Essential Category Weight[1]

Jeffrey A. Strom
University of Wisconsin, Madison
strom@math.wisc.edu

Introduction

Everybody's favorite theorem on Lusternik-Schnirelmann category is Eilenberg's cup product theorem: if there is a nonzero N-fold cup product in $\widetilde{h}^*(X)$, then $\mathrm{cat}(X) > N$. Unfortunately, this theorem often fails to detect the full category of a space X; this is the case, for example, when X is a lens space L_p (with p odd) or $X = Sp(n)$.

In [4], Fadell and Husseini introduced the concept of the *category weight* of a cohomology class, written $\mathrm{cwgt}(u)$. The key properties of category weight were that $1 \leq \mathrm{cwgt}(u) < \mathrm{cat}(X)$, and that $\mathrm{cwgt}(u_1 \cdots u_n) \geq \sum \mathrm{cwgt}(u_i)$. Thus, Eilenberg's theorem follows as the trivial case of the product formula, and much better lower bounds on $\mathrm{cat}(X)$ are made possible. One of the main theorems of [4] shows the effect of cohomology operations on category weight. As a consequence, they show that there are indecomposable classes with $\mathrm{cwgt}(u) > 1$.

In this paper, we generalize category weight to arbitrary maps $f : X \longrightarrow Y$, and expand the results of [4] and [11]. We also observe in the first two sections that $\mathrm{cwgt}(f)$ (or $\mathrm{cwgt}(u)$) is not a satisfactory concept in one important respect: it can be altered dramatically by composition with a homotopy equivalence. This makes $\mathrm{cwgt}(f)$ very difficult to compute..

We introduce a new concept, the *essential category weight* of a map $f : X \longrightarrow Y$ ($E(f)$) which also satisfies $1 \leq E(f) < \mathrm{cat}(X)$ and a product formula, and which is preserved by homotopy equivalences.

This paper is the beginning of a theory of (essential) category weight. This theory unifies and strengthens previous results on Lusternik-Schnirelmann category and leads to new theorems. The main results of the general theory are Theorem 3.7 and Theorem 5.1. Theorem 3.7 is extremely important when considering the effect of cohomology operations.

Theorem 3.7 If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, then

$$\mathrm{cwgt}(g \circ f) \geq E(g) \cdot \mathrm{cwgt}(f),$$

and

$$E(g \circ f) \geq E(g) \cdot E(f).$$

There is a corresponding statement for cohomology classes, in which f represents the cohomology class u, and g represents the cohomology operation θ:

Theorem 4.2 If θ and u are as above, then

$$\mathrm{cwgt}(\theta(u)) \geq E(\theta) \cdot \mathrm{cwgt}(u)$$

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and

$$E(\theta(u)) \geq E(\theta) \cdot E(u).$$

In light of this result, we see that the conclusion of Theorem 3.12 of [4] can be changed from $\mathrm{cwgt}(\theta(u)) \geq 2$ to $\mathrm{cwgt}(\theta(u)) \geq 2 \cdot \mathrm{cwgt}(u)$, clearly a substantial improvement.

Theorem 5.1 Let $f : X \longrightarrow K$ and $g : X \longrightarrow L$. Let $\wedge : K \times L \longrightarrow K \wedge L$ be the quotient map. Then

$$E(\wedge \circ (f \times g)) \geq E(f) + E(g),$$

and

$$\mathrm{cwgt}(\wedge \circ (f \times g) \circ d) \geq \mathrm{cwgt}(f) + \mathrm{cwgt}(g).$$

Many of the most useful lower bounds on $\mathrm{cat}(X)$ are of the product formula type. The basic examples of this kind of result are Eilenberg's cup product theorem, Whitehead's theorem that $\mathrm{cat}(X)$ is bounded from below by the nilpotence class of $[X, G]$ (Theorem 2.10 of [16]), and Steenrod's theorem that $\mathrm{cat}(X) > N$ if θ is a normal cohomology operation of N variables and $\theta(u_1, \ldots, u_N) \neq 0$ (see [12]).

Theorem 5.1 will be seen to be the common ancestor of all of the product formula type lower bounds on $\mathrm{cat}(X)$. In fact, each of the theorems mentioned above may be derived from a corollary of Theorem 5.1 by making the trivial observation that if $f \not\simeq *$, then $\mathrm{cwgt}(f) \geq 1$. If it is known that $\mathrm{cwgt}(f_i) > 1$ in any of these situations, then substantially better lower bounds follow immediately. It is here that Theorems 3.7 and 4.2 are particularly useful.

Having established the usefulness of essential category weight for unifying and extending the existing theory, we also show how to obtain new results, by applying essential category weight to Ganea's conjecture. An easy application of Theorem 5.1 yields the following useful result.

Proposition 6.1 Let X be a space with $\mathrm{cat}(X) = N$. If there is a map $f : X \longrightarrow Y$ with $E(f) = N - 1$, and $S^n f \not\simeq *$, then

$$\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$$

for every $0 < k \leq n$.

This leads directly to the following partial solution to Ganea's conjecture, which first appeared as Theorem 3.2 of [14].

Theorem 6.3 Suppose X is n-dimensional, $(p-1)$-connected and $\mathrm{cat}(X) = N$. If

$$N = \left\lfloor \frac{n}{p} \right\rfloor + 1$$

and $n \not\equiv -1 \bmod p$, then

$$\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$$

for all $k > 0$.

The proof amounts to showing that under the conditions imposed, $\widehat{d}_{N-1} : X \longrightarrow X^{(N-1)}$ is stably nontrivial; it is easy to see that $E(\widehat{d}_{N-1}) \geq N$ if $\widehat{d}_N \not\simeq *$.

I would like to thank Sufian Husseini, Edward Fadell and Doug Lepro, who listened to many of these arguments as they were forming. Thanks also to Professor Husseini for suggesting that I study category weight.

1 Background

In this section we establish some notation and recall some results which will figure prominently in this paper.

1.1 Notation

Spaces are pointed, and have the pointed homotopy type of CW complexes. The set of pointed homotopy classes of maps $f : X \longrightarrow Y$ is denoted $[X, Y]$. Basepoints are always denoted by $*$. The constant map is also denoted $*$.

The (reduced) mapping cylinder and mapping cone of the map f are denoted M_f, and C_f respectively. The (reduced) suspension of X is SX, the suspension of f is Sf.

The suspension spectrum of X is denoted $\Sigma(X)$, the image of a map under the suspension spectrum functor is denoted $\Sigma(f)$.

The diagonal map is denoted $d_N : X \longrightarrow X^N$ (or simply d). The map $\wedge : X \times Y \longrightarrow X \wedge Y$ is the canonical quotient map. The composition $\wedge \circ f$ is denoted $\widehat{f}$. The projections from a product are denoted $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$.

The notation $H^*(X)$ means ordinary cohomology with arbitrary coefficients. The dimension of u as a cohomology class is denoted $|u|$ (that is, $|u| = -\text{degree}(u)$).

The greatest integer less than or equal to x is denoted $\lfloor x \rfloor$; the least integer greater than or equal to x is denoted $\lceil x \rceil$.

1.2 Lusternik-Schnirelmann Category

In this section, we recall some basic facts concerning Lusternik-Schnirelmann category. See [10] for a more complete survey.

Definition Let X be a CW complex, and let $f : X \longrightarrow Y$ be a map. The *category* of f, denoted $\text{cat}(f)$, is the least integer N such that

$$X = X_1 \cup \cdots \cup X_N$$

where each X_i is a subcomplex of X, and $f|_{X_i} \simeq *$.

Clearly, if $f \simeq g$, then $\text{cat}(f) = \text{cat}(f')$; if $g : Y \longrightarrow Z$, then $\text{cat}(g \circ f) \leq \text{cat}(f)$. If g is a homotopy equivalence, then $\text{cat}(g \circ f) = \text{cat}(f)$, for

$$\text{cat}(f) = \text{cat}(g^{-1} \circ (g \circ f)) \leq \text{cat}(g \circ f).$$

Definition Let $i : A \hookrightarrow X$ be the inclusion of a subcomplex. The *category* of A in X is $\text{cat}_X(A) = \text{cat}(i)$. We write $\text{cat}(X) = \text{cat}_X(X)$.

It is equivalent to say $\text{cat}_X(A)$ is the least integer N such that

$$A = A_1 \cup \cdots \cup A_N,$$

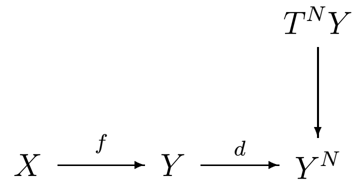
where each A_i is a subcomplex of X, and A_i is contractible to a point in X.

Write

$$T^N X = \{(x_1, \ldots, x_N) \in X^N \mid \text{at least one } x_i = *\}.$$

The following result is due to Berstein and Ganea (see Proposition 1.8 of [2]).

Proposition 1.1 Let $f : X \longrightarrow Y$. Then $\mathrm{cat}(f) \leq N$ if and only if there is a lift (up to homotopy) in the diagram



The following upper bound on $\mathrm{cat}(f)$ follows easily from the cellular approximation theorem (see Theorem 6.8 of [15]).

Proposition 1.2 If $f : X \longrightarrow Y$ where Y is $(p-1)$-connected, and X is n-dimensional, then $\mathrm{cat}(f) \leq \left\lfloor \frac{n}{p} \right\rfloor + 1$. In particular, if $A \subseteq X$, then

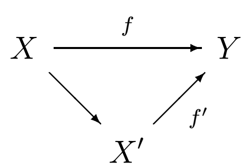
$$\mathrm{cat}_X(A) \leq \left\lfloor \frac{\dim(A)}{p} \right\rfloor + 1.$$

There is a related concept, which plays an important role in this paper.

Definition Let X be a CW complex. The *geometric category* of X is the least integer N such that $X = X_1 \cup \cdots \cup X_N$ where each X_i is a contractible subcomplex. We write $\mathrm{gcat}(X) = N$.

The usefulness of gcat comes from the following result, due to Berstein and Ganea (see Propostion 1.7 of [2]).

Theorem 1.3 Let $f : X \longrightarrow Y$. Then $\mathrm{cat}(f) \leq N$ if and only if f factors as in the diagram



in which $\mathrm{gcat}(X') \leq N$.

1.3 Spectra

Many of our applications use cohomology. We will state our results for cohomology classes in the greatest generality possible, and state results for ordinary cohomology as corollaries or examples. Therefore, we must give a brief account of generalized cohomology theories and spectra. See [1] or [15] for definitions and details.

For any spectrum F, there is a (reduced) cohomology theory defined on spectra by $F^*E = [E, F]$. If X is a CW complex, we define

$$\widetilde{F}^*(X) = F^*(\Sigma(X)) = [\Sigma(X), F]$$

where $\Sigma(X)$ is the suspension spectrum of X. The Brown Representation Theorem states that every cohomology theory arises in this way (see Theorem 9.12 of [15]).

Theorem 1.4 Let $\widetilde{h}^*$ be a reduced cohomology theory. Then there is a loop spectrum L and a class $\iota \in h^*(L)$ so that the natural transformation

$$\widetilde{L}^*(E) = [E, L] \xrightarrow{\epsilon} \widetilde{h}^*(E)$$

given by $f \mapsto f^*\iota$ is an isomorphism for every spectrum E.

Theorem 1.5 Any spectrum is equivalent to a loop spectrum. If $E = \Sigma(X)$ and F is a loop spectrum, then

$$\widetilde{F}^n(X) = [\Sigma(X), F]_{-n} \cong [X, F_n].$$

The inclusion $i_n : \Sigma(L_n) \hookrightarrow L$ gives us classes $\iota_n = i_n^*(\iota)$, and the map

$$[X, L_n] \xrightarrow{\epsilon_n} L^n(X)$$

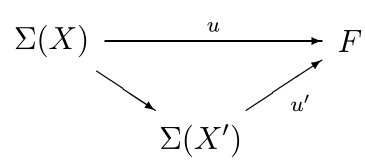
given by $f \mapsto f^*(\iota_n)$ is an isomorphism.

This observation can be used to reduce questions about spectra to questions about spaces.

Definition The *category* of a map $u : \Sigma(X) \longrightarrow F$ is the least N such that $X = X_1 \cup \cdots \cup X_n$, where each X_i is a subcomplex of X, and $u|_{\Sigma(X_i)} = 0$.

Lemma 1.6 Let $u : \Sigma(X) \longrightarrow F$, and let L be a loop spectrum equivalent to F. Suppose $u = \epsilon(f)$. Then $\mathrm{cat}(u) = \mathrm{cat}(f)$.

Corollary 1.7 $\mathrm{cat}(u) \leq N$ if and only if u factors as in the diagram



where $\mathrm{gcat}(X') \leq N$.

2 Category Weight

In the first part of this section, we give the definition of the *category weight* of a map $f : X \longrightarrow Y$, and prove some basic properties. In the second part, we give a procedure for finding maps $f : X \longrightarrow Y$ with large category weight.

2.1 Definition and Basic Properties

In this section, we give the definitions of the category weight of a map of spaces or spectra. We also give some examples and easy estimates. The category weight of a cohomology class first appeared in [4], and has been further studied in [11]. The category weight of a map is a simple generalization of their idea.

Definition Let X be a CW complex, and let $f : X \longrightarrow Y$ with $f \not\simeq *$. The *category weight* of f ($\mathrm{cwgt}(f)$) is the largest integer N such that $f|_A \simeq *$ whenever $\mathrm{cat}_X(A) \leq N$.

Notice that we do *not* require that Y be a CW complex.

To talk about the category weight of a map, the map must first be nontrivial. Whenever the symbol $\mathrm{cwgt}(f)$ appears, we are tacitly assuming that $f \not\simeq *$.

Clearly, it is always true that

$$1 \leq \mathrm{cwgt}(f) < \mathrm{cat}(X).$$

It is easy to see that if $f \simeq g$, then $\mathrm{cwgt}(f) = \mathrm{cwgt}(g)$.

We can also define the category weight of a cohomology class. Since any cohomology theory is represented by a spectrum F, it is equivalent to define the category weight of a map $u : \Sigma(X) \longrightarrow F$, where F is an arbitrary spectrum. (The difficulty with defining $\mathrm{cwgt}(u)$ for a map $u : E \longrightarrow F$ of arbitrary spectra is that there is no obvious way to define the $\mathrm{cat}_E(A)$ where A is a subspectrum of E.)

Definition Let X be a CW complex and let $u : \Sigma(X) \longrightarrow F$ be a (nonzero) map where F is an arbitrary spectrum. The *category weight* of u is the largest integer N such that $u|_{\Sigma(A)} = 0$ whenever $\mathrm{cat}_X(A) \leq N$.

Just as above, when we write $\mathrm{cwgt}(u)$, we are assuming $u \neq 0$. Also,

$$0 \leq \mathrm{cwgt}(u) \leq \mathrm{cat}(X) - 1,$$

and $\mathrm{cwgt}(u) = 0$ if and only if $u \notin \widetilde{h}^*(X)$. For this reason, we will usually consider classes $u \in \widetilde{h}^*(X)$.

Category weight gives us a new method of computing lower bounds on the category of a CW complex X. If $f : X \longrightarrow Y$ and $\mathrm{cwgt}(f) = N$, then $\mathrm{cat}(X) > N$; similarly for cohomology classes. Of course, this begs the question of how to find such maps and classes.

It is possible to define category weight of elements of $F(X)$, where $F : CW \longrightarrow Set_*$ is any contravariant homotopy functor from the category of CW complexes to the category of pointed sets, as follows. If $\alpha \in F(X)$, then $\mathrm{cwgt}(\alpha)$ is the largest integer N such that $F(i)(u) = *$ whenever $i : A \hookrightarrow X$ and $\mathrm{cat}_X(A) \leq N$. It is clear that when F is a cohomology theory or the functor $F = [-, Y]$, this definition reduces to the ones given above.

Many of our results could be proved for very general functors of this type, but in all of our examples, the functor F can be expressed $F(X) = [X, Y]$ for an appropriate space Y. Therefore, we will state our results in terms of maps.

Example 2.1 Let h^* be a cohomology theory with products. For any $u_i \in \widetilde{h}^*(X)$,

$$u_1 \cdots u_N|_A = 0$$

for any $A \subseteq X$ with $\mathrm{cat}_X(A) \leq N$, by Eilenberg's theorem. Therefore, $\mathrm{cwgt}(u_1 \cdots u_N) \geq N$.

It is possible to express the category weight of a map $\Sigma(X) \longrightarrow E$ in terms of category weight of maps of spaces.

Proposition 2.2 Let F be a spectrum, and suppose L is a loop spectrum equivalent to F. Let $\iota_n \in \widetilde{F}^n(L_n)$ be as in Theorem 1.5. If $u = f^*\iota_n$, then $\mathrm{cwgt}(f) = \mathrm{cwgt}(u)$.

Proof
First suppose $\mathrm{cwgt}(f) = N$, and let $i_A : A \hookrightarrow X$, with $\mathrm{cat}_X(A) \leq N$. The diagram

$$\begin{array}{ccc} [X, L_n] & \xrightarrow[\cong]{\epsilon} & \widetilde{F}^n(X) \\ \downarrow {\scriptstyle i_A^{\#}} & & \downarrow {\scriptstyle i_A^*} \\ [A, L_n] & \xrightarrow[\cong]{\epsilon} & \widetilde{F}^n(A) \end{array}$$

commutes, so

$$i_A^*(u) = i_A^*(\epsilon(f)) = \epsilon(i_A^{\#}(f)) = \epsilon(f \circ i_A) = \epsilon(*) = 0.$$

Thus, $u|_A = 0$, and so $\mathrm{cwgt}(u) \geq \mathrm{cwgt}(f)$.

Now suppose that $\mathrm{cwgt}(u) = N$, and let $A \subseteq X$ with $\mathrm{cat}_X(A) \leq N$. Then

$$0 = i_A^*(u) = \epsilon(i_A^{\#}(f)) = \epsilon(f \circ i_A).$$

Since ϵ is an isomorphism, it's injective, and so $f \circ i_A \simeq *$. ∎

It is natural to ask about the category weight of a composition; indeed, this is the motivation for most of the results in this paper. The following is our first result in this direction.

Proposition 2.3 Let $f : X \longrightarrow Y$, and $g : Y \longrightarrow Z$, and $i : A \hookrightarrow X$. Then
1. $\mathrm{cwgt}(g \circ f) \geq \mathrm{cwgt}(f)$;
2. $\mathrm{cwgt}(f \circ i) \geq \mathrm{cwgt}(f)$.

The analagous results hold for maps of spectra. If $u : \Sigma(X) \longrightarrow F$ and $v : F \longrightarrow E$, then

1′. $\mathrm{cwgt}(v \circ u) \geq \mathrm{cwgt}(u)$;
2′. $\mathrm{cwgt}(u \circ \Sigma(i)) \geq \mathrm{cwgt}(u)$.

Proof
The proofs are easy. ∎

We conclude this section with some useful estimates for $\mathrm{cwgt}(f)$ and $\mathrm{cwgt}(u)$.

In Proposition 2.4, we require that X be a CW complex. It is not enough that X simply be homotopy equivalent to a CW complex, as we will see in Example 2.10 below.

Proposition 2.4 (cf. [11], Proposition 1.11) Let X be a CW complex. Suppose $\pi_r(Y) = 0$ for $r > m$ and that X is $(p-1)$-connected, and let $f : X \longrightarrow Y$. Then

$$\mathrm{cwgt}(f) \leq \left\lfloor \frac{m}{p} \right\rfloor.$$

If $u \in F^*(X)$, then

$$\mathrm{cwgt}(u) \leq \left\lfloor \frac{|u| + m}{p} \right\rfloor.$$

Proof
To prove the first assertion, it suffices to find a subcomplex $A \subseteq X$ with $\mathrm{cat}_X(A) \leq \left\lfloor \frac{m}{p} \right\rfloor + 1$ so that $u|_A \neq 0$.

Since $\pi_r(Y) = 0$ for $r > m$, the inclusion of the m skeleton $i : X^{(m)} \hookrightarrow X$ induces an injection

$$i^* : [X, Y] \longrightarrow [X^{(m)}, Y].$$

Since $f \not\simeq *$, $f|_{X^{(m)}} \not\simeq *$. This proves the first assertion, since

$$\mathrm{cat}\left(X^{(m)}\right) \leq \left\lfloor \frac{m}{p} \right\rfloor + 1$$

by Proposition 1.3.

To prove the second assertion, let L be a loop spectrum equivalent to F, and write $|u| = n$. Then u corresponds to a map

$$f : X \longrightarrow L_n,$$

where $\pi_r(L_n) \cong \pi_{r-n}(F)$ for all r. In particular, $\pi_r(L_n) = 0$ for $r > m+n$. The first assertion now implies the second, using Proposition 2.2. ∎

It follows from Proposition 2.4 that if $\pi_*(F)$ is bounded above, and $u \in \widetilde{F}^*(X)$, then $\mathrm{cwgt}(u) < \infty$. Rudyak has observed that if $\mathrm{cwgt}(u) = \infty$, then u must be a phantom class; the same is true for maps. It is possible to show, for example, that if G is a finite dimensional topological group and $f : BG \longrightarrow Y$ is a phantom map, then $\mathrm{cwgt}(f) = \infty$. In particular, this is true for the phantom maps $f : \mathbb{CP}^\infty \longrightarrow S^3$ described by Gray in [7].

Lastly, we describe the effect of basic arithmetic operations on the category weight of cohomology classes.

Proposition 2.5 Let $u, v \in \widetilde{F}^*(X)$. Then

$$\mathrm{cwgt}(u+v) \geq \min\left(\mathrm{cwgt}(u), \mathrm{cwgt}(v)\right);$$

if $\mathrm{cwgt}(u) \neq \mathrm{cwgt}(v)$, then strict equality holds. For any $a \in \widetilde{F}^*(S^0)$,

$$\mathrm{cwgt}(au) \geq \mathrm{cwgt}(u).$$

Proof
The proof is easy. ∎

2.2 Existence of Maps with Maximal Category Weight

In this section we will produce maps $f : X \longrightarrow S^n$ with $\mathrm{cwgt}(f) = \mathrm{cat}(X) - 1$ for suitable spaces X. The basic observation is Theorem 2.6.

Theorem 2.6 Suppose that $X = Y \cup_g D^n$ (we do not require that $\dim(Y) < n$). Let

$$f : X \longrightarrow X/Y = S^n$$

be the quotient map. If g is surjective, then $\mathrm{cwgt}(f) = \mathrm{cat}(X) - 1$.

Proof
If $A \subseteq X$ with $\mathrm{cat}_X(A) < \mathrm{cat}(X)$, then $A \neq X$. Since g is surjective, $D^n \not\subseteq A$, and since A is a subcomplex, it is closed in X, so A misses an interior point of D^n. Therefore $f|_A : A \longrightarrow S^n$ is not surjective, and so $f|_A \simeq *$. ∎

Corollary 2.7 Let X be as above and suppose $u = f^*(v) \in \widetilde{F}^n(X)$. Then $\mathrm{cwgt}(u) = \mathrm{cat}(X) - 1$.

Proof
This follows immediately from Theorem 2.6 and Proposition 2.3. ∎

The next theorem shows that Theorem 2.6 applies to any compact manifold. Recall that any compact connected n-dimensional manifold can be given a CW decomposition with exactly one n-cell.

Theorem 2.8 Let M be a compact connected n-manifold, and give it a CW decomposition with precisely one n-cell. Let

$$f : M \longrightarrow M/M^{(n-1)} = S^n$$

be the quotient map. Then $f \not\simeq *$, and $\mathrm{cwgt}(f) = \mathrm{cat}(M) - 1$. Also, if M is R-orientable, and $u \in H^n(M; R)$, then $\mathrm{cwgt}(u) = \mathrm{cat}(M) - 1$.

Proof
Since every point of an n-manifold has an n-dimensional neighborhood, the attaching map for the n cell must be surjective. By Theorem 2.6, $\mathrm{cwgt}(f) = \mathrm{cat}(M) - 1$ if $f \not\simeq *$.

To see that $f \not\simeq *$, observe that if M is R-orientable, then

$$f^* : H^n(S^n; R) \longrightarrow H^n(M; R)$$

is an isomorphism. Since every compact manifold is $\mathbb{Z}/2$-orientable, it follows that $f \not\simeq *$.

If $u \in H^n(M; R)$, and M is R-orientable, then $u = f^*v$, so $\mathrm{cwgt}(u) = \mathrm{cat}(M) - 1$ by Corollary 2.7. ∎

Example 2.9 Consider $\mathbb{RP}^{2n-1}$. By Corollary 2.8, if $u \in H^{2n-1}(\mathbb{RP}^{2n-1}; \mathbb{Z})$ then $\mathrm{cwgt}(u) = 2n - 1$, even though it is not decomposable as a product.

The condition that g be surjective is not usually preserved by homotopy. This is our first indication that changes within homotopy type can change the category weight of a map. The next example is a dramatic demonstration of this phenomenon.

Example 2.10 Let X be any finite CW complex, and write $\mathrm{cat}(X) = N$. Let $S^n \subseteq \mathbb{R}^{n+1}$ as usual, and let $q : S^n \longrightarrow [-1, 1]$ be the projection on the first coordinate.

Since X is a finite CW complex, there is a surjective map $r : [-1, 1] \longrightarrow X$ (by the Hahn-Mazurkiewicz Theorem, Theorem 3.30 in [9]). We can even assume that both q and r preserve basepoints. Let $f = r \circ q : S^n \longrightarrow X$. Then f is clearly nullhomotopic, and so there is a homotopy equivalence

$$h : X \cup_f D^{n+1} \xrightarrow{\simeq} X \vee S^{n+1}.$$

Let σ generate $\widetilde{H}^*(S^{n+1}; G) \subseteq \widetilde{H}^*(X \vee S^{n+1}; G)$, and let $\tau = h^*(\sigma)$. Since f is surjective, Corollary 2.7 tells us that $\mathrm{cwgt}(\tau) = N - 1$. On the other hand, since $\sigma|_{S^{n+1}} \neq 0$, and $\mathrm{cat}_{X \vee S^{n+1}}(S^{n+1}) \le \mathrm{cat}(S^{n+1}) = 2$, $\mathrm{cwgt}(\sigma) = 1$.

Taking $n = 1$ and $\mathrm{cat}(X) > 3$ in the above example gives us a counterexample to the assertion which results from letting X in Proposition 2.4 be simply homotopy equivalent to a CW complex.

This procedure can be adapted to find maps $f : X \longrightarrow S^n$ whose category weight detects the full category of X.

We need the following lemma.

Lemma 2.11 Let X be a finite CW complex. Then every map $f : S^n \longrightarrow X$ is homotopic to a surjective map.

Proof
Let $h : [-1, 1] \longrightarrow X$ be a surjective pointed map (such a map exists by the Hahn-Mazurkiewicz Theorem). Since $[-1, 1]$ is contractible, $h \simeq *$. Now let $f : S^n \longrightarrow X$ represent the homotopy class α. Then $\alpha = \alpha + 0$ is represented by the composition

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{f \vee h} X \vee X \xrightarrow{\text{fold}} X.$$

Since h is surjective, so is this representative of the class α. ∎

Next we recall a result from [3]. Suppose X is a simply connected CW complex with homology groups $H_n(X;\mathbb{Z}) = H_n$, and let A be the space constructed as follows. Let $A_1 = *$. Given A_{n-1}, there is a cofibration

$$M(H_n, n-1) \xrightarrow{j_{n-1}} A_{n-1} \longrightarrow A_n$$

where $M(H_n, n-1)$ is the Moore space with

$$\widetilde{H}_q(M(H_n, n-1);\mathbb{Z}) = \begin{cases} H_n & \text{if } q = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

and the map j_{n-1} induces the zero map in homology (in general there are many choices for j_{n-1}). Then $A = \cup A_n$. According to Theorem 2.1 of [3], the maps j_k can be chosen so that $A \simeq X$.

It follows that if $H_q(X) = 0$ for $q \geq n$ and $H_{n-1}(X)$ is free (so that we may take $M(H_{n-1}, n-2)$ to be a wedge of spheres), then X is homotopy equivalent to a CW complex with no cells of dimension n or higher.

Proposition 2.12 If X is homotopy equivalent to a simply connected finite CW complex, then there is a space $X' \simeq X$ and a map $f : X' \longrightarrow S^n$ such that $\mathrm{cwgt}(f) = \mathrm{cat}(X) - 1$.

Proof

First, replace X by a homotopy equivalent CW complex, and assume that it has been given a CW decomposition with no cells of dimension greater than n. We may assume that there is no CW complex homotopy equivalent to X which does not have n dimensional cells.

Write $Y = X^{(n-1)}$, so $X = Y \cup_g D_1^n \cup \cdots \cup D_k^n$. Let $A = Y \cup D_1^n$ and $B = Y \cup D_2^n \cup \cdots \cup D_k^n$. We will prove that the map

$$f : X \longrightarrow X/B = S^n$$

is detected by homology in some coefficients. An easy argument using the Mayer-Vietoris exact homology sequence shows that it is sufficient to show that the map

$$f' : A \longrightarrow A/Y = S^n$$

is detected by homology in some coefficients.

If we give S^n the cellular decompostion $S^n = * \cup D^n$, we see that the map of cellular chain complexes induced by f' is

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_n(A) = \mathbb{Z} & \xrightarrow{\partial_n} & C_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & C_n(S^n) = \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Therefore, it suffices to show that $H_n(A) \neq 0$ for some coefficients.

Suppose that $H_n(A) = 0$ for any coefficients. Then $H_{n-1}(A)$ must be free. By Theorem 2.1 of [3], X is homotopy equivalent to an $(n-1)$ dimensional CW complex, which contradicts our assumption that there is no CW complex homotopy equivalent to X which does not have any n dimensional cells.

Therefore, $f_* \neq 0$ for some coefficients; it follows that $f \not\simeq *$. To finish the proof, we use Lemma 2.11 to replace the attaching map $g : S^{n-1} \longrightarrow Y$ for D_1^n by a map $g' : S^{2n-1} \longrightarrow B$ which is surjective, and which is homotopic in B to g. Then

$$X \simeq B \cup_g D_1^n \simeq B \cup_{g'} D_1^n = X',$$

and $f' : X' \longrightarrow S^n$ is nontrivial, and hence $\mathrm{cwgt}(g') = \mathrm{cat}(X) - 1$ by Theorem 2.6. ∎

I am grateful to Chuck McGibbon for bringing [3] to my attention.

3 Essential Category Weight

We saw in Example 2.10 that composition with a homotopy equivalence can dramatically alter the category weight of a map. We constructed a space $Y \simeq X \vee S^n$ in which the the category weight of the cohomology class in $H^*(Y)$ corresponding to the generator of $H^*(S^n)$ was larger than it 'should' have been. In Example 3.1, we find a space $Y \simeq \mathbb{RP}^{2n-1}$ in which the category weight of a generator in $H^{2n-1}(Y;\mathbb{Z})$ is smaller than it 'should' be.

These examples motivate us to look for a concept similar to category weight, but better behaved under composition. The result is essential category weight, denoted $E(f)$.

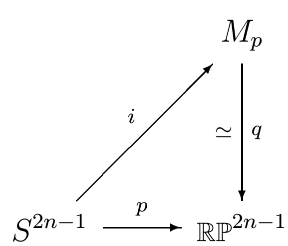
In 3.1, we give the definitions, some examples, and a list of equivalent definitions. In 3.2, we examine in detail the behavior of $E(f)$ under composition.

3.1 Definition and Basic Properties

We begin with Example 3.1 which serves as motivation for the concept of essential category weight.

Example 3.1 Let $p : S^{2n-1} \longrightarrow \mathbb{RP}^{2n-1}$ be the standard double cover. Let $\mu \in H^{2n-1}(\mathbb{RP}^{2n-1};\mathbb{Z})$ and $\sigma \in H^*(S^{2n-1};\mathbb{Z})$ be generators. Then $p^*(\mu) = 2\sigma$. Recall from Example 2.9 that $\mathrm{cwgt}(\mu) = 2n-1$.

Now consider M_p, the mapping cylinder of the map p. The diagram



commutes. Let $\mu' = q^*(\mu)$. Then $\mu'|_{i(S^{2n-1})} = p^*(\mu) = 2\sigma \neq 0$. Since $\mathrm{cat}_{M_p}(S^{2n-1}) \leq \mathrm{cat}(S^{2n-1}) = 2$, $\mathrm{cwgt}(\mu') = 1$. This shows that category weight can be altered dramatically by composition with a homotopy equivalence.

We can do even worse. Define an inclusion $j : M_p \hookrightarrow \mathbb{RP}^{2n-1} \times D^{2n}$ by

$$j : [x, t] \mapsto (p(x), tx)\,.$$

Let $\mu'' = p^*_{\mathbb{RP}^{2n-1}}(\mu)$. Then $\mu''|_{M_p} = \mu'$. By Lemma 2.3, $\mathrm{cwgt}(\mu'') \leq \mathrm{cwgt}(\mu') = 1$, so $\mathrm{cwgt}(\mu'') = 1$. We have obtained arbitrarily large drops in category weight by taking a product with a disk.

This phenomenon is actually quite general. If $f : X \longrightarrow Y$ and $g : Z \longrightarrow X$ such that $\mathrm{cwgt}(f \circ g) \leq N$, then the maps $f' : M_g \longrightarrow Y$ and $f'' : X \times CZ \longrightarrow Y$ satisfy $\mathrm{cwgt}(f') \leq N$ and $\mathrm{cwgt}(f'') \leq N$. If Z can be embedded in some sphere, CZ can be replaced by D^n for large enough n.

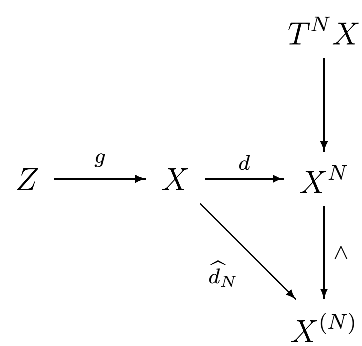
Thus, category weight is not preserved by even very straightforward homotopy equivalences. This shortcoming motivates the concept of essential category weight.

Definition Let $f : X \longrightarrow Y$ with $f \not\simeq *$. The *essential category weight* of f is

$$E(f) = \min\left(\mathrm{cwgt}\,(f \circ g)\ \mid g : Z \longrightarrow X\right).$$

In this definition, the spaces Z must be CW complexes; we do *not* require that X or Y be CW complexes.

Example 3.2 If $\widehat{d}_N \not\simeq *$, then $E(\widehat{d}_N) \geq N$. To see this, it suffices to show that $\widehat{d}_N \circ g \simeq *$ whenever $\mathrm{cat}(g) \leq N$ (by Proposition 3.5, below). Let $g : Z \longrightarrow X$ with $\mathrm{cat}(g) \leq N$. Then we have the following commutative diagram.



Since $\mathrm{cat}(g) \leq N$, there is a lift up to homotopy into $T^N X$ by Proposition 1.1. Therefore $\widehat{d}_N \circ g \simeq *$.

Just as for category weight, there are several equivalent ways to define the essential category weight of a cohomology class. The definition we give applies to maps between arbitrary spectra; this generality will be useful later when we discuss cohomology operations.

Definition Let $f : E \longrightarrow F$ be a map of spectra, and n be an integer. The *degree n essential category weight of f* is

$$E(f, n) = \min\left(\mathrm{cwgt}\,(f \circ g) \;\mid g : \Sigma(Z) \longrightarrow F \text{ is a map of degree } -n\right).$$

A cohomology class $u \in F^n(X)$ corresponds to a map $u : \Sigma(X) \longrightarrow F$ of degree $-n$. The *essential category weight* of u is

$$E(u) = \min\left(\mathrm{cwgt}\,(g^*(u)) \;\mid g : Z \longrightarrow X\right).$$

Notice that, according to our convention, these minimums are taken over all g such that $f \circ g \not\simeq *$ or $g^* u \neq 0$, respectively. As before, the spaces Z must be CW complexes.

It follows immediately that $E(u) \geq E(u, 0)$. It would follow immediately from the Freyd's generating hypothesis that $E(f, n) = 1$ for any map f between finite spectra.

Example 3.3 Let F be a ring spectrum. Let $f : Z \longrightarrow X$, and let $u_i \in F^*(X)$. Then

$$f^*(u_1 \cdots u_N) = (f^* u_1) \cdots (f^* u_N),$$

and $\mathrm{cwgt}(f^*(u_1) \cdots f^*(u_N)) \geq N$ by Example 2.1. Therefore

$$E(u_1 \cdots u_N) \geq N.$$

We can express the essential category weight of a map of spectra in terms of the essential category weight of a map of spaces.

Proposition 3.4 Let $u : E \longrightarrow F$ be a map of spectra, and let L, K be loop spectra equivalent to E and F respectively. The map u corresponds to a sequence of maps $f_n : L_n \longrightarrow K_{n+k}$. With this notation,

$$E(u, n) = E(f_n).$$

If $u : \Sigma(X) \longrightarrow F$ is of degree $-n$, then u corresponds to a map $f : X \longrightarrow K_n$. Then

$$E(u) = E(f).$$

Proof

The proof is strictly parallel to the proof of Proposition 2.2. ∎

Using Proposition 3.4, we see that every result for essential category weight of maps will have an analogue for essential category weight of maps of spectra: simply replace the maps of spectra in question by maps of spaces and apply the result for spaces.

Clearly $E(f) \leq \mathrm{cwgt}(f)$ and similarly for cohomology classes. Also

$$1 \leq E(f) \leq \mathrm{cwgt}(f) < \mathrm{cat}(X);$$

just as for cwgt, $E(u) = 0$ if and only if $u \notin \widetilde{F}^*(X)$. In fact, replacing cwgt with E in each of the results of Section 2.1 yields a true statement for essential category weight. To prove these statements, take minimums over the given proofs.

The next result is very useful for computing essential category weight.

Proposition 3.5 Let $f : X \longrightarrow Y$. The following are equivalent:

1. $E(f) \geq N$;
2. $f \circ g \simeq *$ whenever $\mathrm{cat}(g) \leq N$;
3. $f \circ g \simeq *$ whenever $g : Z \longrightarrow X$ with $\mathrm{gcat}(Z) \leq N$;
4. $\mathrm{cwgt}(f \circ g) \geq N$ whenever $g : X' \simeq X$.

Let $u : E \longrightarrow F$ be a map of spectra. The following are equivalent:

1′. $E(u, n) \geq N$;

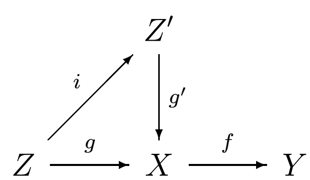
2′. $g^* u = 0$ whenever $g : \Sigma(Z) \longrightarrow E$ is a map of degree $-n$ and $\mathrm{cat}(g) \leq N$;

3′. $g^* u = 0$ whenever $g : \Sigma(Z) \longrightarrow E$ is a map of degree $-n$ and $\mathrm{gcat}(Z) \leq N$;

4′. (assuming $E = \Sigma(X)$ for some CW complex X) $\mathrm{cwgt}(g^* u) \geq N$ whenever $g : X' \simeq X$.

Proof

Let $g : Z \longrightarrow X$ with $\mathrm{cat}(g) \leq N$. By Theorem 1.3, g factors through a space Z' with $\mathrm{gcat}(Z') \leq N$, so that the diagram



commutes up to homotopy. Since $E(f) \geq N$, $\mathrm{cwgt}(g' \circ f) \geq N$. Since $\mathrm{cat}(Z') \leq \mathrm{gcat}(Z') = N$, $g' \circ f \simeq *$, and so $f \circ g \simeq *$.

Suppose now that $f \circ g \simeq *$ whenever $\mathrm{cat}(g) \leq N$, and let $g : Z \longrightarrow X$ with $\mathrm{gcat}(Z) \leq N$. By Theorem 1.4, $\mathrm{cat}(g) \leq N$, so $f \circ g \simeq *$ by 2.

Now assume that $f \circ g \simeq *$ whenever $g : Z \longrightarrow X$ with $\mathrm{gcat}(Z) \leq N$. Let $f : Z \longrightarrow X$, and let $A \subseteq Z$ with $\mathrm{cat}_Z(A) \leq N$. The inclusion map $i_A : A \hookrightarrow Z$ has $\mathrm{cat}(i_A) = \mathrm{cat}_Z(A) \leq N$, and so it factors through a space A' with $\mathrm{gcat}(A') \leq N$. Since $(f \circ g)|_A$ factors through A', $f \circ g|_A \simeq *$. Therefore $E(f) \geq N$.

We have now shown that 1, 2, and 3 are equivalent. Finally, we prove that 4 is equivalent to 1.

Clearly

$$E(f) = \min\left(\mathrm{cwgt}\,(f \circ g)\ \mid g : Z \longrightarrow X\right) \leq \min\left(\mathrm{cwgt}\,(f \circ g)\ \mid g : X' \xrightarrow{\simeq} X\right).$$

It remains to prove the reverse inequality.

Let $g : Z \longrightarrow X$ such that $\mathrm{cwgt}(f \circ g) = E(f) = k$. Then there is a subspace $A \subseteq Z$ such that $\mathrm{cat}_Z(A) = k + 1$, and $f \circ g|_A \not\simeq *$. Let $X' = M_g$, and let $p : M_g \longrightarrow X$ be the standard retraction, which is a homotopy equivalence. Then $\mathrm{cat}_{X'}(A) \leq \mathrm{cat}_Z(A) = k + 1$, and $f \circ p|_A = f \circ g|_A \not\simeq *$, so $\mathrm{cwgt}(p \circ f) \leq k$.

The proof that $1'$, $2'$, $3'$, and $4'$ are equivalent is an easy adaptation of the proof just given, using Proposition 3.4.

For example, to prove that $1'$ implies $2'$, we proceed as follows. Let K, L be loop spectra equivalent to E and F, respectively. Then

$$\widetilde{E}^n(X) \cong [X, L_n]$$

as in Theorem 1.5. If $g : \Sigma(X) \longrightarrow E$ corresponds to $h : X \longrightarrow L_n$, then $\mathrm{cat}(g) = \mathrm{cat}(h)$. Also, the map u corresponds to a map $f : K_n \longrightarrow L_m$, and $E(u, n) = E(f)$.

Therefore, $u \circ g : \Sigma(X) \longrightarrow F$ corresponds to the map $f \circ h$. Since $E(f) \geq N$ and $\mathrm{cat}(h) \leq N$, we are done, using our proof that 1 implies 2. ∎

3.2 Behavior Under Composition

Having defined essential category weight with the hope of getting better compostion properties, our next task is to describe those composition properties. The main result of this section is Theorem 3.7.

Before we can prove Theorem 3.7, we must establish some numerical relations between $\mathrm{cat}(f)$ and $\mathrm{cwgt}(f)$. The basic principle is that in a perfect world, $\mathrm{cat}(f) \cdot \mathrm{cwgt}(f)$ would be equal to $\mathrm{cat}(X)$.

Proposition 3.6 Let $f : X \longrightarrow Y$, and let $A \subseteq X$. Then

$$\mathrm{cat}(f|_A) \leq \left\lceil \frac{\mathrm{cat}_X(A)}{\mathrm{cwgt}(f)} \right\rceil$$

and

$$\mathrm{cwgt}(f) \leq \left\lfloor \frac{\mathrm{cat}_X(A)}{\mathrm{cat}(f|_A) - 1} \right\rfloor .$$

The corresponding results hold when $f : \Sigma(X) \longrightarrow F$ is a map of spectra: if $u : \Sigma(X) \longrightarrow F$, and $A \subseteq X$, then

$$\mathrm{cat}(u|_{\Sigma(A)}) \leq \left\lceil \frac{\mathrm{cat}_X(A)}{\mathrm{cwgt}(u)} \right\rceil$$

and

$$\mathrm{cwgt}(u) \leq \left\lfloor \frac{\mathrm{cat}_X(A)}{\mathrm{cat}(u|_{\Sigma(A)}) - 1} \right\rfloor .$$

Proof
Suppose $\mathrm{cwgt}(f) = p$ and $\mathrm{cat}_X(A) = n$. Write $A = U_1 \cup \cdots \cup U_n$, with each U_i closed and contractible in X. Let $V_1 = U_1 \cup \cdots \cup U_p$, $V_2 = U_{p+1} \cup \cdots \cup U_{2p}$, and so on, up to

$$V_{\lceil n/p \rceil} = U_{p(\lceil n/p \rceil - 1)+1} \cup \cdots \cup U_n .$$

Observe that $\mathrm{cat}_X(V_i) \leq p$ for each i, so $f|_{V_i} \simeq *$. Since there are precisely $\lfloor n/p \rfloor$ of them, this proves the first formula.

The second formula follows by arithmetic, and the statements for maps of spectra follow using Propositions 2.2 and 3.4. ∎

Next we relate the category weight and essential category weight of a composition of two maps to the category weight and essential category weight of the individual maps.

Theorem 3.7 Let $g : X \longrightarrow Y$, $f : Y \longrightarrow Z$. Then

$$\mathrm{cwgt}(f \circ g) \geq E(f) \cdot \mathrm{cwgt}(g)$$

and

$$E(f \circ g) \geq E(f) \cdot E(g).$$

If $u : \Sigma(X) \longrightarrow F$ and $v : F \longrightarrow G$, then

$$\mathrm{cwgt}(v \circ u) \geq E(v, |u|) \cdot \mathrm{cwgt}(u), \qquad E(v \circ u) \geq E(v, |g|) \cdot E(u).$$

If $u : E \longrightarrow F$, $v : F \longrightarrow G$, then

$$E(v \circ u, n) \geq E(v, n + |u|) \cdot E(u, n).$$

Proof
Write $E(f) = p$, $\mathrm{cwgt}(g) = q$, and let $A \subseteq X$, where $\mathrm{cat}_X(A) \leq pq$. By Proposition 3.6

$$\mathrm{cat}(g|_A) \leq \left\lceil \frac{\mathrm{cat}_X(A)}{\mathrm{cwgt}(g)} \right\rceil \leq \left\lceil \frac{pq}{q} \right\rceil = p.$$

Since $E(f) = p$,

$$(f \circ g)|_A = f \circ g|_A \simeq *$$

by Proposition 3.5. This proves that $\mathrm{cwgt}(f \circ g) \geq pq$.

The second formula is a formal consequence of the first. The proofs for maps of spectra are analagous. ∎

In the next section, we will use Theorem 3.7 to compute the category weight of some cohomology classes by showing that they are in the image of cohomology operations whose representatives $\theta \in F^*(F)$ have $E(\theta, n) > 1$.

The hoped for invariance of essential category weight under compostion follows immediately from theorem 3.7.

Corollary 3.8 Let $f : X \longrightarrow Y$, and $g : Z \longrightarrow X$. Then

$$E(f \circ g) \geq E(f).$$

The corresponding results for maps of spectra also holds: if $u : E \longrightarrow F$ and $v : F \longrightarrow G$

$$E(v \circ u, n) \geq E(v, n + |u|);$$

if $u : \Sigma(X) \longrightarrow F$, and $f : F \longrightarrow G$, then $E(f \circ u) \geq E(f)$. Let $f : X \longrightarrow Y$, and let $g : Z \longrightarrow X$ be a homotopy equivalence, then $E(f) = E(f \circ g)$. Similarly, if $u : E \longrightarrow F$ and $g : G \xrightarrow{\simeq} E$, then $E(g^* u) = E(u)$.

Proof
The inequalities follow from Theorem 3.7 on observing that if $f \not\simeq *$, then $E(f) \geq 1$.

To prove invariance under homotopy equivalences, it suffices to show that $E(f) \geq E(f \circ g)$. Let h be a homotopy inverse for g. Then

$$E(f) = E\left((f \circ g) \circ h\right) \geq E(f \circ g).$$

The proof for maps of spectra is identical. ∎

It follows that if g is a homotopy equivalence, then $E(g) = 1$.

We also obtain the following invariance of essential category weight under the *inverses* of maps induced by inclusions.

Proposition 3.9 Let X be a CW complex, and let $g : X^{(m+1)} \longrightarrow Y$. If $\pi_r(Y) = 0$ for $r > m$, then the map g extends to a map

$$f : X \longrightarrow Y,$$

and $E(f) = E(g)$. If $\pi_r(F) = 0$ for $r > m$, and $u \in F^*(X^{(m)})$, then there is a class $v \in F^*(X)$ such that $v|_{X^{(m)}} = u$, and $E(v) = E(u)$.

Proof
By basic obstruction theory, the map

$$[X, Y] \longrightarrow [X^{(m+1)}, Y]$$

induced by the inclusion $X^{(m+1)} \hookrightarrow X$ is an isomorphism for any space X. This proves that the map g extends to a map $f : X \longrightarrow Y$.

To prove $E(f) = E(g)$, let $h : Z \longrightarrow X$, where $\mathrm{cat}(h) \leq E(g)$. By cellular approximation, we may assume that the map h is cellular. Since the diagram

$$\begin{array}{ccc}
[X, Y] & \xrightarrow{\ h^*\ } & [Z, Y] \\
\cong \big\downarrow & & \big\downarrow \cong \\
[X^{(m+1)}, Y] & \xrightarrow{(h|_{Z^{(m+1)}})^*} & [Z^{(m+1)}, Y]
\end{array}$$

commutes, it suffices to show that $(h|_{Z^{(m+1)}})^* \circ g \simeq *$. Since $\mathrm{cat}(h|_{Z^{(m+1)}}) \leq \mathrm{cat}(h) \leq E(g)$, this follows immediately from assertion 2 of Proposition 3.5.

The statement for cohomology classes follows from the statement for maps, using Propositions 2.2 and 3.4. ∎

4 Essential Category Weight in Cohomology

In this section we will describe the effect of cohomology operations on the (essential) category weight of cohomology classes.

Recall that a cohomology operation is a natural transformation $\theta : F^n(X) \longrightarrow E^m(X)$. The *essential category weight of the cohomology operation* θ is

$$E(\theta) = \min \left(\mathrm{cwgt}(\theta(u)) \mid u \in \widetilde{F}^*(X) \right).$$

Let δ denote the connecting homomorphism in the cohomology long exact sequence. A *stable* cohomology operation is a collection $\theta = \{\theta_n\}$ of cohomology operations $\theta_n : F^n(X) \longrightarrow E^m(X)$ such that $\theta_n \circ \delta = \delta \circ \theta_{n-1}$. If θ is a stable operation, we define

$$E(\theta, n) = \min \left(\mathrm{cwgt}(\theta(u)) \mid |u| = n \right) = E(\theta_n).$$

Now we need to recall some basic classifications of cohomology operations. Let E and F be spectra, and let $\iota : L \longrightarrow F$ be an equivalence, where L is a loop spectrum. Then we have maps ι_n defined by the composition

$$\Sigma(L_n) \hookrightarrow L \stackrel{\iota}{\longrightarrow} F.$$

The map

$$\epsilon : [X, L_n] \longrightarrow \widetilde{F}^n(X)$$

given by $\epsilon : f \mapsto f^* \iota_n$ is an isomorphism. It is easy to see that if $O_n = \{\theta : \widetilde{F}^n(X) \longrightarrow \widetilde{E}^*(X)\}$ (that is, the cohomology operations with domain $F^n(X)$) then the map $O_n \longrightarrow \widetilde{E}^*(L_n)$ given by $\theta \mapsto \theta(\iota_n)$ is an isomorphism. In particular, if K is a loop spectrum equivalent to E, then the cohomology operations $\theta : F^n(X) \longrightarrow E^m(X)$ are in one to one correspondence with $\widetilde{E}^m(L_n) \cong [L_n, K_m]$.

Theorem 4.1 With the notation as above, if $f : L_n \longrightarrow K_m$ corresponds to the cohomology operation θ, then

$$E(\theta) = E(f).$$

If θ is a stable operation corresponding to the map $f : F \longrightarrow E$, then

$$E(\theta, n) = E(f, n).$$

Proof
This follows from Proposition 3.4. ∎

From now on, we adopt the convention that if $\theta : F^n(X) \longrightarrow E^m(X)$ is a cohomology operation defined only in dimension n, then $E(\theta, n) = E(\theta)$, and $E(\theta, k)$ is meaningless for $k \neq n$.

A straightforward application of Theorem 3.7 yields the following important observation.

Theorem 4.2 For any class $u \in F^n(X)$, and any cohomology operation $\theta : F^*(X) \longrightarrow E^*(X)$,

$$\mathrm{cwgt}(\theta(u)) \geq E(\theta, n) \cdot \mathrm{cwgt}(u)$$

and

$$E(\theta(u)) \geq E(\theta, n) \cdot E(u).$$

Proof

Let L be a loop spectrum equivalent to F, and let $\theta \in \widetilde{E}^*(L_n)$ be the class corresponding to the operation θ. Suppose u corresponds to the map $u : E(X) \longrightarrow F$ of degree $-n$. Then $\theta(u)$ corresponds to the composition

$$\Sigma(X) \xrightarrow{u} \Sigma(L_n) \xrightarrow{\theta_n} E.$$

The theorem now follows immediately from Theorem 3.7. ∎

Recall that Fadell and Husseini suggested that we say that an operation θ is *universal in dimension* n *for category weight* k if $\mathrm{cwgt}(\theta(u)) \geq k$ for every $u \in \widetilde{h}^n(X)$. In our terminology, this says $E(\theta, n) \geq k$. In view of Theorem 4.2, we see that the conclusion to their Theorem 3.12 can be improved from $\mathrm{cwgt}(\theta(u)) \geq 2$ to $\mathrm{cwgt}(\theta(u)) \geq 2 \cdot \mathrm{cwgt}(u)$.

Next, we show that a given stable cohomology operation θ will have $E(\theta, n) > 1$ for at most one integer n; we also give a useful characterization of this dimension n.

Theorem 4.3 (cf. Theorems 3.6 and 3.12 of [4]) Let $\theta \in E^k(F)$ be a stable operation of degree k. Then $E(\theta, n) > 1$ if and only if $\theta(\widetilde{F}^{n-r}(X)) = 0$ for any CW complex X and any $r > 0$.

Proof

Suppose there is some $r > 0$ such that $\theta(\widetilde{F}^{n-r}(X)) \neq 0$. Then $\theta(\widetilde{F}^n(S^r X)) \neq 0$. Since $\mathrm{cat}(S^r X) = 2$, $E(\theta, n) = 1$. Thus, if $E(\theta, n) > 1$, then θ is identically 0 in every dimension less than n.

Now we assume that $\theta(\widetilde{F}^{n-r}(X)) = 0$ for all $r > 0$, and show that $E(\theta, n) > 1$. By Proposition 3.5, it suffices to show that $\theta(\widetilde{F}^n(Z)) = 0$ for any CW complex Z with $\mathrm{gcat}(Z) \leq 2$. Let $Z = A \cup B$ with both A and B contractible subcomplexes. Then we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{F}^{n-1}(A \cap B) & \underset{\cong}{\xrightarrow{\delta}} & \widetilde{F}^n(Z) & \longrightarrow & 0 \\
 & & \Big\downarrow \theta & & \Big\downarrow \theta & & \\
0 & \longrightarrow & \widetilde{E}^{n-1+k}(A \cap B) & \underset{\cong}{\xrightarrow{\delta}} & \widetilde{E}^{n+k}(Z) & \longrightarrow & 0
\end{array}$$

from the Mayer Vietoris exact sequence. Since $\theta(\widetilde{F}^{n-1}(A \cap B)) = 0$, θ vanishes on $\widetilde{F}^n(Z)$. ∎

We now introduce a notation for the first dimension in which a stable cohomology class is nonzero. For a stable cohomology operation $\theta \in F^*(F)$, let

$$d(\theta) = \min\left(n \mid \theta\left(F^n(X)\right) \neq 0 \text{ for some } X\right).$$

Thus,

$$E(\theta, n) = \begin{cases} 1 & \text{if } n \neq d(\theta) \\ > 1 & \text{if } n = d(\theta). \end{cases}$$

Observe that in the case θ is an admissible monomial in Steenrod powers, $d(\theta)$ is equal to the *excess* ($e(\theta)$) of the monomial (see [13]). Thus, we recover Theorems 3.6 and 3.12 of [4].

Next we see that for some spectra, the condition $d(\theta) > -\infty$ is obvious.

Lemma 4.4 Let F be a spectrum with $\pi_r(F) = 0$ for $r > m$ and let θ be a stable cohomology operation. Then there is an integer $d(\theta)$ such that $\theta(F^r(X)) = 0$ for any CW complex X and any $r < d(\theta)$.

Proof
By induction on the skeleta of X, $F^r(X) = 0$ for $r < -m$. ∎

Example 4.5 We can use this result to compute of the category of the lens spaces. Let $L = S^{2n-1}/\mathbb{Z}_p$, where p is an odd prime. The (ordinary) cohomology ring with $\mathbb{Z}_p$ coefficients is

$$H^*(L; \mathbb{Z}_p) = \Lambda(x) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[y]/(y^n),$$

where $|x| = 1$ and $y = \beta(x)$. Then $E(y) \geq 2$, since β is a stable operation, and $d(\beta) = e(\beta) = 1$. The class $xy^{n-1} \in H^*(L; \mathbb{Z}_p)$ is nonzero, and

$$E(xy^{n-1}) \geq 1 + (n-1)2 = 2n - 1,$$

so $\text{cat}(L) \geq 2n$. On the other hand, $\dim(L) = 2n - 1$, so $\text{cat}(L) \leq 2n$ by Proposition 1.3. thus $\text{cat}(L) = 2n$.

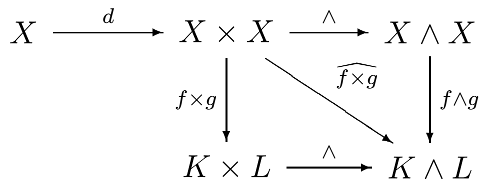
5 Product Formulae

As we observed in the introduction, many of the most useful lower bounds on $\text{cat}(X)$ are of the product formula type. In this section, we realize each of these results as consequences of a single statement concerning (essential) category weight. This main theorem is Theorem 5.1.

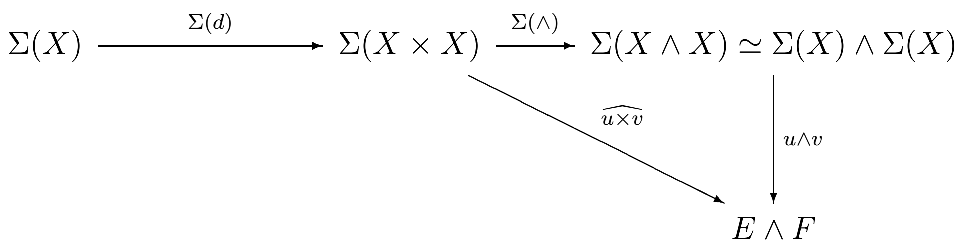
In 5.1, we prove Theorem 5.1 and some corollaries. In 5.2, we use Theorem 5.1 to prove Theorem 5.6, which is our generalization of Steenrod's theorem on normal cohomology operations (see [12]). In 5.3, we derive Whitehead's result on the nilpotence of $[X, G]$ (see [16]) and Eilenberg's cup product lower bound on $\text{cat}(X)$ as corollaries of category weight formulas which are in turn corollaries of Theorem 5.6.

5.1 Product Formulae For Maps

Let $f : X \longrightarrow K$ and $g : X \longrightarrow L$. Then we have the diagram



Let X be a space and let E and F be spectra. Suppose $u : X \longrightarrow E$ and $v : X \longrightarrow F$. Then we have an analagous diagram



The results of this section all follow from Theorem 5.1, which describes the category weight and essential category weight of various maps in these diagrams.

Theorem 5.1 Let $f : X \longrightarrow K$ and $g : X \longrightarrow L$. Then

$$E(\widehat{f \times g}) \geq E(f) + E(g)$$

and

$$\operatorname{cwgt}\left((\widehat{f \times g}) \circ d\right) \geq \operatorname{cwgt}(f) + \operatorname{cwgt}(g).$$

The analagous results hold for maps of spectra. If $u : \Sigma(X) \longrightarrow E$, and $v : \Sigma(X) \longrightarrow F$, then

$$E(\widehat{u \times v}) \geq E(f) + E(g)$$

and

$$\operatorname{cwgt}\left((\widehat{u \times v}) \circ \Sigma(d)\right) \geq \operatorname{cwgt}(u) + \operatorname{cwgt}(v).$$

Proof
Write $E(f) = p$, $E(g) = q$ and suppose $\operatorname{gcat}(Z) \leq p + q$. Write $Z = A \cup B$ with $\operatorname{gcat}(A) \leq p$ and $\operatorname{gcat}(B) \leq q$. Let $h : Z \longrightarrow X \times X$. Then

$$(f \times g) \circ h = (f \circ h) \times (g \circ h) \circ d.$$

By assumption, $f \circ h \simeq f'$, where $f'(A) = *$, and $g \circ h \simeq g'$, where $g'(B) = *$. Therefore,

$$(f \times g) \circ h = (f \circ h) \times (g \circ h) \circ d \simeq (f' \times g') \circ d.$$

Since $((f' \times g') \circ d)(Z) \subseteq K \vee L$,

$$\widehat{(f \times g)} \circ h \simeq \wedge \circ (f' \times g') \circ d = *.$$

The proof of the second assertion is analagous. Let $\mathrm{cwgt}(f) = p$ and $\mathrm{cwgt}(g) = q$, and let $A \subseteq X$ with $\mathrm{cat}_X(A) \leq p+q$. By writing $A = B \cup C$ with $\mathrm{cat}_X(B) \leq p$ and $\mathrm{cat}_X(C) \leq q$ and proceeding as above, we can find $h' \simeq (f \times g) \circ d$ such that $h'(A) \subseteq K \vee L$.

To prove the statement for maps of spectra, let K and L be loop spectra equivalent to E and F respectively. Then the maps u and v correspond to maps (of spaces) $u' : X \longrightarrow K_n$ and $v' : X \longrightarrow L_m$. Observe that the diagram

$$\begin{array}{ccccccc}
\Sigma(X \times X) & \longrightarrow & \Sigma(X \wedge Y) & \xrightarrow{\simeq} & \Sigma(X) \wedge \Sigma(Y) & \overset{u \wedge v}{\searrow} & \\
\downarrow{\scriptstyle \Sigma(u' \times v')} & & \downarrow{\scriptstyle \Sigma(u' \wedge v')} & & \downarrow & & \\
\Sigma(L_n \times K_m) & \xrightarrow{\Sigma(\wedge)} & \Sigma(L_n \wedge K_m) & \xrightarrow{\simeq} & \Sigma(L_n) \wedge \Sigma(K_m) & \xrightarrow{j} & E \wedge F
\end{array}$$

commutes, so $\widehat{u \times v} = (\widehat{u' \times v'})^*(j)$. Since $E(u,n) = E(u')$ and $E(v,m) = E(v')$, we are done by the first case and Proposition 3.4.

The proof of the second assertion for maps of spectra is similar. ∎

Observe that the statement $E(\widehat{(f \times g)} \circ d) \geq E(f) + E(g)$ follows immediately from Theorem 5.1 and Corollary 3.8.

Corollary 5.2 Let $f : X \longrightarrow K$, $g : Y \longrightarrow L$. Then

$$E(\widehat{f \times g}) \geq E(f) + E(g).$$

The statement remains true for X, Y, K and L spectra.

Proof

Apply Theorem 5.1 to $f \circ p_X : X \times Y \longrightarrow K$ and $g \circ p_Y : X \times Y \longrightarrow L$. Since $E(f \circ p_X) \geq E(f)$ and $E(g \circ p_Y) \geq E(g)$,

$$E(\widehat{f \times g}) = E\left((f \circ p_X) \times (g \circ p_Y) \circ d\right) \geq E(f \circ p_X) + E(g \circ p_Y) \geq E(f) + E(g),$$

which completes the proof of the first assertion. The proof of the statement for spectra is identical. ∎

The following lemma is very useful when it comes to applying Corollary 5.2.

Lemma 5.3 Let $\wedge : X \times Y \longrightarrow X \wedge Y$, and let Z be a space. Then

$$\wedge^* : [X \wedge Y, Z] \longrightarrow [X \times Y, Z]$$

is weakly injective (in the sense that $\wedge^*(f) = *$ if and only if $f = *$).

Proof

This is Lemma 1.2 of [14]. ∎

The next Proposition is very useful when trying to apply Corollary 5.2

Proposition 5.4 The map $\widehat{f \times g} \not\simeq *$ if and only if $f \wedge g \not\simeq *$.

Proof

The diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times g} & K \times L \\ \downarrow{\scriptstyle \wedge} & & \downarrow{\scriptstyle \wedge} \\ X \wedge Y & \xrightarrow{f \wedge g} & K \wedge L \end{array}$$

is commutative. Proposition 5.4 now follows immediately from Lemma 5.3. ∎

We immediately derive the following result, which was first observed by Fox (see [5]).

Corollary 5.5 Let X and Y be spaces such that $X \wedge Y \not\simeq *$. Then

$$\mathrm{cat}(X \times Y) \geq 3.$$

Proof

Let $\wedge : X \times Y \longrightarrow X \wedge Y$. Since $1_{X \wedge Y} = 1_X \wedge 1_Y \not\simeq *$, $E(\wedge) \geq 2$ by Corollary 5.2. ∎

It follows from Corollary 5.5 that Ganea's $X \times S^k$ conjecture holds for any space with $\mathrm{cat}(X) = 2$ and $SX \not\simeq *$.

5.2 Operations of Several Variables

In [12], Steenrod introduced the concept of a *normal* cohomology operation of several variables, and he proved that if θ is a normal cohomology operation of n variables, then (in our language)

$$\mathrm{cwgt}\,(\theta(u_1, \ldots, u_n)) \geq n.$$

This suggests the more general formula

$$\mathrm{cwgt}\,(\theta(u_1, \ldots, u_n)) \geq \sum \mathrm{cwgt}(u_i)$$

and its corollary

$$E\left(\theta(u_1,\ldots,u_n)\right) \geq \sum E(u_i).$$

We will prove a result of this form. Unfortunately, we will have to use a slightly stronger definition of a normal operation than the one Steenrod uses. On the other hand, our definition has the added generality of applying to any functor of the form $F(X) = [X, Y]$ and not just to cohomology theories.

Let

$$\theta : [X, F_1] \times \cdots \times [X, F_n] \longrightarrow [X, F]$$

be a natural transformation. There is a natural equivalence

$$[X, F_1] \times \cdots \times [X, F_n] \xrightarrow{\cong} [X, F_1 \times \cdots \times F_n]$$

given by $(f_1, \ldots, f_n) \mapsto (f_1 \times \cdots \times f_n) \circ d$. This gives us a natural transformation

$$\theta' : [X, F_1 \times \cdots \times F_n] \longrightarrow [X, F]$$

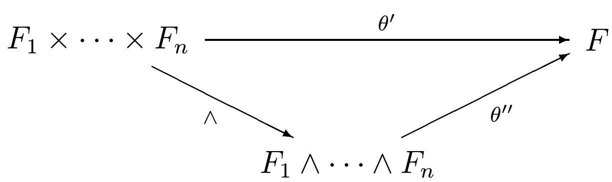
which is induced by the map

$$\theta' = \theta'(1) : F_1 \times \cdots \times F_n \longrightarrow F.$$

Definition Let $\theta : [X, F_1] \times \cdots \times [X, F_n] \longrightarrow [X, F]$ be a natural transformation. We say θ is *normal* if the map

$$\theta' : F_1 \times \cdots \times F_n \longrightarrow F$$

factors as in the diagram.



Notice that, using our definition, if any $f_i \simeq *$, then $\theta(f_1, \ldots, f_n) = 0$. It is this property that Steenrod used as the definition of a normal operation.

We now give our generalization of Steenrod's result.

Theorem 5.6 If θ is normal (in our sense), then

$$\text{cwgt}\left(\theta(f_1,\ldots,f_n)\right) \geq \sum \text{cwgt}(f_i).$$

Similarly,

$$E\left(\theta(f_1,\ldots,f_n)\right) \geq \sum E(f_i).$$

The analagous statement is true for spectra.

Proof
Write $f = f_1 \times \cdots \times f_n$. By definition,

$$\begin{aligned}\theta(f_1, \ldots, f_n) &= \theta' \circ (f_1 \times \cdots \times f_n) \circ d \\ &= \theta'' \circ \widehat{f} \circ d.\end{aligned}$$

Thus, Theorem 5.6 follows directly from Theorem 5.1. ∎

5.3 Commutators and Cup Products

We now show how Whitehead's theorem on the nilpotence of $[X, G]$ and the classical cup product lower bound on $\text{cat}(X)$ follow from useful corollaries of Theorem 5.6.

We begin with the following result, which sharpens Whitehead's theorem (see [16]). Recall that a space G is *grouplike* if it satisfies the group axioms up to pointed homotopy. If G is grouplike, $[X, G]$ has a natural group structure.

Corollary 5.7 Let G be grouplike, let $f, g \in [X, G]$, and denote the commutator of f and g by $[f, g]$. Then

$$\text{cwgt}([f, g]) \geq \text{cwgt}(f) + \text{cwgt}(g)$$

and

$$E([f, g]) \geq E(f) + E(g).$$

Proof
This operation is induced by the commutator map $[\,,] : G \times G \longrightarrow G$. Since $[G, 1] = [1, G] = 1$, $[\,,]$ factors through $(G \times G)/(G \vee G) = G \wedge G$, and hence is normal. ∎

Corollary 5.8 If $\text{cat}(X) \leq N$, then $[X, G]$ is nilpotent with nilpotence class at most $N - 1$.

Proof
By Corollary 5.7, all N-fold commutators have category weight at least N. Since $\text{cat}(X) \leq N$, they are trivial. ∎

The cup length formulae for category weight and essential category weight also follow from Theorem 5.6.

Corollary 5.9 (cf. Theorem 3.5 [4]) Let $u_i \in F_i^*(X)$, and consider

$$u_1 \wedge \cdots \wedge u_n \in (F_1 \wedge \cdots \wedge F_n)^*(X).$$

Then

$$\text{cwgt}(u_1 \wedge \cdots \wedge u_n) \geq \sum \text{cwgt}(u_i)$$

and

$$E(u_1 \wedge \cdots \wedge u_n) \geq \sum E(u_i).$$

Proof
Since the cup product is the map

$$\Sigma(X) \xrightarrow{\widehat{d}} \Sigma(X \wedge \cdots \wedge X) \simeq \Sigma(X) \wedge \cdots \wedge \Sigma(X) \xrightarrow{u_1 \wedge \cdots \wedge u_n} F_1 \wedge \cdots \wedge F_n$$

this is just a restatement of Theorem 5.1. Alternatively, since θ is a evidently a normal operation, the corollary follows from Theorem 5.6. ∎

It follows that any N-fold cup product will vanish on spaces with $\text{cat}_X(A) \leq N$, since $\text{cwgt}(u_i) \geq 1$ for any $u_i \in \widetilde{F}^*(X)$.

Just as in Corollary 5.2 the corollary to the previous theorem yields a generalization for essential category weight.

Corollary 5.10 Let $u_i \in F_i^*(X_i)$; then

$$u_1 \times \cdots \times u_n = (p_1^* u_1) \wedge \cdots \wedge (p_n^* u_n) \in (F_1 \wedge \cdots \wedge F_n)^*(X_1 \times \cdots \times X_n)$$

satisfies

$$E(u_1 \times \cdots \times u_n) \geq \sum E(u_i).$$

Proof
This is just a restatement of Corollary 5.2. ∎

It is not true in general that $\text{cwgt}(u \times v) \geq \text{cwgt}(u) + \text{cwgt}(v)$, as the following example shows.

Example 5.11 Consider $\mathbb{RP}^{2n-1}$ as Example 3.1 (also continue the notation of Example 3.1). Let

$$\widetilde{\mu} = p^*_{\mathbb{RP}^{2n-1}}(\mu) \in H^*(\mathbb{RP}^{2n-1} \times S^{2n}; \mathbb{Z}).$$

The inclusion $D^{2n} \subseteq S^{2n}$ (as the Northern hemisphere, say) induces an inclusion $\mathbb{RP}^{2n-1} \times D^{2n} \subseteq \mathbb{RP}^{2n-1} \times S^{2n}$, and $\widetilde{\mu}|_{\mathbb{RP}^{2n-1} \times D^{2n}} = \mu''$. By Proposition 2.3, $\text{cwgt}(\widetilde{\mu}) \leq \text{cwgt}(\mu'') = 1$, so $\text{cwgt}(\widetilde{\mu}) = 1$. In fact, we know that the inclusion

$$S^{2n-1} \subseteq M_p \subseteq \mathbb{RP}^{2n-1} \times D^{2n} \subseteq \mathbb{RP}^{2n-1} \times S^{2n}$$

induces $\widetilde{\mu} \mapsto 2\sigma$. If $\tau \in H^{2n}(S^{2n}, \mathbb{Z})$ generates $H^{2n}(S^{2n})$ and $\widetilde{\tau} = p^*_{S^{2n}}(\tau)$, then $\mu \times \tau = \widetilde{\mu} \cdot \widetilde{\tau}$, and so

$$\mu \times \tau|_{S^{2n-1} \times S^{2n}} = 2\sigma \times \tau \neq 0.$$

Since $\text{cat}(S^{2n-1} \times S^{2n}) = 3$, $\text{cwgt}(\mu \times \tau) = E(\mu \times \tau) = 2$.

6 Application to Ganea's Conjecture

In this section, we show how to apply the results of this chapter to Ganea's $X \times S^k$ conjecture.

Recall that for any CW complex X,

$$\text{cat}(X) \leq \text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y) - 1.$$

(See [James] for a proof.) In the case $Y = S^k$, we have

$$\text{cat}(X) \leq \text{cat}(X \times S^k) \leq \text{cat}(X) + 1.$$

The only known examples where the lower bound holds are cases where X and Y are P and Q local for complementary sets P and Q of primes. This serves as motivation for Ganea's conjecture (see Problem 2 in [6]) that

$$\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$$

for all spaces X.

The conjecture is trivial to prove if the category of X is represented by cup length; that is, if there is a nonzero $(\mathrm{cat}(X) - 1)$-fold cup product in the cohomology of X. Of course, it is not always true that the category of X is represented by cup length.

On the other hand, we have seen in Proposition 2.13 that for (many) finite CW complexes X, the category of X can be represented by category weight (in particular, the category of a compact manifold is represented by category weight by Theorem 2.8). Unfortunately, this is not enough, as Example 5.11 shows.

The appropriate generalization of cup length is essential category weight.

Proposition 6.1 Let X be a space with $\mathrm{cat}(X) = N$, and suppose there is a map $f : X \longrightarrow Y$ such that $E(f) = N - 1$, and that $S^n f \not\simeq *$. Then

$$\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$$

for all $0 < k \leq n$. If u is a cohomology class with $E(u) = N - 1$, then

$$\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$$

for all $k > 0$.

Note Rudyak calls a cohomology class such that $E(u) = \mathrm{cat}(X) - 1$ a *detecting class*, and proves the second part of this proposition (see Theorem 2.5 of [11]).

Proof

Since a cohomology class is represented by a stably nontrivial map, the second part follows from the first.

Consider the map $\widehat{f \times 1_{S^k}} : X \times S^k \longrightarrow S^k Y$. By Theorem Corollary 5.2,

$$E(\widehat{f \times 1_{S^n}}) \geq E(f) + E(1_{S^n}) = E(f) + 1,$$

assuming $\widehat{f \times 1_{S^n}} \not\simeq *$. By Proposition 5.4, it suffices to show that $f \wedge 1_{S^n} = S^k f$ is nontrivial. This is the case because $S^n f \not\simeq *$, and $n \geq k$. ∎

Proposition 6.1 shows how to fit the main result of [14] into the general framework of category weight. Say X is n-dimensional and $(p-1)$-connected. We have seen in Example 3.2 that a good candidate for a map with $E(f) = N - 1$ is

$$\widehat{d} : X \longrightarrow X^{(N-1)}.$$

Suppose $N \geq \frac{n+4}{2p}$; then since $X^{(N-1)}$ is $(Np-1)$ connected, $\widehat{d}_N$ will be stably nontrivial if it is nontrivial. Therefore, if $N \geq \frac{n+4}{2p}$, it suffices to show that $\widehat{d}_{N-1} \not\simeq *$ or, equivalently, that $\mathrm{cat}(X) = \mathrm{wcat}(X)$ (recall that $\mathrm{wcat}(X)$ is the least integer N such that $\widehat{d}_N \simeq *$).

The following is Theorem 2.2 of [14].

Theorem 6.2 If X is n-dimensional and $(p-1)$-connected, (with $p > 1$) and

$$\mathrm{cat}(X) = \left\lfloor \frac{n}{p} \right\rfloor$$

and $n \not\equiv -1 \bmod p$, then

$$\mathrm{cat}(X) = \mathrm{wcat}(X).$$

Proof
The conditions imposed allow us to use the Blakers-Massey Excision Theorem to conclude that the sequence

$$[X, T^{N-1}X] \longrightarrow [X, X^{N-1}] \xrightarrow{\wedge_*} [X, X^{(N-1)}]$$

is exact. By assumption, $d \in [X, X^{N-1}]$ does not lift into $[X, T^{N-1}X]$, so $\wedge_*(d) = \widehat{d} \not\simeq *$. ∎

Now Theorem 3.3 of [14] follows immediately from Proposition 6.1.

Theorem 6.3 If X is n-dimensional and $(p-1)$-connected (with $p > 1$)

$$\mathrm{cat}(X) = \left\lfloor \frac{n}{p} \right\rfloor + 1$$

and $n \not\equiv -1 \bmod p$, then

$$\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$$

for every $k > 0$.

Proof
The argument given above does not apply to certain spaces with $\mathrm{cat}(X) = 2$; in these cases the conjecture follows from Corollary 5.5. ∎

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University of Wisconsin-Madison
Madison, WI 53706
U. S. A.