Miller Spaces and Spherical Resolvability of Finite Complexes

Jeffrey Strom

Abstract

We show that if K is a nilpotent finite complex, then ΩK can be built from spheres using fibrations and homotopy (inverse) limits. This is applied to show that if $\mathrm{map}_*(X, S^n)$ is weakly contractible for all n, then $\mathrm{map}_*(\Sigma X, K)$ is weakly contractible for any nilpotent finite complex K.

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Discussion of Results

A **Miller space** is a CW complex X with the property that the space of pointed maps from X to K, written $\mathrm{map}_*(X, K)$, is weakly contractible for every nilpotent finite complex K. They are named for Haynes Miller, who proved in [12] that the spaces $B\,\mathbb{Z}/p$ are all Miller spaces; in fact, he proved that $\mathrm{map}_*(B\,\mathbb{Z}/p, K)$ is weakly contractible for every finite dimensional CW complex K.

In the stable category, one can define a Miller spectrum by requiring that the mapping spectrum $F(X, K) \simeq *$ for every finite spectum K. Since cofibrations and fibrations are the same in the stable category, a finite spectrum K with m cells is the fiber in a fibration $K \longrightarrow L \longrightarrow S^n$ in which L has only $m-1$ cells; in the terminology of [5, 11], this means that K is *spherically resolvable with weight* m. An easy induction shows that X is a Miller spectrum if and only if $F(X, S^n) \simeq *$ for every n.

Our goal is to prove the following unstable analog of this observation: if $\text{map}_*(X, S^n)$ is weakly contractible for all n, then ΣX is a Miller space. The proof of the stable version is not available to us because cofibrations are not fibrations, unstably. To prove our result, it is necessary to determine the extent to which a finite complex can be constructed from spheres in a more general way, i.e., by arbitrary homotopy (inverse) limits [3] and by extensions by fibrations.

To be more precise, we require some new terminology. We call a nonempty class $\mathcal{R}$ of spaces a **resolving class** if it is closed under weak equivalences and pointed homotopy (inverse) limits (all spaces and homotopy limits will be pointed). It is a **strong resolving class** if it is further closed under extensions by fibrations, i.e., if whenever $F \longrightarrow E \longrightarrow B$ is a fibration with $F, B \in \mathcal{R}$, then $E \in \mathcal{R}$.

Resolving classes are dual to closed classes as defined in [4] and [6, p. 45].

Notice that every resolving class $\mathcal{R}$ contains $*$ (cf. [6, p. 47]). From this, it follows that if $F \longrightarrow E \longrightarrow B$ is a fibration with $E, B \in \mathcal{R}$, then $F \in \mathcal{R}$. Similarly, if $A_\alpha \in \mathcal{R}$ for each α then the *categorical product* $\Pi_\alpha A_\alpha \in \mathcal{R}$ also. The *weak product* $\widetilde{\Pi}_\alpha A_\alpha$ is the homotopy colimit of the finite subproducts; if for each i only finitely many of the groups $\pi_i(A_\alpha)$ are nonzero, then the weak product has the same weak homotopy type as the categorical product.

Let $\mathcal{S}$ be the smallest resolving class that contains S^n for each n, and let $\overline{\mathcal{S}}$ be the smallest strong resolving class that contains S^n for each n. We say that a space K is **spherically resolvable** if $\Omega^k K \in \overline{\mathcal{S}}$ for some k. This concept is related to, but not the same as, the notion of spherical resolvability described in [5, 11]. We list some other important examples of (strong) resolving classes below.

Examples

(a) If $f : A \longrightarrow B$ is any map then the class of all f-local spaces is a resolving class [6, p. 5]. This includes, for example, the class of all spaces with $\pi_i(X) = 0$ for $i > n$, or all h_*-local spaces, where h_* is a homology theory.

(b) If P is a set of primes, then the class of all P-local spaces is a strong resolving class.

(c) If $f : W \longrightarrow *$, then the class of all f-local spaces is a strong resolving class [6, p. 5]. This includes, for example the class $\{K^+\}$, where K^+ denotes the Quillen plus construction on K [6, p. 27].

(d) More generally, if F is a covariant functor that commutes with homotopy limits (and fibrations) and $\mathcal{R}$ is a (strong) resolving class, then the class $\{K \mid F(K) \in \mathcal{R}\}$ is also a (strong) resolving class. This applies, for example to the functor $F(K) = \mathrm{map}_*(X, K)$.

(e) The class $\{K \mid K \text{ is weakly contractible}\}$ is a strong resolving class.

Our proofs will proceed by induction on a certain kind of cone length [1]. Let $\mathcal{F}$ denote the collection of all finite type wedges of spheres. The **$\mathcal{F}$-cone length** $\mathrm{cl}_{\mathcal{F}}(K)$ of a space K is the least integer n for which there are cofibrations $S_i \longrightarrow K_i \longrightarrow K_{i+1}$, $0 \le i < n$, with $K_0 \simeq *$, $K_n \simeq K$ and each $S_i \in \mathcal{F}$. If no such n exists, then $\mathrm{cl}_{\mathcal{F}}(K) = \infty$. Clearly every finite complex K has $\mathrm{cl}_{\mathcal{F}}(K) < \infty$.

With these preliminaries in place, we can state our main result.

Theorem 1 *If K is a nilpotent space with $\mathrm{cl}_{\mathcal{F}}(K) = n < \infty$, then*

(a) $\Omega K \in \overline{\mathcal{S}}$, *and*

(b) $\Omega^n K \in \mathcal{S}$.

In particular, every nilpotent finite complex K is spherically resolvable in our sense.

Our application to Miller spaces follows from the following more general consequence of Theorem 1.

Theorem 2 *Let $\mathcal{R}$ be a strong resolving class, and assume that X has the property that $\mathrm{map}_*(X, S^n) \in \mathcal{R}$ for each n. Then $\mathrm{map}_*(\Sigma X, K) \in \mathcal{R}$ for each nilpotent space K with $\mathrm{cl}_{\mathcal{F}}(K) < \infty$.*

This result has many corollaries; we list a few.

Corollary 3 *Let X be a space and P a set of primes.*

(a) *If* $\mathrm{map}_*(X, S^n)$ *is weakly contractible for all* n*, then* $\mathrm{map}_*(\Sigma X, K)$ *is weakly contractible for all nilpotent* K *with* $\mathrm{cl}_{\mathcal{F}}(K) < \infty$*. In other words,* X *is a Miller space.*

(b) *If* $\mathrm{map}_*(X, S^n)$ *is* P*-local for all* n*, then* $\mathrm{map}_*(\Sigma X, K)$ *is* P*-local for all nilpotent* K *with* $\mathrm{cl}_{\mathcal{F}}(K) < \infty$*.*

We end by making the surprising observation that a (non-nilpotent, of course) finite complex can be a Miller space!

Example Let A be a connected 2-dimensional acyclic finite complex. (The classifying space of the Graham-Higman group [7] is such a space; so is the space obtained by removing a point from a non-simply connected 3-dimensional Poincaré sphere). Since $\pi_1(A)$ is equal to its commutator subgroup, there are no nontrivial homomorphisms from $\pi_1(A)$ to any nilpotent group. It follows that if $f : A \longrightarrow K$ with K a nilpotent finite complex, then $\pi_1(f) = 0$ and so f factors through $q : A \longrightarrow A/A_1 \simeq \bigvee S^2$. Since $[A, S^2] \cong H^2(A) = 0$, we conclude $f \simeq *$. Thus A is a Miller space.

One of the key points in the proof of Theorem 1 is an explicit description of the homotopy fiber of a map $B \longrightarrow B \cup CA$, which we state as Proposition 5. In an appendix, we use this result to give simple proofs of the Blakers-Massey excision theorem [2] and the James splitting of $\Sigma\Omega\Sigma X$ [10].

I would like to thank Robert Bruner and Charles McGibbon for suggesting I think about Miller spaces. This work owes much to McGibbon in particular – Theorem 3(a) was conjectured in joint work with him. Thanks to Bill Dwyer for directing me to the result of [9], which is the key to Proposition 4; thanks are also due to Daniel Tanré for bringing Proposition 5 to my attention.

1 Proof of Theorem 1

We begin with two supporting results.

Proposition 4 *Let* K *be a connected nilpotent space, let* $\mathcal{R}$ *be a resolving class and let* F *be a functor that preserves fibrations and homotopy (inverse) limits over diagrams indexed on* Δ_n *and* L*. If* $F(\bigvee_{i=1}^m \Sigma K) \in \mathcal{R}$ *for each* m*, then* $F(K) \in \mathcal{R}$*.*

Proof This follows from a result of Hopkins [9, p. 222], which says that K is homotopy equivalent to the homotopy (inverse) limit of a tower

$$A^0 \longleftarrow A^1 \longleftarrow \cdots \longleftarrow A^n \longleftarrow \cdots$$

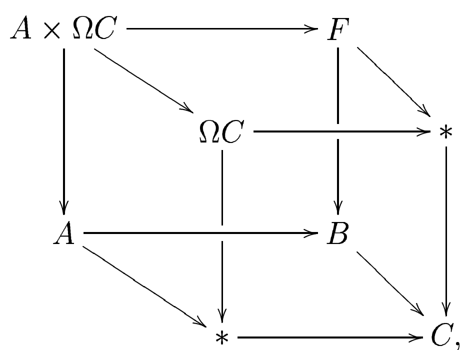
of spaces, each of which is a homotopy (inverse) limit over Δ_n of a diagram of spaces of the form $\bigvee_{i=1}^m \Sigma K$. □

It follows that to show that $\Omega^k K \in \mathcal{R}$ it suffices to show that $\Omega^k(\bigvee_{i=1}^m \Sigma K) \in \mathcal{R}$ for all m; more generally, if $F(\Omega^k(\bigvee_{i=1}^m \Sigma K)) \in \mathcal{R}$ for all m then $F(\Omega^k K) \in \mathcal{R}$.

Proposition 5 *Let $A \longrightarrow B \longrightarrow C$ be a cofibration, and let F be the homotopy fiber of $B \longrightarrow C$. Then*

$$\Sigma F \simeq \Sigma A \vee (\Sigma A \wedge \Omega C).$$

Proof Convert each of the maps $A \xrightarrow{*} C$, $B \longrightarrow C$ and $C \xrightarrow{=} C$ to a fibrations. The total spaces and fibers form the commutative diagram



in which the bottom square is a homotopy pushout. A result of V. Puppe [13] shows that the top square is also a homotopy pushout. Hence, the cofiber ΣF of the map $F \longrightarrow *$ has the same homotopy type as the cofiber of $A \times \Omega C \longrightarrow \Omega C$, namely $\Sigma A \vee (\Sigma A \wedge \Omega C)$, as can be seen from the diagram

$$\begin{array}{ccccc} A \times \Omega C & \longrightarrow & \Omega C & \longrightarrow & \Sigma F \\ \downarrow & \text{pushout} & \downarrow & & \| \\ A & \xrightarrow{*} & A * \Omega C & \longrightarrow & \Sigma A \vee (\Sigma A \wedge \Omega C). \end{array}$$

□

Proof of Theorem 1 Notice that the assumption on X implies that X is connected; we may therefore assume that K is also connected.

We prove both assertions in parallel, by induction on $\mathrm{cl}_{\mathcal{F}}(K)$. If $\mathrm{cl}_{\mathcal{F}}(K) = 1$ then K is homotopy equivalent to a connected finite type wedge of spheres. Then each $\bigvee_{i=1}^m \Sigma K$ is a simply-connected finite type wedge of spheres, and the Hilton-Milnor theorem [8, 14] shows that $\Omega(\bigvee_{i=1}^m \Sigma K)$ is homotopy equivalent to the weak product $\widetilde{\Pi}\Omega S^{n_\alpha} \in \mathcal{S}$. Since the wedge is of finite type and simply connected, all but finitely many factors are not i-connected for each i, so the weak product has the same homotopy type as the categorical product. Proposition 4 proves both assertions in the initial case.

Now assume that both statements are known for all nilpotent spaces with $\mathcal{F}$-cone length less than n, and that K is nilpotent with $\mathrm{cl}_{\mathcal{F}}(K) = n$. By Proposition 4, it is enough to show that $\Omega^n(\bigvee_{i=1}^m \Sigma^2 K) \in \mathcal{S}$ and $\Omega(\bigvee_{i=1}^m \Sigma^2 K) \in \overline{\mathcal{S}}$ for each m.

Write $V = \bigvee_{i=1}^m \Sigma^2 K$. Notice that $\mathrm{cl}_{\mathcal{F}}(\bigvee_{i=1}^m K) \leq \mathrm{cl}_{\mathcal{F}}(K)$, and the double suspension of an $\mathcal{F}$-cone decomposition of $\bigvee_{i=1}^m K$ is an $\mathcal{F}$-cone decomposition of V. Thus we may assume that V has an $\mathcal{F}$-cone decomposition $S_i \longrightarrow V_i \longrightarrow V_{i+1}$, $0 \leq i < n$ with each S_i and V_i simply-connected. Therefore, we have a cofibration $L \longrightarrow V \longrightarrow W$ with L simply-connected, $\mathrm{cl}_{\mathcal{F}}(L) < n$ and W a simply-connected finite type wedge of spheres.

Let F denote the homotopy fiber of $V \longrightarrow W$. Consider the fibration sequences

$$\Omega F \longrightarrow \Omega V \longrightarrow \Omega W \qquad \text{and} \qquad \Omega^n V \longrightarrow \Omega^n W \longrightarrow \Omega^{n-1} F.$$

Since $\Omega^k W \in \mathcal{S} \subseteq \overline{\mathcal{S}}$ for all $k > 0$, it suffices to show that $\Omega F \in \overline{\mathcal{S}}$ and that $\Omega^{n-1} F \in \mathcal{S}$.

Now we use Proposition 5 to determine the homotopy type of ΣF:

$$\Sigma F \simeq \Sigma L \vee (L \wedge \Sigma\Omega W) \simeq L \wedge \left(\bigvee_\alpha S^{n_\alpha}\right)$$

which is a finite type wedge of suspensions of L. If we smash an $\mathcal{F}$-cone length decomposition of L with the space $\bigvee_\alpha S^{n_\alpha}$ we obtain an $\mathcal{F}$-cone length decomposition for ΣF – in other words, $\mathrm{cl}_{\mathcal{F}}(\Sigma F) < n$ and, more importantly, $\mathrm{cl}_{\mathcal{F}}(\bigvee_{i=1}^l \Sigma F) < n$ for each l.

By the inductive hypothesis, $\Omega(\bigvee_{i=1}^{l} \Sigma F) \in \overline{\mathcal{S}}$ and $\Omega^{n-1}(\bigvee_{i=1}^{l} \Sigma F) \in \mathcal{S}$ for each l. Since L, V and W are each simply-connected, so is F, and Proposition 4 implies that $\Omega F \in \overline{\mathcal{S}}$ and that $\Omega^{n-1} F \in \mathcal{S}$, as desired. $\square$

2 Proof of Theorem 2 and Corollary 3

Proof of Theorem 2 Let $\mathcal{M}$ be the class of all spaces Y such that $\mathrm{map}_*(X, Y) \in \mathcal{R}$; we have already seen that $\mathcal{M}$ is a strong resolving class. Since $S^n \in \mathcal{M}$ for each n by assumption, it follows that $\overline{\mathcal{S}} \subseteq \mathcal{M}$. By Theorem 1, $\mathcal{M}$ contains ΩK for every nilpotent space K with $\mathrm{cl}_{\mathcal{F}}(K) < \infty$. $\square$

Proof of Corollary 3 Only statement 3 needs proof. Let $\mathcal{M}$ be the class of all spaces K for which $\mathrm{map}_*(X, \widehat{K})$ is weakly contractible. By assumption $S^n \in \mathcal{M}$ for each n. Since the Sullivan completion functor commutes with fibrations and homotopy (inverse) limits indexed on Δ_n and L,

3 Appendix: Applying Proposition 5

This section is essentially an advertisement for Proposition 5. Note that the proof of Proposition 5 uses only elementary results on homotopy pushouts and pullbacks. Nevertheless, it has many very important consequences, among which are elementary proofs of the Blakers-Massey excision theorem [2] and the James splitting of $\Sigma\Omega\Sigma X$ [10].

Theorem(Blakers-Massey) *Let $A \longrightarrow B \longrightarrow C$ be a cofibration of simply-connected spaces with A $(a-1)$-connected and C $(c-1)$-connected. If F is the homotopy fiber of $B \longrightarrow C$ then the natural map $A \longrightarrow F$ is an $(a+c-2)$-equivalence.*

Proof The assumptions clearly imply that F is simply-connected. The result follows from the fact that the map

$$\Sigma A \longrightarrow \Sigma F \simeq \Sigma A \vee (\Sigma A \wedge \Omega C),$$

is an $(a+c-1)$-equivalence. □

We use the notation $X^{(n)}$ for the n-fold smash product of X with itself.

Theorem(James) *If X is a connected space, then $\Sigma\Omega\Sigma X \simeq \Sigma \bigvee_n X^{(n)}$.*

Proof We apply Proposition 5 to the cofibration $X \longrightarrow * \longrightarrow \Sigma X$. The homotopy fiber of $* \longrightarrow \Sigma X$ is $\Omega\Sigma X$, so we conclude that

$$\begin{aligned} \Sigma\Omega\Sigma X &\simeq \Sigma X \vee (X \wedge (\Sigma\Omega\Sigma X)) \\ &\simeq \Sigma X \vee (\Sigma X \wedge X) \vee (X \wedge X \wedge (\Sigma\Omega\Sigma X))) \simeq \cdots. \end{aligned}$$

The result follows by induction, using the fact that the $X^{(n)}$ is at least $(n-1)$-connected. □

Finally, we observe that our proof of James' theorem reveals that the space $\Sigma(\bigvee_n X^{(n)})$ is a wedge summand of $\Sigma\Omega\Sigma X$, even when X is not connected.

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