THOM SPECTRA OF GENERALIZED BRAID GROUPS

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Abstract. It is proved that Thom spectra of generalized braid groups are the wedges of suspensions over the Eilenberg-MacLane spectrum for $Z/2$. Precise structure of the Thom spectra of the generalized braid groups of the types C and D is obtained. For the generalized braid groups of type C the natural pairing analogous to the pairing of the classical braids is studied. This paring generates the multiplicative structure of the Thom spectrum such that the corresponding bordism theory has the coefficient ring isomorphic to the polynomial ring over $Z/2$ on one generator of dimension one: $Z/2[s]$.

The methods of Algebraic Topology were firstly applied for braid groups by V. I. Arnold [1]. E. Brieskorn [6] generalized the notion of braid group in connection with Coxeter groups. It was proved later by Mark Mahowald [18, 19] and Fred Cohen [10] that the Thom spectrum for the infinite braid group is equivalent to the Eilenberg-MacLane spectrum $K(Z/2)$. Ralph Cohen [11] established connections between Thom spectra for finite braid groups and the Brown-Gitler spectra. Here we are studying Thom spectra for generalized braid groups and proving that for the infinite generalized braid groups the Thom spectra are equivalent to the wedges of suspensions over the Eilenberg-MacLane spectrum $K(Z/2)$. For Thom spectra of finite generalized braid groups relations with the Brown-Gitler spectra are considered.

1. Generalized Braid Groups

Let V be a finite dimensional vector space (dim V = n) with euclidean structure. We denote by W a finite subgroup of $GL(V)$ generated by reflections. We use the terminology and the content of N. Bourbaki [4]. Let $\mathcal{M}$ be the set of hyperplanes such that W is generated by orthogonal reflections with respect to $M \in \mathcal{M}$. We suppose that for any $w \in W$ and any hyperplane $M \in \mathcal{M}$ the hyperplane $w(M)$ belongs to $\mathcal{M}$ and that W acts totally discontinuously on V. The following facts are well known [4].

Proposition 1. *(i) W permutes the chambers of $\mathcal{M}$ transitively.*

(ii) The closure $\bar{A}$ of a chamber A is a fundamental domain of W acting on V.

(iii) If $x \in V$ belongs to $\bar{A}$ its stabilizer is generated by the reflections with respect to the walls of A containing x.

Also there exists a set I and a one to one correspondence of elements of I with the walls of a chamber $A : i \mapsto M_i(A)$, which is called a canonical indexation of the walls of

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a chamber A. Then W is generated by the reflections $w_i = w_i(M_i), i \in I$, satisfying only the following relations

$$(w_i w_j)^{m_{i,j}} = e, \ i,j \in I,$$

where the natural numbers $m_{i,j}$ form the Coxeter matrix of W by which the Coxeter graph $\Gamma(W)$ of W is constructed. We use the following notations of P. Deligne [12]: $\mathrm{prod}(m;x,y)$ denotes the product $xyxy...$ (m factors). The generalized braid group $Br(W)$ of W [12] is defined as a group with generators from the set I and the following relations:

$$\mathrm{prod}\,(m_{i,j};i,j) = \mathrm{prod}\,(m_{j,i};j,i).$$

From this we get the presentation of the group W if we add the relations:

$$w_i^2 = e, \ i \in I.$$

We denote by τ_W the canonical map from $Br(W)$ to W. Classical braids on k strings Br_k are obtained by this construction if $W = A_k = \Sigma_{k+1}$, the symmetric group on $k+1$ symbols. In this case $m_{i,j} = 3$, if $i \neq j$.

Now let $J_1, ..., J_s$ be the sets of vertexes of the connected components of the Coxeter graph of W, W_q is the subgroup of W generated by the reflections $w_i, i \in J_q$, let V_q^0 be the subspace of V consisting of vectors invariant by the action of W_q, V_q is the orthogonal complement of V_q^0 in V, $V_0 = \bigcap_{1\leq q\leq s} V_q^0$. Then from the Proposition 5 ([4], Chapter V, §3.7) we have the following facts.

Proposition 2. *(i) The group W is the direct product of subgroups W_q $(1 \leq q \leq s)$.*

(ii) The vector space V is the direct sum of the orthogonal subspaces $V_1, ..., V_s, V_0$ invariant by the action of W.

If $V_0 = 0$ then each chamber is an open simplicial cone. The classification of irreducible (with connected Coxeter graph) Coxeter groups is well known (Theorem 1, Chapter VI, §4 of [4]). It consists of three infinite series: A, C and D and groups $E_6, E_7, E_8, F_4, G_2, H_3, H_4$ and $I_2(p)$.

Now let us consider the complexification V_C of V and the complexification M_C of $M \in \mathcal{M}$. Let $Y_W = V_C - \bigcup_{M\in\mathcal{M}} M_C$. Then we get from (iii) of Proposition 1 that W acts freely on Y_W. Let $X_W = Y_W/W$, Y_W be a covering of X_W corresponding to the group W. Let $y_0 \in A_0$ be a point in some chamber A_0 and x_0 is its image in X_W. For each $j \in I$, let ℓ_j' be a homotopy class of paths in Y_W starting from y_0 and ending in $w_j(y_0)$ which contains a polygon line with successive vertices: $y_0, y_0 + iy_0, w_j(y_0) + iy_0, w_j(y_0)$. The image ℓ_j of ℓ_j' in X_W is a loop with a base point x_0.

Theorem 1 (E. Brieskorn [6], P. Deligne [12])**.** *(i) The fundamental group $\pi_1(X_W, x_0)$ is generated by ℓ_j satisfying the following relations:*

$$\mathrm{prod}\,(m_{j,k};\ell_j,\ell_k) = \mathrm{prod}\,(m_{k,j};\ell_k,\ell_j).$$

(ii) The universal covering of X_W is contractible, so X_W is $K(\pi;1)$.

If a group W is a direct product of groups W' and W'', then the group $Br(W)$ is a direct product of groups $Br(W')$ and $Br(W'')$. So in the case of Proposition 2 we have that $Br(W) = Br(W_1) \times ... \times Br(W_s)$.

A good reference for generalized braid groups is the survey of V. Ya. Lin [17].

2. Pairings

There exist pairings for symmetric and braid groups $\Sigma_k \times \Sigma_l \to \Sigma_{k+l}$, $\mu : Br_k \times Br_l \to Br_{k+l}$, which commute with the maps $\tau_j : Br_j \to \Sigma_j$. For braid group this pairing may be constructed as the adding of l more strings to the initial k. If σ'_i are generators of Br_k, σ''_j are generators of Br_l and σ_r are generators of $Br(k+l)$, then the map μ can be expressed in the form:

$$\mu(\sigma'_i, e) = \sigma_i,\ 1 \le i \le k-1,$$

$$\mu(e, \sigma''_j) = \sigma_{j+1},\ 1 \le j \le l-1.$$

In terms of Coxeter diagrams it means that we take the vertex, corresponding to σ_k and imbed $Br_k \times Br_l$ into Br_{k+l} in accordance with the inclusion of the $\Gamma(\Sigma_k \times \Sigma_l) = \Gamma(\Sigma_k) \bigcup \Gamma(\Sigma_l)$ into two components of $\Gamma(\Sigma_{k+l}) \setminus \sigma_k$. This permits us to interpret various imbeddings of products of finite Coxeter groups into the group of bigger index. This is true for the corresponding generalized braid groups as well. We take away a vertex in a connected Coxeter graph and obtain different connected components (less or equal than 3), which correspond to irreducible Coxeter groups or braid groups whose direct product is the source of this mapping. For example, we have evident pairings:

$$\mu(C, A) : Br(C_k) \times Br(A_l) \to Br(C_{k+l+1}),$$

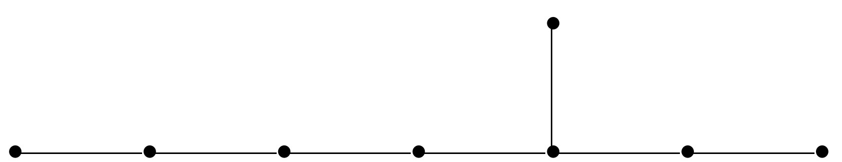
$$\mu(D, A) : Br(D_k) \times Br(A_l) \to Br(D_{k+l+1}) \text{ for any } k \text{ and } l,$$

or pairing

$$\mu(A_3, A_4; E_8) : Br(A_4) \times Br(A_3) \to Br(E_8),$$

which corresponds to the forth horizontal vertex of the Coxeter graph of E_8:

$(\mathbf{E_8})$



Embeddings of groups (not products) can also be expressed in this language. For example, we have an imbedding

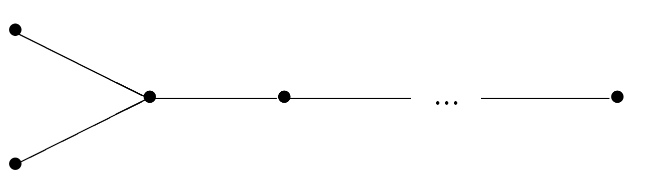
$$\alpha_C : Br(A_{l-1}) \to Br(C_l),$$

and two different imbeddings:

$$\alpha_D : Br(A_{l-1}) \to Br(D_l)$$

in accordance with two different vertices on the one end of the Coxeter graph for D_l:

$(\mathbf{D_l})$



We would like to consider generalized braid group $Br(C_k)$. We depict the Coxeter graph for C_k in the following way:

$(\mathbf{C_n})$

So we have a relation in $Br(C_k)$:

$$w_1 w_2 w_1 w_2 = w_2 w_1 w_2 w_1.$$

Let $Br_{1,n+1}$ be the subgroup of the braid group Br_{n+1} consisting of all elements of Br_{n+1} with the property that the permutations associated with them all leave the number 1 invariant. It means that the end of the first string is again at the first place. W.-L. Chow [9] found the presentation of this group with generators:

$$\sigma_2, ..., \sigma_n, a_2, ..., a_{n+1},$$

where σ_j is the standard generator of the braid group Br_{n+1} and $a_i = \sigma_1^{-1}...\sigma_{i-2}^{-1}\sigma_{i-1}^2\sigma_{i-2}...\sigma_1,\ 2 \leq i \leq n+1$. The elements $\sigma_2, ..., \sigma_n$ generate a subgroup in $Br_{1,n+1}$ isomorphic to Br_n and the elements $a_2, ..., a_{n+1}$ generate a normal free subgroup F_n, so that $Br_{1,n+1}$ is a semi-direct product of Br_n and F_n. The following relation is fulfilled in $Br_{1,n+1}$:

$$\sigma_2 a_2 \sigma_2 a_2 = a_2 \sigma_2 a_2 \sigma_2.$$

So the homomorphism $\phi : Br(C_n) \to Br_{1,n+1}$ can be defined by the formulae:

$$\phi(w_1) = a_2,$$
$$\phi(w_i) = \sigma_i,\ i = 2, ..., n.$$

We shall use the following statement.

Proposition 3. *The map ϕ defines an isomorphism*

$$\phi : Br(C_n) \cong Br_{1,n+1}.$$

Proof. We define the elements $v_i \in Br(C_n)$, $2 \leq i \leq n+1$, by the formula: $v_i = w_{i-1}...w_2 w_1 w_2^{-1}...w_{i-1}^{-1}$. We prove that $\phi(v_i) = a_i$ by induction. For $i = 2$ we have $\phi(v_2) = \phi(w_1) = a_2$. Let it be true for $j < i+1$. We consider $\phi(v_{i+1})$:

$$\phi(v_{i+1}) = \phi(w_i v_i w_i^{-1}) = \sigma_i a_i \sigma_i^{-1} = \sigma_i \sigma_1^{-1}...\sigma_{i-2}^{-1}\sigma_{i-1}^2\sigma_{i-2}...\sigma_1\sigma_i^{-1} =$$

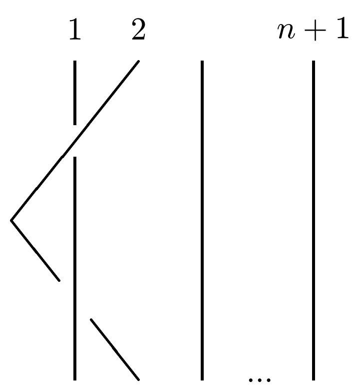
$$= \sigma_1^{-1}...\sigma_{i-2}^{-1}\sigma_i\sigma_{i-1}^2\sigma_i^{-1}\sigma_{i-2}...\sigma_1 = \sigma_1^{-1}...\sigma_{i-2}^{-1}\sigma_{i-1}^{-1}\sigma_{i-1}\sigma_i\sigma_{i-1}^2\sigma_i^{-1}\sigma_{i-2}...\sigma_1 =$$

$$= \sigma_1^{-1}...\sigma_{i-1}^{-1}\sigma_i\sigma_{i-1}\sigma_i\sigma_{i-1}\sigma_i^{-1}\sigma_{i-2}...\sigma_1 = \sigma_1^{-1}...\sigma_{i-1}^{-1}\sigma_i\sigma_i\sigma_{i-1}\sigma_i\sigma_i^{-1}\sigma_{i-2}...\sigma_1 =$$

$$= \sigma_1^{-1}...\sigma_{i-1}^{-1}\sigma_i^2\sigma_{i-1}...\sigma_1 = a_{i+1}$$

The elements $w_2, ..., w_n, v_2, ..., v_{n+1}$ can be taken as generators of $Br(C_n)$ with the same relations as for $\sigma_2, ..., \sigma_n, a_2, ..., a_{n+1}$.

The statement of this proposition is evident from the geometrical point of view. The space X_{C_n} can be interpreted as a space of n different pairs of points of $R^2\backslash 0$, symmetrical with respect to zero [16, 17]. That is the same as simply the space of n different points in $R^2\backslash 0$. The group Br_{n+1} is interpreted as the fundamental group of the space X_{A_n} of $n+1$ different points in R^2. If we consider one point (say 0) to be fixed we get X_{C_n}. For the fundamental group of X_{A_n} it means that the first string must have the same end as its beginning (equal to zero). In this interpretation the first generator of $Br(C_n)$ is the following braid:



We denote by β the homomorphism from $Br(C_n)$ to Br_n defined by the formulae:

$$\beta(w_1) = e,$$
$$\beta(w_i) = \sigma_{i-1}, \text{ for } i > 1.$$

Then we have $\beta\alpha = 1_{Br_n}$. It is known that the group C_k is isomorphic to a wreath product of symmetric group $\Sigma_k = A_{k-1}$ with $Z/2 : C_k \cong \Sigma_k \wr Z/2$. The pairing

$$m_C : C_k \times C_l \to C_{k+l}$$

may be defined using the pairing for the symmetric group

$$\Sigma_k \times \Sigma_l \to \Sigma_{k+l}$$

and a wreath product structure.

Let $z_j, j = 1, ..., n$, be the following elements in $Br(C_n)$: $z_1 = w_1, z_j = w_j...w_2w_1w_2...w_j$, $j = 2, ..., n$.

Lemma 1. *We have the following relations in $Br(C_n)$ between the elements z_j and $w_i, (i,j = 1,...,n)$:*

$$z_j w_i = w_i z_j, i \neq j, j+1$$

$$z_j w_{j+1} z_j w_{j+1} = w_{j+1} z_j w_{j+1} z_j,$$

$$z_i z_j = z_j z_i.$$

Proof. We use the induction by the index of z_j. For $j = 1$ the first two relations are the relations between the elements w_i and the third follows from the relation between the elements w_i. Now let all the relations be true for $j < k$ and consider them for $j = k$. For the first one let us suppose that $i \neq k-1, k, k+1$. Then we have

$$z_k w_i = w_k z_{k-1} w_k w_i = w_k z_{k-1} w_i w_k = w_k w_i z_{k-1} w_k = w_i w_k z_{k-1} w_k = w_i z_k.$$

If $i = k-1$, and $k > 2$, then we obtain

$$z_k w_{k-1} = w_k z_{k-1} w_k w_{k-1} = w_k w_{k-1} z_{k-2} w_{k-1} w_k w_{k-1} = w_k w_{k-1} w_k z_{k-2} w_{k-1} w_k =$$

$$= w_{k-1} w_k w_{k-1} z_{k-2} w_{k-1} w_k = w_{k-1} z_k.$$

If $k = 2$ and $i = 1$ then we get $z_2 w_1 = w_2 w_1 w_2 w_1 = w_1 w_2 w_1 w_2 = w_1 z_2$. Let us consider the second relation. We have:

$$z_k w_{k+1} z_k w_{k+1} = w_k z_{k-1} w_k w_{k+1} w_k w_k z_{k-1} w_k w_{k+1} =$$

$$= w_k z_{k-1} w_k w_{k+1} w_k z_{k-1} w_{k+1}^{-1} w_{k+1} w_k w_{k+1} = w_k z_{k-1} w_k w_{k+1} w_k z_{k-1} w_{k+1}^{-1} w_k w_{k+1} w_k =$$

$$= w_k z_{k-1} w_k w_{k+1} w_k w_{k+1}^{-1} z_{k-1} w_k w_{k+1} w_k = w_k z_{k-1} w_{k+1} w_k w_{k+1} w_{k+1}^{-1} z_{k-1} w_k w_{k+1} w_k =$$

$$= w_k w_{k+1} z_{k-1} w_k z_{k-1} w_k w_{k+1} w_k = w_k w_{k+1} w_k z_{k-1} w_k z_{k-1} w_{k+1} w_k =$$

$$= w_{k+1} w_k w_{k+1} z_{k-1} w_k w_{k+1} z_{k-1} w_k = w_{k+1} w_k z_{k-1} w_{k+1} w_k w_{k+1} z_{k-1} w_k =$$

$$= w_{k+1} w_k z_{k-1} w_k w_{k+1} w_k z_{k-1} w_k = w_{k+1} z_k w_{k+1} z_k.$$

It is sufficient to prove the third relation when $i < j$. So for $j = k$ we suppose at first that $i < k-1$, then we have

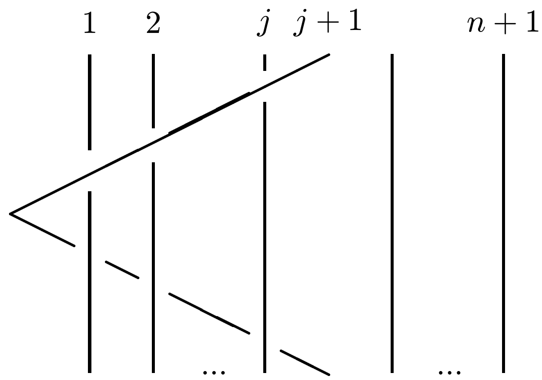
$$z_k z_i = w_k z_{k-1} w_k z_i = w_k z_{k-1} z_i w_k = w_k z_i z_{k-1} w_k = z_i w_k z_{k-1} w_k = z_i z_k.$$

If $i = k-1$ then we obtain

$$z_k z_{k-1} = w_k z_{k-1} w_k z_{k-1} = (\text{by the second relation}) = z_{k-1} w_k z_{k-1} w_k = z_{k-1} z_k.$$

Lemma is proved.

In the geometric interpretation z_j is the following braid:



Let $w'_1, ..., w'_k$ be the generators of $Br(C_k)$ and $w''_1, ..., w''_l$ are the generators of $Br(C_l)$. Then lemma 1 allows us to define a pairing $\mu(C, C) = \mu_C$:

$$\mu(C, C) : Br(C_k) \times Br(C_l) \to Br(C_{k+l})$$

by the formulae:

$$\mu(C, C)(w'_i) = w_i, 1 \leq i \leq k,$$
$$\mu(C, C)(w''_1) = w_{k+1}...w_2 w_1 w_2...w_{k+1},$$
$$\mu(C, C)(w''_j) = w_{k+j}, 1 \leq j \leq l.$$

It is easy to check that this pairing is associative, what means that the following diagram is commutative:

$$\begin{array}{ccc} Br(C_k) \times Br(C_l) \times Br(C_q) & \xrightarrow{\mu_C \times 1} & Br(C_{k+l}) \times Br(C_q) \\ \downarrow 1 \times \mu_C & & \downarrow \mu_C \\ Br(C_k) \times Br(C_{l+q}) & \xrightarrow{\mu_C} & Br(C_{k+l+q}). \end{array}$$

It agrees with the pairing for the Coxeter groups $m_C : C_k \times C_l \to C_{k+l+1}$, so we have a commutative diagram

$$\begin{array}{ccc} Br(C_k) \times Br(C_l) & \xrightarrow{\tau_C \times \tau_C} & C_k \times C_l \\ \downarrow \mu_C & & \downarrow m_C \\ Br(C_{k+l}) & \xrightarrow{\tau_C} & C_{k+l}. \end{array}$$

It also agrees with the pairing $Br(C_k) \times Br_l \to Br(C_{k+l})$ through the canonical inclusion $Br_l \to Br(C_l)$. It is also easy to check the commutativity of the diagram for the homomorphism α :

$$\begin{array}{ccc} Br_k \times Br_l & \xrightarrow{\alpha \times \alpha} & Br(C_k) \times Br(C_l) \\ \downarrow \mu & & \downarrow \mu_C \\ Br_{k+l} & \xrightarrow{\alpha} & Br(C_{k+l}). \end{array}$$

But there is no analogous commutativity for $\beta : Br(C_k) \to Br_k$. To see this let $k = 2$, then $m(\beta \times \beta)(e, w_1'') = m(e, e) = e$ and $\beta\mu(e, w_1'') = \beta(z_3) = \beta(w_3 w_2 w_1 w_2 w_3) = \sigma_3 \sigma_2^2 \sigma_3 \neq e$. So the homomorphism β does not agree with the pairings.

Geometrically the pairing for the braids of the series C can be described in the following way. We map $R^2 \backslash 0$ (with k different points) diffeomorphically onto open disk of radius $k+1/2$ without zero $D_{k+1/2} \backslash 0$ in such a way that the points with coordinates $(1, 0), ..., (k, 0)$ map onto themselves and we map $R^2 \backslash 0$ (with l different points) diffeomorphically onto $R^2 \backslash D_{k+1/2}$ in such a way that the points with coordinates $(1, 0), ..., (l, 0)$ map onto the points $(k + 1, 0), ..., (k + l, 0)$. This map

$$R^2 \backslash 0 \times R^2 \backslash 0 \to R^2 \backslash 0$$

is equivalent to the map of configuration spaces, described by Viktor Vassiliev [21]:

$$X(k) \times X(l) \to X(k + l)$$

where the space X can be presented in the form $X = Y \times R$. Then for the fundamental groups we obtain our pairing. Considering the generalized braid groups of the type C as the subgroups of the ordinary braid groups our pairing can be described as putting $k + 1$ strings of the first group instead of the zero string of the second group.

Let us consider the group $Br_k \wr Z/2$ which can be viewed as a semi-direct product of Br_k with $Z/2 \oplus ... \oplus Z/2$ (k copies) where Br_k acts on $Z/2 \oplus ... \oplus Z/2$ by permutations. We denote by s_1 the element $(a, e, ..., e) \in Z/2 \oplus ... \oplus Z/2$, where a is a generator of $Z/2$, and we denote by $s_2, ..., s_k$ the standard generators of Br_k. Then we have a relation:

$$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1.$$

We define a homomorphism

$$\gamma : Br(C_k) \to Br_k \wr Z/2$$

by the formula

$$\gamma(w_i) = s_i.$$

This homomorphism does not agree with the pairings (μ_C and the pairing determined by the wreath product structure).

Now we would like to consider the direct limits of finite Coxeter groups. We denote by $\mathcal{W}$ the category whose objects are finite Coxeter groups and morphisms are the inclusions $W' \mapsto W$, corresponding to inclusions of Coxeter graphs $\Gamma' \mapsto \Gamma$. We call by a chain a subcategory $\mathcal{E}$ of $\mathcal{W}$ which is a well ordered countable set and such that the total number of connected components of Coxeter graphs of the elements of $\mathcal{E}$ is bounded by some natural number $N_{\mathcal{E}}$ (for a subgroup W' of W we consider Γ' as a subgraph of Γ).

Definition 1. *We call by a limit Coxeter group W_∞ such an infinite group that there exists a chain $\mathcal{E}$ for which W_∞ is equal to the direct limit of $\mathcal{E}$.*

If we take as $\mathcal{E}$ the groups from one of the series A, C or D with canonical inclusions as morphisms we obtain A_∞, C_∞ or D_∞ as the corresponding limit Coxeter groups.

Proposition 4. *(i) The limit Coxeter group W_∞ is isomorphic to a direct product of a finite number (greater or equal than one) of groups of type A_∞, C_∞ or D_∞ and of a finite number of finite Coxeter groups.*

Proof. It follows from the fact that W_∞ must be infinite and its Coxeter graph is to have finite number of components.

Pairings described above generate the pairings of limit Coxeter groups and corresponding braid groups, for example

$$\mu(C,A) : Br(C_\infty) \times Br(A_\infty) \to Br(C_\infty),$$

$$\mu(C,C) : Br(C_\infty) \times Br(C_\infty) \to Br(C_\infty),$$

$$\mu(D,A) : Br(D_\infty) \times Br(A_\infty) \to Br(D_\infty).$$

For general limit Coxeter group W_∞ we have different pairings with $Br(A_\infty) = Br_\infty$ depending on the copy of one of the infinite groups of types A_∞, C_∞ or D_∞ for which this pairing is taken

$$\mu(W,A) : Br(W_\infty) \times Br(A_\infty) \to Br(W_\infty).$$

3. Construction of Thom Spectra for Generalized Braid Groups

From the construction of a finite Coxeter group W we have the inclusion into the orthogonal group $O(n)$ acting in V:

$$\nu_W : W \to O(n),$$

which can be involved into the following commutative diagram:

$$\begin{array}{ccccc}
Br(W) & \xrightarrow{\tau_W} & W & \longrightarrow \\
\| & & \| & \\
Br(W_1)\times\cdots\times Br(W_s) & \xrightarrow{\tau_1\times\cdots\times\tau_s} & W_1\times\cdots\times W_s & \longrightarrow
\end{array}$$

$$\begin{array}{cc}
\longrightarrow & O(n) \\
 & \uparrow \\
\longrightarrow & O(n_1)\times\cdots\times O(n_s)\times O(n_0).
\end{array}$$

This commutative diagram generates the commutative diagram of classifying spaces:

$$\begin{array}{ccccc}
BBr(W) & \xrightarrow{B\tau_W} & BW & \longrightarrow \\
\| & & \| & \\
BBr(W_1)\times\cdots\times BBr(W_s) & \xrightarrow{B\tau_1\times\cdots\times B\tau_s} & BW_1\times\cdots\times BW_s & \longrightarrow
\end{array}$$

$$\xrightarrow{\nu_W} \qquad BO(n)$$

$$\uparrow$$

$$\xrightarrow{\nu_1\times\cdots\times\nu_s} BO(n_1)\times\cdots\times BO(n_s)\times BO(n_0).$$

This commutative diagram generates in its turn the commutative diagram of Thom spectra:

$$\begin{array}{ccccc} MBr(W) & \xrightarrow{M\tau_W} & MW & \longrightarrow \\ \| & & \| & \\ MBr(W_1)\wedge\cdots\wedge MBr(W_s)\wedge S_0^n & \xrightarrow{M\tau_1\wedge\cdots\wedge M\tau_s\wedge 1} & MW_1\wedge\cdots\wedge MW_s\wedge S_0^n & \longrightarrow \end{array}$$

$$\xrightarrow{M\nu_W} \qquad MO(n)$$

$$\uparrow$$

$$\xrightarrow{M\nu_1\wedge\cdots\wedge M\nu_s\wedge\kappa} MO(n_1)\wedge\cdots\wedge MO(n_s)\wedge MO(n_0),$$

where κ is the inclusion of the sphere S^{n_0} into the Thom space: $S^{n_0}\to MO(n_0)$. The composition of the maps τ_W and ν_W classifies the bundle

$$Y_W\times_W R^n \ \to X_W,$$

which has its Thom space $Y_W \ltimes_W B^n/S^n$ where B^n is a unit ball and $\ltimes$ denotes the half smash product: $A\ltimes B = A\times B/A\times b_0$, $b_0\in B$ is the base point. For the series C it is equivalent to $Y_{C_n}\ltimes_{C_n} S^{1\,(n)}$, where $S^{1\,(n)}$ denotes the n-fold smash product of S^1 on which C_n acts by permutations between copies of S^1 and by complex conjugation on each S^1. For the series D the Thom space is equivalent to $Y_{C_n}\ltimes_{D_n} S^{1\,(n)}$, where D_n also acts on $S^{1\,(n)}$ by permutations between copies of S^1 and by complex conjugation on each S^1, but according to of the description of the group D_n the number of conjugations must be even. If the Coxeter graph of a group consists of one point ($A_1=\Sigma_2=Z/2$), then $Br_2=Z$ and we have $B\Sigma_2=RP^\infty$, $BBr_2=S^1$ and $M\Sigma_2=S^{-1}(RP^\infty)$, $MBr_2=S^{-1}(RP^2)$ and the map $M\tau$ is the canonical inclusion.

4. Computation of Thom Spectra for Limit Groups

Using the procedure described above we get Thom spectra MW_∞ and $MBr(W_\infty)$ for a limit Coxeter group and corresponding infinite braid group. Pairings of Coxeter and braid groups generate the pairings of Thom spaces and spectra (which we shall denote by the same symbol μ). For $W_\infty=A_\infty$ it was proved by Mark Mahowald [18, 19] and Fred Cohen [10] that MBr_∞ is multiplicatively isomorphic to the Eilenberg-MacLane spectrum $K(Z/2)$. The pairing described earlier induce on MW_∞ module structures over $M\Sigma_\infty$. So MW_∞ has at least one module structure over $M\Sigma_\infty$ and the same way $MBr(W_\infty)$ has at least one module structure over MBr_∞. Let $\eta : S^0 \to MBr_\infty$ be the unit map of the spectrum MBr_∞. The composition of $1 \wedge \eta$ and μ:

$$MBr(W_\infty) \wedge S^0 \to MBr(W_\infty) \wedge MBr_\infty \to MBr(W_\infty)$$

is equal to the identity map of $MBr(W_\infty)$. This follows from the fact that the composition:

$$W_k = W_k \times A_0 \to W_k \times A_l \to W_{k+l+1}$$

is equal to the inclusion $W_k \to W_{k+l+1}$. The same is true for $MBr(W_k)$. Hence the spectrum $MBr(W_\infty)$ is a direct summand in $MBr(W_\infty) \wedge K(Z/2)$ and it is itself a wedge of Eilenberg-MacLane spectra. The spaces X_W are connected, so $\pi_0(MBr(W_\infty)) = Z/2$.

Shaun Bullet studied in [8] Thom spectra and corresponding bordism theories for the following groups: $\Sigma_\infty, \Sigma_\infty \wr Z/2 = C_\infty, Br_\infty \wr Z/2$. It was proved by him that these bordism theories are multiplicative and that $M\Sigma_*, M(\Sigma \wr Z/2)_*$ and $M(Br \wr Z/2)_*$ are polynomial algebras over $Z/2$. He also proved that the canonical map induces the injective multiplicative morphism of cobordism theories:

$$M\Sigma^*(\ ) \to M(\Sigma \wr Z/2)^*(\ ),$$

such that the composition

$$M\Sigma^*(\ ) \to M(\Sigma \wr Z/2)^*(\ ) \to MO^*(\ ),$$

and the map

$$M(Br \wr Z/2)^*(\ ) \to MO^*(\ )$$

are surjective. The spectrum $M\Sigma_\infty$ is equivalent to the wedge of Eilenberg-MacLane spectra $K(Z/2)$. Being a module over $M\Sigma_\infty$ a Thom spectrum MW_∞ for the limit Coxeter group W_∞ becomes a module over $K(Z/2)$ as well. So MW_∞ is also a wedge of Eilenberg-MacLane spectra $K(Z/2)$. As a result we have proved the following theorem.

Theorem 2. *The Thom spectra $MBr(W_\infty)$ and MW_∞ for the limit Coxeter groups are equivalent to the wedges of Eilenberg-MacLane spectrum $K(Z/2)$, $\pi_0(MBr(W_\infty)) = Z/2$.*

From the cofibre sequence

$$S^1 \to S^1 \to RP^2 \to \dots\ ,$$

where the first map is a multiplication by 2 we obtain

Corollary 1. *If the Coxeter graph of W_∞ contains an isolated vertex, $W_\infty = W' \times (Z/2)$, then $MBr(W_\infty) = MBr(W') \wedge SMBr(W')$, where S, denotes a suspension over a spectrum.*

Now let us consider the Thom spectra for the groups C_∞ and D_∞. We would like to know the number of summands $K(Z/2)$ in each dimension for these spectra. This means to know modules $\pi_*(MBr(C_\infty)) = MBr(C_\infty)_*$ and $\pi_*(MBr(D_\infty)) = MBr(D_\infty)_*$. We use the knowledge of cohomology of the braid groups of the type C and D and then the Thom isomorphism. These cohomologies with coefficients in Z were computed by V. Goriunov [15, 16]. Namely there are expressions for the cohomologies of generalized braids in terms of classical ones:

$$H^q(Br(C_\infty); Z) = \bigoplus_{i=0}^{\infty} H^{q-i}(Br_\infty; Z),$$

$$H^q(Br(D_\infty); Z) = H^q(Br_\infty; Z) \oplus [\bigoplus_{i=0}^{\infty} H^{q-2i-3}(Br_\infty; Z/2)].$$

The formula for the cohomologies of $Br(C_\infty)$ may be also proved using the proposition 3 and the fact that the cohomologies of $Br_{1,n+1}$ are isomorphic to the cohomologies of Br_{n+1} with coefficients in the Coxeter representation X_{n+1} [10], [21].

$$H^*(Br_{1,n+1}; Z) \cong H^*(Br_{n+1}; X_{n+1}).$$

Representation X_{n+1} is defined by the composition

$$Br_{n+1} \to \Sigma_{n+1} \to Aut\, Z^{n+1},$$

Σ_{n+1} acts on the basis of Z^{n+1} by permutations.

Theorem 3. *The Thom spectra $MBr(C_\infty)$ and $MBr(D_\infty)$ are equivalent to the following wedges of the Eilenberg-MacLane spectra*

$$MBr(C_\infty) = \bigvee_{i=0}^{\infty} S^i K(Z/2),$$

$$MBr(D_\infty) = K(Z/2) \vee [\bigvee_{i=0}^{\infty} S^{2+i} K(Z/2)].$$

The pairing defined for the braid groups of type C induces a multiplicative structure (probably not commutative) for the theory $MBr(C_\infty)_*(\ )$. So $MBr(C_\infty)_*$ has a ring structure which we would like to consider. We take a circle S^1 with its standard imbedding in R^{n+1}. Its normal bundle is trivial, so the corresponding classifying map

$$\xi_n : S^1 \to BO(n)$$

is homotopic to zero. Now we take a fibration

$$f_n : BC_n \to BO(n)$$

homotopic to the canonical map and analogously a fibration

$$\psi : BBr(C_n) \to BC_n,$$

so that the composition

$$f_n\psi = f'_n : BBr(C_n) \to BO(n)$$

is a fibration homotopic to the canonical map from $BBr(C_n)$ to $BO(n)$. We have

$$H_1(BBr(C_n); Z) = Br(C_n)/[Br(C_n), Br(C_n)] = Z \oplus Z,$$

$$H_1(B(C_n); Z) = Br(C_n)/[C_n, C_n] = Z/2 \oplus Z/2,$$

and the map $H_1(\psi)$ is the canonical projection. We consider a map $g' : S^1 \to BBr(C_n)$, such that in homology the generator of $H_1(S^1; Z)$ maps by $H_1(g')$ to some generator v of $H_1(BBr(C_n); Z)$ and such that the composition

$$f'_n g' : S^1 \to BBr(C_n) \to BO(n)$$

is homotopic to zero. We take $g : S^1 \to BBr(C_n)$ as a map homotopic to g' and such that $f'_n g = \xi_n$. The map g defines a $(BBr(C_n), f'_n)$-structure on S^1, and the map ψg defines a (BC_n, f_n)-structure on S^1 [20]. Let $w' \in H^1(BBr(C_n); Z)$ be the element dual to $v \in H_1(BBr(C_n); Z)$ and w is the reduction mod 2 of w'. By our construction the characteristic number of S^1 with $(BBr(C_n), f'_n)$-structure which corresponds to w is nonzero element of $Z/2$. So bordism class of S^1 may be considered as a generator of $MBr(C_\infty)_1$ and its reduction from $BBr(C_n)$ to $B(C_n)$ is a nonzero element of $(MC_\infty)_1$. The ring $(MC_\infty)_*$ is a free algebra over $Z/2$. So we proved the following theorem.

Theorem 4. *The coefficient ring $MBr(C_\infty)_*$ of the bordism theory corresponding to the braid group of type C is a polynomial algebra from one generator s in dimension 1:*

$$MBr(C)_* \cong Z/2[s].$$

It is possible to prove theorem 4 by studing the Hopf algebra structure on the cohomologies of $Br(C_\infty)$ as it was done by D. B. Fuks [13] for the ordinary braid group. One more way of proving theorems 3 and 4 is to use the results of D. B. Fuks [14] that the "quillenisation" of $K(Br(C_\infty), 1)$ is equal to $\Omega^2 S^3 \times \Omega S^2$ nd the "quillenisation" of $K(Br(D_\infty), 1)$ is equal to $\Omega^2 S^3 \times F$, where F is a homotopy fibre of a map of degree 2 from S^3 to S^3.

Corollary 2. *The image of the ring $MBr(C_\infty)_*$in the unoriented cobordism ring is equal to zero in positive dimensions.*

Remark. In the unoriented cobordism ring $MO_2 = Z/2,\ MO_3 = 0$. So the canonical map to unoriented cobordism for the bordism groups of the braids of type D

$$MBr(D_\infty)_* \to MO_*,$$

is neither monomorphism nor epimorphism.

5. Thom Spectra of Groups of Finite Type

Let us consider Thom spectra, corresponding to braid groups of finite Coxeter groups. We have seen that these spectra are smash products of the spectra for irreducible Coxeter groups. Thom spectra MBr_k were studied by E. Brown and F. Peterson [7] and Ralph Cohen [11]. In particular, it was proved, that MBr_k is 2-equivalent to the Brown-Gitler spectrum $B([k/2])$.

We denote by t_W the Thom class of the spectrum $MBr(W)$:

$$t_W : MBr(W) \to K(Z/2).$$

Let $MBr_n \to MBr(C_n)$ be the map induced by the imbeddings of Coxeter graphs described earlier. The composition:

$$MBr_n \to MBr(C_n) \to MO(n) \to MO \to K(Z/2),$$

where the last map is the Thom class of MO, is equal to the Thom class of MBr_n. The analogous compositions for the series D and E:

$$MBr_n \to MBr(D_n) \to MO(n) \to MO \to K(Z/2),$$

$$MBr_n \to MBr(E_n) \to MO(n) \to MO \to K(Z/2),\ n = 6, 7, 8$$

are equal to the Thom class of MBr_n. So we get that the homomorphisms induced in cohomology:

$$H^*(MBr(C_n); Z/2) \to H^*(MBr_n); Z/2),$$

$$H^*(MBr(D_n); Z/2) \to H^*(MBr_n); Z/2),$$

$$H^*(MBr(E_n); Z/2) \to H^*(MBr_n); Z/2),\ n = 6, 7, 8,$$

are epimorphisms. Using the results of Ralph Cohen [11] we obtain the following theorem.

Theorem 5. *If X is any CW complex then the maps for bordism theories $MBr(C_n)_*(\ )$, $MBr(D_n)_*(\ )$ and $MBr(E_n)_*(\ )$, induced by the Thom class t :*

$$MBr(C_n)_q(X) \to H_q(X; Z/2),$$

$$MBr(D_n)_q(X) \to H_q(X; Z/2),$$

$$MBr(E_n)_q(X) \to H_q(X; Z/2),\ n = 6, 7, 8,$$

are epimorphisms for $q \le 2[n/2] + 1$, where $[a]$ denotes the integer part of a.

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