The Lusternik-Schnirelmann Category of $S^1_{\mathbb{Q}} \times S^1$ and $S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}}$

JEFFREY STROM

Dartmouth College

Email: `Jeffrey.Strom@Dartmouth.edu`

Abstract

We answer a question of Rudyak by showing that $\mathrm{cat}(S^1_{\mathbb{Q}} \times S^1) = \mathrm{cat}(S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}}) = 3$. The second formula shows that $X = S^1_{\mathbb{Q}}$ is an example of a space for which $\mathrm{cat}(X \times X) < 2\,\mathrm{cat}(X)$. These calculations are derived from a general formula for the category weight of elements of $H^*(BG;\pi)$ that is of independent interest.

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The Lusternik-Schnirelmann **category** of a map $f : X \longrightarrow Y$ between CW complexes is the least integer n for which X has a cover $\{A_0, \ldots, A_n\}$ by subcomplexes with the property that $f|_{A_i} \simeq *$ for each i [1]. The category of a space X, $\mathrm{cat}(X)$, is the category of the identity map id_X. A classical result due to Bassi [4, Thm. 9] shows that, for two CW complexes X and Y, $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$. Until recently, the only known cases in which this formula was not an equality involved torsion phenomena in homology. In [5], Ganea asked whether $\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$ for every space X. That this is true has come to be known as Ganea's conjecture. The rational version of the conjecture, which can be stated $\mathrm{cat}(X_{\mathbb{Q}} \times S^k_{\mathbb{Q}}) = \mathrm{cat}(X_{\mathbb{Q}}) + 1$ when $X \times S^k$ is simply connected, was proved for simply connected spaces by Jessup and Hess in [10, 7] (we denote by $X_{\mathbb{Q}}$ the rationalization of the space X). More recently, counterexamples to the conjecture have been constructed by Iwase [8], by Stanley [17], and others. In the wake of these counterexamples, there has been some interest in the related problem of finding space X such that $\mathrm{cat}(X \times X) < 2\,\mathrm{cat}(X)$ [12].

In this note we use category weight techniques to compute the Lusternik-Schnirelmann category of the spaces $S^1_{\mathbb{Q}} \times S^1$ and $S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}}$, thereby answering a question asked by Y. Rudyak [15]. The calculation shows that $S^1_{\mathbb{Q}}$ satisfies the Ganea conjecture, and also has the property that $\mathrm{cat}(X \times X) < 2\,\mathrm{cat}(X)$.

Recall that the **category weight** of a map $f : X \longrightarrow Y$ is the least integer n such that $f \circ g \simeq *$ for every $g : Z \longrightarrow X$ with $\mathrm{cat}(g) < n$. We write $\mathrm{wgt}(f) = n$ if the category weight of f is n, and $\mathrm{wgt}(f) = \infty$ if there is no such integer. The category weight of a cohomology class is obtained from the isomorphism $H^n(X;\pi) \cong [X, K(\pi, n)]$. Clearly $\mathrm{wgt}(f)$ is a lower bound for $\mathrm{cat}(X)$, provided f is nontrivial. This concept has also been called *strict category weight* (see Rudyak [13, 14, 11]) or *essential category weight* [18, 19].

We will make use of a well-known alternative characterization of the category of a map $f : X \longrightarrow Y$. For each space Y and $n \geq 0$ there is a fibration $p_n : G_n(Y) \longrightarrow Y$ with the property that $\mathrm{cat}(f) \leq n$ if and only if f has a lift into $G_n(Y)$ [6]; these are known as the Ganea fibrations. If $X \simeq BG$ for some discrete group G then the map p_n is homotopically equivalent to the inclusion $B_nG \hookrightarrow BG \simeq X$ [20], see also [18]. The following result is a basic property of category weight [13, 14, 18, 19].

Theorem 1 *Let $f : X \longrightarrow Y$, and let $p_n : G_n(X) \longrightarrow X$ be the n^{th} Ganea fibration. Then $\mathrm{wgt}(f) \geq n$ if and only if $f \circ p_n \simeq *$.*

This follows immediately from the definitions. The following corollary, which can be found in [18, Cor. 79] and implicitly in the proof of [11, Thm. 4.1], has proved useful in differential geometry; in fact, it is an essential ingredient in the proof of the Arnold conjecture for certain special symplectic manifolds [11, 16].

Corollary 2 *If G is a discrete group and $u \in H^n(BG;\pi)$, then $\mathrm{wgt}(u) = n$.*

Proof Since G is a discrete group, $G_n(BG) \simeq B_nG$ which is $(n-1)$-dimensional, and hence has trivial cohomology in dimensions $\geq n$. □

This corollary generalizes to finite-dimensional groups: if G is a d-dimensional topological group and $u \in H^n(BG;\pi)$, then $\mathrm{wgt}(u) \geq \frac{n}{d+1}$. This shows, for example, that every nonzero class $u \in H^{4n}(\mathbb{HP}^m;\pi)$ has $\mathrm{wgt}(u) = n$, even without a cup product structure.

Corollary 2 immediately implies a result of Eilenberg and Ganea: if G is a discrete group then the category of BG is bounded below by its cohomological dimension [2]. Since $S^1_{\mathbb{Q}}$ is a $K(\mathbb{Q},1)$ and $H^2(S^1_{\mathbb{Q}};\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Q},\mathbb{Z})$ which is isomorphic to $\mathbb{R}$ as rational vector spaces, it follows that $\mathrm{cat}(S^1_{\mathbb{Q}}) \geq 2$; since $S^1_{\mathbb{Q}}$ is homotopy equivalent to a 2-dimensional space we see that $\mathrm{cat}(S^1_{\mathbb{Q}}) = 2$. Therefore, Ganea's conjecture predicts

$$\mathrm{cat}(S^1_{\mathbb{Q}} \times S^1) = \mathrm{cat}(S^1_{\mathbb{Q}}) + 1 = 3.$$

Since $H^*(S^1_{\mathbb{Q}})$ is torsion free, the general product formula which motivated Ganea's conjecture predicts that $\mathrm{cat}(S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}})$ is equal to 4.

With these preliminaries in place, we state and prove our main theorem.

Theorem 3 $\mathrm{cat}(S^1_{\mathbb{Q}} \times S^1) = \mathrm{cat}(S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}}) = 3$.

Proof Notice first that it follows from Bassi's formula that $\mathrm{cat}(S^1_{\mathbb{Q}} \times S^1) \leq 3$, and since $S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}} \simeq (S^1 \times S^1)_{\mathbb{Q}}$ can be constructed as a 3-dimensional CW complex, $\mathrm{cat}(S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}}) \leq 3$ as well. Now we have from the universal coefficient formula

$$H^3(S^1_{\mathbb{Q}} \times S^1;\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \otimes \mathbb{Z} \cong \mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \cong \mathbb{R} \neq 0$$

and

$$H^3(S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}};\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \cong \mathbb{R} \neq 0.$$

Since $S^1_{\mathbb{Q}} \times S^1 \simeq K(\mathbb{Q} \times \mathbb{Z}, 1)$ and $S^1_{\mathbb{Q}} \times S^1_{\mathbb{Q}} \simeq K(\mathbb{Q} \times \mathbb{Q}, 1)$, the result follows immediately from Corollary 2. $\square$

Remark In fact, a similar argument shows that $\mathrm{cat}((S^1_{\mathbb{Q}})^n) = n + 1 < 2n$.

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