Torsion in the Cohomology of Mapping Spaces

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Abstract

It is conjectured that under certain hypotheses, $H^*(\mathrm{Map}(BV, X);)$ is a free Z/p^m module when $H^*(X; Z/p^m)$ is a free Z/p^m module. We will discuss attempts by the author to prove this conjecture as well as other related results.

Contents

Chapter 1

Introduction

Since the mid 1980's, a popular topic in algebraic topology has been the study and applications of the T-functors defined by Jean Lannes, which take a category called $\mathcal{U}$ (or $\mathcal{K}$) to itself. The history of these functors finds its roots in Haynes Miller's work with the space of maps from an infinite dimensional space to another space and his answer to the Sullivan conjecture. One real strength of the T-functors lies in the property that there is a natural map

$$T_V H^*(X; Z/p) \to H^*(\mathrm{Map}(BV, X); Z/p) \tag{1.1}$$

which is quite often an isomorphism. (By $\mathrm{Map}(X, Y)$, we mean the space of unbased maps from X to Y given the compact open topology.) It is this isomorphism which interests us, so Chapter 3 includes a discussion which summarizes many of the known conditions for which the natural map is an isomorphism.

The author believes the following to be true:

Conjecture 5.1.1 *Suppose X is a space such that the natural map (1.1) is an isomorphism. If $H^*(X; Z/p^m)$ is a free Z/p^m module, then $H^*(Map(BV, X); Z/p^m)$ is a free Z/p^m module.*

Let Z_p denote the p-adic integers. If the previous conjecture is true, the following would be a corollary

Conjecture 5.1.5 *Suppose that X is a space as in the previous theorem. If $H^*(X; Z_p)$ has no torsion, then $H^*(Map(BV, X); Z_)$ also has no torsion.*

This result would combine with a theorem of Dwyer and Zabrodsky to show the next proposition. Clarence Wilkerson has informed me that he and Bill Dwyer have observed this result to be true and from the context of his note it appears that they used Borel's Theorem [Bo].

Proposition 5.2.3 *Suppose that G is a connected compact Lie group and that $g \in G$ is an element of order p. Let $\mathcal{Z}_G(g)$ be the centralizer of g in G. If $H^*(BG; Z)$ has no p torsion, then neither does $H^*(B\mathcal{Z}_G(g); Z)$, i.e. Conjecture 5.1.3 holds for BG.*

The main result of this paper is the reduction of Conjecture 5.1.1 to a technical conjecture, which is mentioned stated later in this introduction. The reduction utilizes the (mod p) Bockstein spectral sequence and the evaluation map $\varepsilon_X : BV \times \mathrm{Map}(BV, X) \to X$. It is the evaluation map which induces the natural map I have been referring to. The mathematics behind the case $m = 2$ has proven to be the most interesting and may be of independent interest to many readers. In order to show that case, I had to explore the internal structure of T_V. We will restrict our attention in this paper to $V = Z/p$ and set $T = T_V$, since it can be shown quite easily that $T_W = T^{dimW}$.

Using a map from M to $H^*(BZ/p) \otimes TM$ which is analogous to the evaluation map, one can define (Definition 3.2.1) a family of natural additive operations $\{t_s\}_{s \in Z}$ from a module M to TM. The operation t_s sends an element $x \in M_n$ to an element $t_s x \in (TM)_{n-s}$. We will also give two alternate definitions for this family of operations, each of which has its advantages and disadvantages. We now summarize many of the results of Chapter 3 with the following theorem.

Theorem 1.0.1 *Let M be an object of $\mathcal{U}$.*

a. *If M has a presentation as an $\mathcal{A}$-module with generators $\{x_\gamma\}$ and relations $\{r_\beta\}$, then TM has a presentation with generators $\{t_i x_\gamma\}$ and relations $\{t_j r_\beta\}$.*

b. *The Steenrod algebra and the operations $\{t_s\}$ interact as follows:*

   (i) *Suppose $p = 2$. Then $t_s Sq^k = \sum_i \binom{s-i}{i} Sq^{k-i} t_{s-i}$.*

   (ii) *Suppose $p > 2$. Then*

$$t_s P^k = \sum_i \binom{[\frac{s}{2}] - i(p-1)}{i} P^{k-i} t_{s-2i(p-1)}$$

and

$$\begin{aligned} t_{2r}\beta &= \beta t_{2r} + t_{2r-1} \\ t_{2r+1}\beta &= \beta t_{2r+1}. \end{aligned}$$

   (iii) *Let p be any prime and let Q_k be the k^{th} Milnor Bockstein. Then*

$$\begin{aligned} t_{2r} Q_k &= Q_k t_{2r} + t_{2r-2p^k+1} \\ t_{2r+1} Q_k &= Q_k t_{2r+1} \end{aligned}$$

c. *If $\{x_\gamma\}_{\gamma \in \Gamma}$ is a set of elements which generate M as a vector space, then $\{t_s x_\gamma\}_{\gamma, s}$ generate TM as a vector space.*

Parts a and c constitute Theorem 3.2.5, while part b is Theorem 3.2.7. Part c can also be viewed as a corollary of parts a and b.

The case $m = 2$ in Conjecture 5.1.1 is the case $k = 1$ of the following theorem. The cases $k \geq 2$ help inspire Conjectures 6.1.1 and 6.1.2. The Milnor Bocksteins $\{\mathrm{Q}_k\}$ are important elements in the Steenrod algebra and they are discussed in Chapter 2.

Theorem 3.2.8 *If the k^{th} Milnor Bockstein is zero on all of an unstable module M, it is zero on all of TM.*

As for the cases $m > 2$, we reduce Conjecture 5.1.1 to the following conjecture.

Conjecture 5.1.3 *For any space X, define*

$$t_{r,s} : H^*(X; Z/p^s) \to H^{*-r}(Map(BZ/p, X); Z/p^s)$$

by $t_{r,s}(a) = \varepsilon_X^*(a)/h_{r,s}$, where / denotes the slant product and $h_{r,s}$ is non-zero in $H^r(BZ/p; Z/p^s)$. Under the hypothesis on X in Conjecture 5.1.1, if the s^{th} Bockstein is zero on $H^*(X; Z/p^s)$, then $t_{2r-1,s} \equiv 0$ for all r.

Furthermore, we can show that this conjecture is true if there exist certain operations, analogous to the Milnor Bocksteins, from mod Z/p^s cohomology to itself. The properties required of these operations are discussed in full in chapter 5.

We now outline the rest of this dissertation. Chapter 2 consists of a review of the prerequisite knowledge needed about the Steenrod algebra and the related unstable categories, as well as the homology and cohomology of $\mathrm{B}Z/p$ and the Bockstein spectral sequence. In chapter 3, the T-functor and its properties are introduced, and we explore some of the internal structure of the T-functor. Chapter 4 explores some Künneth relations that we will need. In chapter 5, we reduce Conjecture 5.1.1 to Conjecture 5.1.3 and examine some of its corollaries. Chapter 6 discusses various conjectures and potential conjectures analogous to Conjecture 5.1.1, including some involving the connective Morava K-theories, a generalized cohomology theory useful in homotopy theory.

Chapter 2

Background

In this chapter, we will review much of background knowledge needed for the rest of this paper. In section 1, we review the Steenrod algebra and some of the related categories. In section 2, we recall the definition of the Bockstein homomorphism and its effects on the cohomology of BZ/p. In section 3, we discuss the Bockstein spectral sequence.

2.1 The Steenrod Algebra

Let $\mathcal{A}$ denote the mod p Steenrod algebra, the algebra of all natural stable transformations from mod p cohomology to itself, or equivalently the mod p cohomology of the mod p spectrum HZ/p. The algebra $\mathcal{A}$ is the quotient of the free graded associative F_p-algebra with unit generated on elements:

- Sq^i of degree i, if $p = 2$,
- β of degree 1 and
- P^i of degree $2i(p-1)$, $i > 0$, if p an odd prime

by the ideal generated by the Adem relations

$$\mathrm{Sq}^i\mathrm{Sq}^j - \sum_{0}^{[i/2]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

for all positive integers i and j such that $i < 2j$ if $p = 2$ or $\beta^2 = 0$ and the Adem relations

$$\mathrm{P}^i\mathrm{P}^j - \sum_{0}^{[i/p]} (-1)^{i+k} \binom{(p-1)(j-k)-1}{i-pk} \mathrm{P}^{i+j-k}\mathrm{P}^k$$

for all positive integers i and j such that $i < pj$ and

$$\begin{aligned} \mathrm{P}^i\beta\mathrm{P}^j \quad & - \sum_{0}^{[i/p]}(-1)^{i+k}\binom{(p-1)(j-k)}{i-pk}\beta\mathrm{P}^{i+j-k}\mathrm{P}^k \\ & - \sum_{0}^{[(i-1)/p]}(-1)^{i+k-1}\binom{(p-1)(j-k)-1}{i-pk-1}\mathrm{P}^{i+j-k}\beta\mathrm{P}^k \end{aligned}$$

for all positive integers i and j such that $i \leq pj$ if p is an odd prime.

The Steenrod algebra is a Hopf algebra, with the coproduct $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ given by

$$\begin{aligned} \delta(\mathrm{Sq}^k) &= \textstyle\sum_{i+j=k} \mathrm{Sq}^i \otimes \mathrm{Sq}^j \text{ if } p = 2; \\ \delta(\beta) &= \beta \otimes 1 + 1 \otimes \beta, \\ \text{and } \delta(\mathrm{P}^k) &= \textstyle\sum_{i+j=k} \mathrm{P}^i \otimes \mathrm{P}^j \text{ if } p > 2, \end{aligned}$$

where Sq^0 and P^0 denote the units of $\mathcal{A}$ for $p = 2$ and p is odd, respectively. The coproduct allows us to define an $\mathcal{A}$ module structure on the tensor product of two $\mathcal{A}$ modules M and N by utilizing the formula

$$(\theta \otimes \theta')(m \otimes n) = (-1)^{|\theta'||m|}\theta m \otimes \theta' n$$

for all $\theta, \theta' \in \mathcal{A}$, $m \in M$ and $n \in N$.

2.1.1 Instability

The mod p cohomology of a space X has a certain property as a $\mathcal{A}$ module which is called *instability*. More precisely, for X any space:

- $\mathrm{Sq}^i x = 0$ if $x \in H^*(X)$, $i > |x|$ and $p = 2$;
- $\beta^\epsilon \mathrm{P}^i x = 0$ if $x \in H^*(X)$, $\epsilon + 2i > |x|$, $e = 0, 1$ and $p > 2$,

where $|x|$ denotes the degree of x.

Any $\mathcal{A}$ module M which satisfies this property is called **unstable.**

Note that any unstable module is trivial in negative degrees since we have identified Sq^0 (resp. P^0) with the identity operation.

Unstable algebras

The mod p cohomology of a space X is naturally a graded communative algebra over $\mathcal{A}$ satisfying two properties:

K1:

$$\begin{aligned} \mathrm{Sq}^i(xy) &= \textstyle\sum_j (\mathrm{Sq}^j x)(\mathrm{Sq}^{i-j} y) \text{ for } x, y \in H^*(X) \text{ if } p = 2, \\ \mathrm{P}^i(xy) &= \textstyle\sum_j (\mathrm{P}^j x)(\mathrm{P}^{i-j} y), \\ \beta(xy) &= (\beta x)y + (-1)^{|x|} x\beta y \text{ for } x, y \in H^*(X) \text{ if } p > 2. \end{aligned}$$

The property (K1) is known as the "Cartan formula".

K2:

$$\begin{aligned} \mathrm{Sq}^{|x|}x &= x^2 \text{ for any } x \in H^*(X) \text{ if } p = 2, \\ \mathrm{P}^{|x|/2}x &= x^p \text{ for any } x \text{ with even degree in } H^*(X) \text{ if } p > 2. \end{aligned}$$

This leads to the following definition.

Definition 2.1.1 *Suppose an unstable module K has maps $\mu : K \otimes K \to K$ and $\eta : F_p \to K$ which give K a communative, unital, F_p-algebra structure. K is then called an* **unstable algebra** *if properties K1 and K2 hold.*

The Unstable Categories

Denote the category whose objects are graded unstable $\mathcal{A}_p$-modules and whose morphisms are $\mathcal{A}_p$-linear maps of degree zero $\mathcal{U}_p$, or, if p is understood, $\mathcal{U}$ and let $\mathcal{K}_p$, or simply $\mathcal{K}$, denote the category whose objects are unstable algebras and whose morphisms are $\mathcal{A}_p$-linear algebra homomorphisms. We concentrate on the category $\mathcal{U}$.

Two main properties of $\mathcal{U}$ are that it is abelian and that it has enough projectives. This latter property is implied by

Proposition 2.1.2 *[SE] [MP] There is an unique (up to isomorphism) unstable module $F(n)$ with a class i_n in degree n such that the natural transformation from $Hom_{\mathcal{U}}(F(n), M)$ to M^n defined by $f \mapsto f(i_n)$ is an equivalence of functors.*

The functor $M \mapsto M^n$ is right exact, so $F(n)$ is projective. $F(n)$ is often called the *free* unstable module on a generator of degree n.

2.1.2 Examples

Preliminary Remark

As already mentioned, the mod p cohomology of a space X is an example of a module over the mod p Steenrod algebra. Before continuing with specific examples, we now remark about the nature of Sq^1 for the case of $p = 2$ and β for the case $p > 2$. On the mod p cohomology of a space, this operation is the same as the *Bockstein homomorphism* induced by the short exact sequence

$$0 \to Z/p \to Z/p^2 \to Z/p \to 0.$$

(Recall that any short exact sequence of groups induces a long exact sequence in both homology and cohomology.)

BZ/p

The mod 2 cohomology of $\mathrm{B}Z/2$ is one example of an unstable algebra of particular interest in the study of the Steenrod algebra and this paper. $H^*(\mathrm{B}Z/2)$ is the polynomial algebra $\mathsf{F}_2[x]$, where x is the generator in degree 1. The action of $\mathcal{A}$ is completely determined by properties K1 and K2.

The mod p cohomology of $\mathrm{B}Z/p$ is also an unstable algebra of interest. $H^*(\mathrm{B}Z/p)$ is the tensor product of an exterior algebra $E(u)$ with generator u of degree 1 and a polynomial algebra $\mathsf{F}_p[v]$ with generator v of degree 2. The action is determined by the two properties K1 and K2 and by the fact that β is the Bockstein homomorphism, i.e. $\beta u = v$.

The importance of these examples in this paper will become clear. Their importance in the study of the Steenrod algebra deserves some comment now.

- Let us define $P(n)$ to be the product of n copies of $\mathrm{B}Z/2$ and let w be the product $x_1 \ldots x_n$, where x_i is the degree 1 generator of the cohomology of the i^{th} copy of $\mathrm{B}Z/2$ in $P(n)$. The map $\mathcal{A}_2 \rightarrow H^*(P(n); Z/2)$ given by $\theta \mapsto \theta w$ is a monomorphism in degrees less than or equal to n. There is a similar statement for odd primes.

- Milnor [Mil] uses the cohomology of $\mathrm{B}Z/p$ and its finite n^{th} skeletons to determine the structure of the dual of $\mathcal{A}_p$, which he then uses to study $\mathcal{A}_p$.

In chapter 3, we will need to know the homology structure of $\mathrm{B}Z/p$. $\mathrm{B}Z/p$ is a Hopf space, a topological space with a group structure, and this group structure induces a product on the homology of $\mathrm{B}Z/p$. For $p = 2$, the mod 2 homology of $\mathrm{B}Z/2$ is a divided power algebra: an algebra with additive generators $\{a_i\}$ indexed by a subset of the non-negative integers and product given by $a_i a_j = \binom{i+j}{i} a_{i+j}$. In particular, it is a divided power algebra on the non-zero elements $h_r \in H_r(\mathrm{B}Z/2; Z/2)$. For odd primes, let $u \in H^1(\mathrm{B}Z/p; Z/p)$ and $v \in H^2(\mathrm{B}Z/p; Z/p)$ be as before, when we described the mod p cohomology of $\mathrm{B}Z/p$. Then the mod p homology of $\mathrm{B}Z/p$ is the tensor product of an exterior algebra generated by an element h_1 and of a divided power algebra generated by elements $h_2, h_4, h_6, \ldots, h_{2i}, \ldots$, where $h_1 \in H_1(\mathrm{B}Z/p; Z/p)$ and $h_{2k} \in H_{2k}(\mathrm{B}Z/p; Z/p)$ are such that $< h_1, u >= 1 \in Z/p$ and $< h_{2k} , v^k >= 1 \in Z/p$.

2.1.3 The Milnor Bocksteins

In the paper that we just referenced, Milnor introduces the elements $\mathrm{Q}_i \in \mathcal{A}^{2p^i-1}$, which we shall use heavily in this paper (Theorem 3.2.8, et al). We now recall their definition and some of their properties.

Definition 2.1.3 *For $p = 2$ ($p > 2$), set $Q_0 = Sq^1$ (β) and define Q_i by $Q_i = [Q_{i-1}, Sq^{2^i}]$ ($Q_i = [Q_{i-1}, P^{p^i}]$), where $[\theta, \theta'] = \theta\theta' - (-1)^{|\theta||\theta'|}\theta'\theta$*

Among the properties of the Milnor Bocksteins are:

- They commute with one other and their square is trivial, i.e. the subalgebra of $\mathcal{A}$ generated by them is an exterior algebra.
- They are derivations, i.e. $\mathrm{Q}_i(xy) = \mathrm{Q}_i(x)y + x\mathrm{Q}_i(y)$.
- If $x \in H^*(\mathrm{B}Z/2; Z/2)$, then $\mathrm{Q}_i x = x^{2^{i+1}}$ and therefore $\mathrm{Q}_i x^k = \binom{k}{1} x^{k+2^{i+1}-1}$. Similarly, if $u^\epsilon v^k \in H^{2k+\epsilon}(\mathrm{B}Z/p; Z/p)$, then

$$\mathrm{Q}_i u^\epsilon v^k = \begin{cases} 0 & \text{if } \epsilon = 0 \\ v^{k+p^i} & \text{if } \epsilon = 1 \end{cases} \tag{2.1}$$

- $\mathcal{A}\mathrm{Q}_i\mathcal{A} = \sum_{j \geq i} \mathcal{A}\mathrm{Q}_j$.

For proofs and more details on the Steenrod algebra and the categories $\mathcal{U}_p$ and $\mathcal{K}_p$, see [McC] [Mil] [MT] [SE] [Sw] and [Sch].

2.2 The Bockstein homomorphisms and BZ/p

2.2.1 The Bockstein homomorphisms

Proposition 2.2.1 *For any short exact sequence of abelian groups,*

$$\mathcal{E}\colon\ 0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

and any space X, there are long exact sequences

$$\cdots H_{n+1}(X;C) \xrightarrow{d} H_n(X;A) \xrightarrow{i_*} H_n(X;B) \xrightarrow{p_*} H_n(X;C) \xrightarrow{d} H_{n-1}(X;A) \cdots$$

and

$$\cdots H^{n-1}(X;C) \xrightarrow{d} H^n(X;A) \xrightarrow{i_*} H^n(X;B) \xrightarrow{p} H^n(X;C) \xrightarrow{d} H^{n+1}(X;A) \cdots.$$

The homomorphism d is commonly called the Bockstein homomorphism associated to the given short exact sequence.

2.2.2 The Bockstein homorphisms on BZ/p

In this paper, we will need to understand various Bockstein homomorphisms on the cohomology of BZ/p. A necessary first step is to know the cohomology of BZ/p with various coefficients. The integral homology or cohomology of BZ/p can be found either by computing it or by referring to sources such as [Wh]. In either case, we find that

$$H_n(\mathrm{B}Z/p) \cong \begin{cases} Z & n = 0 \\ Z/p & n > 0 \text{ and odd} \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

and

$$H^n(\mathrm{B}Z/p) \cong \begin{cases} Z & n = 0 \\ Z/p & n > 0 \text{ and even} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

An application of the universal coefficient theorem gives us:

$$H_n(\mathrm{B}Z/p; Z/p^i) \cong \begin{cases} Z/p^i & n = 0 \\ Z/p & n > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

$$H^n(\mathrm{B}Z/p; Z/p^i) \cong \begin{cases} Z/p^i & n = 0 \\ Z/p & n > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

It is an easy exercise using (2.4), (2.5) and induction on i to show:

Proposition 2.2.2 *For the short exact sequence*

$$0 \to Z/p^j \to Z/p^k \to Z/p^{k-j} \to 0$$

and the space BZ/p, the Bockstein homomorphism for homology

$$d : H_i(BZ/p; Z/p^{k-j}) \ \to \ H_{i-1}(BZ/p; Z/p^j)$$

is a group isomorphism for i even and the zero homomorphism for i odd, while the Bockstein homomorphism for cohomology

$$d : H^i(BZ/p; Z/p^{k-j}) \ \to \ H^{i+1}(BZ/p; Z/p^j)$$

is a group isomorphism for i odd and the zero homomorphism for i even.

2.3 The Bockstein spectral sequence

Let X be any space. The mod p *Bockstein spectral sequence (BSS)* on X is a natural singly graded spectral sequence perhaps best defined by using the exact couple :

$$\begin{tikzcd} H^*(X;Z) \arrow[rr, "i_*"] & & H^*(X;Z) \arrow[dl, "\rho_*"] \\ & H^*(X;Z/p). \arrow[ul, "\beta"] & \end{tikzcd}$$

induced from the exact sequence $0 \to Z \xrightarrow{i} Z \xrightarrow{\rho} Z/p \to 0$, where β is the associated Bockstein. The BSS converges to $(H^*(X;Z)/Torsion) \otimes Z/p$. Unlike most other spectral sequences, this ones indetermediate steps tell us something useful: A cycle which survives to E^r_n and is contained in the image of d_r comes from a direct summand Z/p^r of $H^n(X;Z)$. For more on exact couples and spectral sequences, see [McC],[MT] or [Wh]; for more on the BSS, see [B] or [MT].

There are, however, technical difficulties in using this form of the BSS for our purposes, which we will not elaborate on. In [MT], an alternate procedure is given. While for simplicity [MT] only discusses it for $p = 2$, a careful check shows the analogous result for odd primes also holds. It was this version that we used. A near equivalent to this version arises as follows, where we look at the spectral sequence in terms of spectra. Recall that an element in $H^*(X; R)$ can be thought of as a map, unique up to homotopy, from the suspension spectrum of X to the Eilenberg-MacLane spectrum with coefficients R, HR.

The short exact sequence $0 \to Z \stackrel{\times p}{\to} Z \to Z/p \to 0$ induces a fibration of spectra $HZ \stackrel{\times p}{\to} HZ \to HZ/p$, which provides us with the exact couple mentioned above and with a long cofibration/fibration sequence (either since we are discussing spectra):

$$\ldots \stackrel{j_1}{\to} \Sigma^{-1}HZ/p \stackrel{k_1}{\to} HZ \stackrel{i_1}{\to} HZ \stackrel{j_1}{\to} HZ/p \stackrel{k_1}{\to} \Sigma HZ \ldots$$

where i_1 is alternate notation for the map $\times p$, j_1 is the "reduction" to HZ/p, k_1 is the map to the cofiber of j_1 and $\sum$ is the suspension functor (and $\sum^{-1}$ the "desuspension" functor). The first differential d_1 is the composite $j_1 k_1$. There are similar fibration sequences arising from short exact sequences $0 \to Z \stackrel{\times p^r}{\to} Z \to Z/p^r \to 0$ which also induce (co)fibration sequences

$$\ldots \stackrel{j_r}{\to} \Sigma^{-1}HZ/p \stackrel{k_r}{\to} HZ \stackrel{i_r}{\to} HZ \stackrel{j_r}{\to} HZ/p^r \stackrel{k_r}{\to} \Sigma HZ \ldots,$$

where the various maps are defined analogously to the case $r = 1$ discussed above. At each stage of the spectral sequence, an element lifts if the differential is zero modulo some indetermancy. The m^{th} differential of the Bockstein spectral sequence, defined on lifts of elements from the case $r = 1$, is $d_m = j_m k_m$. One sees the liftings arise by chasing the following commutative diagram of maps.

$$\begin{array}{ccccccc}
\ldots \to & HZ & \stackrel{\times p^r}{\to} & HZ & \stackrel{j_r}{\to} & HZ/p^r & \to \ldots \\
& \uparrow \times p & & \| & & \uparrow \rho_r & \\
\ldots \to & HZ & \stackrel{\times p^{r+1}}{\to} & HZ & \stackrel{j_{r+1}}{\to} & HZ/p^{r+1} & \to \ldots
\end{array}$$

In what follows below, we will define the differentials in our viewpoint of the Bockstein spectral sequence by $\beta_r = \rho_1 \rho_2 \ldots \rho_{r-1} d_r$. The advantage of this viewpoint is that β_r can be seen as the Bockstein homomorphism arising from the short exact sequence

$$0 \to Z/p \to Z/p^{r+1} \to Z/p^r \to 0.$$

All of what follows can be shown by diagram chasing. The diagram on the last page of this chapter has been provided for the reader's convienence.

Summary of how the Bockstein spectral sequence works from this perspective: For any space X, let $\hat{a} \in H^n(X; Z/p^i)$ be an element which maps to an element $a \in H^n(X; Z/p)$ under the reduction map $Z/p^i \to Z/p$. If $\beta_i \hat{a} = 0$ modulo indetermancy, then there is an element of $H^n(X; Z/p^{i+1})$ which hits $\hat{a}$ under the homomorphism ρ_i and a under the composite. We can refer to such an element by calling it a *lift* of a to the $i+1$ step. Continuing in this way, one can see that for any n, if an element $a \in H^n(X; Z/p)$ has a lift for all i, then it is either in the image of β_k for some k or a is a non-zero permanent cycle, i.e. it is the image of an additive generator of an integral direct summand of $H^n(X; Z)$. If at the i^{th} stage, $\beta_i \hat{a}$ is not zero modulo indetermancy, then a is the image of an additive generator of a Z/p^i direct summand of $H^n(X; Z)$.

We end this chapter with some special cases and consequences of the Bockstein spectral sequences.

Theorem 2.3.1 *Let X be a space whose integral cohomology is of finite type (in each dimension, the cohomology is finitely generated).*

a. $H^*(X; Z)$ has no p-torsion if and only if $H^*(X; Z/p^m)$ is a free Z/p^m-module for all $m \geq 1$.

b. The following are equivalent:

i) $H^*(X; Z/p^m)$ is a free Z/p^m-module.

ii) $\beta_1, \ldots, \beta_{m-1}$ are identically zero on the lifts of elements from $H^*(X; Z/p)$.

iii) The reduction map $\rho : Z/p^m \to Z/p$ induces an epimorphism from $H^*(X; Z/p^m)$ to $H^*(X; Z/p)$.

Furthermore, if any of the above hold, the free Z/p^m module $H^*(X; Z/p^m)$ is additively generated by lifts of elements of $H^*(X; Z/p)$.

Proof: Part a and the case i $\Leftarrow$ ii of part b is a diagram chase. The case i $\Leftrightarrow$ iii is a diagram chase using the universal coefficient theorem and the definition of Tor. Recall that the universal coefficient theorem says that

$$0 \to H^n(X; Z) \otimes R \to H^n(X; R) \to Tor(H^{n+1}(X; Z), R) \to 0,$$

while to define $\mathrm{Tor}(R, R')$ one takes an exact sequence

$$0 \to K \xrightarrow{f} F \xrightarrow{g} R \to 0$$

where K and F are free and tensoring the sequence with R'. $\mathrm{Tor}(R, R')$ is then defined as the kernel of $f \otimes 1_{R'}$. The case i $\Rightarrow$ ii of part b can also be done as a diagram chase or as follows: If $H^*(X; Z/p^m)$ is a free Z/p^m module, the universal coefficient theorem implies that $H^*(X; Z/p^{m-l})$ is a free Z/p^{m-l} module for all l such that $0 \leq l \leq m-1$. A diagram chase of either the Bockstein spectral sequence diagram or a diagram involving the universal coefficient theorem applied twice yields that the reduction map $\rho : Z/p^m \to Z/p^{m-1} \to 0$ induces

$$H^*(X; Z/p^m) \xrightarrow{\rho_*} H^*(X; Z/p^{m-1}) \to 0$$

under the hypothesis. The reduction map fits inside the short exact sequence that induces β_{m-1}, hence β_{m-1} has all of $H^*(X; Z/p^{m-1})$ as its kernel. Thus there is an induction argument to show the case i $\Rightarrow$ ii of part b.

The question arises that this is modulo indetermancy. The stronger statement made is accurate. The indetermancy for

$$\beta_{m-1} : H^*(X; Z/p^{m-1}) \to H^*(X; Z/p)$$

comes the map from $H^*(X; Z/p^{m-2})$ to $H^*(X; Z/p^{m-1})$ induced by the inclusion of Z/p^{m-2} into Z/p^{m-1}. By naturality, the composite of the inclusion and β_{m-1}, which sends $H^*(X; Z/p^{m-2})$ to $H^*(X; Z/p)$ must be either β_{m-2} or the zero map, but β_{m-2} is zero, by induction. Actually, it can be shown that the composite must be β_{m-2}, see [MT]. Thus i $\Leftrightarrow$ ii.

As for the claim about lifts additively generating $H^*(X; Z/p^m)$, we have the long exact sequence of cohomology induced by

$$0 \to Z/p^{m-1} \to Z/p^m \to Z/p \to 0,$$

which tells us that $H^*(X; Z/p^m)$ is generated by elements which either are the image of elements from $H^*(X; Z/p^{m-1})$ or map to non-zero elements of $H^*(X; Z/p)$. However, no direct summand of $H^*(X; Z/p^m)$ is generated by an element in the image of $H^*(X; Z/p^{m-1})$, since $H^*(X; Z/p^{m-1})$ and its image are annihilated by p^{m-1} and $H^*(X; Z/p^m)$ is a free Z/p^m module. Hence $H^*(X; Z/p^m)$ is generated by lifts. $\square$

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Chapter 3

The T-functor and its structure

In this chapter, we introduce Lannes' functor T and study the action of the mod p Steenrod algebra $\mathcal{A}_p$ on the image of M under the functor T, TM. We will see that knowledge of the $\mathcal{A}_p$-module structure of M has a lot to say about the $\mathcal{A}_p$-module structure of TM.

3.1 The T-functor

Let V be an elementary abelian p-group, or a finite dimensional vector space over F_p. The functor T_V is the left adjoint to tensoring with $H^*(BV) = H^*(V)$ in the category $\mathcal{U}_p$, i.e. T_V is that unique functor such that for any two objects M and N in $\mathcal{U}_p$, $\mathrm{Hom}_{\mathcal{U}_p}(T_V M, N) \cong \mathrm{Hom}_{\mathcal{U}_p}(M, H^*(V) \otimes N)$. There is a similarly defined left adjoint $T_V^{'}$ in the category $\mathcal{K}_p$, and if $M \in \mathcal{K}_p$, then Lannes has shown that as modules $T_V^{'} M \cong T_V M$.

3.1.1 The existence of T

Jean Lannes cites in [L3] three different approaches to showing the existence of T_V. Each approach has its advantages and disadvantages in showing various results and properties. One method which is of particular interest in this paper is that of J. F. Adams [Ad3]. We now describe it.

Let $\mathcal{M}$ denote the category whose objects are (stable) modules over $\mathcal{A}_p$ and whose morphisms are the $\mathcal{A}_p$-linear maps of degree zero. Note that $\mathcal{U}$ is a full subcategory of $\mathcal{M}$. Suppose that M and N are objects of $\mathcal{M}$ and furthermore suppose that $N^n = 0$ for n less than some integer n_0. Let $H^* = H^*(V) =$

$H^*(BV)$. There is an isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(M, H^* \otimes N) \cong \mathrm{Hom}_{\mathcal{M}}(M \otimes H_*, N),$$

where H_* denotes the dual of H^*, i.e. $H_* \cong H_*(BV)$ (viewed as being in negative degrees) with $\mathcal{A}_p$-module structure on H_* given by

$$\theta.\ t = t.\ \chi\ \theta \text{ for } t \in H_* \text{ and } \theta \in \mathcal{A}_p$$

where $\chi : \mathcal{A}_p \to \mathcal{A}_p$ denotes the canonical anti-isomorphism.

The forgetful functor $\Sigma^\infty : \mathcal{U} \to \mathcal{M}$ has a left adjoint denoted by Ω^∞ and referred to by Lannes in [L3] as the *destablization* functor. Explicitly, if M is an object of $\mathcal{M}$, then $\Omega^\infty M$ is the *universal unstable quotient* of M. For example, $\Omega^\infty(\Sigma^n \mathcal{A}_p)$ is $F(n)$. The adjunction then implies that we can define $T_V M$ as $\Omega^\infty(M \otimes H_*)$.

To close this section, we point out that since Ω^∞ is the left adjoint of the forgetful functor from the category of unstable modules to the category of all modules over $\mathcal{A}_p$, we can observe that $\Omega^\infty M$ is the universal unstable quotient of M, i.e. the largest quotient of M which is unstable.

3.1.2 Properties of T

In all that follows, we will use T to mean $T_{Z/p}$.

Theorem 3.1.1 *The functor T_V has the following properties [L1], [L2], [L3]:*

(a) $T_V \cong T^{dimV}$.

(b) T is exact. (It is right exact since it is a left adjoint.) Thus T takes direct sums to direct sums.

(c) T commutes with colimits.

(d) T commutes with tensor products.

(e) Suppose that M is a locally finite module, i.e. if for all $m \in M$, the submodule generated by m is finite. Then $TM \cong M$.

We will not prove this theorem here, one reason being that to repeat a proof here would roughly double the length of this paper. However, do note that some of it follows from the fact that T is a left adjoint. Part (a) automatically gives us some extensions of the results included herein which we do not specifically mention. Throughout the sequel, we restrict our attention to T.

One property of T of importance in this chapter is contained in the following proposition.

Proposition 3.1.2 *For any n, the module $T(F(n))$ is isomorphic to the direct sum $\bigoplus_{i=0}^{n} F(i)$.*

Proof: Let $H^* = H^*(\mathrm{B}Z/p; Z/p)$. Then for all unstable modules M,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}_p}(TF(n), M) &\cong \mathrm{Hom}_{\mathcal{U}_p}(F(n), H^* \otimes M) \\ &\cong (H^* \otimes M)^n \\ &\cong \bigoplus_{r=0}^{n} M^{n-r} \\ &\cong \bigoplus_{r=0}^{n} \mathrm{Hom}_{\mathcal{U}_p}(F(n-r), M) \\ &\cong \mathrm{Hom}_{\mathcal{U}_p}(\bigoplus_{r=0}^{n} F(n-r), M). \end{aligned} \tag{3.1}$$

Alternately setting M equal to $TF(n)$ and $\bigoplus_{r=0}^{n} F(n-r)$, we see by standard arguments that $TF(n)$ and $\bigoplus_{r=0}^{n} F(n-r)$ are isomorphic. □

One additional property mentioned earlier in this paper is now repeated here in greater detail. Consider the evaluation map

$$\varepsilon_X : \mathrm{B}Z/p \times \mathrm{Map}(\mathrm{B}Z/p, X) \to X$$

defined by taking the point (x, f) to $f(x)$. This map induces a homomorphism from $H^*(X; Z/p)$ to $H^*(\mathrm{B}Z/p; Z/p) \otimes H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p)$, which is an element of $\mathrm{Hom}_{\mathcal{U}_p}(H^*(X; Z/p), H^*(\mathrm{B}Z/p; Z/p) \otimes H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p))$ and gives us an element η in $\mathrm{Hom}_{\mathcal{U}_p}(TH^*(X; Z/p), H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p))$. Under many different sets of conditions, η, which is the natural map mentioned throughout this paper, is an isomorphism. It is this property which many consider the most important, as it allows one to study $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p)$ in a new way. We now include a discussion of many of these sets of conditions. Throughout this discussion, assume that all cohomology, unless otherwise stated, is mod p cohomology.

Let $\hat{X}$ denote the Bousfield-Kan p-completion of X [BK].

Theorem 3.1.3 *[L3] Let X be a space such that both the unstable $\mathcal{A}_p$-algebra $H^*(X)$ and $T_V H^*(X)$ are of finite type and $T_V H^*(X)$ is trivial in degree* 1. *Then the natural map*

$$T_V H^*(X) \to H^*(Map(BV, \hat{X}))$$

is an isomorphism.

Theorem 3.1.4 *[DZ] Let X be a space such that the unstable $\mathcal{A}_p$-algebra $H^*(X)$ is of finite type. If $\pi_1 X$ is a finite p-group, then the canonical map*

$$H^*(Map(BV, \hat{X})) \to H^*(Map(BV, X))$$

is an isomorphism.

The nature of the maps in these two theorems give us one set of sufficient conditions, i.e. the composite of the maps in these two theorems give us the natural map $\eta : T_V H^*(X) \to H^*(\mathrm{Map}(\mathrm{B}Z/p, X))$ defined previously.

Another set of conditions for which η is an isomorphism is given to us by the work of Dror-Farjoun and Smith [DS]

Theorem 3.1.5 *Suppose that X is a connected nilpotent p-local space which has only finitely many non-trivial homotopy groups. If $\pi_n(X)$ is finite for all n, then η is an isomorphism.*

There is a third set of conditions, which we have chosen to mention last, because we wished to state more concrete conditions first.

Theorem 3.1.6 *[L3] Suppose X and Y are two spaces, each with mod p cohomology of finite type, and suppose there is a map $\omega : BV \times Y \to X$. Then the following two conditions are equivalent:*

a. the unstable $\mathcal{A}_p$-algebra homomorphism $T_V H^*(X) \to H^*(Y)$, adjoint to $\omega^* : H^*(X) \to H^*(BV) \otimes H^*(Y)$, is an isomorphism;

b. the map $\hat{Y} \to Map(BV, \hat{X})$ induced by ω is a homotopy equivalence.

This almost says that if a space smells and behaves like $\mathrm{Map}(BV, X)$, then it is $\mathrm{Map}(BV, X)$.

3.2 The internal structure of T

To study the structure of the module TM when we understand the structure of M, we need tools to examine the individual elements of TM. We choose to present Adams' proof of existence rather than referring the reader to other proofs because Adams' approach has the advantage that we can view individual elements quite easily. Let h_r be an additive generator of $H_r(\mathrm{B}Z/p; Z/p)$ and let m denote an element of M. Then $h_r \otimes m \in H_*(\mathrm{B}Z/p; Z/p) \otimes M$ projects to an element $[h_r \otimes m] \in \Omega^\infty\left(H_*(\mathrm{B}Z/p; Z/p) \otimes M\right)$. We will often abuse notation and simply write $h_r \otimes m$ when we mean the class $[h_r \otimes m]$. In [Ad3], Adams gives necessary and sufficient conditions for $\sum_r h_r \otimes m_r \in H_*(\mathrm{B}Z/p; Z/p) \otimes M$ to map to zero in $\Omega^\infty\left(H_*(\mathrm{B}Z/p; Z/p) \otimes M\right)$

3.2.1 Another useful definition

Let α be defined as that map in $\mathrm{Hom}_{\mathcal{U}_p}(M, H^* \otimes TM)$ which is the image of the identity from TM to TM under the adjunction isomorphism $\mathrm{Hom}_{\mathcal{U}_p}(M, H^* \otimes TM) \cong \mathrm{Hom}_{\mathcal{U}_p}(TM, TM)$.

Definition 3.2.1 *(a) Let $p = 2$. For $m \in M^n$ and r any integer, define $t_r m$ by $\alpha(m) = \sum_r (x^r \otimes t_r m)$. Note that $t_r m$ is well-defined.*

(b) *Let p be an odd prime. For $m \in M^n$, r any integer, $\epsilon = 0$ or 1, u an additive generator of $H^1(Z/p)$ and $v = \beta u \in H^2(Z/p)$, define $t_{2r+\epsilon}m$ by $\alpha(m) = \sum_{r,\epsilon=0,1}(u^\epsilon v^r \otimes t_{2r+\epsilon}m)$. Note that $t_s m$ depends on the choice of u, but as u is unique up to multiplication by elements of $Z/p^* = Z/p - \{0\}$, so is $t_s m$.*

Remarks

(a) For $p = 2$ ($p > 2$), if $r < 0$ or if $r > |m|$ ($r > [|m|/2]$), $t_r m = 0$ ($t_{2r+\epsilon}m = 0$)

(b) Throughout the rest of the paper, we will assume that $t_{2r+\epsilon}$ is well-defined for odd primes.

The following theorem holds.

Theorem 3.2.2 *$t_r m = [h_r \otimes m]$, where h_r is chosen as in chapter 2.*

Before beginning the proof, we make the following observation.

Lemma 3.2.3 *Let U, V be two finite dimensional vector spaces over F_p and let V^* be the dual of V. Furthermore, let the set $S \subset V$ be a basis for V. The map $\phi \in Hom_{\mathsf{F}_p}(U, V^* \otimes V \otimes U)$ adjoint to the identity on $U \otimes V$ is given by*

$$\phi(u) = \sum_{v \in S} v^* \otimes v \otimes u.$$

Proof: The isomorphism $\Psi : \mathrm{Hom}_{\mathsf{F}_p}(V \otimes U, V \otimes U) \to \mathrm{Hom}_{\mathsf{F}_p}(U, V^* \otimes V \otimes U)$ is defined by $\Psi(f)(u) = \sum_{v \in S} v^* \otimes f(v \otimes u)$ $\square$
Proof of the theorem: The theorem follows from the lemma, the definition of $t_r m$, the fact that $H_*(\mathrm{B}Z/p; Z/p)$ is of finite type and Adam's definition of TM. Specifically, we said that $t_r m$ is given by

$$m \mapsto \sum u^\epsilon v^k \otimes t_{2k+\epsilon} m$$

while $TM = \Omega^\infty(H_*(BZ/p; Z/p) \otimes M)$. $\square$

Before continuing, let us return to Proposition 3.1.2. For the next theorem which shows the usefulness of our definitions, we need the following lemma.

Lemma 3.2.4 *There is an isomorphism $\bigoplus_{r=0}^n F(n-r) \to TF(n)$ which sends ι_{n-r} to $t_r \iota_n$*

Proof: Recall the isomorphisms given by 3.1 in the proof of Proposition 3.1.2. Let $M = TF(n)$ and let us chase through through the isomorphisms, starting with the identity map on $TF(n)$. By our definition of t_r, the identity map is sent to the map in $\mathrm{Hom}_{\mathcal{U}_p}(F(n), H^* \otimes TF(n))$ determined by $\iota_n \mapsto \sum u^\epsilon v^k \otimes t_{2k+\epsilon}$. Under the next isomorphism, this map is sent to the element

$\sum u^\epsilon v^k \otimes t_{2k+\epsilon} \in (H^* \otimes TF(n))^n$. The following isomorphism sends this element to the map in $\bigoplus_r \mathrm{Hom}_{\mathcal{U}_p}(F(n-r), TF(n))$ determined by $\iota_{n-r} \mapsto t_r\iota_n$. The lemma then follows. $\square$

Theorem 3.2.5 *a) If M is an object of $\mathcal{U}_p$ with $\{m_\gamma\}$ as a set of homogenous generators of M and $\{r_\beta\}$ as a set of relations on M such that the sets describe a presentation of M, then TM has a presentation with generators $\{t_i m_\gamma\}$ and relations $\{t_j r_\beta\}$.*

b) If the set $\{m_\gamma\}$ additively generates an unstable module M as a vector space, then the set $\{t_s m_\gamma\}$ additively generates TM as a vector space.

Proof: Part a: Note that Lemma 3.2.4 shows part a for $M = F(n)$, which has a single generator ι_n and no relations (except for those forced by unstability), since $TF(n)$ has generators $\{t_i\iota_n\}$ and no relations.

The theorem then follows for all M since specifying sets of generators and relations is equivalent to defining a right exact sequence

$$\bigoplus_\beta F(n_\beta) \overset{\oplus r_\beta}{\to} \bigoplus_\gamma F(n_\gamma) \overset{\oplus m_\gamma}{\to} M \to 0.$$

Since T is right exact and commutes with direct sums, we have

$$\bigoplus_\beta TF(n_\beta) \to \bigoplus_\alpha TF(n_\alpha) \to TM \to 0.$$

Part b: By using the identification of $h_r \otimes m$ with $t_r m$, we see that an element of TM can be written as a sum of elements of the form $t_k m_k$, so it suffices to show that an element $t_r m$ can be written as a sum of elements from the set $\{t_s m_\gamma\}$. Since m is a sum of elements from the set $\{m_\gamma\}$ by hypothesis, part b follows. $\square$

The proof of part a of the previous theorem suggests a third way to define $t_r m$. Recall that $TF(n) \cong \bigoplus_{k=0}^n F(k)$. Define j_k to be the inclusion of $F(k)$ into $TF(n)$. We noted in Chapter 2 that an element $m \in M^n$ is the image of ι_n under some unique map $f : F(n) \to M$. Define the element $\tau_r m \in (TM)^{n-r}$ by the equation $\tau_r m = (Tf) \circ j_{n-r}\iota_n$.

Lemma 3.2.6 *$\tau_r m = t_r m$.*

Proof: We can see from the proof of part a of Theorem 3.2.5 that it suffices to show the theorem for $M = F(n)$. It should be clear that by definition $\tau_r\iota_n = \iota_{n-r}$, and in Theorem 3.2.5, we saw that $t_r\iota_n = \iota_{n-r}$. The lemma follows. $\square$

There is yet another way to define $t_r m$ if for some space X, $M = H^*(X; Z/p)$ and the natural map from $TH^*(X; Z/p)$ to $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p)$ is an isomorphism. For the moment, let X and Y be any two spaces. There exists an *external* product from $H_*(Y;R) \otimes H^*(Y \times X;R)$ to $H^*(X;R)$ called the *slant product*, which takes an element $(a,b) \in H_q(Y;R) \otimes H^p(Y \times X;R)$ to an element $b/a \in H^{p-q}(X;R)$. The slant product is described in detail in many basic homology and cohomology texts.

Now let X be our X, Y be $\mathrm{B}Z/p$ and G be Z/p. Define $t_{r,1}m$ to be the element $\epsilon_X^*(m)/h_r$, where ϵ_X is the evaluation map described in section 3.1. Recalling that ϵ_X^* induces the natural map η from $TH^*(X;Z/p)$ to $H^*(\mathrm{Map}(\mathrm{B}Z/p,X);Z/p)$, we see that $t_{r,1}m = t_r m$ when the natural map is an isomorphism, i.e. we have

$$h_r \otimes \sum (u^\epsilon v^k \otimes t_{2k+\epsilon} m) \mapsto \sum < h_r, u^\epsilon v^k > \otimes t_{2k+\epsilon} m = t_r m.$$

Remark In a later chapter, we will generalize the notion of $t_{r,1}$.

3.2.2 The action of $\mathcal{A}_p$ on TM

The next theorem determines the action of $\mathcal{A}_p$ on TM in terms of the action on M.

Theorem 3.2.7 *(a) Suppose $p = 2$. Then $t_r Sq^k = \sum_i \binom{r-i}{i} Sq^{k-i} t_{r-i}$.*

(b) Suppose $p > 2$. Then

$$t_s P^k = \sum_i \binom{[s/2] - i(p-1)}{i} P^{k-i} t_{s-2i(p-1)}$$

and

$$\begin{aligned} t_{2r}\beta &= \beta t_{2r} + t_{2r-1} \\ t_{2r+1}\beta &= \beta t_{2r+1}. \end{aligned}$$

(c) Suppose p is any prime. Then

$$\begin{aligned} t_{2r} Q_k &= Q_k t_{2r} + t_{2r-2p^k+1} \\ t_{2r+1} Q_k &= Q_k t_{2r+1}. \end{aligned}$$

Proof: Let $m \in M$, M any object of $\mathcal{U}_p$. Since α is an $\mathcal{A}_p$-module homomorphism, we have $\theta\alpha(m) = \alpha(\theta m)$ for all $\theta \in \mathcal{A}_p$. The theorem is basically a consequence of this relation. Assume $p = 2$. On the one hand we have

$$\begin{aligned} \mathrm{Sq}^k \alpha(m) &= \mathrm{Sq}^k \textstyle\sum_r x^r \otimes t_r m \\ &= \textstyle\sum_r \sum_i \mathrm{Sq}^i x^r \otimes \mathrm{Sq}^{k-i} t_r m \\ &= \textstyle\sum_r \sum_i \binom{r}{i} x^{r+i} \otimes \mathrm{Sq}^{k-i} t_r m \end{aligned}$$

while on the other we have $\alpha(\mathrm{Sq}^k m) = \sum_r x^r \otimes t_r \mathrm{Sq}^k m$. If we compare the two equations term by term, part (a) follows. The proof of part (b) is almost the same. To prove part (c), let us again simplify by assuming $p = 2$. Recalling that Q_k is a derivation, we have

$$\begin{aligned} \mathrm{Q}_k\alpha(m) &= \mathrm{Q}_k \textstyle\sum_r x^r \otimes t_r m \\ &= \textstyle\sum_r (\mathrm{Q}_k x^r \otimes t_r m + x^r \otimes \mathrm{Q}_k t_r m \\ &= \textstyle\sum_r (\binom{r}{1} x^{r+2^{k+1}-1} \otimes t_r m + x^r \otimes \mathrm{Q}_k t_r m) \end{aligned}$$

while $\alpha(\mathrm{Q}_k m) = \sum_r x^r \otimes t_r \mathrm{Q}_k m$. Part(c) follows for $p = 2$ and a similar proof works for odd primes. □

Remark It is worth noting that the previous theorem combined with part a of Theorem 3.2.5 yields a proof of part b of Theorem 3.2.5 which does not resort to the identification of $t_r m$ and $[h_r \otimes m]$. For simplicity, let us restrict our attention to the case $p = 2$. The proof for odd primes is very similar. By Theorem 3.2.5a, the set $\{t_s x_\gamma\}$ will generate TM as a module over the Steenrod algebra. Hence it shall suffice to verify part b on the elements of the form $\mathrm{Sq}^k t_r x$, where x is one of the additive generators of M. We use induction on r. By Theorem 3.2.7,

$$\mathrm{Sq}^k t_r x = t_r \mathrm{Sq}^k x + \sum_{i>0} \binom{r-i}{i} \mathrm{Sq}^{k-i} t_{r-i} x.$$

For $r = 0$, one has $\mathrm{Sq}^k t_0 x = t_0 \mathrm{Sq}^k x$, and $\mathrm{Sq}^k x$ can be written in terms of the specified additive generators of M. So the result follows by the additivity of t_0. If the result has been shown for values less than r, then the right summand of the equation above can be written in terms of the set $\{t_s x_\gamma\}$. All that is left to do is to write $t_r \mathrm{Sq}^k x$ in term of the set. Again $\mathrm{Sq}^k x$ can be written in terms of the specified additive generators of M. The additivity of t_r lets us complete the induction argument.

The next theorem is in some ways the main theorem of this chapter in the context of this dissertation and questions that arise from it motivate much of the rest of this paper.

Theorem 3.2.8 *Suppose M is an object of $\mathcal{U}_p$ with the property that the n^{th} Milnor Bockstein Q_n is zero on M^i for all $i \geq j$ for some j. Then Q_n is zero on $(TM)^i$ for all $i \geq j$.*

Proof: By Theorem 3.2.7, we can see that our hypothesis implies that $\mathrm{Q}_n t_{2r-1} m = 0$ and $\mathrm{Q}_n t_{2r} m = -t_{2r-2p^n+1} m$ for all appropriate r and all $m \in M$ such that $|m| \geq j$. We will further show that our hypothesis implies that $t_{2l-1} m = 0$ for all $m \in M$ of high enough dimension and all $l \in \mathcal{N}$. Then we will have shown that $\mathrm{Q}_n t_{2r} m = -t_{2r-2p^n+1} m = 0$, completing the proof.

Observe that the definition of the Milnor Bocksteins implies that if Q_n is zero on elements of dimension larger than j in an $\mathcal{U}_p$-module M, then Q_k is zero on such elements of M for all $k \geq n$. Thus for all $k \geq n$, $\mathrm{Q}_k t_{2r} m = -t_{2r-2p^k+1} m$, which implies that $\mathrm{Q}_k t_{2r+2p^k-2} m = -t_{2r-1} m$. The dimension of $t_{2r+2p^k-2} m$ is negative ($|m| - 2r - 2p^k + 2$) for large enough k, hence $t_{2r+2p^k-2}m = 0$ and if $k \geq n$, $-t_{2r-1}m = Q_k t_{2r+2p^k-2}m = 0$. By choosing $k \geq \max(n, \log_p((|m| - 2r + 2)/2) + 1)$, we are done. □

Corollary 3.2.9 *If the natural map η from $TH^*(X; Z/p)$ to $H^*(Map(BZ/p, X); Z/p)$ is an isomorphism and $H^*(X; Z)$ has no Z/p direct summands, i.e. β (Sq^1 if $p = 2$) is zero on $H^*(X; Z/p)$, then $H^*(Map(BZ/p, X); Z)$ has no Z/p direct summands.*

Proof: It follows from the fact that β is the first differential of the Bockstein spectral sequence (see Chapter 2). □

Remarks

(a) Theorem 3.2.8 is a generalization of the following unpublished observation of Nick Kuhn:

Observation 3.2.10 (Kuhn) *Suppose M is an object in $\mathcal{U}_2$ and suppose that $Sq^1 \equiv 0$ on M. Then $Sq^1 \equiv 0$ on TM.*

This case of Theorem 3.2.8 can be shown by observing that the doubling functor Φ and the suspension functor Σ both commute with the T functor.

(b) It is worth noting that the following lemma, which is cited in [Sch], is very similar to Theorem 3.2.8 in style of proof. I came across this lemma only days after proving Theorem 3.2.8.

Lemma 3.2.11 *Let p be an odd prime and $T'(-)$, the left adjoint to $H^*(CP^\infty; Z/p) \otimes (-)$ in $\mathcal{U}'$, the subcategory of $\mathcal{U}_p$ whose objects are concentrated in even dimensions. If M is in $\mathcal{U}'$, then $TM \cong T'M$ in $\mathcal{U}_p$.*

3.2.3 The operations on products

Recall that $\mathcal{K}_p$ is the full subcategory of $\mathcal{U}_p$ whose objects are the unstable $\mathcal{A}_p$ algebras . Then for $M \in \mathcal{K}_p$, $\alpha : M \to H^* \otimes TM$ is an algebra homomorphism, hence $\alpha(mn) = \alpha(m)\alpha(n)$. For $p = 2$, this is the same as

$$\sum x^r \otimes t_r(mn) = (\sum x^i \otimes t_i m)(\sum x^j \otimes t_j n),$$

which implies that

$$t_r mn = \sum_{i+j=r} (t_i m)(t_j n).$$

If $p > 2$, we have

$$\sum u^\epsilon v^r \otimes t_{2r+\epsilon}(mn) = (\sum u^{\epsilon'} v^i \otimes t_{2i+\epsilon'} m)(\sum u^{\epsilon''} v^j \otimes t_{2j+\epsilon''} n),$$

which yields

$$t_{2r}mn = \sum_{i+j=r} (t_{2i}m)(t_{2j}n), \text{ (recall that } u^2 = 0)$$

and

$$t_{2r+1}mn = \sum_{i+j=r} \{(t_{2i+1}m)(t_{2j}n) + (-1)^{|m|}(t_{2i}m)(t_{2j+1}n)\}.$$

Summarizing:

Theorem 3.2.12 *Let M be an object of $\mathcal{K}_p$. Then*

(a) If $p = 2$, $t_r mn = \sum_{i+j=r}(t_i m)(t_j n)$.

(b) If $p > 2$,

$$t_{2r}mn = \sum_{i+j=r} (t_{2i}m)(t_{2j}n)$$

and

$$t_{2r+1}mn = \sum_{i+j=r} \{(t_{2i+1}m)(t_{2j}n) + (-1)^{|m|}(t_{2i}m)(t_{2j+1}n)\}.$$

3.2.4 The algebra structure of the operations

In this section, we examine some of the additional structure of the t_i's. In particular, let V and W be any two vector spaces. For any homomorphism η from V to W, there exists a natural transformation $\eta_!$ from the functor T_V to the functor T_W. This functor is defined as follows: η induces a homomorphism from $H^*(W)$ to $H^*(V)$, which in turn induces maps

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{U}_p}(M, H^*(W) \otimes N) & \cong & \mathrm{Hom}_{\mathcal{U}_p}(T_W M, N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{U}_p}(M, H^*(V) \otimes N) & \cong & \mathrm{Hom}_{\mathcal{U}_p}(T_V M, N). \end{array}$$

Hence there is a natural map induced by η from $T_V M$ to $T_W M$ for all M. Using Adams' proof of the existence of $T_V M$, we can give this map more precisely. Our map η induces a homomorphism η_* from $H_*(BV)$ to $H_*(BW)$, which in turn gives a map $\eta_* \otimes 1_M$ from $H_*(BV) \otimes M$ to $H_*(BW) \otimes M$. Thus understanding the map η_* allows us to understand the map $T_V M \to T_W M$ induced by $\eta : V \to W$

Let us examine what happens when we restrict our attention to the case where $V = Z/p \times Z/p$, $W = Z/p$ and the homomorphism η is addition. Recalling that $T_V \cong T^2$, we see a way of composing two t_* operations. The resulting algebra then can be studied. More specifically, the addition homomorphism induces a homomorphism from $H_*(\mathrm{B}Z/p) \otimes H_*(\mathrm{B}Z/p)$ to $H_*(\mathrm{B}Z/p)$ given by

$h_i \otimes h_j \mapsto \binom{i+j}{i} h_{i+j}$. This allows us to think of the t_* operations as operations from TM to TM since

$$\eta_! t_r(t_s m) = \eta_![h_r \otimes [h_s \otimes m]] = [\eta_*(h_r \otimes h_s) \otimes m].$$

Thus we could redefine, if we choose to do so, the operations t_* as operations from $T-$ to $T-$, and the "old" t_r would become the inclusion of the module into $T-$, followed by the "new" t_r.

Recall the definition of a divided power algebra: it is an algebra with additive generators $\{a_i\}$ indexed by a subset of the non-negative integers and product given by $a_i a_j = \binom{i+j}{i} a_{i+j}$.

Theorem 3.2.13 *a) When $p = 2$, the t_* operations form a divided power algebra over $Z/2$. The algebra is generated by the operations t_{2^i} for integers $i \geq 0$ with $t_{2^i} t_{2^i} = 0$. More precisely, $t_i t_j = \binom{i+j}{i} t_{i+j}$.*

b) When p is an odd prime, the t_* operations with even index form a divided power algebra over Z/p. The algebra of all t_i is generated by the operations t_1 with $t_1 t_1 = 0$ and the operations t_{2p^i} for integers $i \geq 0$ and $(t_{2p^i})^p = 0$. More precisely,

(i) $t_{2i} t_{2j} = \binom{i+j}{i} t_{2(i+j)}$

(ii) $t_{2i} t_{2j+1} = \binom{i+j}{i} t_{2(i+j)+1}$

(iii) $t_{2i+1} t_{2j} = -\binom{i+j}{i} t_{2(i+j)+1}$

(iv) $t_{2i+1} t_{2j+1} = 0$

Proof: By our choice of V and W and homomorphism η, we see that the induced map $BV = \mathrm{B}Z/p \times \mathrm{B}Z/p \to BW = \mathrm{B}Z/p$ is the usual product on the Hopf space $\mathrm{B}Z/p$, which in turn gives us the usual product

$$\mu : H_*(\mathrm{B}Z/p) \otimes H_*(\mathrm{B}Z/p) \to H_*(\mathrm{B}Z/p)$$

on the homology. The theorem follows (see chapter 2). □

The theorem has the following interesting corollary.

Corollary 3.2.14 *Let m be an element of an unstable module M. When $p = 2$, if $t_i m = 0$ and $\binom{r}{i}$ is odd, then $t_r m = 0$. Now suppose p is odd. If $t_1 m = 0$, then $t_{2k+1} m = 0$. If $t_{2i} m = 0$ and $\binom{r}{i}$ is not zero mod p, then $t_{2r} m = 0$.*

For emphasis' sake, realize that this implies that *if $t_1 m = 0$, then $t_{2k+1} m = 0$.*

Chapter 4

Künneth relations

A classic problem in the study of homology and cohomology theories is finding conditions under which one can compute $E_*(X \times Y)$ or $E^*(X \times Y)$ in terms of E_*X and E_*Y or E^*X and E^*Y. In this chapter, we will examine situations where

$$E^*X \otimes_{E^*} E^*Y \cong E^*(X \times Y)$$

which do not seem to be mentioned in other literature.

Let us restrict our attention to cohomology theories E^* which have products and satisfy the wedge axiom, i.e.

$$E^*(\vee_\alpha X_\alpha) \cong \prod_\alpha E^*(X_\alpha).$$

This allows us to reformulate many questions about cohomology theories into questions about ring spectra ([Ad1] [Ad2] [Sw]). A familiar theorem is

Theorem 4.0.1 *[Ad1] [Ad2] [Sw] If either E^*X is a finitely generated free right E^* module or E^*Y is a finitely generated free left E^* module, then*

$$E^*X \otimes_{E^*} E^*Y \cong E^*(X \times Y).$$

Another general situation which is known is the case $E^* = HF^*$, F a field, and either HF^*X or HF^*Y is of finite type, i.e., each dimension contains only a finite number of generators. If this is our case, then

$$HF^*X \otimes_F HF^*Y \cong HF^*(X \times Y).$$

Assuming, without loss of generality, that HF^*Y is the one that is of finite type, we claim that what is crucial about this last example is that

- HF^*Y is a free module over $HF^*(pt) \cong F$, and

- HF^* is bounded below, or *coconnective*: for all n less than a fixed integer, 0 in this case, and for all spaces W, $HF^nW = 0$.

Specifically, we can show the following:

Theorem 4.0.2 *Suppose E is a coconnective ring spectrum. If either E^*X is a free right E^* module of finite type or E^*Y is a free left E^* module of finite type, then*

$$E^*X \otimes_{E^*} E^*Y \cong E^*(X \times Y).$$

Hence by restricting the ring spectra we consider, we can loosen the requirement on the "finiteness" of E^*X or E^*Y.

To show the theorem, we need an algebraic lemma.

Lemma 4.0.3 *If $j \geq 1$ is finite and $\{M_\alpha\}$ is a collection of right modules over a ring R, then*

$$(\prod M_\alpha) \otimes_R R^j \cong \prod (M_\alpha \otimes_R R^j)$$

Proof:

$$\begin{aligned}
(\prod M_\alpha) \otimes_R R^j &\cong \bigoplus_{j \text{ copies}} (\prod M_\alpha) \otimes_R R \\
&\cong \bigoplus_{j \text{ copies}} \prod M_\alpha \\
&\cong \prod \bigoplus_{j \text{ copies}} M_\alpha \\
&\cong \prod M_\alpha \otimes_R R^j
\end{aligned}$$

□

Proof of Theorem 4.0.2: The proof follows the style used to prove Theorem 4.0.1. Let us recall the style. Assume that E^*Y is a free left E^* module of finite type, the argument for E^*X being a free right module of finite type is similar. We regard Y as fixed and X as a variable. We have two functors

$$F_1(-) = E^*(-) \otimes_{E^*} E^*Y, \quad F_2(-) = E^*(- \times Y)$$

Since E is a ring spectrum, we have the product map $E \wedge E \to E$ which induces a natural transformation from F_1 to F_2. One then shows that each is a cohomology theory satisfying the wedge axiom and that the two theories take on the same values on a point, with the natural transformation inducing an isomorphism. If E^*Y is finitely generated, this is no problem. However, since tensor products do not in general commute with products, we usually have trouble showing that F_1 satisfies the wedge axiom. However, E^* is a coconnective theory, and this is just enough to work.

We write E^*Y as $\prod_i N_i$, where N_i is the free module on the generators of E^*Y in dimension i. Thus $(\prod M_\alpha)\otimes N_i \cong \prod(M_\alpha \otimes N_i)$ by the previous lemma. Let $\{X_\alpha\}$ be an arbitrary collection of spaces and let $M_\alpha = E^*(X_\alpha)$. So what we need to show is that

$$(\prod M_\alpha) \otimes_{E^*} E^*Y \cong \prod(M_\alpha \otimes_{E^*} E^*Y).$$

There is a map Φ from $(\prod M_\alpha) \otimes_{E^*} E^*Y$ to $\prod(M_\alpha \otimes_{E^*} E^*Y)$, given by $\Phi((m_\alpha)\otimes n) = (m_\alpha \otimes n)$, where $m_\alpha \in M_\alpha$ and $n \in E^*Y$. We can define a map Ψ going the other way as follows. For each i for which E^iY is non-zero, let $\{b_{i,r}\}_r$ be a set of linearly independent generators for N_i. An element in $[\prod(M_\alpha \otimes_{E^*} E^*Y)]^k$ can be written in the form

$$(\sum_i \sum_r m_{\alpha,i,r} \otimes b_{i,r}),$$

where $m_{\alpha,i,r} \in M_\alpha^{k-i}$. We would like to have

$$\Psi\ ((\sum_i \sum_r m_{\alpha,i,r}\ \otimes b_{i,r})) = \sum_i \sum_r (m_{\alpha,i,r} \otimes b_{i,r}).$$

It remains to show that the sum $\sum_i \sum_r (m_{\alpha,i,r} \otimes b_{i,r})$ has only finite many non-zero summands. Since for each i, $\{b_{i,r}\}$ is a finite set, it suffices to show that there is only a finite number of i's such that $m_{\alpha,i,r}$ can be non-zero. Since we are working with a coconnective theory, there is an integer N for which E^*W is zero in dimensions less than N. Thus there is a smallest value for i. The coconnective hypothesis also implies that there is a smallest value for $k-i$ such that $m_{\alpha,i,r}$ can be non-zero, which in turn implies that for each k, there is a largest value of i. Therefore, the sum $\sum_i \sum_r (m_{\alpha,i,r} \otimes b_{i,r})$ has only finite many non-zero summands. One then observes that Φ and Ψ are inverses of one another and completes the proof as Theorem 4.0.1 is completed in [Sw], [Ad1] or [Ad2]. $\square$

Corollary 4.0.4 *If $H^*(Y;R)$ is a free R module of finite type and R is finite, then*

$$H^*(X \times Y; R) \cong H^*(X;R) \otimes_R H^*(Y;R).$$

Chapter 5

The integral cohomology

In this chapter, we reduce Conjecture 5.1.1 to a technical conjecture.

5.1 Bocksteins and $H^*(\mathbf{Map}(\mathbf{B}Z/p, X); Z)$

We would like to show:

Conjecture 5.1.1 *Suppose X is a space such that $T_V H^*(X; Z/p)$ is of finite type and the natural map (1.1) from $T_V H^*(X; Z/p)$ to $H^*(Map(BV, X); Z/p)$, induced by the evaluation map, is an isomorphism. If $H^*(X; Z/p^i)$ is a free Z/p^i module, then $H^*(Map(BV, X); Z/p^i)$ is a free Z/p^i module.*

Since $T_V = T^{dim\, V}$, it would suffice to show the theorem for $V = Z/p$. The case $i = 2$ follows from Theorem 3.2.8, since β_1 is the 0^{th} Milnor Bockstein.

Suppose we knew the result for all i less than some integer s. That would imply that the first $s - 1$ Bocksteins are zero and that

$$H^*(\mathrm{B}Z/p \times \mathrm{Map}(\mathrm{B}Z/p, X); Z/p^i) \cong H^*(\mathrm{B}Z/p; Z/p^i) \otimes H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p^i)$$

for all i less or equal to s (Corollary 4.0.4). We wish to show that β_s is zero on $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p^s)$ if β_s is zero on $H^*(X; Z/p^s)$.

We introduce some notation. Let $\varepsilon_X : \mathrm{B}Z/p \times \mathrm{Map}(\mathrm{B}Z/p, X) \to X$ be the evaluation map on X and let us abuse notation by letting ε_X^* represent the induced homomorphism from $H^*(X; R)$ to $H^*(\mathrm{B}Z/p \times \mathrm{Map}(\mathrm{B}Z/p, X); R)$ for any commutative ring and also the composite homomorphism

$$\begin{aligned} &H^*(X; Z/p^i) \to \\ &\quad H^*(\mathrm{B}Z/p \times \mathrm{Map}(\mathrm{B}Z/p, X); Z/p^i) \xrightarrow{\cong} \\ &\quad H^*(\mathrm{B}Z/p; Z/p^i) \otimes H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p^i) \end{aligned}$$

for all i, $1 \leq i \leq s$. The context will indicate which ε_X^* is meant. For a homogenous element $b \in H^*(X; Z/p^i)$, denote its image under ε_X^* as $\sum x_k \otimes t_k^i b$, where x_k is a generator of $H^k(\mathrm{B}Z/p; Z/p^i)$ and $t_k^i b$ is an element of $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/2^i)$, well-defined up to a multiple of an element of order p. We let ι and ρ be defined by the exact sequence

$$0 \to Z/p^{s-1} \xrightarrow{\iota} Z/p^s \xrightarrow{\rho} Z/p \to 0$$

and define ι_* and ρ_* to be the induced homomorphisms.

By Theorems 3.2.5 and 3.2.8, $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p)$ is additively generated by elements of the form $t_{2r}m$, where m belongs to a set of additive generators for $H^*(X; Z/p)$. Thus, for $i \leq \ s$, $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p^i)$ will be generated by lifts of the $t_r m$'s. Theorem 2.3.1 and our hypothesis tell us that $\beta_s = 0$ on the lifts of elements from $H^*(X; Z/p)$. Consider an arbitrary element $a \in H^*(X; Z/p^s)$ which is a lift of some element $\rho_* a$ in $H^*(X; Z/p)$. If $\beta_s a = 0$, then its image under ε_X^*, $\beta_s\ \varepsilon_X^* a$, is zero, since all maps under consideration are natural. One easily sees that $\varepsilon_X^* a$ is a lift of $\varepsilon_X^* \rho_* a$ (again, forgive the abuse of notation), and Theorem 3.2.8 implies that $\varepsilon_X^* \rho_* a = \sum v^r \otimes t_{2r} \rho_* a$ (x^{2r} if $p = 2$). Therefore, if $\varepsilon_X^* a = \sum_k x_k \otimes t_k^s a$, $t_{2r}^s a$ is a lift of $t_{2r} \rho_* a$ and $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p^i)$ is generated by elements of the form $t_{2r}^i b$, $b \ \in \ H^*(X; Z/p^i)$.

Let us examine the consequences of $\beta_s \sum (x_k \otimes t_k^s a) = 0$. Since β_s is a differential and additive, we have

$$\begin{aligned}
\beta_s \sum (x_k \otimes t_k^s a) &= \sum \beta_s (x_k \otimes t_k^s a) \\
&= \sum (\beta_s x_k \otimes \rho_* t_k^s a + \rho_* x_k \otimes \beta_s t_k^s a) \\
&= \sum (v^r \otimes \rho_* t_{2r-1}^s a + v^r \otimes \beta_s t_{2r}^s a) \\
&= \sum v^r \otimes (\rho_* t_{2r-1}^s a + \beta_s t_{2r}^s a) \\
(&= \sum x^{2r} \otimes (\rho_* t_{2r-1}^s \beta_s t_{2r}^s a) \text{ if } p = 2).
\end{aligned}$$

Examining the last sum term by term, we see that $\beta_s t_{2r}^s a = -\rho_* t_{2r-1}^s a$ for all r. Since a was a lift of an arbitrary element, the result would follow if $\rho_* t_{2r-1}^s a = 0$ for all r and all a. We have assumed that $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p^s)$ is a free Z/p^s module, thus this last statement is equivalent to showing that $t_{2r-1}^m a$ is a multiple of p for all r and all a.

We now discuss some equivalents to this last condition. Choose a non-zero element $h_{r,s} \ \in \ H_r(\mathrm{B}Z/p; Z/p^s)$ in dimension $-r$ in the way that was discussed in chapter 2. For any space X, *define*

$$t_{r,s} : H^*(X; Z/p^s) \to H^{*-r}(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p^s)$$

by $t_{r,s}(a) = \varepsilon_X^*(a)/h_{r,s}$, where / denotes the slant product. (In the odd prime case, this is uniquely defined only up to a unit multiple, i.e. choice of $h_{1,s}$. See chapter 2.)

The operations $t_{r,s}$ are related to the t^s_r operations (when they exist) as follows:

Lemma 5.1.2 $t_{r,s}a = p^{s-1}t^s_r a$

Proof: Under our usual set of hypotheses, with $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p^s)$ shown to be a free Z/p^s module, we have

$$\varepsilon^*_X(a)/h_{r,s} = (\sum x_k \otimes t^s_k a)/h_{r,s} =< h_{r,s}, x_r > \otimes t^s_r a.$$

Since $< h_{r,s}, x_r >= p^{s-1} \in Z/p^s$, i.e. it has order p, $t_{r,s}(a) = p^{s-1}\ t^s_r\ a$. $\square$

Conjecture 5.1.3 *Under the usual hypotheses, if* $\beta_s \equiv 0$ *on* $H^*(X; Z/p^s)$*, then* $t_{2r-1,s} \equiv 0$.

The above discussion shows:

Theorem 5.1.4 *Conjecture 5.1.3 implies Conjecture 5.1.1*

Attempts to prove the Conjecture

To simplify the exposition, we limit ourselves to the prime 2; analogous statements hold for odd primes.

The Bockstein homomorphism $\beta_s : HZ/2^s \to \Sigma HZ/2$ is associated to the short exact sequence

$$0 \to Z/2 \to Z/2^{s+1} \to Z/2^s \to 0$$

and it has a lift to a Bockstein homomorphism $\tilde{\beta}_s : HZ/2^s \to \Sigma HZ/2^s$ associated to the short exact sequence

$$0 \to Z/2^s \to Z/2^{2s} \to Z/2^s \to 0$$

with the lift given by the diagram

$$\begin{array}{ccccccccc} 0 & \to & Z/2^s & \to & Z/2^{2s} & \to & Z/2^s & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & Z/2 & \to & Z/2^{s+1} & \to & Z/2^s & \to & 0. \end{array}$$

If $\beta_s \equiv 0$ on $H^*(X; Z/2^s)$ and $a \in H^*(X; Z/2^s)$ is an element of order 2^s, then we can show that

$$\tilde{\beta}_s t_{2r,s}(a) = t_{2r-1,s}(a) + t_{2r,s}(\tilde{\beta}_s(a)) = t_{2r-1,s}(a).$$

This would seem to suggest that we might be able to imitate the proof of Theorem 3.2.8 to prove that $t_{2r-1,s}(a) = 0$ for all r and all $a \in H^*(X; Z/p^s)$. To do this, it would suffice to show the existence of a set of operations

$$\mathrm{Q}_{k,s} :\ HZ/2^s \ \to\ \Sigma^{|\mathrm{Q}_{k,s}|}\ HZ/2^s$$

with the properties that

(a) $|\mathrm{Q}_{k+1,s}| > |\mathrm{Q}_{k,s}|$

(b) $\mathrm{Q}_{k,s}(ab) = (\mathrm{Q}_{k,s}a)b + a(\mathrm{Q}_{k,s}b)$

(c) $\mathrm{Q}_{k,s}h_{2r,s} = h_{2r-|\mathrm{Q}_{k,s}|,s}$; $\mathrm{Q}_{k,s}h_{2r-1,s} = 0$
which is equivalent to:

(c') $\mathrm{Q}_{k,s}x_{2r-1} = x_{2r-1+|\mathrm{Q}_{k,s}|}$; $\mathrm{Q}_{k,s}x_{2r} = 0$

(d) If $\tilde{\beta}_s \equiv 0$ on the cohomology of a space X, then $\mathrm{Q}_{k,s} \equiv 0$ on the cohomology of X.

We conclude this section with another conjecture, which is a corollary of Conjecture 5.1.1. Let Z_p represent the p-adic integers.

Conjecture 5.1.5 *Suppose X is a space such that the natural map (1.1) from $T_V H^*(X; Z/p)$ to $H^*(Map(BV, X); Z/p)$, induced by the evaluation map, is an isomorphism and suppose that $T_V H^*(X; Z/p)$ is of finite type. If $H^*(X; Z_p)$ is a free Z_p module, then $H^*(Map(BV, X); Z_p)$ is a free Z_p module.*

5.2 Plausibility: Application to Lie Groups

In this section, we will provide further evidence Conjecture 5.1.1 is true. This is done by examining two theorems, one by Dwyer and Zabrodsky, the other by Borel.

To state Dwyer and Zabrodsky's result, we first need to make some definitions. A map $f : X \to Y$ is defined to be a *strong mod p equivalence* if f induces

(a) an isomorphism $\pi_0 X \xrightarrow{\cong} \pi_0 Y$,

(b) an isomorphism $\pi_1(X, x) \xrightarrow{\cong} \pi_1(Y, f(x))$ for any $x \in X$ and

(c) an isomorphism
$$H_*(\tilde{X}_x; Z/p) \xrightarrow{\cong} H_*(\tilde{Y}_{f(x)}; Z/p),$$
where $\tilde{X}_x$ is the universal cover of the component of X containing x and $\tilde{Y}_{f(x)}$ is the universal cover of the component of Y containing $f(x)$.

See [DZ].

Consider the equivalence relation $\sim$ on the set of group homomorphisms from π to G given by $\eta \sim \theta$ if there is an $h \in G$ such that $\eta(a) = h\theta(a)h^{-1}$ for all $a \in \pi$. Let $\mathrm{Rep}(\pi, G)$ be the set of equivalence classes under this relation. If we let $Z_G(\eta)$ denote the centralizer in G of the image of η, then we have:

Theorem 5.2.1 ([DZ]) *If π is a finite p-group and G is a compact Lie group, then there is a natural map*

$$f : \coprod_{\eta \in Rep(\pi,G)} BZ_G(\eta) \to Map(B\pi, BG)$$

which is a strong mod p equivalence.

An elementary abelian p subgroup of a compact Lie group G is called *toral* if it can be embedded in a torus of G. Borel's theorem follows.

Theorem 5.2.2 ([Bo]) *Let G be a connected compact Lie group and let BG be its classifying space. The following are equivalent:*

a) The integral cohomology of G has no p torsion.

b) The integral cohomology of BG has no p torsion.

c) Every elementary abelian p group in G is toral.

d) Every elementary abelian p group of rank ≤ 3 in G is toral.

If G is such a Lie group, certain subgoups of G will inherit property c; among these are the centralizers of elements of order p. Hence we have the following theorem.

Theorem 5.2.3 *Let G be a compact Lie group whose integral cohomology is p-torsion free and of finite type. If V is an elementary abelian p-group and η is a homorphism from V to G, then $H^*(BZ_G(\eta); Z)$ is p-torsion free. In particular, if g is an element of order p, then $H^*(BZ_G(g); Z)$ is p-torsion free.*

This theorem, combined with Theorem 5.2.1, implies that Conjecture 5.1.5 holds for connected compact Lie groups.

We remark that Borel's proof of his theorem uses the classification of compact Lie groups.

Chapter 6

Speculations

6.1 Connective Morava K-theory

The style of proof proposed for Conjecture 5.1.1 suggests that there may similar results with other cohomology theories. An example is the n^{th} connective Morava k-theory $k(n)$, $n > 0$ [JW] [WSW]. The coefficient ring of $k(n)$ is $F_p[v_n]$, where the dimension of v_n is $-2(p^n - 1)$. There is a fibration

$$\Sigma^{2(p^n-1)}k(n) \xrightarrow{v_n} k(n) \to HZ/p$$

of spectra which gives rise to a Bockstein spectral sequence much like the ordinary mod p Bockstein spectral sequence. In studying this spectral sequence, one sees fibrations

$$\Sigma^{2r(p^n-1)}k(n) \xrightarrow{v_n^r} k(n) \to E_r,$$

where the theory associated to E_r has the coefficient ring $E_r^* \cong F_p[v_n]/(v_n^r)$.

As before, we assume that X is a space such that the natural map from $TH^*(X; Z/p)$ to $H^*(\mathrm{Map}(BZ/p, X); Z/p)$ is an isomorphism.

Conjecture 6.1.1 *If $E_r^*(X)$ is a free module over the coefficient ring E_r^*, then $E_r^*(Map(BZ/p, X))$ is also a free module over E_r^*.*

The conjecture is true for $r = 1$, as $E_1 = HZ/p$. I can show it is true for $r = 2$ (this is a corollary of Theorem 3.2.8 in my summary). The proof should be analogous to the proof of Theorem 5.1.1. If the conjecture is true, some version of the following conjecture would seem to be a corollary.

Conjecture 6.1.2 *If $k(n)^*(X)$ is a free module over the ring $k(n)^*$, then so is $k(n)^*(Map(BZ/p, X))$.*

These conjectures motivate the style of arguments and proofs in much of this dissertation. The generality of the statement of Theorem 3.2.8 and the spectra style proof of Theorem 4.0.2 are examples of this.

6.2 The cohomology of Map(B$Z/p, X)_f$

A map $f : \mathrm{B}Z/p \to X$ lies in some path component of $\mathrm{Map}(\mathrm{B}Z/p, X)$; we call this component $\mathrm{Map}(\mathrm{B}Z/p, X)_f$. The T functor technology provides us a way to compute the mod p cohomology of $\mathrm{Map}(\mathrm{B}Z/p, X)_f$ under our usual hypothesis that the natural map η defined in chapter 3 from $TH^*(X; Z/p)$ to $H^*(\mathrm{Map}(\mathrm{B}Z/p, X); Z/p)$ is an isomorphism.

The map f or any other map in $\mathrm{Map}(\mathrm{B}Z/p, X)_f$ induces a homomorphism $f^* : H^*(X; Z/p) \to H^*(\mathrm{B}Z/p; Z/p)$, unique since $\mathrm{Map}(\mathrm{B}Z/p, X)_f$ is path connected and a path in this space of maps is a homotopy between the two maps which are the endpoints. Since $H^*(\mathrm{B}Z/p; Z/p) \cong H^*(\mathrm{B}Z/p; Z/p) \otimes \mathsf{F}_p$, f^* induces a map $TH^*(X; Z/p) \to \mathsf{F}_p$. The last map is actually a map from $(TH^*(X; Z/p))^0$ to F_p, which we will label $\tilde{f}$. Defining $T_f H^*(X; Z/p)$ to be $TH^*(X; Z/p) \otimes_R \mathsf{F}_p$, where $R = (TH^*(X; Z/p))^0$ and the action across the tensor product is provided by $\tilde{f}$, it follows that the natural map η induces a map

$$\eta_f : T_f H^*(X; Z/p) \to H^*(\mathrm{Map}(\mathrm{B}Z/p, X)_f; Z/p).$$

Furthermore, if η is an isomorphism, so is η_f.

A natural question that arises is that if Conjecture 5.1.1 and the other conjectures are true, are there similar statements with more restrictive hypotheses which yield similar results for the path component $\mathrm{Map}(\mathrm{B}Z/p, X)_f$?

Two preliminary steps in this direction are provided by the next two results.

The elements in $TH^*(X)^0$

Proposition 6.2.1 *Let X be a space and $f : BZ/p \to X$ be a map. We have that $\tilde{f}(t_{|m|}m)$ is non-zero if and only if $f^*(m) \in H^*(BZ/p)$ is non-zero.*

Proof: The map

$$f^* : H^*(X) \to H^*(\mathrm{B}Z/p) \cong H^*(\mathrm{B}Z/p) \otimes \mathsf{F}_p$$

sends m to the element $f^*(m) \otimes 1$. Recalling how Adams showed the existence of T (see chapter 3), we see that f^* induces a map

$$H_*(\mathrm{B}Z/p) \otimes H^*(X) \to \mathsf{F}_p$$

which sends $h_{|m|} \otimes m$ to 1. The proposition follows. □

Therefore we have a beachhead for our questions.

Dwyer-Wilkerson's approach

W. Dwyer and C. Wilkerson in [DW] have shown another way of computing $T_f H^*(X)$ that the author is not yet sure how to make compatible with his approach of viewing T locally. However, using Dwyer-Wilkerson's approach, we can make a statement like Theorem 3.2.8. Let us examine their approach, which seems to be inspired by Smith theory. We will restrict our discussion to the immediately applicable cases. See [DW] for the more general statements.

Let f be a map that as before maps from $\mathrm{B}Z/p$ to X. The ring homomorphism $f^* : H^*(X; Z/p) \to H^*(\mathrm{B}Z/p; Z/p)$ provides us a way of making $H^*(\mathrm{B}Z/p; Z/p)$ into a (left) module over $H^*(X; Z/p)$, with the structure given by $a \cdot b = f^*(a)b$. Now define S_f to be the multiplicative set generated by those elements in $H^2(X; Z/p)$ that are in the image of the Bockstein β, (Sq^1 if $p = 2$) of those elements of $H^1(X; Z/p)$ which map nonzero under f^*. Let $I = \ker f^*$.

Theorem 6.2.2 *[DW] If $H^*(BZ/p; Z/p)$ is finitely generated as a module over $H^*(X; Z/p)$, then*

$$T_f H^*(X; Z/p) \cong Un\ ((S_f^{-1} H^*(X; Z/p))_{\hat{I}})$$

where $Un\ M$ is the largest unstable submodule of M for M a module over $\mathcal{A}_p$.

Suppose we wish to know when the Bockstein β will be zero on such a $T_f H^*(X; Z/p)$. We see from the equation

$$\beta m = \beta(sm/s) = (\beta s)m/s + s(\beta m/s) = s(\beta m/s),$$

that $\beta m/s = (\beta m)/s$ which will equal zero $(0/1)$ when there is a $s_m \in S_f$ such that $s_m \beta m = 0$. If for all $m \in H^*(X; Z/p)$ there exists such a $s_m \in S_f$, then $\beta = 0$ on $T_f H^*(X; Z/p)$.

For more on: localizations and completions of rings, see [Mat]; the action of the Steenrod algebra on the localizations and completions, see [Si] and [Wi].

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