

GENERIC REPRESENTATIONS OF ORTHOGONAL GROUPS: THE FUNCTOR CATEGORY \mathcal{F}_{quad}

CHRISTINE VESPA

ABSTRACT. In this paper, we define the functor category \mathcal{F}_{quad} associated to \mathbb{F}_2 -vector spaces equipped with a quadratic form. We show the existence of a fully-faithful, exact functor $\iota : \mathcal{F} \rightarrow \mathcal{F}_{quad}$, which preserves simple objects, where \mathcal{F} is the category of functors from the category of finite dimensional \mathbb{F}_2 -vector spaces to the category of all \mathbb{F}_2 -vector spaces. We define the subcategory \mathcal{F}_{iso} of \mathcal{F}_{quad} , which is equivalent to the product of the categories of modules over the orthogonal groups; the inclusion is a fully-faithful functor $\kappa : \mathcal{F}_{iso} \rightarrow \mathcal{F}_{quad}$ which preserves simple objects.

Keywords: functor categories; quadratic forms over \mathbb{F}_2 ; Mackey functors; representations of orthogonal groups over \mathbb{F}_2 .

INTRODUCTION

In recent years, one of the functor categories which has been particularly studied is the category $\mathcal{F}(p)$ of functors from the category \mathcal{E}^f of finite dimensional \mathbb{F}_p -vector spaces to the category \mathcal{E} of all \mathbb{F}_p -vector spaces, where \mathbb{F}_p is the prime field with p elements. This category is connected to several areas of algebra and some examples of these can be found in [4]. The category $\mathcal{F}(p)$ is closely related to the general linear groups. An important application of $\mathcal{F}(p)$ is given in [5], where the four authors proved that this category is very useful for the study of the stable cohomology of the general linear groups with suitable coefficients. They showed that the calculation of certain extension groups in the category \mathcal{F} determines some stable cohomology groups of general linear groups. One of the motivations of the work presented here is to construct and study a category \mathcal{F}_{quad} which could play a similar role for the stable cohomology of the orthogonal groups.

In this paper, we restrict to the prime $p = 2$; the techniques can be applied in the odd prime case, but the case $p = 2$ presents features which make it particularly interesting. Henceforth, we will suppose that $p = 2$ and we will denote the category $\mathcal{F}(2)$ by \mathcal{F} .

After some recollections on the theory of quadratic forms over \mathbb{F}_2 , we give the definition of the category \mathcal{F}_{quad} . In order to have a good understanding of the category \mathcal{F}_{quad} , we seek to classify the simple objects of this category. The first important result of this paper is the following theorem.

Theorem. *There is a functor*

$$\iota : \mathcal{F} \rightarrow \mathcal{F}_{quad}$$

which satisfies the following properties:

- (1) ι is exact;
- (2) ι preserves tensor products;
- (3) ι is fully-faithful;
- (4) if S is a simple object of \mathcal{F} , $\iota(S)$ is a simple object of \mathcal{F}_{quad} .

Remark. Using a study of the projective functors of the category \mathcal{F}_{quad} , we will show in [?] that $\iota(F)$ is a thick subcategory of \mathcal{F}_{quad} .

To study a particular family of functors of \mathcal{F}_{quad} , the isotropic functors, we define the subcategory \mathcal{F}_{iso} of \mathcal{F}_{quad} which is related to \mathcal{F}_{quad} by the following theorem:

Theorem. *There is a functor*

$$\kappa : \mathcal{F}_{iso} \rightarrow \mathcal{F}_{quad}$$

which satisfies the following properties:

- (1) κ is exact;
- (2) κ preserves tensor products;
- (3) κ is fully-faithful;
- (4) if S is a simple object of \mathcal{F}_{iso} , $\kappa(S)$ is a simple object of \mathcal{F}_{quad} .

We obtain the classification of the simple objects of the category \mathcal{F}_{iso} from the following theorem.

Theorem. *There is a natural equivalence of categories*

$$\mathcal{F}_{iso} \simeq \prod_{V \in \mathcal{S}} \mathbb{F}_2[O(V)] - mod$$

where \mathcal{S} is a set of representatives of isometry classes of quadratic spaces (possibly degenerate).

Apart from the previous theorem, the results of this paper are contained in the Ph.D. thesis of the author [13], although several results are presented here from a more conceptual point of view.

The author wants to thank her PhD supervisor, Lionel Schwartz, as well as Geoffrey Powell and Aurélien Djament for their useful comments and suggestions on a previous version of this paper.

1. QUADRATIC SPACES OVER \mathbb{F}_2

We recall the definition and the classification of quadratic forms over the field \mathbb{F}_2 . We refer the reader to [9] for details.

1.1. Definitions. Let V be a \mathbb{F}_2 -vector space of finite dimension. A quadratic form over V is a function $q : V \rightarrow \mathbb{F}_2$ such that $q(x + y) + q(x) + q(y) = B(x, y)$ is a bilinear form. As a direct consequence of the definition, we have that the bilinear form associated to a quadratic form is alternating.

The radical of a quadratic space (V, q_V) is the subspace of V given by

$$\text{Rad}(V, q_V) = \{v \in V \mid \forall w \in V \ B(v, w) = 0\}$$

where $B(-, -)$ is the bilinear form associated to the quadratic form. A quadratic space (V, q_V) is non-degenerate if $\text{Rad}(V, q_V) = 0$.

1.2. Non-degenerate quadratic forms.

1.2.1. Classification. In this paragraph, we recall the classification of non-degenerate quadratic forms. The classification of non-singular alternating forms implies that a non-degenerate quadratic space over \mathbb{F}_2 has even dimension and has a symplectic basis.

The space H_0 is the non-degenerate quadratic space of dimension two with symplectic basis $\{a_0, b_0\}$, and quadratic form determined by:

$$\begin{array}{rcl} q_0 : H_0 & \rightarrow & \mathbb{F}_2 \\ & a_0 & \mapsto 0 \\ & b_0 & \mapsto 0. \end{array}$$

The space H_1 is the non-degenerate quadratic space of dimension two with symplectic basis $\{a_1, b_1\}$, and quadratic form determined by:

$$\begin{aligned} q_1 : H_1 &\rightarrow \mathbb{F}_2 \\ a_1 &\mapsto 1 \\ b_1 &\mapsto 1. \end{aligned}$$

The spaces H_0 and H_1 are not isometric, whereas the spaces $H_0 \perp H_0$ and $H_1 \perp H_1$ are isometric. The non degenerate quadratic spaces of dimension $2m$, for $m \geq 1$, are classified by the following result.

Proposition 1.1. [9] *Let $m \geq 1$ be an integer.*

- (1) *The quadratic spaces $H_0^{\perp m}$ and $H_1 \perp H_0^{\perp(m-1)}$ are not isometric.*
- (2) *A quadratic space of dimension $2m$ is isometric to either $H_0^{\perp m}$ or $H_1 \perp H_0^{\perp(m-1)}$.*

The two spaces $H_0^{\perp m}$ and $H_1 \perp H_0^{\perp(m-1)}$ are distinguished by the Arf invariant introduced in [1]. Observe that $\text{Arf}(H_0^{\perp m}) = 0$ and $\text{Arf}(H_1 \perp H_0^{\perp(m-1)}) = 1$.

1.2.2. *The category \mathcal{E}_q .*

Definition 1.2. *Let \mathcal{E}_q be the category having as objects finite dimensional \mathbb{F}_2 -vector spaces equipped with a non degenerate quadratic form and with morphisms linear maps which preserve the quadratic forms.*

Observe that a linear map which preserves the quadratic forms preserves the underlying bilinear form, but the converse is, in general, false. The following proposition summarizes straightforward but important results about the category \mathcal{E}_q .

Proposition 1.3. (1) *The morphisms of \mathcal{E}_q are injective linear maps. Consequently they are monomorphisms.*

- (2) *The category \mathcal{E}_q does not admit push-outs or pullbacks.*

Example 1.4. *The diagram $V \leftarrow \{0\} \rightarrow V$, where $V \neq \{0\}$ does not admit a push-out in \mathcal{E}_q .*

To resolve this difficulty, we define the notion of a pseudo push-out in \mathcal{E}_q .

1.2.3. *Pseudo push-out.* To define the pseudo push-out in \mathcal{E}_q , we need the following remark, which uses the non-degeneracy of the quadratic form in an essential way.

Remark 1.5. *For f an element of $\text{Hom}_{\mathcal{E}_q}(V, W)$, we have $W \simeq f(V) \perp V'$, where V' is the orthogonal space to W in the space V . As the spaces V and $f(V)$ are isometric, the spaces W and $V \perp V'$ are also. We will write*

$$f : V \rightarrow W \simeq V \perp V'.$$

Definition 1.6. *For $f : V \rightarrow W \simeq V \perp V'$ and $g : V \rightarrow X \simeq V \perp V''$ morphisms in \mathcal{E}_q , the pseudo push-out of f and g , denoted by $W \perp_V X$, is the object $V \perp V' \perp V''$ of \mathcal{E}_q .*

We give, in the following proposition, the principal properties of the pseudo push-out.

Proposition 1.7. *Let $f : V \rightarrow W \simeq V \perp V'$ and $g : V \rightarrow X \simeq V \perp V''$ be morphisms in \mathcal{E}_q , the pseudo push-out of f and g satisfies the following properties.*

- (1) *There exists a commutative diagram of the form*

$$\begin{array}{ccc} V & \xrightarrow{g} & X \\ f \downarrow & & \downarrow \\ W & \longrightarrow & X \perp_V W \end{array}$$

- (2) $W \perp_V X \simeq X \perp_V W$;
- (3) if $V \simeq V'$ then $W \perp_V X \simeq W \perp_{V'} X$;
- (4) *Associativity*: $(X \perp_V W) \perp_Z Y \simeq X \perp_V (W \perp_Z Y)$;
- (5) *Unit*: $V \perp_V W \simeq W$.

Remark 1.8. *By the first point of the previous proposition, the pseudo push-out occurs in a commutative diagram*

$$\begin{array}{ccc} V & \longrightarrow & V \perp V'' \\ \downarrow & & \downarrow \\ V \perp V' & \longrightarrow & V \perp V' \perp V'' \end{array}$$

which is equivalent to the orthogonal sum of the diagram

$$\begin{array}{ccc} \{0\} & \longrightarrow & V'' \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V' \perp V'' \end{array}$$

with V . Hence, the pseudo push-out can be considered as a generalization of the orthogonal sum.

1.3. Degenerate quadratic forms. The previous section implies that, in particular, all quadratic spaces of odd dimension are degenerate. We begin by considering the quadratic spaces of dimension one.

Notation 1.9. For $\alpha \in \{0, 1\}$, let (x, α) be the quadratic space of dimension one generated by x such that $q(x) = \alpha$.

1.3.1. *Classification.* We have the following classification:

Theorem 1.10. [9]

- (1) Every quadratic space over \mathbb{F}_2 has an orthogonal decomposition

$$V \simeq H \perp \text{Rad}(V)$$

such that H is non-degenerate and $\text{Rad}(V)$ is isometric to either $(x, 0)^{\perp r}$ or $(x, 1)^{\perp r}$, where r is the dimension of $\text{Rad}(V)$.

- (2) Let $V \simeq H \perp \text{Rad}(V)$ and $V \simeq H' \perp \text{Rad}(V)$ be two decompositions of V , if $\text{Rad}(V) \simeq (x, 0)^{\perp r}$ for $r \geq 0$, then $H \simeq H'$.
- (3) Let H and H' be two non-degenerate quadratic forms such that $\dim(H) = \dim(H')$ then, for all $r > 0$,

$$H \perp (x, 1)^{\perp r} \simeq H' \perp (x, 1)^{\perp r}.$$

Remark 1.11. *The third point of the previous theorem, is implied by the isometry:*

$$H_0 \perp (x, 1) \simeq H_1 \perp (x, 1).$$

This exhibits one of the particularities of quadratic forms over \mathbb{F}_2 : the “non-degenerate part” of a quadratic form is not unique in general, not even up to isometry.

1.3.2. *The category $\mathcal{E}_q^{\text{deg}}$.*

Definition 1.12. Let $\mathcal{E}_q^{\text{deg}}$ be the category having as objects finite dimensional \mathbb{F}_2 -vector spaces equipped with a (possibly degenerate) quadratic form and with morphisms, injective linear maps which preserve the quadratic forms.

Remark 1.13. *The hypothesis that the morphisms are linear injective maps is essential for later considerations.*

The following proposition underlines one of the important difference between \mathcal{E}_q and $\mathcal{E}_q^{\text{deg}}$.

Proposition 1.14. *The category $\mathcal{E}_q^{\text{deg}}$ admits pullbacks.*

Nevertheless, $\mathcal{E}_q^{\text{deg}}$ does not contain all push-outs.

2. DEFINITION OF THE CATEGORY \mathcal{F}_{quad}

We have emphasized the fact that all the morphisms of \mathcal{E}_q are monomorphisms. The constructions which link the category \mathcal{F} to the stable homology of the general linear groups use, in an essential way, the existence of retractions in the category \mathcal{E}^f . Therefore, to consider analogous constructions in the quadratic case, we would like to add formally retractions to the category \mathcal{E}_q . For this, we define the category \mathcal{T}_q inspired by the construction of the category of co-spans introduced by Bénabou in [2].

2.1. The category \mathcal{T}_q .

2.1.1. *The categories of spans and co-spans.* We will begin by recalling the construction of Bénabou.

Remark 2.1. *Our principal interest is in the category \mathcal{F}_{quad} . The construction of this category uses a generalization of the category of co-spans, which we have chosen to present rather than spans.*

Definition 2.2. [2] *Let \mathcal{D} be a category equipped with push-outs, the category $\text{coSpan}(\mathcal{D})$ is defined in the following way:*

- (1) *the objects of $\text{coSpan}(\mathcal{D})$ are those of \mathcal{D} ;*
- (2) *for A and B two objects of $\text{coSpan}(\mathcal{D})$, $\text{Hom}_{\text{coSpan}(\mathcal{D})}(A, B)$ is the set of equivalence classes of diagrams in \mathcal{D} of the form $A \xrightarrow{f} D \xleftarrow{g} B$, for the equivalence relation which identifies the two diagrams $A \xrightarrow{f} D \xleftarrow{g} B$ and $A \xrightarrow{u} D' \xleftarrow{v} B$ if there exists an isomorphism $\alpha : D \rightarrow D'$ such that the following diagram is commutative*

$$\begin{array}{ccc}
 & B & \\
 & \downarrow g & \\
 A & \xrightarrow{f} D & \\
 & \searrow u & \swarrow v \\
 & & D'
 \end{array}$$

The morphism of $\text{Hom}_{\text{coSpan}(\mathcal{D})}(A, B)$ represented by the diagram $A \xrightarrow{f} D \xleftarrow{g} B$ will be denoted by $[A \xrightarrow{f} D \xleftarrow{g} B]$;

- (3) *the composition is given by:*
for two morphisms $T_1 = [A \rightarrow D \leftarrow B]$ and $T_2 = [B \rightarrow D' \leftarrow C]$,

$$T_2 \circ T_1 = [A \rightarrow D \amalg_B D' \leftarrow C].$$

By duality, Bénabou gives the following definition.

Definition 2.3. *Let \mathcal{D} be a category equipped with pullbacks. The category $\text{Sp}(\mathcal{D})$ is defined by: $\text{Sp}(\mathcal{D}) \simeq \text{coSpan}(\mathcal{D}^{\text{op}})^{\text{op}}$.*

Example 2.4. *The category $\text{Sp}(\mathcal{E}_q^{\text{deg}})$ is defined, since $\mathcal{E}_q^{\text{deg}}$ admits pullbacks by proposition 1.14.*

2.1.2. *The category \mathcal{T}_q .* By proposition 1.3 neither the category of Spans nor coSpans are defined for \mathcal{E}_q . However, we observe that the universality of the push-out plays no role in the definition of the category $\text{coSpan}(\mathcal{D})$. So, by definition 1.6 of the pseudo push-out of \mathcal{E}_q we can give the following definition.

Definition 2.5. *The category $\hat{\mathcal{T}}_q$ is defined in the following way:*

- (1) *the objects of $\hat{\mathcal{T}}_q$ are those of \mathcal{E}_q ;*
- (2) *for V and W two objects of $\hat{\mathcal{T}}_q$, $\text{Hom}_{\hat{\mathcal{T}}_q}(V, W)$ is defined in the same way as for the category $\text{coSpan}(\mathcal{D})$ and we will use the same notation for the morphisms;*
- (3) *the composition is given by:*
for $T_1 = [V \rightarrow X_1 \leftarrow W]$ and $T_2 = [W \rightarrow X_2 \leftarrow Y]$,

$$T_2 \circ T_1 = [V \rightarrow X_1 \underset{W}{\perp} X_2 \leftarrow Y]$$

where $X_1 \underset{W}{\perp} X_2$ is the pseudo push-out.

The elementary properties of the pseudo push-out given in 1.7 show that the composition is well-defined and associative. Thus, the above defines a category.

To define the category \mathcal{T}_q , we consider the following relation on the morphisms of $\hat{\mathcal{T}}_q$.

Definition 2.6. *For V and W two objects of $\hat{\mathcal{T}}_q$, the relation \mathcal{R} on $\text{Hom}_{\hat{\mathcal{T}}_q}(V, W)$ for V and W objects of $\hat{\mathcal{T}}_q$, is defined by:*
for $T_1 = [V \rightarrow X_1 \leftarrow W]$ and $T_2 = [V \rightarrow X_2 \leftarrow W]$, two elements of $\text{Hom}_{\hat{\mathcal{T}}_q}(V, W)$, $T_1 \mathcal{R} T_2$ if there exists a morphism α of \mathcal{E}_q such that the following diagram is commutative

$$\begin{array}{ccc} & W & \\ & \downarrow & \\ V & \longrightarrow & X_1 \\ & \searrow & \downarrow \alpha \\ & & X_2 \end{array}$$

We will denote by \sim the equivalence relation on $\text{Hom}_{\hat{\mathcal{T}}_q}(V, W)$ generated by the relation \mathcal{R} .

Lemma 2.7. *The composition in $\hat{\mathcal{T}}_q$ induces an application:*

$$\circ : \text{Hom}_{\hat{\mathcal{T}}_q}(V, W) / \sim \times \text{Hom}_{\hat{\mathcal{T}}_q}(U, V) / \sim \rightarrow \text{Hom}_{\hat{\mathcal{T}}_q}(U, W) / \sim .$$

Proof. By the properties of the pseudo push-out given in Proposition 1.7 we verify that:

- (1) if $T_1 \mathcal{R} T_2$, then $(T_3 \circ T_1) \mathcal{R} (T_3 \circ T_2)$;
- (2) if $T_3 \mathcal{R} T_4$, then $(T_1 \circ T_3) \mathcal{R} (T_1 \circ T_4)$.

□

Thanks to the previous lemma, we can give the following definition.

Definition 2.8. *Let \mathcal{T}_q be the category having as objects the objects of \mathcal{E}_q and with morphisms $\text{Hom}_{\mathcal{T}_q}(V, W) = \text{Hom}_{\hat{\mathcal{T}}_q}(V, W) / \sim$.*

For convenience we will use the same notation for the morphisms of \mathcal{T}_q as for those of $\hat{\mathcal{T}}_q$.

2.1.3. *Properties of the category \mathcal{T}_q .* We have the following important property.

Proposition 2.9. *For $f : V \rightarrow W$ a morphism in the category \mathcal{E}_q , we have the following relation in \mathcal{T}_q*

$$[W \xrightarrow{\text{Id}} W \xleftarrow{f} V] \circ [V \xrightarrow{f} W \xleftarrow{\text{Id}} W] = \text{Id}_V.$$

In particular, $[W \xrightarrow{\text{Id}} W \xleftarrow{f} V]$ is a retraction of $[V \xrightarrow{f} W \xleftarrow{\text{Id}} W]$.

Proof. It is a direct consequence of the definition of \mathcal{T}_q . \square

To close this paragraph we give two useful constructions in the category \mathcal{T}_q .

Definition 2.10 (Transposition). *The transposition functor, $tr : \mathcal{T}_q^{op} \rightarrow \mathcal{T}_q$, is defined on objects by $tr(V) = V$ and on morphisms by:*

$$tr(f) = tr([V \rightarrow X \leftarrow W]) = [W \rightarrow X \leftarrow V],$$

for V and W objects of \mathcal{T}_q^{op} and f an element of $\text{Hom}_{\mathcal{T}_q^{op}}(W, V)$.

Observe that the transposition functor is involutive.

Proposition 2.11 (Orthogonal sum). *There exists a bifunctor $\perp : \mathcal{T}_q \times \mathcal{T}_q \rightarrow \mathcal{T}_q$, called the orthogonal sum, defined on objects by:*

$$\perp(V, W) = V \perp W$$

and on morphisms by:

$$\perp([V \xrightarrow{f} X \xleftarrow{g} W], [V' \xrightarrow{f'} X' \xleftarrow{g'} W']) = [V \perp V' \xrightarrow{f \perp f'} X \perp X' \xleftarrow{g \perp g'} W \perp W'].$$

This bifunctor gives \mathcal{T}_q the structure of a symmetric monoidal category, with unit $\{0\}$.

Proof. It is straightforward to verify that \perp is a well-defined bifunctor, and that it is associative, symmetric and that the object $\{0\}$ of \mathcal{T}_q is a unit for \perp . \square

2.2. **Definition and properties of the category \mathcal{F}_{quad} .** As the category \mathcal{T}_q is essentially small, we can give the following definition.

Definition 2.12. *The category \mathcal{F}_{quad} is the category of functors from \mathcal{T}_q to \mathcal{E} .*

Remark 2.13. *By analogy with the classical definition of Mackey functors, given in [12] and due, originally, to Dress [3], and with the work of Lindner in [8], we can view the category \mathcal{F}_{quad} as the category of generalized Mackey functors over \mathcal{E}_q . Note that, in our definition, we consider all the functors from \mathcal{T}_q to \mathcal{E} and not only the additive functors, unlike [12].*

By classical results about functor categories and by the Yoneda lemma, we obtain the following theorem.

Theorem 2.14. (1) *The category \mathcal{F}_{quad} is abelian.*

(2) *The tensor product of vector spaces induces a structure of symmetric monoidal category on \mathcal{F}_{quad} .*

(3) *For any object V of \mathcal{T}_q , the functor $P_V = \mathbb{F}_2[\text{Hom}_{\mathcal{T}_q}(V, -)]$ is a projective object and there is a natural isomorphism:*

$$\text{Hom}_{\mathcal{F}_{quad}}(P_V, F) \simeq F(V)$$

for all objects F of \mathcal{F}_{quad} .

The set of functors $\{P_V | V \in \mathcal{S}\}$ is a set of projective generators of \mathcal{F}_{quad} , where \mathcal{S} is a set of representatives of isometry classes of non-degenerate quadratic spaces. In particular, the category \mathcal{F}_{quad} has enough projective objects.

The transposition functor of \mathcal{T}_q allows us to give the following definition.

Definition 2.15. *The duality functor of \mathcal{F}_{quad} is the functor $D : \mathcal{F}_{quad}^{op} \rightarrow \mathcal{F}_{quad}$ given by:*

$$DF = -^* \circ F \circ tr^{op}$$

for F an object of \mathcal{F}_{quad} , $-^*$ the duality functor from \mathcal{E}^{op} to \mathcal{E} and tr the transposition functor of \mathcal{T}_q defined in 2.10.

The following proposition summarizes the basic properties of the duality functor D .

Proposition 2.16. (1) *The functor D is exact.*

(2) *The functor D is right adjoint to the functor D^{op} , i.e. we have a natural isomorphism:*

$$\mathrm{Hom}_{\mathcal{F}_{quad}}(F, DG) \simeq \mathrm{Hom}_{\mathcal{F}_{quad}^{op}}(D^{op}F, G) \simeq \mathrm{Hom}_{\mathcal{F}_{quad}}(G, DF).$$

(3) *For F an object of \mathcal{F}_{quad} with values in finite dimensional vector spaces, the unit of the adjunction between \mathcal{F}_{quad} and \mathcal{F}_{quad}^{op} , $F \rightarrow DD^{op}F$, is an isomorphism.*

A straightforward consequence of the second point of the last proposition and theorem 2.14 is:

Corollary 2.17. *The category \mathcal{F}_{quad} has enough injective objects.*

3. CONNECTION BETWEEN \mathcal{F} AND \mathcal{F}_{quad}

Recall that \mathcal{F} is the category of functors from \mathcal{E}^f to \mathcal{E} , where \mathcal{E}^f is the full subcategory of \mathcal{E} having as objects the finite dimensional spaces. The main result of this section is the following theorem.

Theorem 3.1. *There is a functor*

$$\iota : \mathcal{F} \rightarrow \mathcal{F}_{quad}$$

which satisfies the following properties:

- (1) ι is exact;
- (2) ι preserves tensor products;
- (3) ι is fully-faithful;
- (4) if S is a simple object of \mathcal{F} , $\iota(S)$ is a simple object of \mathcal{F}_{quad} .

To define the functor ι of the last theorem we need to define the forgetful functor $\epsilon : \mathcal{T}_q \rightarrow \mathcal{E}^f$ which can be viewed as an object of the category \mathcal{F}_{quad} , by composing with $\mathcal{E}^f \hookrightarrow \mathcal{E}$.

3.1. The forgetful functor ϵ of \mathcal{F}_{quad} .

3.1.1. Definition.

Notation 3.2. *We denote by $\mathcal{O} : \mathcal{E}_q \rightarrow \mathcal{E}^f$ the functor which forgets the quadratic form.*

Proposition 3.3. *There exists a functor $\epsilon : \mathcal{T}_q \rightarrow \mathcal{E}^f$ defined by $\epsilon(V) = \mathcal{O}(V)$ and*

$$\epsilon([V \xrightarrow{f} W \perp W' \xleftarrow{g} W]) = p_g \circ \mathcal{O}(f)$$

where p_g is the orthogonal projection from $W \perp W'$ to W .

The proof of this proposition relies on the following straightforward property of the pseudo push-out of \mathcal{E}_q .

Lemma 3.4. *For a pseudo push-out diagram in \mathcal{E}_q*

$$\begin{array}{ccc} V & \xrightarrow{g} & V \perp V'' \\ f \downarrow & & \downarrow l \\ V \perp V' & \xrightarrow{k} & V \perp V' \perp V'' \end{array}$$

we have the following relation in \mathcal{E}^f :

$$\mathcal{O}(g) \circ p_f = p_l \circ \mathcal{O}(k)$$

where p_f and p_l are the orthogonal projections associated to, respectively, f and l .

Proof of the proposition. It is straightforward to check that the functor ϵ is well defined on the classes of morphisms of \mathcal{T}_q . To show that ϵ is a functor, we verify that

$$\epsilon([V \xrightarrow{\text{Id}} V \xleftarrow{\text{Id}} V]) = \text{Id}$$

and the relation for the composition is a direct consequence of the above lemma. \square

3.1.2. *The fullness of ϵ .* The aim of this section is to prove the following proposition:

Proposition 3.5. *The functor $\epsilon : \mathcal{T}_q \rightarrow \mathcal{E}^f$ is full.*

Proof. Let (V, q_V) and (W, q_W) be two objects of \mathcal{T}_q and $f \in \text{Hom}_{\mathcal{E}^f}(\epsilon(V, q_V), \epsilon(W, q_W))$ be a linear map from V to W . We prove, by induction on the dimension of V , that there is a morphism in \mathcal{T}_q : $T = [V \xrightarrow{\varphi} X = W \perp Y \xleftarrow{\psi} W]$ such that $\epsilon(T) = f$. The proof is based on the idea that, for a sufficiently large space X , we can obtain all the linear maps.

As the quadratic space V is non-degenerate, we know that it has even dimension.

To start the induction, let (V, q_V) be a non-degenerate quadratic space of dimension two, with symplectic basis $\{a, b\}$ and $f : V \rightarrow W$ be a linear map. We verify that the following linear map:

$$\begin{aligned} g_1 : V &\rightarrow W \perp H_1 \perp H_0 \simeq W \perp \text{Vect}(a_1, b_1) \perp \text{Vect}(a_0, b_0) \\ a &\mapsto f(a) + (q(a) + q(f(a)))a_1 + a_0 \\ b &\mapsto f(b) + (q(b) + q(f(b)))a_1 + (1 + B(f(a), f(b)))b_0 \end{aligned}$$

preserves the quadratic form. Consequently, the morphism:

$$T = [V \xrightarrow{g_1} W \perp H_1 \perp H_0 \leftrightarrow W]$$

is a morphism of \mathcal{T}_q such that $\epsilon(T) = f$.

Let V_n be a non-degenerate quadratic space of dimension $2n$, $\{a_1, b_1, \dots, a_n, b_n\}$ be a symplectic basis of V_n and $f_n : V_n \rightarrow W$ be a linear map. By induction, there exists a map :

$$\begin{aligned} g_n : V_n &\rightarrow W \perp Y \\ a_1 &\mapsto f(a_1) + y_1 \\ b_1 &\mapsto f(b_1) + z_1 \\ \dots &\dots \\ a_n &\mapsto f(a_n) + y_n \\ b_n &\mapsto f(b_n) + z_n \end{aligned}$$

where y_i and z_i , for all integers i between 1 and n , are elements of Y , which preserves the quadratic form and such that:

$$\epsilon([V_n \xrightarrow{g_n} W \perp Y \leftrightarrow W]) = f_n.$$

Let V_{n+1} be a non-degenerate quadratic space of dimension $2(n+1)$, $\{a_1, b_1, \dots, a_n, b_n, a_{n+1}, b_{n+1}\}$ a symplectic basis of V_{n+1} and $f_{n+1} : V_{n+1} \rightarrow W$ a linear map. To define the map g_{n+1} , we will consider the restriction of f_{n+1} over

V_n and extend the map g_n given by the inductive assumption. For that, we need the following space: $E \simeq W \perp Y \perp H_0^{\perp n} \perp H_0^{\perp n} \perp H_1 \perp H_0$, for which we specify the notations for a basis:

$$E \simeq W \perp Y \perp (\perp_{i=1}^n \text{Vect}(a_0^i, b_0^i)) \perp (\perp_{i=1}^n \text{Vect}(A_0^i, B_0^i)) \perp \text{Vect}(A_1, B_1) \perp \text{Vect}(C_0, D_0).$$

We verify that the following map:

$$\begin{array}{lcl} V & \xrightarrow{g_{n+1}} & W \perp Y \perp H_0^{\perp n} \perp H_0^{\perp n} \perp H_1 \perp H_0 \\ a_1 & \mapsto & f(a_1) + y_1 + a_0^1 \\ b_1 & \mapsto & f(b_1) + z_1 + A_0^1 \\ \dots & & \dots \\ a_i & \mapsto & f(a_i) + y_i + a_0^i \\ b_i & \mapsto & f(b_i) + z_i + A_0^i \\ \dots & & \dots \\ a_n & \mapsto & f(a_n) + y_n + a_0^n \\ b_n & \mapsto & f(b_n) + z_n + A_0^n \\ a_{n+1} & \mapsto & f(a_{n+1}) + (q(a_{n+1}) + q(f(a_{n+1})))A_1 + C_0 + \sum_{i=1}^n B(f(a_i), f(a_{n+1}))b_0^i \\ & & + \sum_{i=1}^n B(f(b_i), f(a_{n+1}))B_0^i \\ b_{n+1} & \mapsto & f(b_{n+1}) + (q(b_{n+1}) + q(f(b_{n+1})))A_1 + (1 + B(f(a_{n+1}), f(b_{n+1})))D_0 \\ & & + \sum_{i=1}^n B(f(a_i), f(b_{n+1}))b_0^i + \sum_{i=1}^n B(f(b_i), f(b_{n+1}))B_0^i \end{array}$$

preserves the quadratic form and we have

$$\epsilon([V \xrightarrow{g_{n+1}} W \perp Y \perp H_0^{\perp n} \perp H_0^{\perp n} \perp H_1 \perp H_0 \leftrightarrow W]) = f$$

which completes the inductive step. \square

3.2. Proof of theorem 3.1. Theorem 3.1 is a consequence of a general result about functor categories which we recall in Proposition A.2 of the appendix. But, as the functor ϵ is not essentially surjective, we can not apply directly the proposition to the category \mathcal{F} . Consequently, we introduce a category \mathcal{F}' equivalent to \mathcal{F} .

Definition 3.6. *The category $\mathcal{E}^{f-(\text{even})}$ is the full subcategory of \mathcal{E}^f having as objects the \mathbb{F}_2 -vector spaces of even dimension.*

Notation 3.7. *We denote by \mathcal{F}' the category of functors from $\mathcal{E}^{f-(\text{even})}$ to \mathcal{E} .*

We have the following result:

Proposition 3.8. *The categories \mathcal{F} and \mathcal{F}' are equivalent.*

The proof of this proposition relies on the following standard lemma.

Lemma 3.9. [7] *Let $n > 0$ be a natural integer. Any idempotent linear map $e_{2n-1} : \mathbb{F}_2^{2n} \rightarrow \mathbb{F}_2^{2n}$ of rank $2n - 1$, verifies:*

$$P_{\mathbb{F}_2^{2n-1}}^{\mathcal{F}} \simeq P_{\mathbb{F}_2^{2n}}^{\mathcal{F}} \cdot e_{2n-1}$$

where $P_{\mathbb{F}_2^n}^{\mathcal{F}}(-) = \mathbb{F}_2[\text{Hom}_{\mathcal{E}^f}(\mathbb{F}_2^n, -)]$ is the standard projective object of \mathcal{F} given by the Yoneda lemma.

Proof of proposition 3.8. Let V be an object of \mathcal{E}^f .

If the dimension of V is even, V is an object of $\mathcal{E}^{f-(\text{even})}$.

If the dimension of V is odd $\dim(V) = 2n - 1$, by the previous lemma, $P_{\mathbb{F}_2^{2n-1}}^{\mathcal{F}}$ is a direct summand of $P_{\mathbb{F}_2^{2n}}^{\mathcal{F}}$.

As the category $\mathcal{E}^{f-(\text{even})}$ is a full subcategory of \mathcal{E}^f , we obtain by proposition A.4 of the appendix, the theorem. \square

Proof of theorem 3.1. The functor ι of theorem 3.1 is, by definition, the precomposition functor by the functor $\epsilon : \mathcal{E}_q \rightarrow \mathcal{E}^f$. The two first points of the theorem are clear. As the objects of \mathcal{E}_q are spaces of even dimension, the functor $\epsilon : \mathcal{E}_q \rightarrow \mathcal{E}^f$ factorizes through the inclusion $\mathcal{E}^{f-(\text{even})} \hookrightarrow \mathcal{E}^f$. This induces a functor $\epsilon' : \mathcal{E}_q \rightarrow \mathcal{E}^{f-(\text{even})}$

which is full and essentially surjective. We deduce from proposition A.2 that the functor $- \circ \epsilon' : \mathcal{F}' \rightarrow \mathcal{F}_{quad}$ is fully-faithful and, for a simple object S of \mathcal{F}' , $S \circ \epsilon'$ is simple in \mathcal{F}_{quad} . The theorem follows from proposition 3.8. \square

3.3. Duality. In this section, we prove that the duality defined over \mathcal{F}_{quad} in 2.15 is an extension of the duality over \mathcal{F} , $D : \mathcal{F}^{op} \rightarrow \mathcal{F}$, given by $DF = -^* \circ F \circ (-^*)^{op}$ for F an object of \mathcal{F} and $-^*$ the duality functor from \mathcal{E}^{op} to \mathcal{E} .

Proposition 3.10. *We have the following commutative diagram, up to natural isomorphism*

$$\begin{array}{ccc} \mathcal{F}^{op} & \xrightarrow{\iota^{op}} & \mathcal{F}_{quad}^{op} \\ D \downarrow & & \downarrow D \\ \mathcal{F} & \xrightarrow{\iota} & \mathcal{F}_{quad}. \end{array}$$

Proof. This relies on the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T}_q^{op} & \xrightarrow{\epsilon^{op}} & (\mathcal{E}^f)^{op} \\ tr \downarrow & & \downarrow -^* \\ \mathcal{T}_q & \xrightarrow{\epsilon} & \mathcal{E}^f. \end{array}$$

The commutativity is a consequence of classical results of linear algebra about the duality of vector spaces and the fact that a non-singular bilinear form on a vector space V determines a privileged isomorphism between V and V^* . \square

4. THE CATEGORY \mathcal{F}_{iso}

In this section, we define a subcategory \mathcal{F}_{iso} of \mathcal{F}_{quad} which is, by theorem 4.7, an abelian symmetric monoidal category with enough projective objects. The category \mathcal{F}_{iso} is related to \mathcal{F}_{quad} by the following theorem.

Theorem 4.1. *There is a functor*

$$\kappa : \mathcal{F}_{iso} \rightarrow \mathcal{F}_{quad}$$

which satisfies the following properties:

- (1) κ is exact;
- (2) κ preserves tensor products;
- (3) κ is fully-faithful;
- (4) if S is a simple object of \mathcal{F}_{iso} , $\kappa(S)$ is a simple object of \mathcal{F}_{quad} .

We obtain a classification of the simple objects of \mathcal{F}_{iso} from the following theorem.

Theorem 4.2. *There is a natural equivalence*

$$\mathcal{F}_{iso} \simeq \prod_{V \in \mathcal{S}} \mathbb{F}_2[O(V)] - mod$$

where \mathcal{S} is a set of representatives of isometry classes of objects of \mathcal{E}_q^{deg} .

4.1. Definition of the category \mathcal{F}_{iso} .

4.1.1. *The category $\text{Sp}(\mathcal{E}_q^{deg})$.* Example 2.4 implies that the category $\text{Sp}(\mathcal{E}_q^{deg})$ is defined. In this section, we give some properties of this category which are similar to those given for the category \mathcal{T}_q .

Definition 4.3 (Transposition). *The transposition functor, $tr : \text{Sp}(\mathcal{E}_q^{deg})^{op} \rightarrow \text{Sp}(\mathcal{E}_q^{deg})$, is defined on objects by $tr(V) = V$ and on morphisms by:*

$$\text{tr}(f) = \text{tr}([V \leftarrow X \rightarrow W]) = [W \leftarrow X \rightarrow V],$$

for f an element of $\text{Hom}_{\text{Sp}(\mathcal{E}_q^{deg})^{op}}(W, V)$.

Proposition 4.4 (Orthogonal sum). *There exists a bifunctor $\perp : \mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}}) \times \mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}}) \rightarrow \mathcal{S}_q$, called the orthogonal sum, defined on objects by:*

$$\perp(V, W) = V \perp W$$

and on morphisms by:

$$\perp([V \xleftarrow{f} X \xrightarrow{g} W], [V' \xleftarrow{f'} X' \xrightarrow{g'} W']) = [V \perp V' \xleftarrow{f \perp f'} X \perp X' \xrightarrow{g \perp g'} W \perp W'].$$

This bifunctor gives $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$ the structure of a symmetric monoidal category, with unit $\{0\}$.

4.1.2. *The category \mathcal{F}_{iso} .* As the category $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$ is essentially small, we can give the following definition.

Definition 4.5. *The category \mathcal{F}_{iso} is the category of functors from $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$ to \mathcal{E} .*

Remark 4.6. *The category \mathcal{F}_{iso} is equivalent to the category of Mackey functors from $\mathcal{E}_q^{\mathrm{deg}}$ to \mathcal{E} , by the paper [8].*

As for the category \mathcal{F}_{quad} , we obtain the following theorem.

- Theorem 4.7.**
- (1) *The category \mathcal{F}_{iso} is abelian.*
 - (2) *The tensor product of vector spaces induces a structure of symmetric monoidal category on \mathcal{F}_{iso} .*
 - (3) *For an object V of $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$, the functor $Q_V = \mathbb{F}_2[\mathrm{Hom}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(V, -)]$ is a projective object and there is a natural isomorphism:*

$$\mathrm{Hom}_{\mathcal{F}_{iso}}(Q_V, F) \simeq F(V)$$

for all objects F of \mathcal{F}_{iso} .

The set of functors $\{Q_V | V \in \mathcal{S}\}$ is a set of projective generators of \mathcal{F}_{iso} , where \mathcal{S} is a set of representatives of isometry classes of degenerate quadratic spaces. In particular, the category \mathcal{F}_{iso} has enough projective objects.

Definition 4.8. *The duality functor of \mathcal{F}_{iso} is the functor $D : \mathcal{F}_{iso}^{op} \rightarrow \mathcal{F}_{iso}$ given by:*

$$DF = -^* \circ F \circ tr^{op}$$

for F an object of \mathcal{F}_{iso} , $-^*$ the duality functor from \mathcal{E}^{op} to \mathcal{E} and tr is the transposition functor of $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$ defined in 4.3.

The following proposition summarizes the basic properties of the duality functor D .

- Proposition 4.9.**
- (1) *The functor D is exact.*
 - (2) *The functor D is right adjoint to the functor D^{op} , i.e. we have a natural isomorphism:*

$$\mathrm{Hom}_{\mathcal{F}_{iso}}(F, DG) \simeq \mathrm{Hom}_{\mathcal{F}_{iso}^{op}}(D^{op}F, G) \simeq \mathrm{Hom}_{\mathcal{F}_{iso}}(G, DF).$$

- (3) *For F an object of \mathcal{F}_{iso} with values in finite dimensional vector spaces, the unit of the adjunction between \mathcal{F}_{iso} and \mathcal{F}_{iso}^{op} : $F \rightarrow DD^{op}F$ is an isomorphism.*

A straightforward consequence of the first point of the last proposition and theorem 4.7 is:

Corollary 4.10. *The category \mathcal{F}_{iso} has enough injective objects.*

4.2. An equivalent definition of \mathcal{F}_{iso} . In order to apply proposition A.2 in the appendix to prove theorem 4.1, we will use the same strategy as for theorem 3.1. In other words, we will introduce a category equivalent to \mathcal{F}_{iso} such that κ will be the precomposition functor by a full and essentially surjective functor. First, we give the following definition.

Definition 4.11. *The category \mathcal{S}_q is the full subcategory of $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$ having as objects the non-degenerate quadratic spaces.*

Remark 4.12. *A morphism of \mathcal{S}_q is represented by a diagram $V \xleftarrow{f} D \xrightarrow{g} W$ where V and W are non-degenerate quadratic spaces and D is a possibly degenerate quadratic space.*

The transposition functor and the orthogonal sum defined in the previous section for the category $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$ induce, by restriction, a transposition functor and an orthogonal sum for the category \mathcal{S}_q .

As the category \mathcal{S}_q is, by definition, a full subcategory of the category $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$, we have the existence of a functor λ' from \mathcal{F}_{iso} to $\mathrm{Func}(\mathcal{S}_q, \mathcal{E})$ induced by the inclusion $\mathcal{S}_q \hookrightarrow \mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$. The aim of this section is to show that λ' is an isomorphism.

Theorem 4.13. *There exists a natural isomorphism:*

$$\mathcal{F}_{iso} \simeq \mathrm{Func}(\mathcal{S}_q, \mathcal{E})$$

where $\mathrm{Func}(\mathcal{S}_q, \mathcal{E})$ is the category of functors from \mathcal{S}_q to \mathcal{E} .

To prove the theorem, we require some results about the idempotents of the category $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$.

4.2.1. The idempotents of $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$. We begin this section with the following notation.

Notation 4.14. *Let V be an object of $\mathcal{E}_q^{\mathrm{deg}}$, α and β be morphisms of $\mathrm{Hom}_{\mathcal{E}_q^{\mathrm{deg}}}(D, V)$. We denote by $f_{\alpha,\beta}$ the following morphism of $\mathrm{Hom}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(V, V)$:*

$$[V \xleftarrow{\alpha} D \xrightarrow{\beta} V].$$

Proposition 4.15. (1) *An idempotent of $\mathrm{Hom}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(V, V)$ is of the form*

$$e_\alpha = [V \xleftarrow{\alpha} D \xrightarrow{\alpha} V]$$

where α is an element of $\mathrm{Hom}_{\mathcal{E}_q^{\mathrm{deg}}}(D, V)$, for some D .

- (2) *For α and β two morphisms of $\mathcal{E}_q^{\mathrm{deg}}$ with range V , the idempotents e_α and e_β commute.*
- (3) *For α and β two morphisms of $\mathcal{E}_q^{\mathrm{deg}}$ with range V , the elements $1 + e_\alpha$ and $1 + e_\beta$ are idempotents of $\mathbb{F}_2[\mathrm{Hom}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(V, V)]$ which commute.*

Proof. By definition of $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$, $f_{\alpha,\beta} \circ f_{\alpha,\beta} = [V \leftarrow D' \rightarrow V]$ where D' is the pullback in $\mathcal{E}_q^{\mathrm{deg}}$ of the diagram $D \xrightarrow{\beta} V \xleftarrow{\alpha} D$ and $f_{\alpha,\beta} \circ f_{\alpha,\beta} = f_{\alpha,\beta}$ if and only if there exists an isomorphism $g : D \rightarrow D'$ such that the following diagram is

commutative:

$$\begin{array}{ccccc}
 D & & & & \\
 \searrow \alpha & & & & \\
 & D' & \xrightarrow{g^{-1}} & D & \xrightarrow{\beta} & V \\
 & \downarrow g^{-1} & & \downarrow \alpha & & \\
 & D & \xrightarrow{\beta} & V & & \\
 & \downarrow \alpha & & & & \\
 & V & & & &
 \end{array}$$

This implies that $\alpha = \beta$. Consequently $f_{\alpha,\beta} \circ f_{\alpha,\beta} = f_{\alpha,\beta}$ if and only if $f_{\alpha,\beta} = e_\alpha$.

The second point is straightforward and the last point is a direct consequence of the second one by a standard result about idempotents (noting that $-1 = 1$ in \mathbb{F}_2). \square

4.2.2. *Proof of theorem 4.13.* The proof of theorem 4.13 relies on the following crucial lemma.

Lemma 4.16. *For V an object of $\text{Span}(\mathcal{E}_q^{\text{deg}})$ and $\alpha : A \hookrightarrow V$ a subobject of V in $\mathcal{E}_q^{\text{deg}}$,*

$$Q_V \cdot e_\alpha \simeq Q_A.$$

Proof. Let $\alpha_* : Q_A \rightarrow Q_V$ (respectively $\alpha^* : Q_V \rightarrow Q_A$) be the morphism of \mathcal{F}_{iso} which corresponds by the Yoneda lemma to the element $[V \xleftarrow{\alpha} A \xrightarrow{\text{Id}} A]$ of $Q_V(A)$ (respectively $[A \xleftarrow{\text{Id}} A \xrightarrow{\alpha} V]$ of $Q_A(V)$).

As $[V \xleftarrow{\alpha} A \xrightarrow{\text{Id}} A] \circ [A \xleftarrow{\text{Id}} A \xrightarrow{\alpha} V] = \text{Id}$ we have $\alpha^* \circ \alpha_* = \text{Id}$ and as $[A \xleftarrow{\text{Id}} A \xrightarrow{\alpha} V] \circ [V \xleftarrow{\alpha} A \xrightarrow{\text{Id}} A] = e_\alpha$ we have $\alpha_* \circ \alpha^* = \cdot e_\alpha$. \square

Proof of theorem 4.13. Let A be an object of $\text{Sp}(\mathcal{E}_q^{\text{deg}})$, there exists an object V of \mathcal{S}_q such that A is a subobject of V . By the previous lemma, we deduce that Q_A is a direct summand of Q_V . As the category \mathcal{S}_q is a full subcategory of $\text{Sp}(\mathcal{E}_q^{\text{deg}})$ we obtain, by proposition A.4 of the appendix, the theorem. \square

4.3. **Relation between \mathcal{F}_{iso} and \mathcal{F}_{quad} .** The main result of this section is theorem 4.1, which gives the existence of an exact, fully-faithful functor $\kappa : \mathcal{F}_{iso} \rightarrow \mathcal{F}_{quad}$ which preserves the simple objects. To define the functor κ of this theorem we need to define and study the functor $\sigma : \mathcal{T}_q \rightarrow \mathcal{S}_q$.

4.3.1. *Definition of $\sigma : \mathcal{T}_q \rightarrow \mathcal{S}_q$.*

Proposition 4.17. *There exists a monoidal functor $\sigma : \mathcal{T}_q \rightarrow \mathcal{S}_q$ defined by $\sigma(V) = V$ and*

$$\sigma([V \rightarrow X \leftarrow W]) = [V \leftarrow V \times_X W \rightarrow W]$$

where $V \times_X W$ is the pullback in $\mathcal{E}_q^{\text{deg}}$.

The proof of this proposition relies on the following important lemma.

Lemma 4.18. *For V, W and X objects of \mathcal{E}_q and $V \xrightarrow{f} W, V \xrightarrow{g} X$ morphisms of \mathcal{E}_q , we have:*

$$X \times_{X \perp_W} W \simeq V$$

where $- \times_A -$ is the pullback over A in $\mathcal{E}_q^{\text{deg}}$ and $- \perp_B -$ is the pseudo push-out over B in \mathcal{E}_q .

Proof. It is a straightforward consequence of the definitions of the pseudo push-out and the pullback. \square

Remark 4.19. *The result of the previous lemma explains why we have imposed that the morphisms of the category $\mathcal{E}_q^{\text{deg}}$ are monomorphisms.*

Proof of proposition 4.17. The functor σ is well defined on the classes of morphisms of \mathcal{T}_q . To show that σ is a functor, we verify that

$$\sigma([V \xrightarrow{\text{Id}} V \xleftarrow{\text{Id}} V]) = \text{Id}$$

and, that σ respects composition, which is a direct consequence of the above lemma. Finally, the following consequence of the definition of the pullback and of the orthogonal sum:

for $T_1 = [V_1 \rightarrow X_1 \leftarrow W_1]$ and $T_2 = [V_2 \rightarrow X_2 \leftarrow W_2]$, we have:

$$(V_1 \times_{X_1} W_1) \perp (V_2 \times_{X_2} W_2) \simeq (V_1 \perp V_2) \times_{X_1 \perp X_2} (W_1 \perp W_2)$$

implies that σ preserves the monoidal structures. \square

4.3.2. *The fullness of σ .* The aim of this section is to prove the following proposition.

Proposition 4.20. *The functor σ is full.*

The proof of this proposition relies on the following technical lemma.

Lemma 4.21. *Let D be a degenerate quadratic space of dimension r which has the following decomposition*

$$D = (x_1, \epsilon_1) \perp \dots \perp (x_r, \epsilon_r)$$

where $\epsilon_i \in \{0, 1\}$, H a non-degenerate quadratic space and f an element of $\text{Hom}_{\mathcal{E}_q^{\text{deg}}}(D, H)$. Then there exists elements k_1, \dots, k_r in H and a non-degenerate quadratic space H' such that

$$H = \text{Vect}(f(x_1), k_1) \perp \dots \perp \text{Vect}(f(x_r), k_r) \perp H'$$

and $B(f(x_i), k_i) = 1$ where B is the underlying bilinear form.

Proof. We prove this lemma by induction on the dimension r of the space D .

For $r = 1$, we have $f : (x_1, \epsilon_1) \rightarrow H$ and $f(x_1) = h_1$. As H is, by hypothesis, non-degenerate there exists an element k_1 in H such that $B(h_1, k_1) = 1$. Then, the space $K = \text{Vect}(h_1, k_1)$ is a non-degenerate subspace of H , so we have $H = K \perp K^\perp$, with K^\perp non-degenerate.

Suppose that the result is true for $r = n$. Let $(x_1, \dots, x_n, x_{n+1})$ be linearly independent vectors and $f : (x_1, \epsilon_1) \perp \dots \perp (x_n, \epsilon_n) \perp (x_{n+1}, \epsilon_{n+1}) \rightarrow H$. By restriction, we have

$$f \circ i : (x_1, \epsilon_1) \perp \dots \perp (x_n, \epsilon_n) \rightarrow H$$

and by the inductive assumption, we have the existence of k_1, \dots, k_n in H such that:

$$H = \text{Vect}(f(x_1), k_1) \perp \dots \perp \text{Vect}(f(x_n), k_n) \perp H'$$

We decompose $f(x_{n+1})$ over this basis to obtain the following decomposition

$$f(x_{n+1}) = \sum_{i=1}^n (\alpha_i f(x_i) + \beta_i k_i) + h'$$

where α_i and β_i are elements of \mathbb{F}_2 and h' is an element of H' . As f preserves the quadratic form and, consequently, the underlying bilinear form, we have $\beta_i = B(f(x_{n+1}), f(x_i)) = B(x_{n+1}, x_i) = 0$ for all i . Hence

$$(4.21.1) \quad f(x_{n+1}) = \sum_{i=1}^n \alpha_i f(x_i) + h'$$

As the vectors $(f(x_1), \dots, f(x_n), f(x_{n+1}))$ are linearly independent, by the injectivity of f , we have $h' \neq 0$ and, as H' is non-degenerate, there exists an element k' in H' such that $B(h', k') = 1$. We deduce the following decomposition $H' = \text{Vect}(h', k') \perp H''$.

In the equality (4.21.1), after reordering, we can suppose that

$$\alpha_1 = \dots = \alpha_p = 1$$

and

$$\alpha_{p+1} = \dots = \alpha_n = 0.$$

Then we have,

$$B(f(x_{n+1}), k_i) = 1 \text{ pour } i = 1, \dots, p$$

and

$$B(f(x_{n+1}), k_i) = 0 \text{ pour } i = p + 1, \dots, n.$$

Consequently, by the following decomposition of H

$$H = \perp_{i=1}^p \text{Vect}(f(x_i), k_i + k') \perp_{j=p+1}^n \text{Vect}(f(x_j), k_j) \perp \text{Vect}(f(x_{n+1}), k') \perp H''$$

we obtain the result. \square

Proof of proposition 4.20. We prove that, for a morphism $S = [V \xleftarrow{f} D \xrightarrow{g} W]$ of \mathcal{S}_q , there exists a morphism T in $\text{Hom}_{\mathcal{T}_q}(V, W)$ such that $\sigma(T) = S$.

First we decompose the morphism S as an orthogonal sum of more simple morphisms. By theorem 1.10 we have $D \simeq H \perp \text{Rad}(V)$ such that H is non-degenerate and $\text{Rad}(V)$ is isometric to either $(x, 0)^{\perp r}$ or $(x, 1)^{\perp r}$, where r is the dimension of $\text{Rad}(V)$. By the previous lemma, we can decompose the morphisms f and g in the following form:

$$f : H \perp \text{Rad}(D) \rightarrow H \perp D' \perp V' \simeq V$$

and

$$g : H \perp \text{Rad}(D) \rightarrow H \perp D'' \perp W' \simeq W$$

where D' (respectively D'') is one of the non-degenerate spaces constructed in lemma 4.21 and V' (respectively W') is the orthogonal of $H \perp D'$ (respectively $H \perp D''$). We deduce that :

$$S = [H \leftarrow H \rightarrow H] \perp [V' \leftarrow 0 \rightarrow W'] \perp [D' \leftarrow \text{Rad}(D) \rightarrow D''].$$

Furthermore, again by theorem 1.10

$$[D' \leftarrow \text{Rad}(D) \rightarrow D''] = \perp_i [V_i \leftarrow (x_i, \epsilon_i) \rightarrow W_i]$$

where, by lemma 4.21, V_i and W_i are spaces of dimension two, therefore:

$$S = [H \leftarrow H \rightarrow H] \perp [V' \leftarrow 0 \rightarrow W'] \perp [V_1 \leftarrow (x_1, \epsilon_1) \rightarrow W_1] \perp \dots \perp [V_r \leftarrow (x_r, \epsilon_r) \rightarrow W_r].$$

According to proposition 4.17, it is enough to prove that for each morphism S_α , which appears as a factor in the previous decomposition of S , there exists a morphism of \mathcal{T}_q , T_α , such that $\sigma(T_\alpha) = S_\alpha$.

Obviously, we have

$$\sigma([H \rightarrow H \leftarrow H]) = [H \leftarrow H \rightarrow H]$$

and

$$\sigma([V \rightarrow V \perp W \leftarrow W]) = [V \leftarrow 0 \rightarrow W].$$

For the morphisms $[V \leftarrow (x, \epsilon) \rightarrow W]$, we have to consider several cases.

In the case $\epsilon = 0$, as all the non-zero element x of H_1 verify $q(x) = 1$, we have $\text{Hom}_{\mathcal{E}_q^{\text{deg}}}((x, 0), H_1) = \emptyset$. Consequently, we have to consider only the following morphism:

$$S_1 = [H_0 \xleftarrow{f} (x, 0) \xrightarrow{g} H_0].$$

After composing by an element of $O_2^+ = O(H_0)$, we can suppose that $f(x) = g(x) = a_0$. The morphism

$$T_1 = [H_0 \xrightarrow{f'} H_0 \perp H_0 \xleftarrow{g'} H_0]$$

where, f' is defined by:

$$f'(a_0) = a_0 \text{ and } f'(b_0) = b_0 + a'_0$$

and g' is defined by:

$$g'(a_0) = a_0 \text{ and } g'(b_0) = b_0 + b'_0$$

satisfies $\sigma(T_1) = S_1$.

In the case $\epsilon = 1$, we have to consider three kinds of morphisms.

For

$$S_2 = [H_0 \xleftarrow{f} (x, 1) \xrightarrow{g} H_0],$$

as only the element $a_0 + b_0$ of H_0 verify $q(a_0 + b_0) = 1$, we have $f(x) = g(x) = a_0 + b_0$. The morphism

$$T_2 = [H_0 \xrightarrow{f'} H_0 \perp H_0 \xleftarrow{g'} H_0]$$

where, f' is defined by:

$$f'(a_0) = a_0 + a'_0 \text{ and } f'(b_0) = b_0 + a'_0$$

and g' is defined by:

$$g'(a_0) = a_0 + b'_0 \text{ and } g'(b_0) = b_0 + b'_0$$

satisfies $\sigma(T_2) = S_2$.

For

$$S_3 = [H_1 \xleftarrow{f} (x, 1) \xrightarrow{g} H_1],$$

after composing by an element of $O_2^- = O(H_1)$, we can suppose that $f(x) = g(x) = a_1$. The morphism

$$T_3 = [H_1 \xrightarrow{f'} H_1 \perp H_0 \xleftarrow{g'} H_1]$$

where, f' is defined by:

$$f'(a_1) = a_1 \text{ and } f'(b_1) = b_1 + b_0$$

and g' is defined by:

$$g'(a_1) = a_1 \text{ and } g'(b_1) = b_1 + a_0$$

satisfies $\sigma(T_3) = C_3$.

For

$$S_4 = [H_0 \xleftarrow{f} (x, 1) \xrightarrow{g} H_1],$$

we have $f(x) = a_0 + b_0$ and, after composing by an element of $O_2^- = O(H_1)$, we can suppose that $g(x) = a_1$. The morphism

$$T_4 = [H_0 \xrightarrow{f'} H_1 \perp H_0 \xleftarrow{g'} H_1]$$

where, f' is defined by:

$$f'(a_0) = a_1 + b_1 + a_0 + b_0 \text{ et } f'(b_0) = b_1 + a_0 + b_0$$

and g' is defined by:

$$g'(a_1) = a_1 \text{ et } g'(b_1) = b_1$$

satisfies $\sigma(T_4) = C_4$.

The final possibility results from the previous one by transposition. \square

4.3.3. *Proof of theorem 4.1.* The functor κ of the theorem 4.1 is, by definition, the precomposition functor by $\sigma : \mathcal{T}_q \rightarrow \mathcal{S}_q$. The two first point of the theorem are clear. By the proposition 4.17 the functor σ is full and, by definition, it is essentially surjective. So, the two last points of the theorem 4.1 are direct consequences of the proposition A.2 given in the appendix.

4.3.4. *Duality.*

Proposition 4.22. *We have the following commutative diagram, up to natural isomorphism*

$$\begin{array}{ccc} \mathcal{F}_{iso}^{op} & \xrightarrow{\kappa^{op}} & \mathcal{F}_{quad}^{op} \\ D \downarrow & & \downarrow D \\ \mathcal{F}_{iso} & \xrightarrow{\kappa} & \mathcal{F}_{quad}. \end{array}$$

Proof. This relies on the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T}_q^{op} & \xrightarrow{\sigma^{op}} & (\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}}))^{op} \\ tr \downarrow & & \downarrow tr \\ \mathcal{T}_q & \xrightarrow{\sigma} & \mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}}) \end{array}$$

which is a direct consequence of the definitions. \square

4.4. **The isotropic functors.** In this section we are interested in an important family of functors of \mathcal{F}_{iso} named the isotropic functors; the choice of terminology will be explained below. After giving the definition of these functors we prove that they are self-dual. We will show in the following sections that these functors give rise to a family of projective generators of the category \mathcal{F}_{iso} .

4.4.1. *Definition.* Let $(\mathrm{Id}_V)^*$ be the element of

$$DQ_V(V) = Q_V(V)^* = \mathrm{Hom}(\mathbb{F}_2[\mathrm{End}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(V)], \mathbb{F}_2)$$

defined by:

$$(\mathrm{Id}_V)^*([\mathrm{Id}_V]) = 1 \quad \text{and} \quad (\mathrm{Id}_V)^*([f]) = 0 \quad \text{for all } f \neq \mathrm{Id}_V.$$

We denote by $a_V : Q_V \rightarrow DQ_V$ the morphism of \mathcal{F}_{iso} which corresponds by the Yoneda lemma to the element $(\mathrm{Id}_V)^*$ of $DQ_V(V)$.

Definition 4.23. *The isotropic functor $iso_V : \mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}}) \rightarrow \mathcal{E}$ of \mathcal{F}_{iso} is the image of Q_V by the morphism a_V .*

Notation 4.24. *We denote by K_V the kernel of a_V . We have the following short exact sequence:*

$$0 \rightarrow K_V \rightarrow Q_V \xrightarrow{a_V} iso_V \rightarrow 0.$$

In the following lemma, we give, an explicit description of the vector spaces $K_V(W)$ and $iso_V(W)$, which are elementary consequences of the definition.

Lemma 4.25. *For an object W of $\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$, we have that:*

- $K_V(W)$ is the subvector space of $Q_V(W)$ generated by the elements $[V \leftarrow H \rightarrow W]$ where $H \not\cong V$;
- as a vector space, $iso_V(W)$ is isomorphic to the subspace of $Q_V(W)$ generated by the elements $[V \xleftarrow{\mathrm{Id}} V \rightarrow W]$. Consequently $iso_V(W)$ has basis the set $\mathrm{Hom}_{\mathcal{E}_q^{\mathrm{deg}}}(V, W)$.

Remark 4.26. *Observe that the isomorphism given in the second point is not natural.*

Remark 4.27. *The terminology “isotropic functor” was motivated by the first case considered by the author. For the quadratic space $(x, 0)$, we have:*

$$\text{iso}_{(x,0)}(V) \simeq \mathbb{F}_2[I_V]$$

where $I_V = \{v \in V \setminus \{0\} \mid q(v) = 0\}$ is the isotropic cone of the quadratic space V .

4.4.2. Self-duality.

Remark 4.28. *For simplicity, we will denote in this section, $D^{\circ p}$ by D .*

To begin we recall several definitions.

Definition 4.29. (1) *A morphism $b : F \rightarrow DF$ is self-adjoint if $b = Db \circ \eta_F$, where $\eta_F : F \rightarrow D^2F$ is the unit of the adjunction between D and $D^{\circ p}$.*
 (2) *A functor F is self-dual if there exists an isomorphism*

$$\gamma : F \xrightarrow{\simeq} DF$$

which is self-adjoint.

The main result of this section is the following proposition.

Proposition 4.30. *The isotropic functors of $\mathcal{F}_{\text{quad}}$ are self-dual.*

The proof of this proposition relies on the following lemma.

Lemma 4.31. *Let F be a functor which takes finite dimensional values and $a : F \rightarrow DF$ a self-adjoint morphism, then $\text{im}(a)$ is self-dual.*

Remark 4.32. *This lemma and its proof are direct consequences of lemma 1.2.4 in [10].*

Proof. The morphism a admits the following factorisation by $\text{im}(a)$

$$a : F \xrightarrow{p} \text{im}(a) \xrightarrow{j} DF.$$

We deduce the existence of \tilde{a} which makes the following diagram commutative

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(a) & \longrightarrow & F & \xrightarrow{p} & \text{im}(a) \longrightarrow 0 \\ & & \downarrow & & \downarrow \eta_F & & \downarrow \tilde{a} \\ 0 & \longrightarrow & \text{Ker}(Dj) & \longrightarrow & D^2F & \xrightarrow{Dj} & D(\text{im}(a)) \longrightarrow 0. \end{array}$$

As F has finite dimensional values, the unit of the adjunction $\eta_F : F \rightarrow D^2F$ is an isomorphism. Consequently, by the following commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{p} & \text{im}(a) & \xrightarrow{j} & DF \\ \simeq \downarrow \eta_F & & \downarrow \tilde{a} & & \parallel \\ D^2F & \xrightarrow{Dj} & D(\text{im}(a)) & \xrightarrow{Dp} & DF \end{array}$$

we obtain that \tilde{a} is an isomorphism.

If we dualize the first commutative diagram, we obtain:

$$Dp \circ D\tilde{a} = D\eta_F \circ D^2j.$$

So

$$Dp \circ D\tilde{a} \circ \eta_{\text{im}(a)} \circ p = D\eta_F \circ D^2j \circ \eta_{\text{im}(a)} \circ p = j \circ p = a.$$

We have also

$$Dp \circ \tilde{a} \circ p = Dp \circ Dj \circ \eta_F = Da \circ \eta_F = a.$$

We deduce that $D\tilde{a} \circ \eta_{\text{im}(a)} = \tilde{a}$. \square

Proof of proposition 4.30. By the previous lemma, it is enough to prove that the morphism $a_V : Q_V \rightarrow DQ_V$ is self-dual. By the Yoneda lemma we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{F}_{iso}}(Q_V, DQ_W) & \xrightarrow{f} & \mathrm{Hom}_{\mathcal{F}_{iso}}(Q_W, DQ_V) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathbb{F}_2[\mathrm{Hom}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(W, V)]^* & \xrightarrow{\mathbb{F}_2[tr]^*} & \mathbb{F}_2[\mathrm{Hom}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(V, W)]^* \end{array}$$

where $tr : \mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})^{op} \rightarrow \mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})$ is the transposition functor defined in 4.3 and f is the natural isomorphism given in proposition 4.9. By definition, $a_V : Q_V \rightarrow DQ_V$ corresponds, by the Yoneda lemma, to the element $(\mathrm{Id}_V)^*$ of $DQ_V(V)$. Since $tr(\mathrm{Id}_V) = \mathrm{Id}_V$, we deduce that $a_V = Da_V \circ \eta_{Q_V}$. \square

4.5. Decomposition of the projective objects Q_V of \mathcal{F}_{iso} . The aim of this section is to prove the following theorem.

Theorem 4.33. *For V an object of $\mathcal{E}_q^{\mathrm{deg}}$, we have:*

$$Q_V \simeq \bigoplus_{A \in \mathcal{S}_V} iso_A$$

where \mathcal{S}_V is the set of subobjects of V in $\mathcal{E}_q^{\mathrm{deg}}$, represented by a morphism $\alpha : A \hookrightarrow V$.

The proof of this theorem relies on the following proposition.

Proposition 4.34. *For an object V of $\mathcal{E}_q^{\mathrm{deg}}$, the element E_V of $\mathbb{F}_2[\mathrm{Hom}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(V, V)]$*

$$E_V = \prod_{\substack{\alpha : A \hookrightarrow V \\ A \in \mathcal{S}_V \setminus V}} (1 + e_\alpha)$$

verifies:

- (1) $E_V \cdot E_V = E_V$
- (2) $Q_V \cdot E_V \simeq iso_V$. In particular $\mathrm{Hom}(iso_V, F) \simeq F(E_V) \cdot F(V)$.

Proof. The first point is a direct consequence of proposition 4.15 (3) and (1).

For the second point, we consider the following split short exact sequence:

$$0 \rightarrow Q_V \cdot (1 + E_V) \rightarrow Q_V \rightarrow Q_V \cdot E_V \rightarrow 0$$

and we recall that we have by notation 4.24 the following short exact sequence:

$$0 \rightarrow K_V \rightarrow Q_V \xrightarrow{a_V} iso_V \rightarrow 0.$$

By the 5-lemma, to obtain the result, it is sufficient, to prove that:

$$Q_V \cdot (1 + E_V) \simeq K_V.$$

By expanding $1 + E_V$, we obtain that

$$1 + E_V = \prod_{\substack{\gamma : A \hookrightarrow V \\ A \in \mathcal{R}_V}} e_\gamma$$

where \mathcal{R}_V is a subset of $\mathcal{S}_V \setminus V$. Consequently

$$Q_V \cdot (1 + E_V) \subset \sum_{\substack{\gamma : A \hookrightarrow V \\ A \in \mathcal{S}_V \setminus V}} Q_V \cdot e_\gamma = K_V$$

where the last equality is a direct consequence of lemma 4.25.

On the other side, for $\gamma : A \hookrightarrow V$ with $A \in \mathcal{S}_V \setminus V$, we have

$$(Q_V \cdot e_\gamma) \cdot E_V = Q_V \cdot e_\gamma \cdot (1 + e_\gamma) \prod_{\substack{\alpha : A \hookrightarrow V \\ \alpha \neq \gamma \\ A \in \mathcal{S}_V \setminus V}} (1 + e_\alpha) = 0$$

where the first equality follows from proposition 4.15 (3).

Consequently $Q_V \cdot e_\gamma \subset Q_V \cdot (1 + E_V)$ and

$$K_V = \sum_{\substack{\gamma : A \hookrightarrow V \\ A \in \mathcal{S}_V \setminus V}} Q_V \cdot e_\gamma \subset Q_V \cdot (1 + E_V).$$

We deduce that

$$Q_V \cdot (1 + E_V) \simeq K_V. \quad \square$$

An important consequence of the previous proposition is given in the following corollary.

Corollary 4.35. *For V and W objects of $\mathcal{E}_q^{\text{deg}}$, we have:*

$$\text{Hom}_{\mathcal{F}_{iso}}(iso_V, iso_W) \simeq \begin{cases} \mathbb{F}_2[O(V)] & \text{if } W \simeq V \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By lemma 4.25 an element of $iso_W(V)$ is represented by a linear sum of

$$[W \leftarrow W \xrightarrow{f} V].$$

- If $W \not\simeq V$. We have

$$(1 + e_f)[W \leftarrow W \xrightarrow{f} V] = [W \leftarrow W \xrightarrow{f} V] + [W \leftarrow W \xrightarrow{f} V] = 0$$

as the idempotents $(1 + e_f)$ commute by proposition 4.15, we deduce that:

$$\text{Hom}(iso_V, iso_W) = iso_W(E_V) iso_W(V) = 0$$

where the first equality is given by proposition 4.34 (2).

- If $W \simeq V$. For $\alpha : A \hookrightarrow V$ with $A \in \mathcal{S}_V \setminus V$ we have:

$$e_\alpha[W \leftarrow W \rightarrow V] = [W \leftarrow A \rightarrow V] \in K_W(V).$$

So $iso_W(e_\alpha)[W \leftarrow W \rightarrow V] = 0$ and

$$\text{Hom}(iso_V, iso_W) = iso_W(E_V) iso_W(V) = iso_W(V) \simeq \text{Hom}_{\mathcal{E}_q^{\text{deg}}}(W, V) \simeq \mathbb{F}_2[O(V)]$$

by lemma 4.25. □

Proposition 4.36. *For $\alpha : A \hookrightarrow V$ a subobject of V , the idempotent E_α defined by:*

$$E_\alpha = e_\alpha \sum_{\substack{\beta : B \hookrightarrow A \\ B \in \mathcal{S}_A \setminus A}} (1 + e_\beta)$$

verifies

$$Q_V \cdot E_\alpha \simeq iso_A.$$

Proof. By the proposition 4.15, E_α is clearly an idempotent. The result follows from proposition 4.34 and lemma 4.16. □

Proof of theorem 4.33. By the proof of 4.34, there exists an exact sequence

$$\bigoplus_{\substack{\gamma : A \hookrightarrow V \\ A \in \mathcal{S}_V}} Q_V \cdot e_\gamma \rightarrow Q_V \rightarrow iso_V \rightarrow 0$$

hence a complex

$$\bigoplus_{\substack{\gamma : A \hookrightarrow V \\ A \in \mathcal{S}_V}} iso_A \rightarrow Q_V \rightarrow iso_V \rightarrow 0$$

by proposition 4.36. We deduce from proposition 4.15 that the idempotents given in the proposition 4.36 are orthogonal. Consequently the map

$$\bigoplus_{\substack{\gamma : A \hookrightarrow V \\ A \in \mathcal{S}_V}} iso_A \rightarrow Q_V$$

is injective. Furthermore, by lemma 4.25, for an object X of $\text{Sp}(\mathcal{E}_q^{\text{deg}})$

$$Q_V(X) \simeq iso_V(X) \oplus \left(\bigoplus_{\substack{\alpha : W \hookrightarrow V \\ W \in \mathcal{S}_V \setminus V}} iso_W(X) \right).$$

We deduce that the previous complex is a short exact sequence, which is split by proposition 4.34. \square

By theorem 4.7 and the self-duality of the isotropic functors given in proposition 4.30, we obtain the following corollary of theorem 4.33.

Corollary 4.37. *The set of functors $\{iso_V | V \in \mathcal{S}\}$ is a set of projective generators (resp. injective cogenerators) of \mathcal{F}_{iso} , where \mathcal{S} is a set of representatives of isometry classes of possibly degenerate quadratic spaces.*

4.6. Proof of theorem 4.2. In this section we prove the equivalence between the category \mathcal{F}_{iso} and the product of categories of modules over the orthogonal groups.

Proof of theorem 4.2. In [11] we have the following result (Corollary 6.4, p. 103).

For any abelian category \mathcal{C} the following assertions are equivalent:

- (1) The category \mathcal{C} has arbitrary direct sums and $\{P_i\}_{i \in I}$ is a set of projective generators of finite type of \mathcal{C} .
- (2) The category \mathcal{C} is equivalent to the subcategory $\text{Func}^{add}(\mathcal{P}^{op}, \mathcal{A}b)$ of $\text{Func}(\mathcal{P}^{op}, \mathcal{A}b)$ having as objects the functors satisfying $F(f+g) = F(f) + F(g)$ where f and g are morphisms of $\text{Hom}_{\mathcal{P}^{op}}(V, W)$ and \mathcal{P} is the full subcategory of \mathcal{C} having as objects $\{P_i | i \in I\}$.

Let \mathcal{C} be the category \mathcal{F}_{iso} . By corollary 4.37, the set of functors $\{iso_V | V \in \mathcal{S}\}$ is a set of projective generators of \mathcal{F}_{iso} , where \mathcal{S} is a set of representatives of isometry classes of degenerate quadratic spaces. By proposition 4.34 (2) iso_V is a direct summand of Q_V ; as Q_V is of finite type, we deduce that iso_V is of finite type. Consequently, by the previous result we obtain that

$$\mathcal{F}_{iso} \simeq \text{Func}^{add}(\mathcal{P}^{op}, \mathcal{A}b)$$

where \mathcal{P} is the full subcategory of \mathcal{F}_{iso} having as objects the isotropic functors. By corollary 4.35, $\text{Hom}_{\mathcal{F}_{iso}}(iso_V, iso_W) = 0$ if $V \not\cong W$. Consequently

$$\text{Func}^{add}(\mathcal{P}^{op}, \mathcal{A}b) \simeq \prod_{V \in \mathcal{O}(V)} \text{Func}^{add}(\mathcal{I}so_V^{op}, \mathcal{A}b)$$

where $\mathcal{I}so_V$ is the full subcategory of \mathcal{F}_{iso} with one object, the functor iso_V . By corollary 4.35, $\text{Hom}_{\mathcal{F}_{iso}}(iso_V, iso_V) = \mathbb{F}_2[O(V)]$; we deduce that

$$\mathcal{F}_{iso} \simeq \prod_{V \in \mathcal{S}} \mathbb{F}_2[O(V)] - mod.$$

□

APPENDIX A. PROPERTIES OF THE PRECOMPOSITION FUNCTOR

In this appendix we list several results about functor categories. We have chosen to provide proofs since, even if these results are well-known, they are not easy to find in the literature.

We are interested in the following question:

Let \mathcal{C} and \mathcal{D} be two categories, \mathcal{A} be an abelian category, $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $- \circ F : \text{Func}(\mathcal{D}, \mathcal{A}) \rightarrow \text{Func}(\mathcal{C}, \mathcal{A})$ be the precomposition functor, where $\text{Func}(\mathcal{C}, \mathcal{A})$ is the category of functors from \mathcal{C} to \mathcal{A} .

When the functor F has a property \mathcal{P} , what can we deduce for the precomposition functor?

Before giving three answers in the following propositions, we recall that, by [6], $\text{Func}(\mathcal{C}, \mathcal{A})$ and $\text{Func}(\mathcal{D}, \mathcal{A})$ are abelian categories as the category \mathcal{A} is abelian. Furthermore, we remark that the precomposition functor is exact.

Proposition A.1. *If F is essentially surjective, then $- \circ F$ is faithful.*

Proof. As the precomposition functor is exact, it is sufficient to prove that, if H is an object of $\text{Func}(\mathcal{D}, \mathcal{A})$ such that $H \circ F = 0$, then $H = 0$. For an object D of \mathcal{D} , there exists an object C of \mathcal{C} such that $F(C) \simeq D$ as F is essentially surjective. So

$$H(D) \simeq H(F(C)) = H \circ F(C) = 0.$$

□

Proposition A.2. *If F is full and essentially surjective, then:*

- (1) *the precomposition functor is fully-faithful;*
- (2) *any subobject of an object in the image of the precomposition functor is isomorphic to an object in the image of the precomposition functor;*
- (3) *the image by the precomposition functor of a simple functor of $\text{Func}(\mathcal{D}, \mathcal{A})$ is a simple functor of $\text{Func}(\mathcal{C}, \mathcal{A})$.*

Proof. (1) According to proposition A.1, the functor $- \circ F$ is faithful.

For the fullness, we consider two objects G and H of $\text{Func}(\mathcal{D}, \mathcal{A})$ and α an element of $\text{Hom}_{\text{Func}(\mathcal{C}, \mathcal{A})}(G \circ F, H \circ F)$. We want to prove that there exists a morphism β of $\text{Hom}_{\text{Func}(\mathcal{D}, \mathcal{A})}(G, H)$ such that $\beta \circ F = \alpha$.

Let D be an object of \mathcal{D} , as F is essentially surjective, we can chose an object C of \mathcal{C} such that there exists an isomorphism:

$$\phi : F(C) \rightarrow D.$$

We define an element $\beta(D)$ of $\text{Hom}_{\mathcal{A}}(G(D), H(D))$ by the following composition

$$G(D) \xrightarrow{G(\phi^{-1})} G \circ F(C) \xrightarrow{\alpha(C)} H \circ F(C) \xrightarrow{H(\phi)} H(D).$$

Now, we show that β defines a natural transformation from G to H .

Let $f : D \rightarrow D'$ be an element of $\text{Hom}_{\mathcal{D}}(D, D')$ and $\phi : F(C) \rightarrow D$ and $\phi' : F(C') \rightarrow D'$ be the isomorphisms associated to the choices of C and C' by the essential surjectivity of F . As the functor F is full, there exists an element g of $\text{Hom}_{\mathcal{C}}(C, C')$ such that

$$(A.2.1) \quad F(g) = \phi'^{-1} \circ f \circ \phi.$$

Moreover, the following diagram is commutative

$$\begin{array}{ccc}
G(D) & \xrightarrow{\beta(D)} & H(D) \\
G(\phi^{-1}) \downarrow & & \downarrow H(\phi^{-1}) \\
G \circ F(C) & \xrightarrow{\alpha(C)} & H \circ F(C) \\
G \circ F(g) \downarrow & & \downarrow H \circ F(g) \\
G \circ F(C') & \xrightarrow{\alpha(C')} & H \circ F(C') \\
G(\phi') \downarrow & & \downarrow H(\phi') \\
G(D') & \xrightarrow{\beta(D')} & H(D')
\end{array}$$

because the higher square (respectively lower) commute by the definition of $\beta(D)$ (respectively $\beta(D')$) and the commutativity of the square in the center is a consequence of the naturality of α . Since

$$G(\phi') \circ GF(g) \circ G(\phi^{-1}) = G(\phi' \circ F(g) \circ \phi^{-1}) = G(f)$$

and

$$H(\phi') \circ HF(g) \circ H(\phi^{-1}) = H(\phi' \circ F(g) \circ \phi^{-1}) = H(f)$$

where the first equalities rise from the functoriality of G and H and the second ones rise from the relation A.2.1, we deduce that β is natural.

- (2) Let G be an object of $\text{Func}(\mathcal{D}, \mathcal{A})$ and H be a subobject of $G \circ F$. As in the first point, for an object D of \mathcal{D} , by the essential surjectivity of F , we can chose an object C of \mathcal{C} such that there exists an isomorphism:

$$\phi : F(C) \rightarrow D.$$

We define an object of \mathcal{A} by:

$$K(D) := H(C).$$

As in the first point, for $f : D \rightarrow D'$ a morphism of $\text{Hom}_{\mathcal{D}}(D, D')$, we denote by $\phi : F(C) \rightarrow D$ and $\phi' : F(C') \rightarrow D'$ the isomorphisms associated to the choices of C and C' by the essential surjectivity of F . Since the functor F is full, there exists an element g of $\text{Hom}_{\mathcal{C}}(C, C')$ such that

$$(A.2.2) \quad F(g) = \phi'^{-1} \circ f \circ \phi.$$

We define a morphism

$$K(f) : K(D) \rightarrow K(D')$$

by $K(f) = H(g)$. We obtain the following commutative diagram:

$$\begin{array}{ccccccc}
K(D) & \xlongequal{\quad} & H(C) & \xrightarrow{\quad} & G \circ F(C) & \xrightarrow{G(\phi)} & G(D) \\
K(f) \downarrow & & H(g) \downarrow & & G \circ F(g) \downarrow & & G(f) \downarrow \\
K(D') & \xlongequal{\quad} & H(C') & \xrightarrow{\quad} & G \circ F(C') & \xrightarrow{G(\phi')} & G(D')
\end{array}$$

Since the horizontal arrows of the diagram are monomorphisms, the definition of $K(f)$ does not depend of the choice of the morphism g in A.2.2. The functor K , thus defined, satisfies $K \circ F \simeq H$. Moreover, we deduce from the commutativity of the above diagram that K is a subfunctor of G .

- (3) Let S be a simple functor of $\text{Func}(\mathcal{D}, \mathcal{A})$ and $(- \circ F)(S) = S \circ F$ its image by $- \circ F$. Suppose that $S \circ F$ is not simple, so that there exists a proper subfunctor H of $S \circ F$. By the proof of the point (2), of the proposition, there exists a subfunctor K of S such that:

$$H \simeq K \circ F.$$

Since H is a proper subfunctor of $S \circ F$, K is a proper subfunctor of S . This contradicts the simplicity of S . \square

Remark A.3. *The constructions used in the proposition are closely related to the left Kan extension. We have preferred to give an explicit proof rather than a slicker one using the Kan extension.*

In the following proposition, we take $\mathcal{A} = \text{Mod}_A$ the category of right A -modules where A is a commutative ring with unit. By the Yoneda lemma, we know that the functor

$$P_X^{\mathcal{C}} : \mathcal{C} \xrightarrow{\text{Hom}_{\mathcal{C}}(X, -)} \mathcal{E}ns \xrightarrow{A[-]} \text{Mod}_A$$

is a projective object of $\text{Func}(\mathcal{C}, \text{Mod}_A)$, where $A[-]$ is the linearisation functor. We have, in this case, the following proposition.

Proposition A.4. *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is full and, for all objects D of \mathcal{D} , there exists an object C of \mathcal{C} such that $P_D^{\mathcal{D}}$ is a direct summand of $P_{FC}^{\mathcal{D}}$ then:*

- (1) *the precomposition functor $- \circ F$ is faithful;*
- (2) *a morphism σ of $\text{Func}(\mathcal{D}, \text{Mod}_A)$ such that $(- \circ F)(\sigma)$ is an isomorphism, is an isomorphism;*
- (3) *the categories $\text{Func}(\mathcal{D}, \text{Mod}_A)$ and $\text{Func}(\mathcal{C}, \text{Mod}_A)$ are equivalent.*

Proof. (1) As the precomposition functor is exact, it is sufficient to prove that, for H an object of $\text{Func}(\mathcal{D}, \mathcal{A})$ such that $H \circ F = 0$, $H = 0$. Let D be an object of \mathcal{D} , then $H(D) = \text{Hom}(P_D^{\mathcal{D}}, H)$ by the Yoneda lemma. By hypothesis, this is a direct summand of

$$\text{Hom}(P_{FC}^{\mathcal{D}}, H) \simeq H \circ F(C) = 0.$$

Consequently $H(D) = 0$.

- (2) Let $\sigma : F \rightarrow G$ be a morphism of $\text{Func}(\mathcal{D}, \text{Mod}_A)$. We have the following exact sequence:

$$0 \rightarrow \text{Ker}(\sigma) \rightarrow F \xrightarrow{\sigma} G \rightarrow \text{Coker}(\sigma) \rightarrow 0.$$

As $(- \circ H)(\sigma)$ is an isomorphism, $\text{Ker}(\sigma) \circ H = 0$ and $\text{Coker}(\sigma) \circ H = 0$. Consequently, $\text{Ker}(\sigma) = \text{Coker}(\sigma) = 0$ by the argument of the previous point. Hence, σ is an isomorphism.

- (3) By general results, the functor $(- \circ F) : \text{Func}(\mathcal{D}, \text{Mod}_A) \rightarrow \text{Func}(\mathcal{C}, \text{Mod}_A)$ admits a right adjoint, given by Kan extension. We will denote by $(\tilde{-})$ this adjoint. We will prove that $(\tilde{-})$ and $(- \circ F)$ define an equivalence of categories.

As F is full, we have, for all objects W of \mathcal{C}

$$P_{FC}^{\mathcal{D}}(FW) = A[\text{Hom}_{\mathcal{D}}(FC, FW)] = A[\text{Hom}_{\mathcal{C}}(C, W)] = P_C^{\mathcal{C}}(W).$$

It follows that $P_{FC}^{\mathcal{D}} \circ F = P_C^{\mathcal{C}}$. As $P_D^{\mathcal{D}} \circ F$ is a direct summand of $P_{FC}^{\mathcal{D}} \circ F$ by hypothesis, we deduce that $P_D^{\mathcal{D}} \circ F$ is a direct summand of $P_C^{\mathcal{C}}$ and hence projective. Consequently, $(\tilde{-})$ is an exact functor since

$$\tilde{H}(D) \simeq \text{Hom}(P_D^{\mathcal{D}}, \tilde{H}) \simeq \text{Hom}(P_D^{\mathcal{D}} \circ F, H).$$

Let H be an object of $\text{Func}(\mathcal{C}, \text{Mod}_A)$, then there is a sequence of isomorphisms:

$$(- \circ F) \circ (\tilde{-})(H) = \tilde{H}(FZ) \simeq \text{Hom}(P_{FZ}^D \circ F, H) \simeq \text{Hom}(P_Z^C, H) \simeq H(Z).$$

So, the counit of the adjunction

$$(A.4.1) \quad (- \circ F) \circ (\tilde{-}) \rightarrow \text{Id}$$

is an isomorphism.

Consequently, the unit of the adjunction

$$(A.4.2) \quad \text{Id} \rightarrow (\tilde{-}) \circ (- \circ F)$$

induces, by composition with $(- \circ F)$, an isomorphism

$$(A.4.3) \quad (- \circ F) \rightarrow (- \circ F) \circ (\tilde{-}) \circ (- \circ F).$$

So, by the previous point of the proposition, the unit of the adjunction $\text{Id} \rightarrow (\tilde{-}) \circ (- \circ F)$ is an isomorphism. \square

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LABORATOIRE ANALYSE, GÉOMÉTRIE ET APPLICATIONS, UMR 7539, INSTITUT GALILÉE, UNIVERSITÉ PARIS 13, 93430 VILLETANEUSE, FRANCE

E-mail address: vespa@math.univ-paris13.fr