

ON SIMPLICIAL COMMUTATIVE ALGEBRAS WITH VANISHING ANDRÉ-QUILLEN HOMOLOGY

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ABSTRACT. In this paper, we study the André-Quillen homology of simplicial commutative ℓ -algebras, ℓ a field, having certain vanishing properties. When ℓ has non-zero characteristic, we obtain an algebraic version of a theorem of J.-P. Serre and Y. Umeda that characterizes such simplicial algebras having bounded homotopy groups. We further discuss how this theorem fails in the rational case and, as an application, indicate how the algebraic Serre theorem can be used to resolve a conjecture of D. Quillen for algebras of finite type over Noetherian rings, which have non-zero characteristic.

OVERVIEW

Algebraic Serre Theorem. The following topological theorem is due to J.-P. Serre [17] at the prime 2 and to Y. Umeda [19] at odd primes.

Serre's Theorem. *Let X be a nilpotent space such that $H_s(X; \mathbb{F}_p) = 0$ for $s \gg 0$ and each $H_s(X; \mathbb{F}_p)$ is finite dimensional. Then the following are equivalent*

1. $\pi_s(X) \otimes \mathbb{Z}/p = 0$, $s \gg 0$;
2. $\pi_s(X) \otimes \mathbb{Z}/p = 0$, $s \geq 2$.

In [1, 15, 16], M. André and D. Quillen constructed the notion of a homology $D_*(A|R; M)$ for a homomorphism $R \rightarrow A$ of simplicial commutative rings, with coefficients in a simplicial A -module M . These homology groups can be defined as $\pi_*(\mathcal{L}(A|R) \otimes_A M)$ where the simplicial A -module $\mathcal{L}(A|R)$ is called the **cotangent complex** of A over R .

We now propose an algebraic analogue of Serre's Theorem for simplicial augmented ℓ -algebras. To accomplish this we will take simplicial homotopy $\pi_*(-)$ to be the analogue of $H_*(-; \mathbb{F}_p)$ and $H_*^Q(-) = D_*(-|\ell; \ell)$ to be the analogue of $\pi_*(-) \otimes \mathbb{Z}/p$.

Algebraic Serre Theorem. *Let A be a homotopy connected (i.e. $\pi_0 A = \ell$) simplicial supplemented commutative ℓ -algebra, with $\text{char } \ell \neq 0$, such that $\pi_* A$ is a finite graded ℓ -module. Then the following are equivalent*

1. $H_s^Q(A) = 0$, $s \gg 0$;
2. $H_s^Q(A) = 0$, $s \geq 2$.

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We shall prove this theorem by following Serre's original approach in [17]. This will require pooling technical tools such as an analogue of the notion of connected covers of spaces and various ways for making computations of the homotopy and homology of simplicial commutative algebras.

The algebraic Serre theorem cannot hold in general when the ground field has characteristic zero. At the end of §2, we indicate a partial result in the rational case and point to some examples that show that a full version of our theorem cannot hold rationally.

Connections to Quillen's Conjecture. D. Quillen has conjectured that the cotangent complex has certain rigidity properties. In particular, we recall the following, which can be found in [15, (5.7)]:

Quillen's Conjecture. *If A is an algebra of finite type over a Noetherian ring R , such that A has finite flat dimension over R and $\mathrm{fd}_A \mathcal{L}(A|R)$ is finite, then A is a quotient of a polynomial ring by an ideal generated by a regular sequence.*

Earlier results of Lichtenbaum-Schlessinger [9], Quillen [15], and André [1] prove that an R -algebra A is a complete intersection if and only if $\mathrm{fd}_A \mathcal{L}(A|R) \leq 1$. In characteristic 0 the conjecture was proved by Avramov-Halperin [3]. The general case was proved by L. Avramov. Furthermore, Avramov characterized those homomorphisms $R \rightarrow A$ of Noetherian rings having locally finite flat dimension with $\mathrm{fd}_A \mathcal{L}(A|R) < \infty$. See [2] for details.

As a consequence of the Algebraic Serre Theorem, we have the following:

Theorem 0.1. *Quillen's conjecture holds provided the algebra A has non-zero characteristic.*

Proof. Since $\mathrm{fd}_A \mathcal{L}(A|R) \leq N$ if and only if $D_s(A|R; -) = 0$ for $s > N$ then we seek to show that the latter implies $D_s(A|R; -) = 0$ for $s \geq 2$. By [1, (S.30)], it is enough to show that, for each prime ideal $\wp \subset A$, $D_s(A|R; \mathbf{k}(\wp)) = 0$ for $s \geq 2$, where $\mathbf{k}(\wp)$ is the residue field of A_\wp . Since A has non-zero characteristic then each $\mathbf{k}(\wp)$ has prime characteristic. Let ℓ denote a fixed residue field.

Since A is an algebra of finite type over R , then the unit map factors as $R \rightarrow R[X] \xrightarrow{\sigma} A$, with X a finite set and σ a surjection. Since $R \rightarrow R[X]$ is a flat homomorphism, then $D_s(R[X]|R; \ell) \cong D_s(\ell[X]|\ell; \ell) = 0$ for $s \geq 1$, by [1, (4.54, 6.26)]. An application of [1, (5.1)] now implies that $D_s(A|R; \ell) \cong D_s(A|R[X]; \ell)$ for $s \geq 2$. Since $\mathrm{fd}_R A = \mathrm{fd}_{R[X]} A$, by a change-of-rings spectral sequence argument, we may thus assume that $R \rightarrow A$ is surjective.

Let \mathcal{F} be the homotopy pushout over ℓ of $R \rightarrow A$ in the simplicial model category of simplicial commutative R -algebras over ℓ (see [14, 15, 16, 8] for general discussions pertaining to this model structure). Then \mathcal{F} is a connected simplicial supplemented commutative ℓ -algebra with the properties

$$D_*(\mathcal{F}|\ell; \ell) \cong D_*(A|R; \ell)$$

and

$$\pi_*\mathcal{F} \cong \mathrm{Tor}_*^R(A, \ell),$$

the first isomorphism following from the flat base change property for André-Quillen homology [16, (4.7)] while the second follows from an argument utilizing the Kunnet spectral sequence of Theorem 6.b in [14, §II.6].

By the assumption that $\mathrm{fd}_R A < \infty$, it follows that $\pi_*\mathcal{F}$ is a finite graded ℓ -module. The result now follows. \square

Generalizing Quillen’s Conjecture. We propose the following simplicial generalization of Quillen’s conjecture.

Conjecture. *Let R be a Noetherian ring and let A be a simplicial commutative R -algebra with the following properties:*

- (1) $\pi_0 A$ is a Noetherian ring having non-zero characteristic;
- (2) $\pi_* A$ is finite graded as a $\pi_0 A$ -module;
- (3) $\mathrm{fd}_R \pi_* A < \infty$.

Then $D_s(A|R; -) = 0$ for $s \gg 0$ implies $D_s(A|R; -) = 0$ for $s \geq 2$.

Note. The condition on the characteristic of R is clearly needed, as noted above.

A proof of this conjecture can be given when stronger conditions on $\pi_0 A$ are assumed. See [18]. For example, by the same reduction to the algebraic Serre theorem performed in the proof of Theorem 0.1, the following special case can be proved.

Theorem 0.2. *The conjecture holds if property (1) is replaced by the stronger property (1’) $\pi_0 A$ is an algebra of finite type over R .*

Organization of this paper. In the first section, we review the needed notions of the model category structure of simplicial supplemented commutative algebras. In particular, we review the construction and some properties of the homotopy and André-Quillen homology for simplicial commutative algebras. In the next section, we introduce the notion of n -connected envelopes for simplicial commutative algebras which dualizes the notion of n -connected covers of spaces. We then pause to record a crucial splitting result and discuss specific types of simplicial commutative algebras which demonstrate the failure of the algebraic Serre theorem rationally. We then, in the third section, discuss the properties of the Poincaré series for the homotopy of a simplicial commutative algebra. This leads to the last section where we give a proof of the algebraic Serre theorem.

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1. THE HOMOTOPY AND HOMOLOGY OF SIMPLICIAL COMMUTATIVE ALGEBRAS

We now review the closed simplicial model category structure for $s\mathcal{A}_\ell$ the category of simplicial commutative ℓ -algebras augmented over ℓ . We will assume the reader is familiar with the general theory of homotopical algebra given in [14].

We call a map $f : A \rightarrow B$ in $s\mathcal{A}_\ell$ a

- (i) weak equivalence ($\xrightarrow{\sim}$) $\iff \pi_* f$ is an isomorphism;
- (ii) fibration (\twoheadrightarrow) $\iff f$ provided the induced canonical map $A \rightarrow B \times_{\pi_0 B} \pi_0 A$ is a surjection;
- (iii) cofibration (\hookrightarrow) $\iff f$ is a retract of an almost free map [7, p. 23].

Theorem 1.1. [14, 12, 7] *With these definitions, $s\mathcal{A}_\ell$ is a closed simplicial model category.*

For a description of the simplicial structure, see section II.1 of [14]. The details will not be needed for our purposes. Given a simplicial vector space V , over a field ℓ , define its normalized chain complex NV by

$$(1.1) \quad N_n V = V_n / (\text{Im } s_0 + \cdots + \text{Im } s_n)$$

and $\partial : N_n V \rightarrow N_{n-1} V$ is $\partial = \sum_{i=0}^n (-1)^i d_i$. The homotopy groups $\pi_* V$ of V is defined as

$$\pi_n V = H_n(NV), \quad n \geq 0.$$

Thus for A in $s\mathcal{A}_\ell$ we define $\pi_* A$ as above. The Eilenberg-Zilber theorem (see [10]) shows that the algebra structure on A induces an algebra structure on $\pi_* A$.

If we let \mathcal{V} be the category of ℓ -vector spaces, then there is an adjoint pair

$$S : \mathcal{V} \iff \mathcal{A}_\ell : I,$$

where I is the augmentation ideal function and S is the symmetric algebra functor. For an object V in \mathcal{V} and $n \geq 0$, let $K(V, n)$ be the associated Eilenberg-MacLane object in $s\mathcal{V}$ so that

$$\pi_s K(V, n) = \begin{cases} V & s = n; \\ 0 & s \neq n. \end{cases}$$

Let $S(V, n) = S(K(V, n))$, which is an object of $s\mathcal{A}_\ell$ called a **sphere algebra**.

Now recall the following standard result which will be useful for us (see section II.4 of [14]).

Lemma 1.2. *If V is a vector space, A a simplicial commutative algebra, and $[\quad , \quad]$ denotes morphisms in $\text{Ho}(s\mathcal{A}_\ell)$, then the map*

$$[S(V, n), A] \rightarrow \text{Hom}_{\mathcal{V}}(V, I\pi_n A)$$

is an isomorphism. In particular, $\pi_n A = [S(n), A]$, where $S(n) = S(\ell, n)$.

Here \mathcal{V} is the category of vector spaces.

Thus the primary operational structure for the homotopy groups in $s\mathcal{A}_\ell$ is determined by $\pi_*S(V_\bullet)$ for any V_\bullet in $s\mathcal{V}$. By Dold's theorem [6] there is a triple \mathcal{S} on graded vector spaces so that

$$(1.2) \quad \pi_*S(V) \cong \mathcal{S}(\pi_*V)$$

encoding this structure. If $\text{char } \ell = 0$, \mathcal{S} is the free skew symmetric functor and, for $\text{char } \ell > 0$, \mathcal{S} is a certain free divided power algebra (see, for example, [4, 7, 13]).

Recall [16] that given a map of simplicial commutative rings $R \rightarrow S$, there is a functorially defined simplicial S -module $\Omega_{S|R}$ called the **Kaehler differentials of S over R** . Replacing S by a cofibrant simplicial R -algebra model X then the **cotangent complex of S over R** is defined as the cofibrant simplicial S -module

$$\mathcal{L}(S|R) := \Omega_{X|R} \otimes_X S$$

and the **André-Quillen homology of S over R** with coefficients in a simplicial S -module M is defined as

$$D_*(S|R; M) := \pi_*(\mathcal{L}(S|R) \otimes_S M).$$

For A in \mathcal{A}_ℓ , define the indecomposable functor to be $QA = I(A)/I^2(A)$ which is an object of \mathcal{V} . Define the homology functor $H_*^Q(-) : s\mathcal{A}_\ell \rightarrow gr\mathcal{V}$ [7, 8, 12] by

$$H_s^Q(A) = \pi_s QX, \quad s \geq 0,$$

where we choose a factorization

$$\ell \hookrightarrow X \xrightarrow{\sim} A$$

of the unit $\ell \rightarrow A$ as a cofibration and a trivial fibration. This definition is independent of the choice of factorization as any two are homotopic over A (note that every object of $s\mathcal{A}_\ell$ is fibrant). It is straightforward to show [7, (A.1)] that

$$\Omega_{B|\ell} \otimes_B \ell \cong QB,$$

for any augmented ℓ -algebra B , and so

$$H_*^Q(A) = D_*(A|\ell; \ell).$$

We now summarize methods for computing homotopy and André-Quillen homology that we will need for this paper.

- Proposition 1.3.** (1) *If $f : A \xrightarrow{\sim} B$ is a weak equivalence in $s\mathcal{A}_\ell$, then $H_*^Q(f) : H_*^Q(A) \xrightarrow{\cong} H_*^Q(B)$ is an isomorphism. The converse holds provided $I\pi_0 A = 0$, that is, A is **homotopy connected**.*
- (2) *There is a Hurewicz homomorphism $h : I\pi_* A \rightarrow H_*^Q(A)$ such that if A is homotopy connected and $H_s^Q(A) = 0$ for $s < n$ then A is $(n-1)$ -connected and*
- i. $h : \pi_n A \xrightarrow{\cong} H_n^Q(A)$ *is an isomorphism and*
 - ii. $h : \pi_{n+1} A \xrightarrow{\cong} H_{n+1}^Q(A)$ *is a surjection, which is also injective for $n > 1$.*

- (3) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a cofibration sequence in $Ho(s\mathcal{A}_\ell)$. Then: There is a long exact sequence

$$\begin{aligned} \cdots \rightarrow H_{s+1}^Q(C) \xrightarrow{\partial} H_s^Q(A) \xrightarrow{H_*^Q(f)} H_s^Q(B) \\ \xrightarrow{H_*^Q(g)} H_s^Q(C) \xrightarrow{\partial} H_{s-1}^Q(C) \rightarrow \cdots \end{aligned}$$

Proof. For all of these, see [7, IV]. In particular, (2) follows from Quillen's fundamental spectral sequence and the connectivity of Dold's functor \mathcal{S} [6]. \square

2. CONNECTED ENVELOPES

In this section, we construct and determine some properties of a useful tool for studying simplicial algebras.

Given A in $s\mathcal{A}_\ell$, which is homotopy connected, we define its **connected envelopes** to be a sequence of cofibrations

$$A = A(0) \xrightarrow{j_1} A(1) \xrightarrow{j_2} \cdots \xrightarrow{j_n} A(n) \xrightarrow{j_{n+1}} \cdots$$

with the following properties:

- (1) For each $n \geq 1$, $A(n)$ is a n -connected.
- (2) For $s > n$,

$$H_s^Q A(n) \cong H_s^Q A.$$

- (3) There is a cofibration sequence

$$S(H_n^Q A, n) \xrightarrow{f_n} A(n-1) \xrightarrow{j_n} A(n).$$

The existence of a connected envelopes is a consequence of the following :

Proposition 2.1. *Let A in $s\mathcal{A}_\ell$ be $(n-1)$ -connected for $n \geq 1$. Then there exists a map in $s\mathcal{A}_\ell$,*

$$f_n : S(H_n^Q A, n) \rightarrow A,$$

with the following properties

1. f_n is an isomorphism on π_n and H_n^Q ;
2. the homotopy cofibre $M(f_n)$ of $f_n : S(H_n^Q A, n) \rightarrow A$ is n -connected and satisfies $H_s^Q M(f_n) \cong H_s^Q A$ for $s > n$;
3. if $H_s^Q A = 0$, $s \neq n > 0$ then f_n is an isomorphism in $Ho(s\mathcal{A}_\ell)$.

Proof. (1.) By the Hurewicz theorem, Proposition 1.3 (2), the map $h : \pi_n A \rightarrow H_n^Q A$ is an isomorphism. By Lemma 1.2 we have an isomorphism

$$[S(H_n^Q A, n), A] \cong \text{Hom}_\mathcal{V}(H_n^Q A, I\pi_n A).$$

Choosing f_n to correspond to the inverse of h gives the result.

- (2.) This follows from (1.) and the transitivity sequence

$$H_{s+1}^Q M(f_n) \rightarrow H_s^Q S(H_n^Q A, n) \rightarrow H_s^Q A \rightarrow H_s^Q M(f_n).$$

(3.) By (1.), $f_n : S(H_n^Q A, n) \rightarrow A$ is an H_n^Q -isomorphism and hence a weak equivalence by Proposition 1.3(1). The converse follows from the computation

$$H_s^Q S(V, n) = \pi_s Q S(V, n) = \pi_s K(V, n) = V$$

for $s = n$ and 0 otherwise. \square

Applications.

Proposition 2.2. *If there is a cofibration sequence in $s\mathcal{A}_\ell$*

$$S(V, n-1) \rightarrow A \rightarrow S(W, n)$$

for some vector spaces V and W and some $n > 1$, then in $Ho(s\mathcal{A}_\ell)$

$$A \cong S(H_{n-1}^Q A, n-1) \otimes S(H_n^Q A, n).$$

Proof. Proposition 1.3 (3) tells us that $H_s^Q A = 0$ for $s \neq n, n-1$, and there is an exact sequence

$$0 \rightarrow H_n^Q A \rightarrow V \rightarrow W \rightarrow H_{n-1}^Q A \rightarrow 0.$$

Thus A is $n-2$ connected and a connected envelope gives a cofibration

$$S(H_{n-1}^Q A, n-1) \xrightarrow{i} A \xrightarrow{j} S(H_n^Q A, n)$$

for which $H^Q(j)$ is an isomorphism. Lemma 1.2 and Proposition 1.3 (2) give a commutative diagram

$$\begin{array}{ccc} [S(H_n^Q A, n), A] & \xrightarrow{j_*} & [S(H_n^Q A, n), S(H_n^Q A, n)] \\ \cong \downarrow & & \downarrow \cong \\ \text{Hom}(H_n^Q A, \pi_n A) & \xrightarrow{j_*} & \text{Hom}(H_n^Q A, \pi_n S(H_n^Q A, n)) \\ \cong \downarrow h_* & & h_* \downarrow \cong \\ \text{Hom}(H_n^Q A, H_n^Q A) & \xrightarrow{\cong} & \text{Hom}(H_n^Q A, H_n^Q A) \end{array}$$

which shows that j splits up to homotopy. \square

From Proposition 2.1 (3), if $\text{char } \ell = 0$ and V finite-dimensional then $H_*^Q S(V, n) \cong V$ concentrated in degree n and $\pi_* S(V, n)$ is free skew-commutative on a basis of V concentrated in degree n . Thus $\pi_* S(V, n)$ is bounded for any odd n , showing that the algebraic Serre theorem cannot hold rationally. On the other hand, we do have the following

Proposition 2.3. *Let A be a connected simplicial augmented commutative ℓ -algebra, with $\text{char } \ell = 0$, such that $\pi_* A$ is a finite graded ℓ -module. Then if $H_{\text{odd}}^Q = 0$ and $H_s^Q A = 0$ for $s \gg 0$ we can conclude that $I\pi_* A = 0$.*

Proof. 1. Suppose $H_m^Q A \neq 0$ implies that $2r \leq m \leq 2s$. Then $H_m^Q A(2r) \neq 0$ implies that $2(r+1) \leq m \leq 2s$. Furthermore, $H_{\text{odd}}^Q A = 0$ and $\pi_* A(2r)$ is a finite graded ℓ -module by a spectral sequence argument [14, §II.6] (using the fact that $\pi_* S(V, 2r)$ is

finitely-generated polynomial, when V is finite). The result follows by an induction on $s - r$, given that the result is certainly true for $A < r, 1 > \simeq S(2r)$, for any r . \square

Example. Here is another example of a type of rational simplicial algebra with finite homotopy and André-Quillen homology.

Since $\pi_* S(2r) \cong \ell[x_{2r}]$, let $f : S(2rs) \rightarrow S(2r)$ represent x_{2r}^s . Define $A < r, s >$ to be the cofibre of f . Then the cofibration sequence extends to

$$S(2r) \rightarrow A < r, s > \rightarrow S(2rs + 1).$$

The computation of $\pi_*(A < r, s >)$ can be achieved by a Serre spectral sequence argument (see the proof of Lemma 3.1) and the computation of $H_*^Q(A < r, s >)$ can be obtained from Proposition 2.1 (3) using Proposition 1.3 (3). In the end, we obtain

$$\pi_m(A < r, s >) = \begin{cases} \ell & m = 2ri, 0 \leq i < s, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_m^Q(A < r, s >) = \begin{cases} \ell & m = 2r, \\ \ell & m = 2rs + 1, \\ 0 & \text{otherwise.} \end{cases}$$

3. THE POINCARÉ SERIES OF A SIMPLICIAL ALGEBRA

Let A be a homotopy connected simplicial supplemented commutative ℓ -algebra such that $\pi_* A$ is of finite-type. We define its *Poincaré series* by

$$\vartheta(A, t) = \sum_{n \geq 0} (\dim_{\ell} \pi_n A) t^n.$$

If V is a finite-dimensional vector space and $n > 0$ we write

$$\vartheta(V, n, t) = \vartheta(S(V, n), t).$$

Combining the work of [5] with [17, 19], this latter series converges in the open unit disc.

Given power series $f(t) = \sum a_i t^i$ and $g(t) = \sum b_i t^i$ we define the relation $f(t) \leq g(t)$ provided $a_i \leq b_i$ for each $i \geq 0$.

Lemma 3.1. *Given a cofibration sequence*

$$A \rightarrow B \rightarrow C$$

of connected objects in \mathcal{A}_{ℓ} with finite-type homotopy groups, then

$$\vartheta(B, t) \leq \vartheta(A, t)\vartheta(C, t)$$

which is an equality if the sequence is split.

Proof. First, there is a Serre spectral sequence

$$E_{s,t}^2 = \pi_s(C \otimes \pi_t A) \implies \pi_{s+t} B.$$

This follows from Theorem 6(d) in §II.6 of [14], which gives a 1st-quadrant spectral sequence

$$E_{*,*}^2 = \pi_*(B \otimes_A \pi_* A) \Rightarrow \pi_* B,$$

where $\pi_* A$ is an A -module via the augmentation $A \rightarrow \pi_0 A$. Here we can assume our cofibration sequence is a cofibration with cofibre C . Since A is connected, then $B \otimes_A \pi_* A \cong C \otimes \pi_* A$.

Thus we have

$$\vartheta(A, t)\vartheta(C, t) = \sum_n \left(\sum_{i+j=n} \dim_{\ell} E_{i,j}^2 \right) t^n \geq \vartheta(B, t).$$

Finally, if the cofibration sequence is split then the spectral sequence collapses, giving an equality. \square

Now given two power series $f(t)$ and $g(t)$ we say $f(t) \sim g(t)$ provided $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$. Given a Poincaré series $\vartheta(V, n, t)$, for a finite-dimensional ℓ -vector space V and $n > 0$, let

$$\varphi(V, n, t) = \log_p \vartheta(V, n, 1 - p^{-t}).$$

Then the following is a consequence of Théorème 9b in [17] and its generalization to arbitrary non-zero characteristics in [19], utilizing the results of [5] to translate into our present venue.

Proposition 3.2. *For V an ℓ -vector space of finite dimension q and $n > 0$ then $\varphi(V, n, t)$ converges on the real line and*

$$\varphi(V, n, t) \sim qt^{n-1}/(n-1)!.$$

4. PROOF OF THE ALGEBRAIC SERRE THEOREM.

Recall that A is to be a connected simplicial augmented commutative ℓ -algebra with $H_*^Q(A)$ bounded and $\pi_* A$ a finite graded ℓ -module. The approach we take is to mimic the proof of Serre's Theorem in [17]; utilizing higher connected envelopes, in place of higher connected covers, and Poincaré series for homotopy, in place of Poincaré series for homology. Unfortunately, owing to the nature of cofibration sequences, Serre's original proof runs into a glitch at the start in our situation. Fortunately, if we skip the first step and evoke Proposition 2.2, the remainder of Serre's proof works without a hitch.

Proof of the Algebraic Serre Theorem. Let

$$n = \max\{s \mid H_s^Q(A) \neq 0\}.$$

We must show that $n = 1$.

Consider the connected envelope

$$S(H_s^Q(A), s) \rightarrow A(s-1) \rightarrow A(s)$$

for each s . From the theory of cofibration sequences (see section I.3 of [14]) the above sequence extends to a cofibration sequence

$$A(s-1) \rightarrow A(s) \rightarrow S(H_s^Q(A), s+1).$$

Thus, by Lemma 3.1, we have

$$\vartheta(A(s), t) \leq \vartheta(A(s-1), t)\vartheta(H_s^Q(A), s+1, t).$$

Starting at $s = n - 2$ and iterating this relation, we arrive at the inequality

$$\vartheta(A(n-2), t) \leq \vartheta(A, t) \prod_{s=1}^{n-2} \vartheta(H_s^Q(A), s+1, t).$$

Now, $A(n-1) \cong S(H_n^Q(A), n)$ by Proposition 2.1 (3), but, by Proposition 2.2 (1) and Lemma 3.1, we have

$$\vartheta(A(n-2), t) = \vartheta(H_{n-1}^Q(A), n-1, t)\vartheta(H_n^Q(A), n, t).$$

Since $\pi_*(A)$ is of finite-type and bounded then there exists a $D > p$ such that $\vartheta(A, t) \leq D$, in the open unit disc. Combining, we have

$$\vartheta(H_{n-1}^Q(A), n-1, t)\vartheta(H_n^Q(A), n, t) \leq D \prod_{s=1}^{n-2} \vartheta(H_s^Q(A), s+1).$$

Applying a change of variables and \log_p to the above inequality, we get

$$\varphi(H_{n-1}^Q(A), n-1, t) + \varphi(H_n^Q(A), n, t) \leq d + \sum_{s=1}^{n-2} \varphi(H_s^Q(A), s+1).$$

By Proposition 3.2, there is a polynomial $f(t)$ of degree $n-2$, a non-negative integer a , and positive integers b and d such that

$$at^{n-2} + bt^{n-1} \leq d + f(t), \quad t \gg 0$$

which is clearly false for $n > 1$. Thus $n = 1$. The rest of the proof follows from Proposition 2.1 (3). \square

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