THE T-FUNCTOR OF J. LANNES

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The calculation of the homotopy type of the space of continuous maps $Map(X, Y)$ is a fundamental problem of homotopy theory. The set of path components, $\pi_0 Map(X, Y) = [X, Y]$ corresponds to the homotopy classes of such maps. There are relatively few cases for which this information is explicitly known. A major impact of Jean Lannes' work [1] on unstable modules and the T functor has been to expand this knowledge to include many cases in which the sources and targets are classifying spaces of finite and compact Lie groups.

The work of Steenrod and others assigns in a natural way to each topological space X and each prime p an algebraic model consisting of a graded algebra $H^*(X, \mathbb{F}_p) = R^*$ over $\mathbb{F}_p$ and an algebra $\mathcal{A}_p$ of natural operations called the Steenrod algebra. Each $f : X \to Y$ induces an element $f^* \in \mathrm{Hom}_{alg}(H^*(Y, \mathbb{F}_p), H^*(X, \mathbb{F}_p))$ that commutes with the action of $\mathcal{A}_p$. $\mathcal{A}_p$ is a connected graded Hopf algebra acting on the graded algebra R^*.

The hypothesis that R^* is the cohomology of a space imposes an additional "unstable" condition. This is most simply stated if $p = 2$. $\mathcal{A}_2$ is generated as an (non-commutative) algebra by the Steenrod operations $\{\mathcal{S}q^i, i \geq 0\}$, with relations forced by its actions of the cohomology of all topological spaces. For example, $\mathcal{S}q^0 = Id$ and $\mathcal{S}q^1 = \beta$, the mod 2 Bockstein. The unstable condition is then that $Sq^i x_n = 0$ for $i > n$ and $Sq^n x_n = x_n^2$. The algebraic category $\mathcal{K}$ of unstable algebras $\{\mathcal{R}*\}$ over $\mathcal{A}_p$ is thus an approximation to the homotopy category of topological spaces. The larger category $\mathcal{U}$ of unstable modules over $\mathcal{A}_p$ has also proved useful.

For $p > 2$, the structure of $\mathcal{A}_p$ and unstable actions are similar, but slightly more involved. However, in all cases, the set of relations in the Steenrod algebra and the unstable condition are derivable from the known action of $\mathcal{A}_p$ on the cohomology of products of copies of $B\mathbb{Z}/p\mathbb{Z}$. In the following, we omit explicit references to the coefficients.

The relationship of $\pi_0 Map(X, Y)$ to its model $\mathrm{Hom}_{\mathcal{K}}(H^*(Y, \mathbb{F}_p), H^*(X, \mathbb{F}_p))$ is of particular interest. The equivalence

$$Map(X \times Z, Y) \to Map(X, Map(Z, Y))$$

raises the hope that in very favorable cases the map

$$\mathrm{Hom}_{\mathcal{K}}(H^* Map(Z, Y), H^* X) \to \mathrm{Hom}_{\mathcal{K}}(H^* Y, H^* X \otimes H^* Z)$$

might be an isomorphism. That suggests that in the category $\mathcal{K}$, $H^* Map(Z, Y)$ should be approximated by the left adjoint functor to tensoring on the right by $H^* Z$. This motivates Lannes's definition of T:

Definition. If E is a finite dimensional $\mathbb{F}_p$-vector space, then the T-functor $T_E : \mathcal{U} \to \mathcal{U}$ is the left adjoint in $\mathcal{U}$ of the functor $((\_\_\_) \otimes_{\mathbb{F}_p} H^* BV\ ) : \mathcal{U} \to \mathcal{U}$. In the topological case, there is a natural map

$$\lambda_X : T_E H^* X \to H^* Map(BE, X).$$

For general Z, the adjoint to $((\_\_\_) \otimes_{\mathbb{F}_p} H^* Z)$ accounts for only part of the starting page of a Bousfield-Kan unstable Adams spectral sequence for $Map(Z, Y)$. Lannes provides the basic connection to topology by blending the algebraic properties of T_E and $\mathcal{K}$ with the Bousfield-Kan spectral sequence:

Theorem. (Lannes) For many interesting spaces X,

$$H^*Map(BE, X) \approx T_E H^* X.$$

In particular,

$$\pi_0 Map(BE, X) = [BE, X] = \mathrm{Hom}_{\mathcal{K}}(H^*X, H^*BE).$$

For $f : X \to Y$, one has the path component $Map(X, Y)_f$ of functions homotopic to f. The analogous T-construct is

Definition. Each $\varphi \in \mathrm{Hom}_{\mathcal{K}}(R^*, H^*BE)$ induces a T^0_E module structure on $\mathbb{F}_p$ and

$$T_{E,\varphi} R^* = T_E R^* \otimes_{T^0_E} \mathbb{F}_p.$$

The most striking features of T_E are summarized below. To some extent, these were presaged by work of Carlsson and Miller establishing that the $\{H^*BV\}$ are injectives in $\mathcal{U}$.

Theorem (Lannes)**.**

(a) T_E is exact.
(b) T_E respects tensor products, i.e $T_E(M \otimes_{\mathbb{F}_p} N) = T_E M \otimes_{\mathbb{F}_p} T_E N$.
(c) T_E commutes with the p-th power operations in a suitable sense.
(d) T_E maps $\mathcal{K}$ to $\mathcal{K}$.

In principle, $T_E M^*$ can be calculated by using the exactness property and a resolution of M^* by free unstable $\mathcal{A}_p$-modules. In practice, other methods are often more effective:

Examples.

1. If M^* is finite, $T_E M^* = M^*$.
2. If $R^* = H^*BV$, then $T_E R^* = \prod_{\mathrm{Hom}_{grp}(E,V)} H^*BV$, for E and V finite dimensional $\mathbb{F}_p$ vector spaces.
3. If $\tau : R^* \to H^*BE$ in $\mathcal{K}$ is an inclusion, then $T_{E,\tau} R^*$ is the smallest subHopf algebra of H^*BE that contains $\tau(R^*)$.
4. If X is a finite E-complex with fixed point set X^E and $H^*_E X$ the mod p cohomology of the Borel construction, then $T_{E,id} H^*_E X = H^*BE \otimes_{\mathbb{F}_p} H^*X^E$ in $\mathcal{K}$.
5. If G is a compact Lie group, then

$$T_E H^*BG = \prod_{\varphi \,\in\, \mathrm{Hom}_{grp}(E,G)/G-conj} H^*BC_G(\mathrm{im}\,\varphi(E))$$

These examples each have powerful topological consequences. For example, the first and fourth lead to new proofs of the Sullivan conjecture, orignally proved by Miller and Carlsson. The last leads to a new view of the homotopy theory of classifying spaces. Most of the above is referenced in the book of Schwartz [2]. One should also consult the many articles in the past decade by authors such as Lannes, Schwartz, Henn, Zarati, Kuhn, Farjoun-Smith, Morel, Jackowski-McClure-Oliver, Notbohm, Dwyer-Wilkerson, and others.

References

[1] Jean Lannes. Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire. *Inst. Hautes Études Sci. Publ. Math.*, (75):135–244, 1992. With an appendix by Michel Zisman.

[2] Lionel Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture.* University of Chicago Press, Chicago, IL, 1994.

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