305–323, 1975.

Trinity College, Cambridge CB2 1TQ, England

E-mail address: neil@dpmms.cam.ac.uk