

Two Special Cases of Ganea's Conjecture¹

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Abstract Ganea conjectured that for any finite CW complex and any $k > 0$, $\text{cat}(X \times S^k) = \text{cat}(X) + 1$. In this paper we prove two special cases of this conjecture. The main result is the following. Let X be a $(p - 1)$ connected n dimensional CW complex (not necessarily finite). We show that if $\text{cat}(X) = \lfloor \frac{n}{p} \rfloor + 1$ and $n \not\equiv -1 \pmod{p}$ (which implies $p > 1$), then $\text{cat}(X \times S^k) = \text{cat}(X) + 1$. This is proved by showing that $\text{wcat}(X \times S^k) = \text{wcat}(X) + 1$ in a much larger range, and then showing that under the conditions imposed, $\text{cat}(X) = \text{wcat}(X)$. The second special case is an extension of Singhof's earlier result for manifolds.

Introduction

For any CW complex X , and any $k > 0$

$$\text{cat}(X) \leq \text{cat}(X \times S^k) \leq \text{cat}(X) + 1$$

(see [5], Proposition 2.3). Ganea conjectured that $\text{cat}(X \times S^k) = \text{cat}(X) + 1$ for any finite complex X and any $k > 0$ (see [2], Problem 2).

Singhof proved the conjecture for X a compact differentiable or PL manifold and k lying in a bounded (possibly empty) interval depending on the dimension, connectivity and category of X (see [8], Corollary 6.7 or [6], Theorems 1 and 2). More recently, Hess proved that the conjecture holds for simply connected spaces when cat is replaced with cat_0 , the rational version of cat , and $k > 1$ (see [4], Theorem 3). It follows that the conjecture holds for simply connected rational spaces and $k > 1$.

In this paper, we will prove the following.

Theorem 3.2 Let X be a $(p-1)$ connected n dimensional CW complex. If $\text{cat}(X) = \lfloor \frac{n}{p} \rfloor + 1$ (i.e. $\text{cat}(X)$ is as large as possible) and $n \not\equiv -1 \pmod{p}$ (which forces $p > 1$), then

$$\text{cat}(X \times S^k) = \text{cat}(X) + 1$$

for all $k > 0$.

This follows easily from two subsidiary results, which are interesting on their own.

Theorem 2.2 If $\text{cat}(X) = \lfloor \frac{n}{p} \rfloor + 1$ and $n \not\equiv -1 \pmod{p}$, then

$$\text{cat}(X) = \text{wcat}(X).$$

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The proof is a simple application of the Blakers-Massey Excision theorem. In [5], James stated that $\text{cat}(X)$ and $\text{wcat}(X)$ agree in a stable range, but did not give any details. We assume that this is the result he had in mind.

Theorem 3.1 If $\text{wcat}(X) \geq \frac{n+2}{2p} + 1$, then

$$\text{wcat}(X \times S^k) \geq \text{wcat}(X) + 1$$

for all $k > 0$.

With one exception, our result is the first general result on this problem which allows X to be a CW complex without any additional structure (as opposed to a manifold or a rational space). Also, our result holds for arbitrarily large values of k , in contrast to Singhof's theorem.

The exception is the following. In [5] it is proved that if X is $(p-1)$ connected with $\text{cat}(X) = N$ and $\dim(X) = (N-1)p$, then the $(N-1)^{\text{st}}$ power of the fundamental class $\iota \in H^p(X; \pi_p(X))$ is nonzero. It follows that the conjecture holds in this case.

The proof of Theorem 3.1 can be interpreted in terms of product length in an appropriate cohomology theory. Let E and S_k denote the suspension spectra of X and S^k , respectively, and let $\alpha \in \tilde{E}^*(X \times S^k)$ and $\beta \in \tilde{S}_k^*(X \times S^k)$ be the stable classes of $p_1 : X \times S^k \rightarrow X$ and $p_2 : X \times S^k \rightarrow S^k$ respectively. Then

$$\overbrace{\alpha \wedge \cdots \wedge \alpha}^{N-1} \wedge \beta \neq 0 \in (E^{(N-1)} \wedge S_k)^*(X \times S^k).$$

The generalized cohomology version of the classical lower bound on (weak) category given by cup length now finishes the proof.

It is interesting to observe that our proof comes from an analysis of the relationship between category and weak category, while Singhof's proof arises from an examination of the relationship between category and geometric (or strong) category.

In section 4, we will show how the following result for manifolds follows directly from Singhof's theorem.

Corollary 4.2 Let X be a compact differentiable or PL manifold which is $(p-1)$ connected and n dimensional, with $n \geq 4$ and $\text{cat}(X) \geq 4$. If

$$\text{cat}(X) \geq \frac{n + 4p + 3}{2p},$$

then

$$\text{cat}(M \times S^k) = \text{cat}(M) + 1$$

for every $k > 1$. If $p = 1$, then we may take $k = 1$ as well.

Thus, applying Theorem 3.3 to a manifold improves on Singhof's result if X is simply connected and $k = 1$, or if $n \leq 4p - 1$. For example, if $X = \text{Sp}(2)$, Singhof's theorem does not apply, but Theorem 3.3 does apply. It follows that $\text{cat}(\text{Sp}(2) \times S^k) = 5$ for all $k > 0$.

I have recently received a preprint of a paper by Yu Rudyak [7], in which he gives a result similar to Corollary 4.2, with a similar proof.

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1 Preliminaries

In this section, we will describe our notation and recall some basic facts concerning Lusternik-Schnirelman category.

1.1 Notation

Throughout this paper, the space X is a (not necessarily finite) pointed CW complex. We denote the set of pointed homotopy classes of maps $f : X \rightarrow Y$ by $[X, Y]$. We denote the smash product of 2 spaces (or spectra) by $A \wedge B$. The N -fold smash product of A with itself is

$$A^{(N)} = \overbrace{A \wedge A \wedge \cdots \wedge A}^N.$$

We will never refer explicitly to the skeleta of X , so this will not cause any confusion. The canonical projection

$$X_1 \times X_2 \times \cdots \times X_N \rightarrow X_1 \wedge X_2 \wedge \cdots \wedge X_N$$

is denoted by \wedge . The composition $\wedge \circ f$ will be abbreviated \hat{f} . The suspension of the space X or the map f will be denoted SX or Sf , respectively. The loop space of the space X is denoted ΩX (we suppress the base point).

1.2 Lusternik-Schnirelman Category

Next, we recall some basic facts about Lusternik-Schnirelman category. For a more complete survey, see [5].

Definition Let X be a CW complex with base point $*$. Write

$$T^N X = \{(x_1, \dots, x_n) \mid x_i = * \text{ for some } i\} \subseteq X^N.$$

The *category of X* (denoted $\text{cat}(X)$) is the least integer N such that there is a lift up to

homotopy in the diagram

$$\begin{array}{ccc} & & T^N X \\ & & \downarrow \\ X & \xrightarrow{d} & X^N \end{array}$$

There is a related notion, the *weak category* of X (see [3]); $\text{wcat}(X)$ is the least N so that $\hat{d}: X \rightarrow X^{(N)}$ is nullhomotopic. Since $X^{(N)} = X^N/T^N X$, $\text{cat}(X) \geq \text{wcat}(X)$.

It follows immediately from the cellular approximation theorem that if X is n dimensional and $(p-1)$ connected, then $\text{cat}(X) \leq \lfloor \frac{n}{p} \rfloor + 1$ (see [5], Proposition 5.1).

Note 1.1 The conditions that appear in our theorems are of the form $\text{cat}(X) \geq \frac{n+2}{p}$. An easy numerical argument shows that for an integer N

$$\frac{n+2}{p} \leq N \leq \left\lfloor \frac{n}{p} \right\rfloor + 1$$

if and only if $N = \lfloor \frac{n}{p} \rfloor + 1$ and $n \not\equiv -1 \pmod{p}$.

1.3 Homotopy Classes of Maps

Recall that a map $\phi: S \rightarrow T$ of pointed sets is *weakly injective* if $\phi(s) = *$ if and only if $s = *$.

Lemma 1.2 Let $\wedge: X \times Y \rightarrow X \wedge Y$. Then

$$\wedge^*: [X \wedge Y, Z] \rightarrow [X \times Y, Z]$$

is weakly injective for any space Z .

Proof

The sequence

$$[X \times Y, Z] \xleftarrow{\wedge^*} [X \wedge Y, Z] \xleftarrow{\partial^*} [S(X \vee Y), Z] \xleftarrow{(Si)^*} [S(X \times Y), Z]$$

is an exact sequence of pointed sets (see [10], Theorem 2.41). Therefore, it suffices to show that $\partial^* = 0$, or equivalently $(Si)^*$ is surjective.

Since the suspension functor S and the loop space functor Ω are adjoint (see [10], Corollary 2.8), we have the commutative diagram

$$\begin{array}{ccc} [S(X \times Y), Z] & \xrightarrow{\cong} & [X \times Y, \Omega Z] \\ (Si)^* \downarrow & & \downarrow i^* \\ [S(X \vee Y), Z] & \xrightarrow{\cong} & [X \vee Y, \Omega Z] \end{array}$$

Thus it suffices to show that i^* is surjective.

Recall that ΩX is an H space with multiplication given by juxtaposition of loops (see [10], Example 2.15). Given $f \vee g : X \vee Y \rightarrow \Omega X$, define $h : X \times Y \rightarrow \Omega Z$ by the formula $h(x, y) = f(x) \cdot g(y)$. Then

$$i^*(h) = h \circ i = h|_{X \vee Y} \simeq f \vee g.$$

This proves the lemma. ■

2 Category and Weak Category

In this section, we will give a condition under which $\text{cat}(X) = \text{wcat}(X)$. The key ingredient is Lemma 2.1, which is equivalent to the Blakers-Massey Excision Theorem.

Lemma 2.1 Let $A \rightarrow X \xrightarrow{j} B$ be a cofibration, and let $F \rightarrow Y \xrightarrow{q} B$ result from converting j to a fibration. Assume A, X , and B are simply connected. Then there is a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \xrightarrow{j} & B \\ \downarrow i & & \downarrow \simeq g & & \downarrow 1 \\ F & \longrightarrow & Y & \xrightarrow{q} & B \end{array}$$

If A is $(a - 1)$ connected, and B is $(b - 1)$ connected, then i is an $(a + b - 2)$ equivalence.

Proof

The commutativity of the diagram yields an infinite commutative ladder

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & \pi_{r+1}(X, A) & \longrightarrow & \pi_r(A) & \longrightarrow & \pi_r(X) & \longrightarrow & \pi_r(X, A) & \longrightarrow & \cdots \\
 & & \downarrow g_* & & \downarrow i_* & & \downarrow \cong & & \downarrow g_* & & \\
 \cdots & \longrightarrow & \pi_{r+1}(Y, F) & \longrightarrow & \pi_r(F) & \longrightarrow & \pi_r(Y) & \longrightarrow & \pi_r(Y, F) & \longrightarrow & \cdots
 \end{array}$$

By the 5 lemma it suffices to show that g_* is an $(a + b - 1)$ equivalence.

We have the diagram

$$\begin{array}{ccc}
 (X, A) & \xrightarrow{g} & (Y, F) \\
 \downarrow j & & \downarrow q \\
 (B, *) & \xrightarrow{1} & (B, *)
 \end{array}$$

in which q is a homotopy isomorphism. Since all spaces involved are simply connected, j is an $(a + b - 1)$ equivalence by the Blakers-Massey Excision Theorem (Theorem I of [1]). It follows that g is an $(a + b - 1)$ equivalence. ■

Now we can prove the main theorem of this section.

Theorem 2.2 Let X be an n dimensional $(p - 1)$ connected CW complex. If $\text{cat}(X) = \lfloor \frac{n}{p} \rfloor + 1$ and $n \not\equiv -1 \pmod{p}$ (which forces $p > 1$), then

$$\text{cat}(X) = \text{wcat}(X).$$

Proof

Write $N = \text{cat}(X)$. By Note 1.1, the condition on $\text{cat}(X)$ is equivalent to $N \geq \frac{n+2}{p}$.

Let $F \rightarrow E \rightarrow X^{(N-1)}$ result from converting the map $\wedge : X^{N-1} \rightarrow X^{(N-1)}$ to a fibration.

Then we have the diagram

$$\begin{array}{ccccc}
& & T^{N-1}X & \xrightarrow{g} & F \\
& & \downarrow h & & \downarrow j \\
X & \xrightarrow{d} & X^{N-1} & \xrightarrow{i \simeq} & E \\
& \searrow \hat{d} & \downarrow \wedge & \swarrow & \\
& & X^{(N-1)} & &
\end{array}$$

Since $T^{N-1}X$ is $(p-1)$ connected and $X^{(N-1)}$ is $(N-1)p-1$ connected, g is an

$$(N-1)p + p - 2 = Np - 2$$

equivalence by Lemma 2.1. It follows that $g_* : [X, T^{N-1}X] \rightarrow [X, F]$ is a surjection because

$$n = \binom{n+2}{p} p - 2 \leq Np - 2.$$

Suppose $\text{wcat}(X) < \text{cat}(X)$. Then $\hat{d} \simeq *$, and so there is a lift (up to homotopy) λ of $i \circ d$ into F . Let $\kappa \in (g_*)^{-1}(\lambda)$. Then

$$i \circ h \circ \kappa \simeq j \circ g \circ \kappa = j \circ \lambda \simeq i \circ d.$$

Since i is a homotopy equivalence, it follows that $h \circ \kappa \simeq d$; that is, $\text{cat}(X) < N$. This is a contradiction. ■

3 The Main Theorem

In this section and the next, we assume that X is a $(p-1)$ connected n dimensional CW complex.

Theorem 3.1 If $\text{wcat}(X) \geq \frac{n+2}{2p} + 1$ then

$$\text{wcat}(X \times S^k) \geq \text{wcat}(X) + 1$$

for all $k > 0$.

Proof

Write $\text{wcat}(X) = N$; then the map $\hat{d} : X \rightarrow X^{(N-1)}$ is nontrivial. Since $X^{(N-1)}$ is $(N-1)p-1$ connected, and

$$n = 2 \left(\left(\frac{n+2}{2p} + 1 \right) - 1 \right) p - 2 \leq 2(N-1)p - 2$$

this map is stably nontrivial by the Freudenthal Suspension Theorem.

We need to show the map $\hat{d} : X \times S^k \rightarrow (X \times S^k)^{(N)}$ is nontrivial. Consider the commutative diagram

$$\begin{array}{ccccc} X \times S^k & \xrightarrow{d} & (X \times S^k)^N & \xrightarrow{\wedge} & (X \times S^k)^{(N)} \\ & \searrow^{d \times 1} & \downarrow^{p_1 \times \cdots \times p_1 \times p_2} & & \downarrow^{p_1 \wedge \cdots \wedge p_1 \wedge p_2} \\ & & X^{N-1} \times S^k & \xrightarrow{\wedge} & X^{(N-1)} \wedge S^k. \end{array}$$

It is certainly enough to show that the composite

$$X \times S^k \xrightarrow{d \times 1} X^{N-1} \times S^k \xrightarrow{\wedge} X^{(N-1)} \wedge S^k$$

is nontrivial. From the commutative diagram

$$\begin{array}{ccc} X \times S^k & \xrightarrow{d \times 1} & X^{N-1} \times S^k \\ \downarrow^{\wedge} & & \downarrow^{\wedge} \\ X \wedge S^k & \xrightarrow{S^k \hat{d}} & X^{(N-1)} \wedge S^k \end{array}$$

we see that it is enough to show that the composite

$$X \times S^k \xrightarrow{\wedge} X \wedge S^k \xrightarrow{S^k \hat{d}} X^{(N-1)} \wedge S^k$$

represents a nontrivial class in $[X \times S^k, X^{(N-1)} \wedge S^k]$. Since \hat{d} is stably nontrivial,

$$S^k \hat{d} \neq * \in [X \wedge S^k, X^{(N-1)} \wedge S^k].$$

By Lemma 1.2, \wedge^* is injective. This proves the theorem. ■

Remark. It is easy to view this proof in terms of product length using appropriate cohomology theories. We refer the reader to [10], Chapters 8, 9 and 13 for information about generalized cohomology.

Let E and S_k denote the suspension spectra of X and S^k , respectively. Let $p_1 : X \times S^k \rightarrow X$ and $p_2 : X \times S^k \rightarrow S^k$ be the projections, and let $\alpha \in \widetilde{E}^*(X \times S^k)$ and $\beta \in \widetilde{S}_k^*(X \times S^k)$ be the stable classes of the maps p_1 and p_2 , respectively.

The product

$$\overbrace{\alpha \wedge \cdots \wedge \alpha}^{N-1} \wedge \beta \in (E^{(N-1)} \wedge S_k)^*(X \times S^k)$$

is the stable homotopy class of the map

$$((p_1 \wedge \cdots \wedge p_i) \wedge p_2) \circ \hat{d} : X \times S^k \rightarrow X^{(N-1)} \wedge S^k.$$

We saw in the proof of Theorem 3.1 that this map is nontrivial.

In fact it is stably nontrivial. We have seen that \hat{d} is stably nontrivial; let $\delta \in (E^{(N-1)} \wedge S_k)^*(X \wedge S^k)$ denote the corresponding cohomology class. Then $\alpha \wedge \cdots \wedge \alpha \wedge \beta = \wedge^* \delta$ which is nonzero by the cohomology analogue of Lemma 1.2. The existence of an N -fold product in the cohomology of $X \times S^k$ immediately implies $\text{wcat}(X \times S^k) \geq N + 1$.

This argument is easily extended to products of spheres. Let σ_i be the class of the identity map in $(\widetilde{S}_{k_i})^*(S^{k_i})$, and write

$$\beta_i = p_{i+1}^*(\sigma_i) \in (E^{(N-1)} \wedge S_{k_1} \wedge \cdots \wedge S_{k_m})^*(X \times S^{k_1} \times \cdots \times S^{k_m}).$$

It now follows just as above that $\alpha \wedge \cdots \wedge \alpha \wedge \beta_1 \wedge \cdots \wedge \beta_m \neq 0$ and so

$$\text{wcat}(X \times S^{k_1} \times \cdots \times S^{k_m}) \geq \text{wcat}(X) + m.$$

Theorem 3.2 Let X be a n dimensional $(p-1)$ connected CW complex. If $\text{cat}(X) = \lfloor \frac{n}{p} \rfloor + 1$ and $n \not\equiv -1 \pmod{p}$ (forcing $p > 1$), then

$$\text{cat}(X \times S^k) = \text{cat}(X) + 1$$

for all $k > 0$.

Proof

If $n < 2p$, then $\text{cat}(X) \leq 2$, and the result is trivial. If $\text{cat}(X) = 1$, then $X \times S^k \simeq S^k$, so $\text{cat}(X \times S^k) = 2$. If X is nontrivial, there will be a nonzero class $u \in \widetilde{H}^*(X)$, and hence a nontrivial cup product in $H^*(X \times S^k)$. Therefore, we assume $n \geq 2p$; it follows that $\frac{n+2}{p} > \frac{n+2}{2p} + 1$.

By Theorem 2.2, $\text{cat}(X) = \text{wcat}(X)$. By Note 1.1, $\text{cat}(X) \geq \frac{n+2}{p}$, and so

$$\text{wcat}(X) = \text{cat}(X) \geq \frac{n+2}{p} \geq \frac{n+2}{2p} + 1,$$

so Theorem 3.1 applies. Therefore,

$$\text{cat}(X) + 1 \geq \text{cat}(X \times S^k) \geq \text{wcat}(X \times S^k) \geq \text{wcat}(X) + 1 = \text{cat}(X) + 1,$$

and the theorem is proved. ■

Using the remarks following Theorem 3.1, we can extend this argument to show

$$\text{cat}(X \times S^{k_1} \times \cdots \times S^{k_m}) = \text{cat}(X) + m$$

when X satisfies the hypotheses of Theorem 3.2.

Example 3.3 Let $X = \text{Sp}(2)$. Then $n = \dim(\text{Sp}(2)) = 10$, and $\text{Sp}(2)$ is 2 connected, so $p = 3$. By Example 4.4 of [9], $\text{cat}(\text{Sp}(2)) = 4$. Thus

$$\text{cat}(\text{Sp}(2)) = 4 = \left\lfloor \frac{10}{3} \right\rfloor + 1 = 4,$$

so Theorem 3.2 applies. It follows that

$$\text{cat}(\text{Sp}(2) \times S^{k_1} \times \cdots \times S^{k_m}) = 4 + m$$

for all $k_i > 0$.

4 Extending Singhof's Theorem

We continue to assume that X is n dimensional and $(p - 1)$ connected.

The results of [8] and [6] which concern Ganea's conjecture are explicitly stated only for the case $p = 1$. The arguments given apply to larger p , and significantly better results follow immediately. From this point of view, the most complete statement of Singhof's theorem is as follows (cf. [8], Theorems 6.1 and 6.2, and Corollary 6.7 or [6], Theorems 1 and 2).

Theorem (Singhof) If X is an n dimensional $(p - 1)$ connected compact PL or differentiable manifold with $n \geq 4$, $\text{cat}(X) = N \geq 4$, and

$$N \geq \begin{cases} \frac{n+4k+3}{2k} & \text{if } k \leq p \\ \frac{n+k+3p+3}{2p} & \text{if } k \geq p \end{cases}$$

then $\text{cat}(X \times S^k) = N + 1$.

This section is devoted to extending the values of k for which Singhof's theorem applies. For the rest of this section, we assume that X is an n dimensional $(p - 1)$ connected compact PL or differentiable manifold, that $n \geq 4$, and that $N = \text{cat}(X) \geq 4$.

We will show that if Singhof's theorem applies to the manifold X and S^p , then

$$\text{cat}(X \times S^k) = \text{cat}(X) + 1$$

for all $k > 1$. We first show that $\text{cat}(X \times (S^p)^M) = \text{cat}(X) + M$. For each $k > \frac{p}{2}$, Singhof's theorem will apply to $X \times (S^p)^M$ and S^k if M is large enough. It is then a simple matter to conclude that $\text{cat}(X \times S^k) = \text{cat}(X) + 1$.

The case $k \leq \frac{p}{2}$ is more difficult. For this, we choose M large enough that

$$\text{cat}\left(X \times (S^p)^M \times S^{\left\lceil \frac{p}{2} \right\rceil}\right) = \text{cat}(X) + M + 1$$

and such that Singhof's Theorem applies to $X \times (S^p)^M \times S^{\lceil \frac{p}{2} \rceil}$. Then the previous case shows that we may take any $k > \frac{p}{2}$. An easy induction completes the argument.

In [7], Rudyak uses the Singhof's theorem as stated in [8] and an argument similar to ours to prove the case $p = 1$ of our Corollary 4.2 (see [6], Theorem 3.7).

Lemma 4.1 Let X be as above, and suppose

$$N \geq \frac{n + 4p + 3}{2p}.$$

Let $k > \frac{p}{2}$. Write $X' = X \times (S^p)^M \times S^k$, $n' = \dim(X')$, $N' = \text{cat}(X')$, and denote the connectivity of X' by $(p' - 1)$ (of course, these may vary with M). If M is large enough then

$$N' = \text{cat}(X') = N + M + 1$$

and

$$N' \geq \frac{n' + 4p' + 3}{2p'}.$$

Proof

By Singhof's theorem applied to X and S^p ,

$$\text{cat}(X \times S^p) = \text{cat}(X) + 1.$$

Furthermore, since

$$\text{cat}(X \times S^p) = N + 1 \geq \frac{(n + p) + 4p + 3}{2p} = \left(\frac{n + 4p + 3}{2p} \right) + \frac{1}{2},$$

Singhof's theorem also applies to $X \times S^p$ and S^p . Inductively, we obtain

$$\text{cat}(X \times (S^p)^M) = N + M.$$

We have to consider 2 cases. First assume $k \geq p$. To apply Singhof's theorem to $X \times (S^p)^M$ and S^k , we require

$$N + M = \text{cat}(X \times (S^p)^M) \geq \frac{(n + Mp) + k + 3p + 3}{2p}$$

or

$$N \geq \frac{n + k + 3p + 3}{2p} - \frac{M}{2}.$$

Clearly, if M is large enough, this inequality will hold. The case $\frac{p}{2} < k \leq p$ is similar. Now we require

$$N + M = \text{cat}(X \times (S^p)^M) \geq \frac{(n + Mp) + 4k + 3}{2k}$$

or

$$N \geq \frac{n + 4k + 3}{2k} - M \left(1 - \frac{p}{2k} \right).$$

Since $k > \frac{p}{2}$, this inequality will hold for M large enough. This proves the first assertion.

To prove the second assertion in the case $k \geq p$, we require

$$N + M + 1 \geq \frac{(n + Mp + k) + 4p + 3}{2p}$$

or

$$N + 1 \geq \frac{n + k + 4p + 3}{2p} - \frac{M}{2}.$$

In the case $\frac{p}{2} < k \leq p$, we require

$$N + M + 1 \geq \frac{n + Mp + k + 4k + 3}{2k}$$

or

$$N + 1 \geq \frac{n + 5k + 3}{2k} - M \left(1 - \frac{p}{2k}\right).$$

In either case, the condition holds for M large enough. ■

Corollary 4.2 If X is as above, and $N \geq \frac{n+4p+3}{2p}$, then

$$\text{cat}(X \times S^k) = \text{cat}(X) + 1$$

for all $k > 1$. If $p = 1$, we may allow $k = 1$.

Proof

We begin by defining a decreasing sequence of positive integers by $p_1 = p$, and

$$p_{j+1} = \begin{cases} \frac{p_j+1}{2} & \text{if } p_j \text{ is odd} \\ \frac{p_j}{2} + 1 & \text{if } p_j \text{ is even} \end{cases}.$$

It is easy to see that if $p > 1$ this sequence decreases to 2, where it stabilizes and if $p = 1$, it is constantly 1. Notice that $p_{j+1} > \frac{p_j}{2}$.

We now define a sequence of spaces X_j . They will be n_j dimensional, $(p_j - 1)$ connected, and $\text{cat}(X_j) = N_j$. The key properties of the X_j are the following:

1. $X_{j+1} = X_j \times (S^{p_j})^{M_j} \times S^{p_{j+1}}$;
2. $N_{j+1} = N_j + M_j + 1$;
3. $N_j \geq \frac{n_j+4p_j+3}{2p_j}$.

To construct the X_j , first set $X_1 = X$. Given X_j , define X_{j+1} by property 1; by Lemma 4.1 we can find an M_j large enough that 2 and 3 are satisfied.

First we consider the case $p > 1$. Let j be large enough that $p_j = 2$, and consider X_j . By definition,

$$X_j = X \times \left((S^p)^{M_1} \times \prod_{i=2}^{j-1} (S^{p_i})^{M_{i+1}} \right) \times S^2.$$

Write $S = (S^p)^{M_1} \times \prod (S^{p_i})^{M_{i+1}}$ and $s = \text{cat}(S)$. By property 2, we see $N_j = N + s + 1$.

Let $k \geq 2$. By Lemma 4.1, there is an M large enough that

$$\text{cat}\left(X_j \times (S^2)^M \times S^k\right) = N_j + M + 1 = N + s + M + 2.$$

On the other hand,

$$\text{cat}\left((X \times S^k) \times S \times (S^2)^{M+1}\right) \leq \text{cat}(X \times S^k) + s + M + 1.$$

Together, these inequalities imply that $\text{cat}(X \times S^k) \geq N + 1$. It follows that $\text{cat}(X \times S^k) = \text{cat}(X) + 1$.

The case $p = 1$ is easier: we apply Lemma 4.1 directly to X and argue as in the previous paragraph. ■

By increasing the M_j if necessary, we can use the same argument to show that under the conditions of Corollary 4.2,

$$\text{cat}(X \times S^{k_1} \times \cdots \times S^{k_m}) = \text{cat}(X) + m.$$

If $p > 1$, we need to require $k_i \geq 2$, but if $p = 1$, we allow $k_i = 1$ also. We omit the details.

Example 4.3 Let $X = \text{Sp}(2)$, as in Example 3.3. Since

$$\text{cat}(\text{Sp}(2)) = 4 < \frac{10 + 4 \cdot 3 + 3}{2 \cdot 3} = \frac{25}{6},$$

neither Singhof's theorem or Corollary 4.2 apply to $\text{Sp}(2)$.

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