DRAFT-LOOPSPACES AND FINITENESS

CLARENCE W. WILKERSON

Abstract. This expository note began as comments on a shorter note of F. R. Cohen [5]. Cohen's paper is an elegant application of powerful recent results in unstable homotopy theory to a problem of interest to analysts.

Theorem : *(F. R. Cohen,[5]) Let X be a simply connected finite complex which is not contractible and let $\Omega_0^j X$ be the component of the constant map in the j-th pointed loop space of X. If $j \geq 2$, then the Lusternik-Schnirlman category of $\Omega_0^j X$ is not finite.*

This note includes a rederivation of the above theorem using H-space methods of W. Browder from the 60's, [3], [4]. The aim is to reduce the prerequisites for Cohen's theorem to those available after a second course in algebraic topology. We end with a discussion of recent work of Lannes-Schwartz on various notions of finiteness properties and the behavior under looping. The common theme is extensive use of the action of the Steenrod algebra on the cohomology of a topological space.

1. Introduction

The finiteness of the Lusternik-Schnirlman category of a space can be thought of as one possible generalization of a finite complex. It's not difficult to show that finiteness is most often not inherited by loopspaces (see Theorem 1.2 below). Cohen's result shows that this fails for finiteness of the Lusternik-Schnirlman category as well, even if X is finite:

1.1. **Theorem.** *[F. R. Cohen,[5]] Let X be a simply connected finite complex which is not contractible and let $\Omega_0^j X$ be the component of the constant map in the j-th pointed loop space of X. If $j \geq 2$, the Lusternik-Schnirlman category of $\Omega_0^j X$ is not finite.*

A somewhat similar result has long been known – the case $j = 1$:

1.2. **Theorem.** *Let X be a simply connected complex of finite type which is not contractible.*

(a) *(Borel, [2]) If $H^*(X, k)$ is a finite dimensional graded vector space for some field k, then $H^*(\Omega X, k)$ is not finite dimensional.*

Date: May 23, 2006.

(b) *If X has finite Lusternik-Schnirlman category, then ΩX does not have finite Lusternik-Schnirlman category.*

The reader first confronted with this rather wordy introduction might well wonder what the current author could or should add to Cohen's three page note. The answer was originally rooted in a desire for mathematical economy in the prerequisites for the proof. Cohen's proof uses results from Dwyer-Wilkerson which arise from Lannes' T-technology. Thus on the surface at least, the proof depends on recent deep technical advances in unstable homotopy theory. These in turn are grounded in principle in the details of the Bousfield-Kan unstable Adams spectral sequence for function spaces. Most of us would have some difficulty explaining this background to our colleagues or graduate students. Therefore, for the first half of this exposition the goal is to limit the technology to that roughly corresponding to the earlier chapters of Mosher-Tangora [8] – the Serre and Bockstein spectral sequences and Steenrod operations.

This problem demonstrates the immense influence of cohomology algebras and cohomology operations on the evolution during the past half century of unstable homotopy theory. An impressive early demonstration of this power is provided by Serre's beautiful use of the precise knowledge of the action of the Steenrod algebra on the cohomology of Eilenberg-MacLane spaces to give asymptotic information about the homotopy groups of finite complexes.

1.3. **Theorem.** *(Serre, [10]) Let X be a finite simply connected complex such that $H^*(X,\mathbb{F}_p) \neq 0$ in some positive dimension. Then there exist infinitely many distinct positive integers $\{n_i\}$ for which*

$$\pi_{n_i}(X) \otimes \mathbb{F}_p \neq 0.$$

The following formulation of Serre's theorem is very useful:

1.4. **Corollary.** *Let X be a simply connected finite CW complex which is not contractible. Then $\Omega_0^j X$ is not contractible for any $j \geq 0$. If X is a non-contractible simply connected finite CW complex then X is not a finite Postnikov tower.*

The work of Lannes-Schwartz [7] continues this tradition of Serre. We sketch in the last section generalizations of 1.1 and 1.2.

1.5. **Definition** (Lannes-Schwartz, [7]). Let M be a graded unstable $\mathcal{A}_p$-module. If for each $m \in M$, the cyclic submodule $\mathcal{A}_p(m)$ is finite, then M is said to *quasi-bounded* or locally finite over the Steenrod algebra $\mathcal{A}_p$.

It turns out that *quasi-bounded* has better properties for loopspaces than finiteness or finite category:

1.6. **Theorem** (Lannes-Schwartz, [7]). *Let X be a simply connected complex such that $H^*(X, \mathbb{F}_p)$ is non-zero in at least one positive dimension. Suppose further that $H^*(X, \mathbb{F}_p)$ is* quasi-bounded *as a module over the Steenrod algebra $\mathcal{A}_p$. Then*

a) *$H^*(\Omega_0^j X, \mathbb{F}_p)$ and $H^*(\Omega^j X, \mathbb{F}_p)$ are* quasi-bounded *for all $j \geq 0$.*
b) *$\pi_i(X) \otimes \mathbb{F}_p \neq 0$ for infinitely many i.*
c) *$\Omega_0^j X$ is not contractible for any $j \geq 0$.*

Of course, (b) is a form of Serre's earlier result. However the Lannes-Schwartz method generalizes to other interesting inverse connections between the complexity of the mod p cohomology algebras and the existence of infinite dimensional homotopy groups.

These methods imply a stronger version of 1.1:

1.7. **Theorem** (Section 5). *Let X be a simply connected CW complex such $H^*(X, \mathbb{F}_p)$ is finite in each dimension and is* quasi-bounded *over $\mathcal{A}_p$. If $H^*(X, \mathbb{F}_p)$ is non-zero in some positive dimension, then $H^*(\Omega X, \mathbb{F}_p)$ is not a finitely generated algebra over $\mathbb{F}_p$. By iteration, the same holds for each $\Omega_0^j X$ for $j > 0$.*

For the reader with primary interests in homotopy theory, it would well worthwhile to followup on the results of Lannes and Schwartz, using, for example, Schwartz's book, [9]. The non-homotopy theorist might find that the proof of 1.1 using Browder's work is sufficient and requires a much smaller set of prerequisites.

My interest and knowledge about this problem came from conversations with Fred Cohen at M.P.I. in Bonn in the summer of 1997. I thank the M.P.I. for its hospitality. Fred may well regard these comments as a waste of good trees.

2. Preliminaries and the proof of Theorem 1.2

We give two proofs of 1.2. The first is elementary while the second hints at Browder's work. The eventual proof of 1.1 is a blend of these two ideas.

2.1. **Proposition.** *Let X be a simply connected space such that $H^*(X, \mathbb{F}_p)$ is of finite type over $\mathbb{F}_p$. If $H^*(\Omega X, \mathbb{F}_p)$ is finite, then $H^*(X, \mathbb{F}_p)$ is not finite.*

Proof of 1.2 from 2.1. Since ΩX is a connected H-space, $H^*(\Omega X, \mathbb{F}_p)$ is a connected Hopf algebra. The Borel structure theorem for commmuative Hopf algebras of finite type over $\mathbb{F}_p$ [1] implies that $H^*(\Omega X, \mathbb{F}_p)$ as a commutative graded algebra must be a tensor product of exterior algebras, polynomial algebras, and truncated polynomial algebras. If ΩX has finite cup length in cohomology, then the tensor factors of $H^*(\Omega X, \mathbb{F}_p)$ must be a finite number of exterior and truncated polynomial algebras. That is, finite category or finite cup length imply that $H^*(\Omega X, \mathbb{F}_p)$ is finite. □

Elementary proof of 2.1 ala Borel-Serre. Consider the Serre Spectral Sequence of the path space fibration

$$\Omega X \to \mathcal{P}X \to X.$$

The E_2 term is $H^*(X, \mathbb{F}_p) \otimes H^*(\Omega X, \mathbb{F}_p)$ and the differentials go to the right and downward. This spectral sequence converges to the cohomology of the path space $\mathcal{P}X$. Since $\mathcal{P}X$ is contractible, the E_∞ is zero except in bidegree $(0,0)$. If $H^k(X, \mathbb{F}_p) = 0$ for $k > M$, and $H^j(\Omega X, \mathbb{F}_p) = 0$ for $j > N$ then choose the M and N so that $H^M(X, \mathbb{F}_p)$ and $H^N(\Omega X, \mathbb{F}_p)$ are each nonzero. Then the E_2 is concentrated in a rectangle from $(0,0)$ to (M,N). For dimensional reasons, each element of bidegree (M,N) is an infinite cycle and the image of the differentials from other bidegree in (M,N) is trivial. The existence of these non-zero terms in $E_\infty^{M,N}$ contradicts the contractibility of $\mathcal{P}X$. □

2.2. **Proposition** (Second proof of 1.2). *Let X be a simply connected space with finite L.-S. category and $H^*(X, \mathbb{F}_p)$ of finite type and nonzero in at least one positive dimension. Then $H^*(\Omega X, \mathbb{F}_p)$ is not finite.*

Proof. Suppose that $H^*(\Omega X, \mathbb{F}_p)$ is finite and not concentrated in degree 0. Since $H^*(\Omega X, \mathbb{Z})$ has finite type, then the Bockstein specral sequence implies that $H^*(\Omega X, \mathbb{Q})$ is a finite dimensional graded vector space. From the Borel structure theorem, the Euler characteristic $\chi H^*(\Omega X, \mathbb{F}_p) = 0$. Hence $\chi H^*(\Omega X, \mathbb{Q}) = 0$. Since ΩX is connected, this implies that there is at least one odd degree $N > 0$ with $H^N(\Omega X, \mathbb{Q}) \neq 0$. That is, $H^*(\Omega X, \mathbb{Q})$ is a non-trivial exterior algebra on odd-dimensional generators. By the Borel transgression theorem, $H^*(X, \mathbb{Q})$ is thus a finitely generated polynomial algebra. This violates the assumption of finite $L.S.$-category for X. □

There is actually a close connection between the first proof above and Browder's work involving Bockstein spectral sequences and Poincare duality for H-spaces. In the Serre spectral sequence, one deals with differential bigraded algebras and obtains "at the corner" an automatic infinite cycle that never bounds. In the Bockstein spectral sequence for the homology or cohomology of a finite H-space, there is only a single grading so there is no "corner" *per se*. Browder's clever use of filtrations based on the Hopf algebra structure pushes through the basic style of the Serre spectral sequence case, however.

3. Proof of Theorem 1 using Browder's H-space work

We again assume that the category of $\Omega_0^j X$ is finite and derive a contradiction based on finite cup length.

The basic observations in the proof of Theorem 1.2 all apply to the proof of Theorem 1.1 . In particular, $H^*(\Omega_0^j X, \mathbb{F}_p)$ is finite for each prime p, and since it is nontrivial in positive dimensions, it is nontrivial in positive dimensions for all primes. Moreover, $H^*(\Omega_0^j X, \mathbb{Z})$ vanishes above some dimension $d_0 = d_p$.

The proof of Theorem 1.1 is now brutally fast – we are completely anticipated by a 1960 theorem of W. Browder [3]:

3.1**. Theorem.** *(Browder, 1960) If Y is an simply connected H-space such that $H^*(\Omega Y, \mathbb{Z})$ is a finitely generated module over the integers Z, then ΩY is homotopy equivalent to a finite product of circles.*

3.2. *Remark.* Since the first proof of 3.1 in [Browder, [3]] uses information at many primes, the resulting proof of 1.1 does not have a trivial generalization to the p-local or p-complete setting. Later work of Browder on homotopy commutative H-spaces does provide a suitable one prime proof of analogues of 3.1, and thus 1.1, as does the proof in Cohen [5]. At $p = 2$, Hubbuck gives a characterization of homotopy commutative H-spaces with finite cohomology that recovers Browder's theorem at $p = 2$.

In our case $\Omega_0^j X = \Omega^j X[j]$, where $X[j]$ is the connective cover of X with trivial homotopy groups in dimension j and lower. Hence $B\Omega_0^j X = \Omega^{j-1} X[j]$ is simply connected. The remarks in the proof of Theorem 1.2 show that the groups $H^*(\Omega_0^j X, \mathbb{Z})$ are zero for large dimensions, so the hypotheses of Browder's theorem hold and $\Omega_0^j X \approx_{h.e.} S^1 \times \cdots \times S^1$. Hence, contrary to Serre's theorem, $\pi_*(X)$ is not non-zero in infinitely many dimensions. Thus the assumption that the category of $\Omega_0^j X$ is finite must be false.

Sketch of Browder's proof of 3.1. : Using the finite type assumption on $H*(\Omega X, \mathbb{Z})$ yields that $H^*(\Omega X, \mathbb{Z})$ is p-torsion free for almost all primes

p. Hence, for almost all primes, $H^*(X, \mathbb{F}_p)$ is a finitely generated polynomial algebra on even dimensional generators which is a commutative Hopf algebra. Browder then uses Steenrod operations to restrict the degree of the generators to be of the form $\{2p^k\}$ for $p > 2$. Since these dimensions agree with those of the rational cohomology generators, and are valid for all sufficiently large p, the only choice is that each generator of $H^*(X, \mathbb{F}_p)$ is in dimension 2. Since the rational rank of ΩX is r and that is the dimension, in fact $H * (\Omega X, \mathbb{F}_p)$ has no even-dimensional indecomposables. This implies that $H^*(\Omega X, \mathbb{Z})$ is p-torsion free for each p. Hence the composition of $K(\pi_1(\Omega X, 1) \to \Omega X \to K(\pi_1(\Omega X, 1)$ is a homotopy equivalence.

□

4. Quasi-bounded algebras

Neither finiteness or finite L.-S.-category have good inheritance properties under the operation of taking loopspaces. Lannes-Schwartz fill this gap with the notion of *quasi-boundeness* as defined in the introduction and show that this property is inherited under the operation of forming pointed loopspaces.

4.1. *Example.* (1) If X is a finite CW complex, then X has finite category and $H^*(X, \mathbb{F}_p)$ is *quasi-bounded* for any prime p.

(2) The space $Y = \Sigma\mathbb{C}P^\infty$ has category 2 but its mod p cohomology is not *quasi-bounded* for any prime p.

(3) The space ΩS^3 has *quasi-bounded* mod p cohomology for any prime p, but does not have finite category since the cup length in F_p cohomology is not bounded.

4.2. **Theorem** (Lannes-Schwartz [7]). *Let X be a CW complex of finite type.*

(a) *If X is simply connected, then $H^*(X, \mathbb{F}_p)$ quasi-bounded implies that $H^*(\Omega X, \mathbb{F}_p)$ is quasi-bounded .*

(b) *If X is a connected H-space with universal cover Y, then if $H^*(X, \mathbb{F}_p)$ is quasi-bounded , then so is $H^*(Y, \mathbb{F}_p)$.*

(c) *Let X be a simply connected complex of finite type such that $H^*(X, \mathbb{F}_p)$ is quasi-bounded. Then $H^*(\Omega X_0^j, \mathbb{F}_p)$ is quasi-bounded for all $j \geq 0$.*

The next theorem is a generalization of Theorem 1.1:

4.3. **Theorem.** *Let X be a simply connected complex of finite type such $H^*(X, \mathbb{F}_p)$ is quasi-bounded and $H^m(X, \mathbb{F}_p) \neq 0$ for some $m > 0$. Then*

$H^*(\Omega X, \mathbb{F}_p)$ is not finitely generated as an $\mathbb{F}_p$-algebra. Hence ΩX does not have finite cup length or category. By iteration, the same holds for each $\Omega_0^j X$.

The reader should compare Browder's similar statement in the case that X is an H-space.

4.4. *Remark.* Lannes-Schwartz [7] derive many remarkable consequences from their proof that unlike finiteness or finite category, the property of *quasi-bounded* is preserved under taking loopspaces. The proof of 4.2 is quite similar to Cohen's proof of 1.1.

Proof of 4.2. Assume that $H^*(\Omega X, \mathbb{F}_p)$ is a finitely generated algebra. Using the Hopf algebra structure as before, we again conclude that $H^*(\Omega X, \mathbb{F}_p)$ is finite. Next, we appeal to the theory of p-compact groups as developed by Dwyer-Wilkerson [6]. The p-completions of X and ΩX satisfy these conditions. The homotopy fixed point method of [6] produces the existence of a non-trivial map $B\mathbb{Z}/p\mathbb{Z} \to X$ if $H^*(\Omega X, \mathbb{F}_p)$ is non-trivial. By Lannes, such maps are detected by mod p cohomology. We conclude that there is a non-nilpotent element of positive degree in $H^*(X, \mathbb{F}_p)$. But this contradicts the quasi-boundedness assumption on $H^*(X, \mathbb{F}_p)$. Hence the assumption that $H^*(\Omega X, \mathbb{F}_p)$ is finitely generated as an algebra is incorrect. $\square$

There are many mapping space questions suggested by the special case of loopspaces. For example, given CW complexes X and Y, under what conditions is $\pi_*(Map(X, Y))$ non-zero in infinitely many dimensions?

References

[1] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953.

[2] Armand Borel and Jean-Pierre Serre. Impossibilité de fibrer un espace euclidien par des fibres compactes. *C. R. Acad. Sci. Paris*, 230:2258–2260, 1950.

[3] William Browder. Loop spaces of H-spaces. *Bull. Amer. Math. Soc.*, 66:316–319, 1960.

[4] William Browder. Torsion in h-spaces. *Ann. of Math. (2)*, 74:24–51, 1961.

[5] F. R. Cohen. On the Lusternik-Schnirelmann category of an iterated loop space. In *Stable and unstable homotopy (Toronto, ON, 1996)*, pages 39–41. Amer. Math. Soc., Providence, RI, 1998.

[6] W. G. Dwyer and C. W. Wilkerson. Homotopy fixed-point methods for Lie groups and finite loop spaces. *Ann. of Math. (2)*, 139(2):395–442, 1994.

[7] Jean Lannes and Lionel Schwartz. À propos de conjectures de Serre et Sullivan. *Invent. Math.*, 83(3):593–603, 1986.

[8] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory.* Harper & Row Publishers, New York, 1968.

[9] Lionel Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture.* University of Chicago Press, Chicago, IL, 1994.
[10] Jean-Pierre Serre. Cohomologie modulo 2 des complexes d'Eilenberg-MacLane. *Comment. Math. Helv.*, 27:198–232, 1953.

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: wilker@math.purdue.edu