ON FIBREWISE SIMPLICIAL MONOIDS AND MILNOR-CARLSSON'S CONSTRUCTIONS

JIE WU

ABSTRACT. We give a categorical view of Milnor-Carlsson's constructions. The word length filtration is studied. Certain natural maps are combinatorially constructed. An application to the homology of $\Omega(FP^\infty \wedge X)$ is given for $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

1. INTRODUCTION

In [4], G. Carlsson introduced a simplicial group construction which gives a generalization of Milnor's $F(K)$ construction [10]. Roughly speaking, if we construct a simplicial group which is a free product of a simplicial group G over a pointed simplicial set X, then we get a simplicial group construction for $\Omega(BG \wedge X)$, where BG is the classifying space of G. In this article, we give a categorial view of this construction.

Let $\mathcal{C}$ be a category. A fibrewise simplicial object over $\mathcal{C}$, roughly speaking, is a diagram over $\mathcal{C}$ with indices in a simplicial set. This is an abstract view of fibrewise topology [8] or sheaf theory. If the category $\mathcal{C}$ has coproducts, then the abstract F-construction is defined to be certain coadjoint functor from the category of fibrewise simplicial objects over $\mathcal{C}$ to the category of simplicial objects over $\mathcal{C}$. Suppose that there is a functor T from $\mathcal{C}$ to the category of pointed simplicial sets such that T preserves coproducts up to homotopy. Then there is an induced functor $\check{T}$ from the category of fibrewise simplicial objects over $\mathcal{C}$ to the category of pointed bisimplicial sets. Theorem 4.7 shows that $\check{T}$ is homotopy equivalent to $\check{T} \circ F$. Let $\mathcal{C}$ be a category of monoids. Notice that the bar-construction B preserves coproduct up to homotopy [6]. A corollary of this abstrct theorem is the Carlsson theorem.

An application of Carlsson's construciton to homotopy theory is to give a representation of the homotopy groups of simply connected suspension spaces to certain combinatorial groups as centers [15]. Applications of Carlsson's construction to minimal simplicial groups are given in [16]. In this paper, we pay more attention to the geometry of the Carlsson construction. The word length filtration is considered. The resulting cofibres are certain smash product pinched out certain reduced diagonal elements (Proposition 5.2). Our construction in the monoid case is a generalization of the James construction [8]. We construct certain natural map $H_n : \Omega(Y \wedge X) \to$

Research at MSRCI is supported in part by NSF grant DMS-9022140.

$\Omega^n(Y^n \wedge (X^{(n)}/\bar{\triangle}_n))$, which is similar to the James-Hopf map, for any path connected CW-complex Y and any pointed CW-complex X, where $X^{(n)}$ is the n-th fold self smash product of X and $\bar{\triangle}_n = \{(x_1 \wedge \ldots \wedge x_n) \in X^{(n)} \mid x_i = x_{i+1} \text{ for some } i\}$ (Theorem 6.11). A direct application of these natural maps is to give a decomposition of $H_*\Omega(FP^\infty \wedge X)$ for $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $FP_2^\infty = FP^\infty/FP^1$.

Theorem 1.1. *Let $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let X be a pointed space. Suppose that H_* is a multiplicative homology theory such that (1) both $\bar{H}_*(FP^\infty)$ and $\bar{H}_*(FP_2^\infty)$ are free $H_*(pt)$-modules; and (2) the inclusion of the bottom cell $S^d \to FP^\infty$ induces a monomorphism in the homology. Then there is a product filtration $\{F_rH_*\Omega(FP^\infty \wedge X)\}_{r\geq 0}$ of $H_*\Omega(FP^\infty \wedge X)$ such that $F_0 = H_*(pt)$ and*

$$F_r/F_{r-1} \cong \Sigma^{(d-1)r}\bar{H}_*(X^{(r)}/\bar{\triangle}_r),$$

where $d = dim_{\mathbb{R}}F$ and Σ is the suspension. Furthermore, this filatraion is natural for X.

In particular, if $F = \mathbb{R}$, this theorem holds for the mod 2 homology and if $F = \mathbb{C}$ or $\mathbb{H}$, this theorem holds even for integral homology.

The article is organized as follows. In Section 2, we recall some basic properties of the free products of monoids. In Section 3, we recall some basic properties of the bar constructions. The fibrewise simplicial monoids are introduced in Section 4. The word length filtration is studied in Section 5. Theorem 1.1 is given in Section 5 without proof, where Theorem 5.8 is Theorem 1.1. Some natural maps are combinatorially constructed in Section 6. The proof of Theorem 1.1 (Theorem 5.8) is given in Section 7. In Section 8, we give a description of the group ring $\mathbb{F}_p(F^{\mathbb{Z}/p}(X))$.

The author is indebted to helpful discussions with J. C. Moore. The abstract F-construction was suggested by him and he conjectured the main results in Section 4.

2. Free Products of Monoids

In this section, we recall some basic properties of the free products of monoids. Most theorems in this section are well known. We only give the theorems which will be used in next sections.

Notation 2.1. *Let $\{M_x \mid x \in X\}$ be a given set of the monoids M_x, where X is an (indices) set (pointed set).*

We write X^a as the element a in M_x with index x. A word w is an ordered system of elements

$$w = (x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}),$$

where $x_i^{a_i} \in M_{x_i}$. A reduced word w is an ordered system of elements

$$w = (x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n})$$

such that a_i is not the unit element 1 in the monoid M_{x_i} for $i \leq i \leq n$ and $x_i \neq x_{i+1}$ for $1 \leq i \leq n-1$ ($x_i \neq *$ for $1 \leq i \leq n$ if X is a pointed set with a point $*$). In addition, we regard the case $n = 0$ as corresponding to the empty (reduced) word. The set of all of the words is denoted by $\tilde{M}$. Let $M = \tilde{M}/\approx$ denote by the quotient set of $\tilde{M}$ modulo the equivalent relation generated by (1) $(x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}) \approx (x_1^{a_1} \ldots x_{i-1}^{a_{i-1}} x_{i+1}^{a_i+1} \ldots x_n^{a_n})$ if $a_i = 1$ for some $1 \leq i \leq n$ (or $x_i = *$ for some $1 \leq i \leq n$ if x is a pointed set with a point $*$); and (2) $(x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}) \approx (x_1^{a_1} \ldots x_{i-1}^{a_{i-1}} x_i^{a_i a_{i+1}} x_{i+2}^{a_{i+2}} \ldots x_n^{a_n})$ if $x_i = x_{i+1}$ for some i. Furthermore, let M^* denote by the set of all of the reduced words.

Recall that the free product of monoids is defined to be the coproduct of the monoids in the category of monoids and homomorphisms. We will show that there exist natural multiplications in the sets M and M^* such that both M and M^* are free products of the monoids M_x. In the case of groups, the proof can be found out in the standard text book of group theory.

Proposition 2.2. *The binary operation in M induced by juxtaposition make M to be a monoid. Furthermore, M is a free product of the monoids M_x with the natural inclusion $M_x \to M$*

Proof: The juxtaposition given by

$$(x_1^{a_1} \ldots x_n^{a_n})(y_1^{b_1} \ldots y_m^{b_m}) = (x_1^{a_1} \ldots x_n^{a_n} y_1^{b_1} \ldots y_m^{b_m})$$

preserves the equivalent relation. Thus the binary operation is well defined in the quotient set $M = \tilde{M}/\approx$. It is easy to check that this operation is associative with unit element $\emptyset$. The natural inclusion $M_x \to M$ is given by

$$a \to \{(x^a)\}$$

which is a monoid homomorphism. Furthermore, it is straightforward to check that M is a coproduct of the monoids M_x.

Now we define a product in M^* as follows: Let $w = (x_1^{a_1} \ldots x_n^{a_n}$ and $w' = y_1^{b_1} \ldots y_m^{b_m})$ be two reduced words. There exists a unique integer $0 \leq i \leq \min\{n, m\}$ such that (1) $x_{n-j+1} = y_j$ and $a_{n-j+1} \cdot b_j = 1$ for $z \leq j \leq i$ and (2) $x_{n-i} \neq y_{i+1}$, or $x_{n-i} = y_{i+1}$ but $a_{n-i} b_{i+1} \neq 1$. If $x_{n-i} \neq y_{i+1}$, we put

$$w \cdot w' = (x_1^{a_1} \ldots x_{n-i}^{a_{n-i}} y_{i+1}^{b_{i+1}} \ldots y_m^{b_m})$$

. But if $x_{n-i} = y_{i+1}$ and $a_{n-i} b_{i+1} \neq 1$, then

$$w \cdot w' = (x_1^{a_1} \ldots x_{n-i-1}^{a_{n-i-1}} x_{n-i}^{a_{n-i} b_{i+1}} y_{i+2}^{b_{i+2}} \ldots y_m^{b_m}).$$

This product gives a binary operation in M^*. The empty word is the unit element. The proof of associativity is technically complicated. We start with the following lemma which follows directly from the definition.

Lemma 2.3. *Let $w = (x_1^{a_1} \ldots x_n^{a_n})$ and $w' = y_1^{b_1} \ldots y_m^{b_m}$ be reduced words. Then*

$$w \cdot w' = (x_1^{a_1} \cdot ((x_2^{a_2} \ldots x_n^{a_n}) \cdot w') \,.$$

Lemma 2.4. *Let $w = (x_1^{a_1} \dots x_n^{a_n})$ and $w' = y_1^{b_1} \dots y_m^{b_m}$ be reduced words. Then*

$$(w \cdot w') \cdot (x^a) = w \cdot (w' \cdot (x^a))$$

for any reduced word (x^a) of length 1.

Proof: If $n = 1$, the assertion follows from the definition by checking the several cases. For $n > 1$, the assertion follows by induction on n as follows.

$$\begin{aligned}
&(x_1^{a_1} \dots x_n^{a_n}) \cdot \big[(y_1^{b_1} \dots y_m^{b_m}) \cdot (x^a)\big] \\
&= (x_1^{a_1}) \cdot \big[(x_2^{a_2} \dots x_n^{a_n}) \cdot \big[(y_1^{b_1} \dots y_m^{b_m}) \cdot (x^a)\big]\big] \text{ (by Lemma 2.3)} \\
&= (x_1^{a_1}) \cdot \big[\big[(x_2^{a_2} \dots x_n^{a_n}) \cdot (y_1^{b_1} \dots y_m^{b_m})\big] \cdot (x^a)\big] \text{ (by induction)} \\
&= \big[(x_1^{a_1}) \cdot \big[(x_2^{a_2} \dots x_n^{a_n}) \cdot (y_1^{b_1} \dots y_m^{b_m})\big]\big] \cdot (x^a) \text{ (by the case } n = 1) \\
&= \big[(x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}) \cdot (y_1^{b_1} \dots y_m^{b_m})\big] \cdot (x^a) \text{ (by Lemma 2.3).}
\end{aligned}$$

Proposition 2.5. *The product in M^* is associative, that is,*

$$(w \cdot w') \cdot w'' = w \cdot (w' \cdot w'')$$

for any reduced words w, w' and w''.

Proof: let $w = x_1^{a_1} \dots x_n^{a_n})$, let $w' = (y_1^{b_1} \dots y_m^{b_m})$ and let $w'' = (z_1^{c_1} \dots z_p^{c_p})$. The formula holds for the cases $p = 0, 1$. Now we show that the formula holds for any w'' by induction on p. This follows from the following equations.

$$\begin{aligned}
&\big[(x_1^{a_1} \dots x_n^{a_n}) \cdot (y_1^{b_1} \dots y_m^{b_m})\big] \cdot (z_1^{c_1} \dots z_p^{c_p}) \\
&= \big[(x_1^{a_1} \dots x_n^{a_n}) \cdot (y_1^{b_1} \dots y_m^{b_m})\big] \cdot \Big[(z_1^{c_1} \dots z_{p-1}^{c_p-1}) \cdot (z_p^{c_p})\Big] \\
&= \Big[\big[(x_1^{a_1} \dots x_n^{a_n}) \cdot (y_1^{b_1} \dots y_m^{b_m})\big] \cdot (z_1^{c_1} \dots z_{p-1}^{c_p-1})\Big] \cdot (z_p^{c_p}) \text{ (by the case } p = 1) \\
&= \Big[(x_1^{a_1} \dots x_n^{a_n}) \cdot \Big[(y_1^{b_1} \dots y_m^{b_m}) \cdot (z_1^{c_1} \dots z_{p-1}^{c_p-1})\Big]\Big] \cdot (z_p^{c_p}) \text{ (by induction)} \\
&= (x_1^{a_1} \dots x_n^{a_n}) \cdot \Big[\Big[(y_1^{b_1} \dots y_m^{b_m}) \cdot (z_1^{c_1} \dots z_{p-1}^{c_p-1})\Big] \cdot (z_p^{c_p})\Big] \text{ (by the case } p = 1) \\
&= (x_1^{a_1} \dots x_n^{a_n}) \cdot \Big[(y_1^{b_1} \dots y_m^{b_m}) \cdot \Big[(z_1^{c_1} \dots z_{p-1}^{c_p-1}) \cdot (z_p^{c_p})\Big]\Big] \text{ (by the case } p = 1) \\
&= (x_1^{a_1} \dots x_n^{a_n}) \cdot \Big[(y_1^{b_1} \dots y_m^{b_m}) \cdot (z_1^{c_1} \dots z_{p-1}^{c_p-1})\Big].
\end{aligned}$$

Theorem 2.6. *The monoid M^* is a free product of the monoids M_x*

Proof: We have shown that M^* is a monoid. The inclusion $M_x \to M^*$ is given by $a \to (x^a)$. Now it is straight forward to check that M^* satisfies the universal property of the coproducts.

3. Classifying Spaces of Simplicial Monoids

There are many constructions for the classifying spaces of a simplicial monoid. We recall two natural constructions which are used in this paper.

Let M be a simplicial monoid. Define a simplicial set $E(M)$ as follows:

$$E(M)_n = M_n \times M_{n-1} \times \ldots \times M_0$$

with face and degeneracy functions

$$d_j(x_n, \ldots, x_0) = d_j x_n, d_{j-1}x_{n-1}, \ldots, d_0 x_{n-j} \cdot x_{n-j-1}, x_{n-j-2}, \ldots, x_0)$$

and $s_j(x_n, \ldots, x_0) = (s_j x_n, s_{j-1}x_{n-1}, \ldots, s_0 x_{n-j}, x_{n-j}, \ldots, x_0)$ for $0 \leq j \leq n$ and $x_j \in M_j$.

The simplicial set $E(M)$ is a contractible simplicial set with a (right) M-action given by

$$(x_n, \ldots, x_0) \cdot y_n = (x_n y_n, x_{n-1}, \ldots, x_0)$$

for $(x_n, \ldots, x_0 \in E(M)_n$ and $y_n \in M_n$. An easy proof to show that $E(M)$ is a contractible simplicial set is to contract a simplicial map

$$F : C(E(M)) \to E(M),$$

which is an extension of the identity map $1_E(M)$, given by

$$f(0; (x_n, \ldots, x_0)) = (x_n, \ldots, x_0)$$

and

$$f(1; (x_{n-1}, \ldots, x_0)) = (1, x_{n-1}, \ldots, x_0)$$

for the nondegenerate elements in $C(E(M))_n$, where, for any simplicial set X, the cone $C(X)$ is defined by

$$C(X)_n = \{(q; x_{n-q}) \mid q \geq 0, x_{n-q} \in X_{n-q}\}$$

$$d_j(q; x_{n-q}) = \begin{cases} (q; d_{j-q}x_{n-q}) & \text{for } j \geq q \\ (q-1; x_{n-q}) & \text{for } j < q \end{cases}$$

$$\text{and } s_j(q; x_{n-q}) = \begin{cases} (q; s_{j-q}x_{n-q}) & \text{for } j \geq q \\ (q+1; x_{n-q}) & \text{for } j < q, \end{cases}$$

(see [5]; also [12]).

Now the standard bar-construction $B(M)$ of M is defined by

$$B(M) = E(M)/M,$$

the quotient simplicial set of $E(M)$ modulo the M-action. Another bar-construction $B^{\triangle}(M)$ is constructed as follows.

Let M be a simplicial monoid. Define a bisimplicial set $E(M)_{**}$ by

$$E(M)_{n,m} = E(M_n)_m$$

with face and degeneracy functions induced by M and $E(M_n)$ in the canonical way. The simplicial set $E^{\triangle}(M)$ is defined to the diagonal simplicial set of the bisimplicial set $E(M)_{*}*$. Notice that $E(M)_{n,*}$ is contractible for each $n \geq 0$. By a result of A. Bousfield and D. Kan ([3, Lemma 4.2, pp. 335]; also [2]), the diagonal simplicial set $E^{\triangle}(M)$ is also contractible. The (right) M-action on $E^{\triangle}(M)$ is given by

$$(x_n, \ldots, x_0) \cdot y_n = (x_n y_n, x_{n-1}, \ldots, x_0)$$

for $(x_n, \ldots, x_0) \in E^{\triangle}(M)_n = M_n \times \ldots \times M_n$ and $y_n \in M_n$. Now the bar-construction $B^{\triangle}(M)$is defined by

$$B^{\triangle}(M) = E^{\triangle}(M)/M,$$

the quotient simplicial set of $E^{\triangle}(M)$ modeulo the M-action.

The natural transformation $\chi : B^{\triangle} \to B$ is defined by

$$\chi(M) : B^{\triangle}(M) \to B(M)$$

$$\chi(x_{n-1}, \ldots, x_0) = (d_0 x_{n-1}, d_0^2 x_{n-2}, \ldots, d_0^n x_0)$$

for $(x_{n-1}, \ldots, x_0) \in B^{\triangle}(M)_n = M_n \times \ldots \times M_n$.

Proposition 3.1. *The simplicial map $\chi : B^{\triangle}(M) \to B(M)$ is a homotopy equivalence for each simplicial monoid M.*

This theorem can follow from a general theorem given in [14], [13] or [6]. We give an elementary proof as follows.

Proof: Notice that both $\mathbb{Z}(E^{\triangle}(M))$ and $\mathbb{Z}(E^{\triangle}(M))$ produce free resolutions over $\mathbb{Z}(M)$. This shows that the map

$$\chi : B^{\triangle}(M) \to B(M)$$

is a homology equivalence. By the standard excise, the canonical homomorphisms $i_{1}:\pi_0(M) \to \pi_1(B^{\triangle}(M))$ and $i_{2}:\pi_0(M) \to \pi_1(B^{\triangle}(M))$ satisfy the universal condition of the group completion. Furthermore, there is a commutative diagram

$$\begin{array}{ccc} & & \pi_1(B^{\triangle}(M)) \\ & \overset{i_1}{\nearrow} & \\ \pi_0(M) & & \downarrow \chi_* \\ & \underset{i_2}{\searrow} & \\ & & \pi_1(B(M)). \end{array}$$

Thus $\chi_* : \pi_1(B^{\triangle}(M)) \to \pi_1(B(M))$ is an isomorphism and so χ is a weak homotopy equivalence. This shows that χ is a homotopy equivalence which is the assertion.

Recall that, for any pointed simplicial set X, the (reduced) suspension ΣX is defined by $\Sigma X = C(X)/\left(X \cup C(*)\right)$, (rf: [5]; also [12]), where $*$ is the base-point of X. The nondegenerate elements in $\Sigma(X)_n$ are $(1; x_{n-1})$ for nondegenerate elements

$x_{n-1} \in x_{n-1}$. Let Mbe a simplicial monoid. The suspension map $\sigma : \Sigma M \to BM$ is defined by

$$\sigma(1; x_{n-1}) = (x_{n-1}, 1, \ldots, 1) \in B(M)_n = M_{n-1} \times M_{n-2} x \ldots \times M_0$$

for the nondegenerate elements $(1, x_{n-1})$ in ΣX_n.

Proposition 3.2. *(1). The map $\sigma_* : H_*(M) \to H_{*+1}(BM)$ is the homology suspension in the bar spectral sequence for $H_*(BM)$.*

(2). If M is a simplicial group, the map $|\sigma| : |\Sigma M| \to |BM|$ is the adjoint map of $|M| \xrightarrow{\simeq} \Omega|BM|$ up to homotopy.

Proof: (1). is obvious. Now suppose that M is a simplicial group. The twisting function $t : BM \to M, (x_{n-1}, x_{n-2}, \ldots, x_0) \to x_{n-1}$ induces a simplicial homomorphism $\tilde{t} : G(BM) \to M$, where $G(X)$ is Kan's G-construction [9] for a reduced simplicial set X, which is a homotopy equivalence. Notice that $G(\Sigma M) \cong F(M)$, where $F(X)$ is Milnor's F(K)-construction [10], and the composite

$$M \to G(\Sigma M) \xrightarrow{G\sigma} G(BM) \xrightarrow{\tilde{t}} M$$

is the identity map 1_M. The assertion folows.

4. Fibrewise Simplicial Objects

In this section, we introduce fibrewise simplicial monoid to give a categorical understanding of Milnor's F(K)-construction and Carlsson's construction. We start with an abstract construction.

Definition 4.1. *Let $\mathcal{C}$ be a category. A fibrewise simplicial object $\mathcal{A}$ over $\mathcal{C}$ is a diagram of $\mathcal{C}$ over a non-empty simplicial set S.*

More precisely, $\mathcal{A} = \{\mathcal{A}_x | x \in S_n\}_{n\geq 0}$ together with morphisms in $\mathcal{C}$

$$d_j : \mathcal{A}_x \to \mathcal{A}_{d_j x}$$

and

$$s_j : \mathcal{A}_x \to \mathcal{A}_{s_j x}$$

for $0 \leq j \leq n$ and $x \in S_n$ such that the following simplicial identities hold:

$$\begin{aligned} d_i d_j = &\ d_{j-1} d_i \text{ for } i < j \\ d_i s_j = &\ \begin{cases} s_{j-1} d_i \text{ for } i < j \\ \text{identity for } i = j, J+1 \\ s_j d_{i-1} \text{ for } i > j+1 \end{cases} \\ s_i s_j = &\ s_{j+1} s_i \text{ for } i \leq j \end{aligned}$$

Example 4.2. *Let $\mathcal{C}$ be a pointed category and let $\mathcal{S}$ be a simplicial set. Let $\mathcal{A}_x = *$, the point in $\mathcal{C}$, for each $x \in \mathcal{S}$. Then $0(S) = \{\mathcal{A}_x | x \in \mathcal{S}_n\}_{n\geq 0}$ together with canonical face and degeneracy morphisms is a fibrewise simplicial object over $\mathcal{C}$.*

Example 4.3. *Let $\mathcal{C}$ be a category and let X be a simplicial object over $\mathcal{C}$. Let $(X_n)_{pt} = X_n$. Then $X = \{(X_n)_{pt}\}_{n\geq 0}$ is a fibrewise simplicial object $\mathcal{C}$, where $\mathcal{S} = \{pt\}$. Thus the category of simplicial objects over $\mathcal{C}$ embeds into the category of fibrewise simplicial objects over $\mathcal{C}$ in this canonical way.*

The category of fibrewise simplicial objects over $\mathcal{C}$ has enough structure provided $\mathcal{C}$ has. It is straight forward to show that the category of fibrewise simplicial objects over $\mathcal{C}$ is complete (cocomplete, bicomplete) if $\mathcal{C}$ is complete (cocomplete, bicomplete).

Definition 4.4. *Let T be a functor from the category $\mathcal{C}$ to the category of pointed simplicial sets.*

Let $\mathcal{A}$ be a fibrewise simplicial object over $\mathcal{C}$. The bisimplicial set $\check{T}(\mathcal{A})_{**}$ is defined by

$$\check{T}(\mathcal{A})_{n,m} = V_{X\in S_n} T(\mathcal{A}_x)_m$$

where V is the wedge of pointed sets. The face and degeneracy functions are induced by the ones in $\mathcal{A}$ in the canonical way.

The F-construction on the category of fibrewise simplicial objects over $\mathcal{C}$ is defined as follows.

Definition 4.5. *Let $\mathcal{C}$ be a category with coproducts and let $\mathcal{A}$ be a fibrewise simplicial object over $\mathcal{C}$.*

The fibrewise simplicial object over $\mathcal{C}$, $F(\mathcal{A})$, is defined by

$$F(\mathcal{A})_n = \coprod_{x\in\mathcal{S}_n} \mathcal{A}_x$$

where $\coprod$ is the coproduct in the category $\mathcal{C}$. The face and degeneracy functions are inducedby the ones in $\mathcal{A}$ in the canonical way.

An important property of the construction $F(\mathcal{A})$ is that the Seifert-van Kampen theorem holds.

Proposition 4.6. *Suppose that $\mathcal{C}$ is a cocomplete category. Then the construction $F(\mathcal{A})$ preserves the colimits.*

Proof: Let $f.s.\mathcal{C}$ denote by the category of the fibrewise simplicial objects over $\mathcal{C}$. The objects in $f.s.\mathcal{C}$ are fibrewise simplicial objects over $\mathcal{C}$. The morphisms in $f.s.\mathcal{C}$ are the morphisms of the diagrams over $\mathcal{C}$. Let $E : s.\mathcal{C} \to f.s.\mathcal{C}$ denote by the embedding functor given in Example 4.3, where $s.\mathcal{C}$ is the category of simplicial objects over $\mathcal{C}$. Let $\mathcal{A} = \{\mathcal{A}_x | x \in S_n\}_{n\geq 0}$ be an object in $f.s.\mathcal{C}$. The morphisms $\mathcal{A}_x \to F(\mathcal{A})_n = \coprod_{x\in\mathcal{S}_n} \mathcal{A}_x$ gives a natural transformation $\zeta : \mathcal{A} \to EF(\mathcal{A})$. Now let $\{\mathcal{A}_\alpha\}$ be a diagram over $f.s.\mathcal{C}$ and let C be an object over $s.\mathcal{C}$ with a morphism $\{F(\mathcal{A}_a lpha)\} \to C$ of diagrams over $s.\mathcal{C}$. The composite

$$\{\mathcal{A}_\alpha\} \to \{EF(\mathcal{A}_\alpha)\} \to EC$$

is a morphism of the diagrams over $f.s.\mathcal{C}$. Thus there is a unique extension morphism colim $\mathcal{A}_\alpha \to EC$ and so there is an extension morphism $F($ colim $\mathcal{A}_\alpha) \to FEC = C$ which is unique because $F($ colim $\mathcal{A}_\alpha)_n$ is a coproduct of the objects of dimension n in the diagram colim $\mathcal{A}_\alpha$. The assertion follows.

The geometry of $F(\mathcal{A})$is given as follows:

Theorem 4.7. *Let $\mathcal{C}$ be a pointed category with coproducts and let $T : \mathcal{C} \to s.sets$ be a functor, where $s.sets$ is the category of pointed simplicial sets. Suppose that Tpreserves the corproducts up to homotopy. Then*

$$\check{T}(\mathcal{A})_{**} \simeq \check{T}(F(\mathcal{A}))_{**}$$

as bisimplicial sets for any fibrewise simplicial object $\mathcal{A}$ over $\mathcal{C}$.

Proof: The morphism $\mathcal{A} \to EF(\mathcal{A})$, where $E$is the embedding functor, gives a simplicial map $g : \check{T}(\mathcal{A})_{**} \to \check{T}(EF(\mathcal{A}))_{**} = \check{T}(F(\mathcal{A}))_{**}$. Notice that $\check{T}(\mathcal{A})_{n,*} = V_{x\in\mathcal{S}_n} T(\mathcal{A}_x)_*$ and $\check{T}(F(\mathcal{A}))_{n,*} = T(\coprod_{x\in\mathcal{S}_n} \mathcal{A}_x)_*$. The maps $g : \check{T}(\mathcal{A})_{n,*} \to \check{T}(F(\mathcal{A}))_{n,*}$ are homotopy equivalences for each n. By the Bousfied-Kan Theorem [3, Lemma 4.2, pp.335], the map $g : \check{T}(\mathcal{A})_{**} \to \check{T}(F(\mathcal{A}))_{**}$ is a homotopy equivalence which is the assertion.

Now we come back to study fibrewise simplicial monoids, i.e., the fibrewise simplicial objects over the category of monoids.

Definition 4.8. *Let M be a simplicial monoid and let $\mathcal{S}$ be a pointed simplicial set with a base point x_0.*

The simplicial monoid $F^M(S)$is defined by

$$F^M(S)_n = \coprod_{x\in S_n}^{*} (M_n)_x / \approx,$$

where $(M_n)_x$ is a copy of M_n, where $\coprod^*$ is the free product of monoids and where the product equivalent relation $\approx$ is generated by

$$g_{x_o} \approx 1$$

for the elements g in M_n with base point index x_0. The face and degeneracy functions are induced by the homomorphism

$$d_j : (M_n)_x \cong M_n \to M_{n-1} \cong (M_{n-1})_{d_j x}$$

and

$$s_j : (M_n)_x \cong M_n \to M_{n+1} \cong (M_{n+1})_{s_j x}$$

for $0 \le j \le n$.

Theorem 4.9. *There is a homotopy equivalence*

$$BM \wedge S \to BF^M(S)$$

for any pointed simplicial set S and any simplicial monoid M.

Proof: The second coordinate projection $p : M \times S/M \times x_0 \to S$, where x_0 is the base point, induces a fibrewise simplicial monoid $\mathcal{A}$ with $\mathcal{A} = \{(M_n)_x, (1)_{x_0} | x \in S_n - \{x_0\}\}_{n\geq 0}$. By the definition of F^M construction, $F^M(S) \cong F(\mathcal{A})$. Notice that the bar-construction B preserves the coproducts up to homotopy [6]. By Theorem 4.7, there is a homotopy equivalence $\check{B}(\mathcal{A})_{**} \xrightarrow{\simeq} \check{B}F(\mathcal{A})_{**}$. Notice that

$$\check{B}(\mathcal{A})_{n,*} = (V_{x\in S_n} B((M_n)_x)_*)/B\left((M_n)_{x_0}\right)_* \cong B(M_n)_* \wedge S_n$$

for each $N \geq 0$. Thus there is an isomorphism

$$\check{B}(\mathcal{A})_{**} \cong \check{B}(M)_{**} \wedge \tilde{S}_{**},$$

where $\tilde{S}_{**}$ is a bisimplicial set defined by $\tilde{S}_{nm} = S_n$. By taking the diagonal simplicial set, we have a homotopy equivalence

$$B^{\triangle}(M) \wedge S \to B^{\triangle}(F^M(S)).$$

The assertion follows by Proposition 3.1.

Remark 4.10. *(1). If M is the monoid of non-negative integers, then $F^M(S)$ is exact James' construction $J(S)$* [7] *or Milnor's construction $F^+(S)$* [10].

(2). If M is the abelian group $\mathbb{Z}$, then $F^M(S)$ produces a (reduced) free simplicial group generated by $\mathcal{S}$ which is exact Milnor's F(K)-construction [10].

(3). In the case for a group G, the $F^G(S)$ construction here is a special case of Carlsson's construction [4].

Example 4.11. *Let M be a simplicial monoid and let $\mathcal{S}$ be a pointed simplicial set with a right pointed M-action. The simplicial monoid $C^+_M(S)$ is defined by*

$$C^+_M(S) = F^+(S \times M)/\approx$$

where $F^+(X)$ is the reduced free simplicial monoid generated by a pointed simplicial set, i.e. Milnor's F^+-construction, and where the product equivalent relation $\approx$ is generated by requiring the relations

$$[x, g][x, gh] \approx [x, gh]$$

for $x \in S$ and $g, h \in M$. The associate fibrewise simplicial monoid $\mathcal{C}^+_M(S)$ for $C^+_M(S)$ is given by

$$\mathcal{C}^+_M(S) = \left\{C^+_M\left(p^{-1}(a)\right), (1)_{x_0} \mid a \in S_n/M_n - \{p(x_0)\}\right\},$$

where $p : S \to S/M$ is the quotient map and x_0 is the base point. If G is a simplicial group, then the simplicial group $C_G(S)$ is defined to be the group completion of the simplicial monoid $C^+_G(S)$.

The Carlsson Theorem is as follows.

Theorem 4.12 (Carlsson)**.** *Let G be a simplicial group and let $\mathcal{S}$ be a pointed simplicial G-set.*

Then the classifying space $BC_G(S)$ is homotopy equivalent to the homotopy cofibre of the map $S \to EG \times_G S/EG \times_G x_0$, where x_0 is the base point [4, Theorem 9].

We should point out that the group completion UM of a simplicial monoid M, i.e., $U(M)_n = U(M_n)$, is not a homotopy group completion ΩBM of M in general even if M is a connected simplicial monoid.

Example 4.13. *Let M be discrete monoid such that $BM \simeq S^2$. By a result of Fiedorowicz* [6]*, any connected CW-complex has a homotopy type of the classifying space of a discrete monoid. Notice that $\pi_1(BM) = 0$. The group completion $UM = \{1\}$. Now, for any $n > 0$, $F^M(S^n) \simeq \Omega S^{n+2}$. But $UF^M(S^n) \cong F^{U(M)}(S^n) = \{1\}$ for each n.*

For any fibrewise simplicial monoid $\mathcal{A}$, there is a simplicial set $\bar{\mathcal{A}}$ defined by $\bar{\mathcal{A}}_n = U_{x\in S_n}\mathcal{A}_x$, the disjoint union, together with a simplicial surjection $p_{\mathcal{A}} : \bar{\mathcal{A}} \to S$, $g_x \to x$ for $g_x \in \mathcal{A}_x$. The simplicial set $\bar{\mathcal{A}}$ has a property that the face and degeneracy functions, in $\bar{\mathcal{A}}$, restricted to each fibre $p_{\mathcal{A}}^{-1}(x)$ are homomorphiss. Conversely, given a sipmlicial surjection $f : X \to S$ such that each fibre $f^{-1}(x)$ is a moniod and the face and degeneracy functions, in X, restricted to each fibre are homomorphisms of monoids. Then the diagram $\mathcal{A} = \{F^{-1}(x) \mid x \in S_n\}_{n\geq 0}$ is a fibrewise simplicial monoid such that $\bar{\mathcal{A}} = X$. This gives a geometric description.

Proposition 4.14. *Let $\mathcal{A}$ be a fibrewise simplicial monoid. Then the geometric realization $|p_{\mathcal{A}}| : |\mathcal{A}| \to |S|$ is a fibre-wise topologial monoid under the weak topology (rf:* [8]*).*

Proof: It is easy to check that $\mathcal{A}$ is fibrewise simplicial monoid if and only if the simplicial surjection $p_{\mathcal{A}} : \bar{\mathcal{A}} \to S$ has a fibrewise multiplication, i.e., there exists a commutative diagram of simplicial sets

$$\begin{array}{ccc} \bar{\mathcal{A}} \triangle \bar{\mathcal{A}} & \xrightarrow{m} & \bar{\mathcal{A}} \\ \downarrow p_{\mathcal{A}} \triangle p_{\mathcal{A}} & & \downarrow p_{\mathcal{A}} \\ \triangle(S) & \xrightarrow{\cong} & S, \end{array}$$

where $\triangle(S)$ is the diagronal subsimplicial set of $S \times S$ and $\bar{\mathcal{A}} \triangle \bar{\mathcal{A}} = (p_{\mathcal{A}} \times p_{\mathcal{A}})^{-1}(\triangle(S)) \subseteq \bar{\mathcal{A}} \times \bar{\mathcal{A}}$, such that the fibrewise multiplication $m : \bar{\mathcal{A}} \triangle \bar{\mathcal{A}} \to \bar{\mathcal{A}}$ is fibrewise associative with a fibrewise unit element. The assertion follows by applying the geometric realization [11].

Remark 4.15. *(a). Any nontrivial principal G-bundle is not a fibrewise simplicial monoid since any fibrewise simplicial monoid has a cross-section. (b). The construct $F^M(X)$ is the tensor product $X \otimes M$.*

5. The Word Length Filtration of $F^M(X)$

In this section, we study some natural filtrations in the constructions $F(\mathcal{A})$ and $F^M(X)$.

Definition 5.1. *Let $\mathcal{A}$ be a fibrewise simplicial monoid. The word length filtration $\{F_r(\mathcal{A})\}_{r\geq 0}$ on the simplicial monoid $F(\mathcal{A})$ is defined by*

$$F_r(\mathcal{A})_n = \{\omega \in F(\mathcal{A})_n = \amalg^*_{x\in S_n}\mathcal{A}_x \mid \ell(\omega)\leq r\},$$

where $\ell(\omega)$ is the length of the reduced word ω in the free product $\amalg^*_{x\in S_n}\mathcal{A}_x$. The family of the subsimplicial sets, $\{F_r(\mathcal{A})\}_{r\geq 0}$, gives a product filtration of the simplicial monoid $F(\mathcal{A})$ starting with $F_0(\mathcal{A}) = *$ and $F_1(\mathcal{A}) = \mathcal{A}/S$, where $p_{\mathcal{A}}: \bar{\mathcal{A}} \to S$ is the associate simplicial surjection of $\mathcal{A}$ and $\mathcal{S}$ is identified with its zero section.

Proposition 5.2. *Let $\mathcal{A}$ be a fibrewise simplicial monoid with $p_{\mathcal{A}}: \bar{\mathcal{A}} \to S$. Then there is an isomorphism of simplicial sets*

$$F_r(\mathcal{A})/F_{r-1}(\mathcal{A}) \cong (\bar{\mathcal{A}}/S)^{(r)}/D_r$$

where $X^{(r)}$ is the r-fold self smash product of X and where the subsimplicial set D_r of $(\bar{\mathcal{A}}/S)^{(r)}$ is given by

$$D_r = \{a_1\wedge\ldots\wedge a_r \in (\bar{\mathcal{A}}/S)^{(r)} \mid \text{ the indices } p_{\mathcal{A}}(a_i) = p_{\mathcal{A}}(a_{i+1}) \text{ for some } i\}.$$

Proof: Consider the projection map $\varphi: \bar{\mathcal{A}}^r \to F_r(\mathcal{A})$ defined by $\varphi(a_1,\ldots a_r) = a_1\ldots a_r$ for $(a_1,\ldots,a_r)\in\bar{\mathcal{A}}^r = \bar{\mathcal{A}}\times\ldots\times\bar{\mathcal{A}}$. Then the map φ satisfies the property:

$$\varphi(a_1,\ldots,a_r)\in F_{r-1}(\mathcal{A})$$

if $a_i = 1$ for some i or $p_{\mathcal{A}}(a_i) = p_{\mathcal{A}}(a_{i+1})$ for some i. Thus there exists a simplicial map $\bar{\varphi}: (\bar{\mathcal{A}}/S)^{(r)}/D_r \to F_r(\mathcal{A})/F_{r-1}(\mathcal{A})$ such that the diagram

$$\begin{array}{ccc} \bar{\mathcal{A}}^r & \xrightarrow{\varphi} & F_r(\mathcal{A}) \\ \downarrow & & \downarrow \\ (\bar{\mathcal{A}}/S)^{(r)}/D_r & \xrightarrow{\varphi} & F_r(\mathcal{A})/F_{r-1}(\mathcal{A}) \end{array}$$

commutes. By the Reduce Word Theorem (Theorem 2.2), the map

$$\bar{\varphi}: (\bar{\mathcal{A}}/S)^{(r)}/D_r \to F_r(\mathcal{A})/F_{r-1}(\mathcal{A})$$

is an isomorphism, which is the assertion.

Corollary 5.3. *Let X be a pointed simplicial set and let M be a simplicial monoid. Let $\{F^M_r(X)\}_{r\geq 0}$ be the word length filtration of $F^M(X)$. Then there is an isomorphism of simplicial sets*

$$F^M_r(X)/F^M_{r-1}(X) \cong M^{(r)}\wedge (X^{(r)}/\bar{\triangle}_r),$$

where $\bar{\triangle}_r = \{x_1\wedge\ldots\wedge x_r \in X^{(r)} \mid x_i = x_{i+1} \text{ for some } i\ \}$.

Proof: Let $\bar{\mathcal{A}} = M\times X/M\times *$ with the second coordinate projection $p: \bar{\mathcal{A}} \to X$. Then $F(\mathcal{A}) = F^M(X)$. It is easy to check that

$$(\bar{\mathcal{A}}/X)^{(r)}/D_r \cong M^{(r)}\wedge (X^{(r)}/\bar{\triangle}_r).$$

Corollary 5.4. *Let $F=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let X be a pointed CW-complex. Then there exists a simplicial group $F^G(S_*(X))$ such that $|F^G(S_*(X))|$ is homotopy equivalent to $\Omega(FP^\infty\wedge X)$ together with a product filtration $\{F^G_r(S_*(X))\}_{r\geq 0}$ such that $\{F^G_0(S_*(X))=\{1\}$ and*

$$|F^G_r(S_*(X))/F^G_{r-1}(S_*(X))|\simeq\Sigma^{dr}X^{(r)}/\bar{\triangle}_r,$$

where $d=\dim_{\mathbb{R}}F-1$.

By applying the word length filtration, we have

Theorem 5.5. *Let X be a pointed space and let Y be a path connected pointed space. Then, for any homology theory H_*, there exists a spectral sequence $\{E^n_{**}(Y,X)\}_{n\geq 1}$ which has the following properties.*

(1). $\{E^n_{**}(Y,X)\}_{n\geq 1}$ is natural for both X and Y.

(2). $\{E^n_{**}(Y,X)\}_{n\geq 1}$ is convergent to $H_*\Omega(Y\wedge X)$.

(3). $E_pq^1(Y,X)=0$ for $p<0$, $E^1_{0,*}(Y,X)=H_*(pt)$, and $E_pq^1(Y,X)\cong\bar{H}_q\left((\Omega Y)^{(p)}\wedge(X^{(p)}/\bar{\triangle}_p)\right)$ for $p>0$.

(4). If H_* is the ordinary homology theory with coefficients in a ring, then

$$\{E^n_{**}(Y,X)\}_{n\geq 1}$$

is a Hopf algebra spectral sequence.

(5). The homomorphism $E^1_{1,*}(Y,X)\cong\bar{H}_*(\Omega Y\wedge X)\to E^\infty_{1,*}(Y,X)$ is induced, up to isomorphism by the adjoint map of the map $\sigma\wedge X:\Sigma\Omega Y\wedge X\to Y\wedge X$, where $\sigma:\Sigma\Omega Y\to Y$ is the suspension map.

Proof: We may assume that X is a pointed simplicial set and Y is a reduced pointed simplicial set. Let the simplicial group $G=G(Y)$, the Kan G-construction of Y. Then $F^G(X)\simeq\Omega(Y\wedge X)$. The assertions (1) to (4) follow directly fromthe word length filtration of $F^G(X)$. Consider the commutative diagram

$$\begin{array}{llll}
\Sigma G\wedge X & \xrightarrow{\alpha\ \simeq} \Sigma(G\wedge X) & \xrightarrow{\Sigma\theta} & \Sigma F^G(X)\\
\downarrow\sigma\wedge X & & & \downarrow\sigma\\
BG\wedge X & \xrightarrow{\beta} & & BF^G(X)\\
\uparrow X\wedge X & & & \uparrow X\\
B^{\triangle}G\wedge X & \xrightarrow{\beta^{\triangle}} & & B^{\triangle}F^G(X),
\end{array}$$

where

$$\alpha\left((1,g_{n-1}),x_n\right)=((1,g_{n-1}),d_0x_n)$$

for $g_{n-1}\in G_{n-1}$ and $x_n\in X_n$,

$$\theta(g,x)=x^g,\ \text{the element } g \text{ in } G \text{ with index } x,$$

for $g\in G$ and $x\in X$

$$\sigma(1,g_{n-1})=(g_{n-1},1,\dots,1)$$

for $g_{n-1}\in G_{n-1}$ or $F^G(X)_{n-1}$,

$$\beta\left((g_{n-1},\ldots,g_0),x_n\right)=\left((d_0x_n)^g_{n-1},(d_0^2x_n)^{g_{n-2}},\ldots,(d_0^nx_n)^{g_o}\right)$$

for $g_i\in G_i$ and $x_n\in X_n$,

$$\beta^{\triangle}\left((\alpha_{n-1},\ldots,\alpha_0),x_n\right)=(x_n^{\alpha_{n-1}},x_n^{\alpha_{n-2}}\ldots x_n^{\alpha_0})$$

for $\alpha_j\in G_n$ and $x_n\in X_n$, and X is given in Proposition 3.1. Notice that the map X is a homotopy equivalence by Proposition 3.1 and the map $\beta^{\triangle}$ is a homotopy equivalence by Theorem 4.7. By Proposition 3.2, the natural map σ is the suspension. Thus the inclusion map $F^G_1(X)\to F^G(X)$ is the adjoint of the map $\sigma\wedge X:\Sigma\Omega Y\wedge X\to Y$, up to homotopy. The assertion follows.

The following proposition gives the spaces $X^{(r)}/\bar{\triangle}_r$ inductively by using cofibre sequences.

Proposition 5.6. *The spaces $X^{(r)}/\bar{\triangle}_r$ are built inductively as follows:*

$$X^{(1)}/\bar{\triangle}_1\cong X$$

and $X^{(r+1}/\bar{\triangle}_{r+1}$ is the cofibre of the map

$$f_r:X^{(r)}/\bar{\triangle}_r\to(X^{(r)}/\bar{\triangle}_r)\wedge X,$$

where $f_r(x_1,\ldots,x_r)=(x_1,\ldots,x_{r-1},x_r,x_r)$, the diagonal map on the last coordinate.

Proof: The quotient map $(X^{(r)}/\bar{\triangle}_r)\wedge X\to X^{(r+1)}/\bar{\triangle}_{r+1}$ induces a map

$$\left((X^{(r)}/\bar{\triangle}_r)\wedge X\right)/f_r(X^{(r)}/\bar{\triangle}_r)\to X^{(r+1)}/\bar{\triangle}_{r+1}.$$

Now the resulting map is an isomorphism of simplicial sets. The assertion follows.

Proposition 5.7. *Suppose that the space $X=\Sigma Y$, the suspension of a space Y. Then the maps*

$$f_r:X^{(r)}/\bar{\triangle}_r\to X^{(r)}/\bar{\triangle}_r\wedge X$$

are null for each r.

Proof: Notice that $X=\Sigma Y\ S^1\wedge Y$. Define a map

$$\varphi:S^1\wedge Y\wedge I\to(S^1\wedge Y)\wedge(S^1\wedge Y)$$

by setting

$$\varphi(s,y,t)=((s,y),((1-t)s,y))$$

for $0\le s\le 1$, and $y\in Y$. Now define a map

$$\varphi:(X^{(r)}/\bar{\triangle}_r)\wedge I\to(X^{(r)}/\bar{\triangle}_r)\wedge X$$

by setting

$$\varphi_r(x_1,x_2,\ldots,x_{r-1},x_r,t)=(x_1,x_2,\ldots,x_{r-1},\varphi(x_r,t))$$

for $0\leq t\leq 1$ and $x_j\in X$. Notice that $\varphi(x,0)=(x,x)$. The map $\varphi_r\colon (X^{(r)}/\bar{\triangle}_r)\wedge I\to (X^{(r)}/\bar{\triangle}_r)\wedge X$ is an extension of the map $f_r\colon X^{(r)}/\bar{\triangle}_r\to (X^{(r)}/\bar{\triangle}_r)\wedge X$. The assertion follows.

We will show that the spectral sequence induced by the word length filtration of $F^M(X)$ collapses at E^1-terms if the simplicial monoid M is homotopy equivalent to $K(\mathbb{Z}/2,0)$, S^1 or S^3. Further study on the differentials in the word length spectral sequences will be given in [17].

Theorem 5.8. *Let $F=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let X be a pointed space. Suppose that H_* is a multiplicative homology theory such that (1) both $\bar{H}_*(FP^\infty)$ and $\bar{H}_*(FP^\infty_2)$ are free $H_*(pt)$-modules, where $FP^\infty_2=FP^\infty/FP^1$; and (2) the inclusion of the bottom cell $S^d\to FP^\infty$ induces a monomorphism in homology. Then there is a product filtration $\{F_rH_*\Omega(FP^\infty\wedge X)\}_{r\geq 0}$ of $H_*\Omega(FP^\infty\wedge X)$ such that $F_0H_*\Omega(FP^\infty\wedge X)=H_*(pt)$ and*

$$F_rH_*\Omega(FP^\infty\wedge X)/F_{r-1}H_*\Omega(FP^\infty\wedge X)\cong\Sigma^{(d-1)r}\bar{H}_*(X^{(r)}/\bar{\triangle}_r),$$

where $d=\dim_{\mathbb{R}}F$. Furthermore, this filtration is natural for X.

The examples of the homology theory satisfied the conditions (1) and (2) in the Theorem can be found out in any standard text book. The proof of the theorem will be given in Section 7.

6. Natural Extension Maps in the Word Length Filtration of $F^M(X)$

In this section, we construct certain natural maps by using F^M-construction. These maps are similar to James-Hopf maps. But they are not the extension of the James-Hopf maps. A direct application of these maps is to give a proof of Theorem 5.8. The notation x^g in this section means an element g in a monoid (group) with an index x. This new notation helps us to take some calculations in this section.

Let M and N be simplicial monoids and let X and Y be pointed simplicial sets. We define a composition product

$$\odot\colon F^M(X)\times F^N(Y)\to F^N\circ F^M(X\wedge Y)=F^N\left(F^M(X\wedge Y)\right)$$

by setting

$$\begin{aligned}&(x_1^{g_1}\ldots x_s^{g_s})\odot(y_1^{h_1}\ldots y_t^{h_t})\\&=[(x_1\wedge y_1)^{g_1}\ldots(x_s\wedge y_1)^{g_s}]^{h_1}\cdot[(x_1\wedge y_2)^{g_1}\ldots(x_s\wedge y_2)^{g_s}]^{h_2}\cdot[(x_1\wedge y_t)^{g_1}\ldots(x_s\wedge y_t)^{g_t}]^{h_t}\end{aligned}$$

for words $(x_1^{g_1}\ldots x_s^{g_s})\in F^M(X)_n$ and $(y_1^{h_1}\ldots y_t^{h_t})\in F^N(Y)_n$. It is a routine work to check the following statements.

(1). the functions $\odot\colon F^M(X)_n\times F^N(Y)_n\to F^N\circ F^M(X\wedge Y)_n$ are well defined by Proposition 2.2.

(2). the functions $\odot\colon F^M(X)_n\times F^N(Y)_n\to F^N\circ F^M(X\wedge Y)_n$ induce a simplicial map $\odot\colon F^M(X)\times F^N(Y)\to F^N\circ F^M(X\wedge Y)$.

Notice the composition product $\odot: F^M(X)\times F^N(Y)\to F^N\circ F^M(X\wedge Y)$ induces a (reduced) composition product $\odot: F^M(X)\wedge F^N(Y)\to F^N\circ F^M(X\wedge Y)$.

Now let M be a simplicial monoid and let X and Y be pointed simplicial set. The composition product

$$\odot: F^M(X)\times F^N(Y)\to F^M(X\wedge Y)$$

is defined by setting

$$(x_1^{g_1}\dots x_s^{g_s})\odot y=(x_1\wedge y)^{g_1}\dots(x_s\wedge y)^{g_s}$$

for word $(x_1^{g_1}\dots x_s^{g_s})\in F^M(X)$ and $y\in Y$. Notice that the composition product $\odot: F^M(X)\times Y\to F^M(X\wedge Y)$ induces a (reduced) composition product $\odot: F^M(X)\wedge Y\to F^M(X\wedge Y)$. Inductively, we define the composition product

$$\odot:(F^M)^n(X)\wedge Y\to(F^M)^n(X\wedge Y)$$

to be the composite

$$\begin{aligned}(F^M)^n(X)\wedge Y=F^M\left((F^M)^{n-1}(X)A\right)\wedge Y&\xrightarrow{\odot}F^M\left((F^M)^{n-1}(X)\wedge Y\right)\\&\xrightarrow{F^M(\odot)}F^M\left((F^M)^{n-1}(X\wedge Y)\right)=(F^M)^n(X\wedge Y),\end{aligned}$$

where $(F^M)^n=F^M\circ\dots\circ F^M$.

Definition 6.1. *Let M be a simplicial monoid and let X be a pointed simplicial set. The composition product*

$$\odot:(F^M)^n(X)^{(n)}/\bar{\triangle}_n)\times F^M(X)\to(F^M)^{n+1}(X^{(n+1)}/\bar{\triangle}_{n+1}),$$

for $n\geq 1$, is defined to be the composite

$$\begin{aligned}&(F^M)^n(X^{(n)}/\bar{\triangle}_n)\times F^M(X)\xrightarrow{\odot}F^M\left((F^M)^n(X^{(n)}/\bar{\triangle}_n)\wedge X\right)\\&\xrightarrow{F^M(\odot)}F^M\left((F^M)^n(X^{(n)}/\bar{\triangle}_n\wedge X)\right)=(F^M)^{n+1}\left((X^{(n)}/\bar{\triangle}_n)\wedge X\right)\\&\xrightarrow{(F^M)^{n+1}(p)}(F^M)^{n+1}(X^{n+1}/\bar{\triangle}_{n+1}),\end{aligned}$$

where $X^{(n)}$ is the n-fold self smash product of X, where $\bar{\triangle}_1=*$ and $\bar{\triangle}_n=\{x_1\wedge\dots\wedge x_n)\in X^{(n)}\mid x_i=x_{i+1}$ for some $i\}$, and where $p:(X^{(n)}/\bar{\triangle}_n)\wedge X\to X^{(n+1)}/\bar{\triangle}_{n+1}$ is the pinch map.

Definition 6.2. *Let M be a simplicial monoid and let X be a pointed simplicial set. The simplicial maps $h_n: F^M(X)\to(F^M)^n(X^{(n)}/\bar{\triangle}_n)$ are defined inductively as follows. The map $h_1: F^M(X)\to F^M(X/\bar{\triangle}_1)=F^M(X)$ is the identity map and the map*

$$h_n: F^M(X)\to(F^M)^n(X^{(n)}/\bar{\triangle}_n)$$

is defined by the equations

$$h_n(x_1^{g_1}\dots x_s^{g_s})=\begin{cases}1 & x<n\\ (\dots(x_1\wedge\dots\wedge x_n)^{g_1})^{g_2})\dots)^{g_n} & s=n\\ h_n(x_1^{g_1}\dots x_{s-1}^{g_{s-1}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-1}^{g_{s-1}})\odot x_s]^{g_s} & s>n\end{cases}$$

for each word $x_1^{g_1}\dots x_s^{g_s}\in F^M(X)$.

Example 6.3. *The map* $h_2: F^M(X)\to F^M\cdot F^M(X^{(2)}/\bar{\triangle}_2)$ *is given by*

$$h_2(x_1^{g_1}\dots x_s^{g_s})=[(x_1\wedge x_2)^{g_1}]^{g_2}\cdot[(x_1\wedge x_3)^{g_1}\cdot(x_2\wedge x_3)^{g_2}]^{g_3}\dots[(x_1\wedge x_s)^{g_1}\dots x_{s-1}\wedge x_s)^{g_{s-1}}]^{g_s}$$

for each word $(x_1^{g_1}\dots x_s^{g_s})\in F^M(X)$.

Lemma 6.4. *The simplicial maps* $h_n: F^M(X)\to(F^M)^n(X^{(n)}/\bar{\triangle}_n)$ *are well defined.*

Proof: The proof is given by induction on n. The identity map h_1 is already a simplicial map. By checking the relations (1) and (2) in Notations 2.1, it is easy to show that the functions $h_2: F^M(X)_q\to F^M\circ F^M(X^{(2)}/\bar{\triangle}_2)_q$ preserves the relations (1) and (2) for each $q\geq 0$. Thus the functions $h_2: F^M(X)_q\to F^M\circ F^M(X^{(2)}/\bar{\triangle}_2)_q$ are well-defined. By the naturality, these functions preserve the simplicial structure and therefore the simplicial map $h_2: F^M(X)\to F^M\circ F^M(X^{(2)}/\bar{\triangle}_2)$ is well-defined. Now suppose that $h_k: F^M(X)\to(F^M)^k(X^{(k)}/\bar{\triangle}_k)$ is well-defined for $k<n$ with $n>2$. Consider $h_n: F^M(X)\to(F^M)^n(X^{(n)}/\bar{\triangle}_n)$. By the naturality, it suffices to show that

$$h_n: F^M(X)_q\to(F^M)^n(X^{(n)}/\bar{\triangle}_n)_q$$

preserves the relations (1) and (2) in Notations 2.1 for each $z\geq 0$.

It is easy to check that the function h_n preserves the relation (1). To show that the function h_n preserves the relation (2). We start with a second induction on the length s of the word $(x_1^{g_1}\dots x_s^{g_s})\in F^M(X)_q$. If $s\leq n$, it is obvious that the function h_n preserves the relation (2). Suppose that the function h_n preserves the relation (2) for the words in $F^M(X)_q$ with length $<s$ and $s>n$. Let $(x_1^{g_1}\dots x_s^{g_s})$ be a word in $F^M(X)_q$ such that $x_i=x_{i+1}$ for some i. If $i\leq s-2$, then

$$\begin{aligned}h_n(x_1^{g_1}\dots x_s^{g_s}) &= h_n(x_1^{g_1}\dots x_{s-1}^{g_{s-1}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-1}^{g_{s-1}})\odot x_s]^{g_s}\\ &= h_n(x_1^{g_1}\dots x_{i-1}^{g_{i-1}}x_i^{g_ig_{i_1}}x_{i+2}^{g_{i+2}}\dots x_s^{g_s})\end{aligned}$$

by induction.

If $i = s-1$, then

$$\begin{aligned}
h_n(x_1^{g_1}\dots x_s^{g_s}) &= h_n(x_1^{g_1}\dots x_{s-1}^{g_{s-1}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-1}^{g_{s-1}})\odot x_s]^{g_s}\\
&= h_n(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]^{g_{s-1}}\cdot\\
&\quad \big[[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot[h_{n-2}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]^{g_{s-1}}\odot x_s\big]^{g_s}\\
&= h_n(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]^{g_{s-1}}\\
&\quad \big[[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_s]\cdot\big[[h_{n-2}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]^{g_{s-1}}\odot x_s\big]\big]\\
&= h_n(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]^{g_{s-1}}\\
&\quad \Big[[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_s]\cdot\big[[h_{n-2}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]\odot x_s\big]^{g}_{s-1}\Big]^{g_s}\\
&= h_n(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]^{g_{s-1}}\\
&\quad \big[[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_s]\cdot[h_{n-2}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot(x_{s-1}\wedge x_s)]^{g_{s-1}}\big]^{g_s}
\end{aligned}$$

Notice that $x_{s-1} = x_s$. We need the following lemma.

Lemma 6.5. *Let M be a simplicial monoid and let X be a pointed simplicial set. Let $\odot\colon (F^M)^n(X^{(n)}/\bar{\triangle}_n)\wedge X^{(2)} \to (F^M)^n(X^{(n+2)}/\bar{\triangle}_{n+2})$ defined by the composite*

$$(F^M)^n(X^{(n)}/\bar{\triangle}_n)\wedge X^{(2)} \xrightarrow{\odot} (F^M)^n\left((X^{(n)}/\bar{\triangle}_n)\wedge X^{(2)}\right) \xrightarrow{(F^M)^n(p)} (F^M)^n(X^{(n+2)}/\bar{\triangle}_{n+2}),$$

where $p\colon (X^{(n)}/\bar{\triangle}_n)\wedge X^{(2)} \to X^{(n+2)}/\bar{\triangle}_{n+2}$ is the pinch map. Then the element

$$\alpha\odot(x\wedge x) = 1$$

for any element α in $(F^M)^n(X^{(n)}/\bar{\triangle}_n)$ and $x\in X$.

Continuation of the proof of Lemma 6.4:

By Lemma 6.5, the element $h_{n-2}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot(x_{s-1}\wedge x_s) = 1$ in

$$(F^M)^{n-2}(X^{(n)}/\bar{\triangle}_n)$$

and so $[h_{n-2}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot(x_{s-1}\wedge x_s)]^{g_{s-1}} = 1$ in $(F^M)^{n-1}(X^{(n)}/\bar{\triangle}_n)$. Thus

$$\begin{aligned}
h_n(x_1^{g_1}\dots x_s^{g_s}) &= h_n(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]^{g_{s-1}}\\
&= h_n(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\cdot[h_{n-1}(x_1^{g_1}\dots x_{s-2}^{g_{s-2}})\odot x_{s-1}]^{g_{s-1}g_s}\\
&= h_n(x_1^{g_1}\dots x_{s-2}^{g_{s-2}}x_{s-1}^{g_{s-1}g_s}).
\end{aligned}$$

The second induction is finished and therefore the first induction is finished.

Proof of Lemma 6.5:

We consider a slight general case. Let Y be a pointed simplicial set and let the composition product

$$\odot_Y\colon (F^M)^n(Y)\wedge X^{(2)} \to (F^M)^n\left(Y\wedge(X^{(2)}/\bar{\triangle}_2)\right)$$

given by the composite

$$(F^M)^n(Y)\wedge X^{(2)} \xrightarrow{\odot} (F^M)^n(Y\wedge X^{(2)}) \xrightarrow{(F^M)^n(p)} (F^M)^n(Y\wedge X^{(2)}/\bar{\triangle}_2)$$

where $p : Y \wedge X^{(2)} \to Y \wedge X^{(2)}/(Y \wedge \bar{\triangle}_2)$ is the pinch map. If $Y = X^{(n)}/\bar{\triangle}_n$, then the composition product

$$\odot : (F^M)^n(X^{(n)}/\bar{\triangle}_n) \wedge X^{(2)} \to (F^M)^n(X^{(n+2)}/\bar{\triangle}_{n+2})$$

is the composite

$$(F^M)^n(X^{(n)}/\bar{\triangle}_n) \wedge X^{(2)} \xrightarrow{\odot_Y} (F^M)^n \left((X^{(n)}/\bar{\triangle}_n) \wedge (X^{(2)}/\bar{\triangle}_2)\right) \xrightarrow{(F^M)^n(q)} (F^M)^n(X^{(n+2)}/\bar{\triangle}_{n+2}),$$

where $q : (X^{(n)}/\bar{\triangle}_n) \wedge (X^{(2)}/\bar{\triangle}_2) to (X^{(n+2)}/\bar{\triangle}_{n+2}$ is the pinch map. Thus it suffices to show that

$$\alpha \odot_Y (x \wedge x) = 1$$

for any $\alpha \in (F^M)^n(Y)$ and $x \in X$. The proof of this statement is given by induction on n. Suppose that $n = 1$. Let $\alpha = (y_1^{g_1} \dots y_s^{g_s})$ be a word in $F^M(Y)$. Then

$$\alpha \odot_Y (x \wedge x) = (y_1 \wedge x \wedge x)^{g_1} \dots (y_s \wedge x \wedge x)^{g_s} = 1$$

since $y_j \wedge x \wedge x \in Y \wedge \bar{\triangle}_2$ for each j. Suppose that this statement holds for the case $n-1$ with $n > 1$. Let

$$\alpha = (\alpha_1^{g_1} \dots \alpha_s^{g_s})$$

be a word in $(F^M)^n(Y)$, where α_j is an element in $(F^M)^{n-1}(Y)$. Then

$$\alpha \odot_Y (x \wedge x) = ((\alpha_1 \odot_Y (x \wedge x))^{g_1} \dots (\alpha_s \odot_Y (x \wedge x))^{g_s}) = 1$$

since $\alpha_j \odot_Y (x \wedge x) = 1$ for each j. The induction is finished and so the assertion follows.

Now let M be a simplicial monoid and let X be a pointed simplicial set. The simplicial map $\theta : M \wedge X \to F^M(X)$ is defined by setting $\theta(g \wedge x) = x^g$.

Lemma 6.6. *The simplicial map $\theta : M \wedge X \to F^M(X)$ is the adjoint map of the simplicial map $\sigma_M \wedge X : \Sigma M \wedge X \to BM \wedge X \simeq BF^M(X)$ up to homotopy, where the map $\sigma_M : \Sigma M \to BM$ is the suspension defined in Section 3.*

Proof: The assertion follows by the commutative diagram

$$\begin{array}{ccc} \Sigma M \wedge X & \xrightarrow{\alpha \simeq} \Sigma(M \wedge X) \xrightarrow{\Sigma\theta} & \Sigma F^M(X) \\ \downarrow \sigma \wedge X & & \downarrow \sigma \\ BM \wedge X & \xrightarrow{\beta} & BF^M(X), \end{array}$$

where $\alpha((1, g_{n-1}) \wedge x_n) = (1, (g_{n-1} \wedge d_0 x_n))$ for $g_{n-1} \in M_{n-1}$ and $x_n \in X_n$ and

$$\beta(g_{n-1}, \dots, g_0) \wedge x_n) = \left((d_0x_n)^{g_{n-1}}, (d_0^2 x_n)^{g_{n-2}}, \dots, (d_0^n x_n)^{g_0}\right)$$

for $g_i \in M_i$ and $x_n \in X_n$. The simplicial map $\beta : BM \wedge X \to BF^M(X)$ is a homotophy equivalence by the proof of Theorem 5.5.

Notations 6.7. *Let M be a simplicial monoid and let X be a pointed simplicial set. The simplicial maps $\theta_n : M^{(n)} \wedge X \to (F^M)^n(X)$ is defined by setting*

$$\theta_n(g_s \wedge g_{s-1} \wedge \ldots \wedge g_1 \wedge x) = (\ldots(((x)^{g_1})^{g_2})\ldots)^{g_n}$$

for $g_j \in M$ and $x \in X$. The simplicial map $j(M) : M \to G(BM)$ is defined by $j(g_{n-1} = (g_{n-1}, 1, \ldots, 1)$ for $g_{n-1} \in M$, where $G(X)$ is the Kan G-construction of a reduced siplicial set X. It is easy to show that the simplicial map $j(M) : M \to G(BM)$ is the adjoint of the suspension $\sigma : \Sigma M \to BM \simeq BH(BM)$ up to homotopy. Let X be a reduced simplicial set and let Y be a pointed simplicial set. The simplicial map $t : Y \wedge G(X) \to G(Y \wedge X)$ is defined to be the adjoint of $\Sigma(Y \wedge G(X)) \simeq Y \wedge \Sigma G(X) \xrightarrow{Y\wedge\sigma} Y \wedge BG(X) \simeq Y \wedge X$. The simplicial map $t_n : Y \wedge G^n(X) \to G^n(Y \wedge X)$ is defined inductively to be the composite

$$Y \wedge G^n(X) = Y \wedge G(G^{n-1}(X)) \xrightarrow{t} G(Y \wedge G^{n-1}(X)) \xrightarrow{G(t_{n-1})} G^n(Y \wedge X).$$

Definition 6.8. *Let M be a simplicial monoid and let X be a pointed simplicial set. The simplicial maps*

$$\psi_n : (F^M)^n(X) \to G^n\left((BM)^{(n)} \wedge X\right)$$

are defined inductively as follows: The map $\psi_1 : F^M(X) \to G(BM \wedge X)$ is defined to be the composite

$$F^M(X) \xrightarrow{j(F^M(X))} G(BF^M(X)) \xrightarrow{G(\beta)^{-1}} G(BM \wedge X),$$

where the map $G(\beta)^{-1}$ is a monomoty inverse of the simplicial map $G(\beta) : G(BM \wedge X) \xrightarrow{\simeq} G(BF^M(X))$ and where the simplicial map $\beta : BM \wedge X \to BF^M(X)$ is defined in the proof of Lemma 6.6. Now the map $\psi_n : (F^M)^n(X) \to G^n\left((BM)^{(n)} \wedge X\right)$ is defined to be the composite

$$(F^M)^n(X) = F^M\left((F^M)^{n-1}(X)\right) \xrightarrow{F^M(\psi_{n-1})} F^M\left(G^{n-1}\left((BM)^{(n-1)} \wedge X\right)\right)$$

$$\xrightarrow{\psi_1} G(BM \wedge G^{n-1}\left((BM)^{(n-1)} \wedge X\right) \xrightarrow{G(t_{n-1})} G^n\left((BM)^{(n)} \wedge X\right).$$

Lemma 6.9. *The composite*

$$M^{(n)} \wedge X \xrightarrow{\theta_n} (F^M)^n(X) \xrightarrow{\psi_n} G^n\left((BM)^{(n)} \wedge X\right)$$

is the adjoint of the simplicial map

$$\Sigma^n M^{(n)} \wedge X \simeq (\Sigma M)^{(n)} \wedge X \xrightarrow{\sigma^{(n)}\wedge X} (BM)^{(n)} \wedge X,$$

up to homotopy for $n \geq 1$.

Proof: The proof is given by induction on n. If $n = 1$, the assertion follows by the commutative diagram

$$\begin{array}{ccccc} \Sigma F^M(X) & & \xrightarrow{j} & & \Sigma G(BF^M(X)) \xrightarrow{\simeq} \Sigma G(BM \wedge X) \\ \Sigma\theta_1 \uparrow & & & & \downarrow \sigma \\ \Sigma M \wedge X & \xrightarrow{\sigma\wedge X} BM \wedge X & \xrightarrow{\simeq} BF^M(X) & \xrightarrow{\simeq} & BG(BF^M(X)). \end{array}$$

For the general case n, the assertion follows by induction and the following homotopy commutative diagram

$$\begin{array}{ccccccc} \Sigma^n(F^M)^n(X) & \xrightarrow{\Sigma^n F^M(\psi_{n-1})} & \Sigma^n F^M\left(G^{n-1}(Y_{n-1})\right) & \xrightarrow{\Sigma^n\psi_1} & \Sigma^n G\left(BM \wedge G^{n-1}(Y_{n-1})\right) & \longrightarrow & \Sigma^n G^n(Y_n) \\ \uparrow & & \uparrow & & \downarrow \sigma & & \downarrow \sigma \\ \Sigma^n M \wedge (F^M)^{n-1}(X) & \xrightarrow{1\wedge\psi_{n-1}} & \Sigma^n M \wedge G^{n-1}(Y_{n-1})) & \xrightarrow{\sigma\wedge 1} & \Sigma^{n-1} BM \wedge G^{n-1}(Y_{n-1}) & & \\ \uparrow & & \downarrow & & \downarrow & & \\ \Sigma M \wedge \Sigma^{n-1} M^{(n-1)} \wedge X & \xrightarrow{1\wedge\sigma^{(n-1)}} & \Sigma M \wedge Y_{n-1} & \xrightarrow{\sigma\wedge 1} & Y_n & = & Y_n, \end{array}$$

where we have set $Y_{n-1} = (BM)^{(n-1)} \wedge X$ and $Y_n = (BM)^n \wedge X$.

Definition 6.10. *Let M be a simplicial monoid and let X be a pointed simplicial set. The maps*

$$H_n : F^M(X) \to G^n\left((BM)^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n)\right)$$

is defined to be the composite

$$F^M(X) \xrightarrow{h_n} (F^M)^n(X^{(n)}/\bar{\triangle}_n) \xrightarrow{\psi_n} G^n\left((BM)^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n)\right).$$

Theorem 6.11. *The natural maps $H_n : F^M(X) \to G^n\left((BM)^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n)\right)$ is a homotopy extension of the pinch map $F^M_n(X) \to F^M_n(X)/F^M_{n-1}(X)$, i.e., there is a homotopy commutative diagram*

$$\begin{array}{ccc} F^M(X) & \xrightarrow{H_n} & G^n\left((BM)^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n)\right) \\ \uparrow & & \uparrow j^1_n \\ F^M_n(X) & \to F^M_n(X)/F^M_{n-1}(X) & \cong M^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n) \end{array}$$

for $n \geq 1$, where $\{F^M_n(X)\}_{n\geq 0}$ is the word length filtration of $F^M(X)$ and where the simplicial map $j^1_n : M^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n) \to G^n\left((BM)^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n)\right)$ is the adjoint of the multiple suspension.

$$\Sigma^n M^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n) \simeq (\Sigma M)^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n) \xrightarrow{\sigma^{(n)}\wedge 1} (BM)^{(n)} \wedge (X^{(n)}/\bar{\triangle}_n)$$

up to homotopy.

Proof: The assertion follows by the construction of H_n and Lemma 6.9.

Remark 6.12. *The Kan G-construction is actually the loop functor Ω in the homotopy category. For any pointed simplicial set X, the simplicial group $G(X)$ is defined to be the Kan G-construction on the base point connect component of X. Thus all of the simplicial groups $G^n(X) = G \circ \ldots \circ G(X)$ are well defined for any pointed simplicial set X.*

7. Proof of Theorem 1.1 (Theorem 5.5)

Now we are ready to give a proof of Theorem 1.1. We need two lemmas.

Lemma 7.1. *Let $F=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let H_* be a multiplicative homology theory such that (1) both $\bar{H}_*(FP^\infty)$ and $\bar{H}_*(FP^\infty_2)$ are free $H_*(pt)$-modules; and (2) the inclusion of the bottom cell $j: S^d\to FP^\infty$ induces a monomorphism in the homology. Then the homomorphism*

$$(j^{(n)}\wedge X)_*:\bar{H}_*(S^{dn}\wedge X)\to \bar{H}_*\left((FP^\infty)^{(n)}\wedge X\right)$$

is a monomorphism for each n and any space X.

Proof: The proof is given by induction on n. Suppose that $n=1$. Consider the short exact sequence

$$0\to\bar{H}_*(S^d)\to\bar{H}_*FP^\infty\to\bar{H}_*FP^\infty_2\to 0$$

Notice that the $H_*(pt)$-modules $\bar{H}_*(S^d)$, $\bar{H}_*(FP^\infty)$ and $\bar{H}_*(FP^\infty_2)$ are free modules. There is a short exact sequence

$$0\to\bar{H}_*(S^d)\otimes_{H_*(pt)}\bar{H}_*(X)\to\bar{H}_*FP^\infty\otimes_{H_*(pt)}\bar{H}_*(X)\to\bar{H}_*(FP^\infty_2)\otimes_{H_*(pt)}\bar{H}_*(X)\to 0$$

and the isomorphisms $\bar{H}_*(S^d)\otimes_{H_*(pt)}\bar{H}_*(X)\approx\bar{H}_*(S^d\wedge X)$, $\bar{H}_*(FP^\infty)\otimes_{H_*(pt)}\bar{H}_*(X)\approx\bar{H}_*(FP^\infty\wedge X)$ and $\bar{H}_*(FP^\infty_2)\otimes_{H_*(pt)}\bar{H}_*(X)\approx\bar{H}_*(FP^\infty_2\wedge X)$. Thus $j\wedge X)_*:\bar{H}_*(S^d\wedge X)\to\bar{H}_*(FP^\infty\wedge X)$ is a monomorphism. The general case n follows by induction and the following commutative diagram.

$$\begin{array}{ccc}
\bar{H}_*(S^{dn}\wedge X) & \stackrel{(j^{(n)}\wedge 1)_*}{\longrightarrow} & \bar{H}_*\left((FP^\infty)^{(n)}\wedge X\right)\\
\Sigma^d(j^{(n-1)}\wedge 1)_*\downarrow & & ||\\
\bar{H}_*\left(S^d\wedge(FP^\infty)^{(n-1)}\wedge X\right) & \stackrel{\longrightarrow}{(j\wedge 1)_*} & \bar{H}_*\left((FP^\infty)^{(n)}\wedge X\right)
\end{array}$$

Let X be a pointed simplicial set. We denote $\bar{\triangle}_1=*$ and $\bar{\triangle}_n=\{(x_1\wedge\ldots\wedge x_n)\in X^{(n)}\mid x_i=x_{i+1}\text{ for some } i\}$, a subsimplicial set of the n-fold self smash product of X.

Lemma 7.2. *Let G be a simplicial group, let X be a pointed simplicial set and let H_* be a homology theory. Suppose that*

$$(\sigma^{(n)}\wedge 1)_*:\bar{H}_*\left((\Sigma G)^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)\right)\to\bar{H}_*\left((BG)^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)\right)$$

is a monomorphism for each $n\geq 1$, where $\sigma:\Sigma G\to BG$ is the suspension. Then the map

$$(j_n)_*:\bar{H}_*\left(F^G_n(X)\right)\to\bar{H}_*\left(F^G(X)\right)$$

is a monomorphism for each $n\geq 0$, where $j_n:F^G_n(X)\to F^G(X)$ is the injection in the word length filtration $\{F^G_n(X)\}_{n\geq 0}$ of $F^G(X)$.

Proof: The proof is given by induction on n. The assertion holds for the case $n=0$, where $F^G_o(X)=*$. Suppose that $(j_q)_*:\bar{H}_*\left(F^G_q(X)\right)\to\bar{H}_*\left(F^G(X)\right)$ is a monomorphism for $q<n$ and consider $(j_q)_*:\bar{H}_*\left(F^G_n(X)\right)\to\bar{H}_*\left(F^G(X)\right)$. Let $\alpha\in Ker(j_n)_*$. By Theorem 6.11, there is a commutative diagram

$$\begin{array}{ccc}
\bar{H}_*\left(\Sigma^n F^G(X)\right)\xrightarrow{\Sigma^n(H_n)_*}\bar{H}_*\left(\Sigma^n G^n\left((BG)^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)\right)\right. & \longrightarrow & \bar{H}_*\left((BG)^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)\right)\\
\uparrow (j_n)_* & & \uparrow(\sigma^{(n)}\wedge 1)_*\\
\bar{H}_*\left(\Sigma^n F^G_n(X)\right) & \xrightarrow{(p_n)_*} & \bar{H}_*\left(\Sigma^n G^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)\right),
\end{array}$$

where the map

$$p_n:\Sigma^n F^G_n(X)\to\Sigma^n F^G_n(X)/F^G_{n-1}(X)\cong\Sigma^n\left(G^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)\right)\simeq(\Sigma g)^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)$$

is the pinch map. We already assume that

$$(\sigma^{(n)}\wedge 1)_*:\bar{H}_*\left((\Sigma G)^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)\right)\to\bar{H}_*\left((BG)^{(n)}\wedge(X^{(n)}/\bar{\triangle}_n)\right)$$

is a monomorphism. Thus $(p_n)_*(\Sigma^n\alpha)=0$. By the cofibre sequence $F^G_{n-1}(X)\xrightarrow{j_{n-1,n}}F^G_n(X)\to F^G_n(X)/F^G_{n-1}(X)$, there exists an element $\beta\in\bar{H}_*\left(F^G_{n-1}(X)\right)$ such that $j_{n-1,n*}(\beta)=\alpha$. Thus $(j_{n-1})_*(\beta)=(j_n)_*\circ(j_{n-1,n})_*(\beta)=(j_n)_*(\alpha)=0$ By induction, $\beta=0$ and so $\alpha=0$. The assertion follows.

Proof of Theorem 5.8:

Let G be a simplicial group such that $G\simeq\Omega(FP^\infty)$. The assertion follows directly from Lemmas 7.1 and 7.2.

8. Simplicial Group Rings $\mathbb{F}_p\ (F_p(X))$

Let $\mathbb{F}_p$ be a field with p elements, where p is a prime, and let x be a simplicial set. It is well known that $H_*(X;\mathbb{F}_p)\cong\pi_*(\mathbb{F}_p(X))$, where $\mathbb{F}_p(X)$ is the free sipmlicial $\mathbb{F}_p$-module generated by X. In this section, we only give a description of the simplicial group ring $\mathbb{F}_p(F_p(X))$. Further study will be given in [18].

Definition 8.1. *Let V be a (graded) $\mathbb{F}_p$-vector space. The algebra $A^p(V)$ is defined to be the quotient algebra of the tensor algebra $T(V)$ modulo the two sided ideal generated by the elements x^p for $x\in V$. If V is a simplicial $\mathbb{F}_p$-vector space. Then $A^p(V)$ is a simplicial algebra. Let X be a pointed simplicial set. The simplicial algebra $A^p(X)$ is defined to be the quotient simplicial algebra of $A^p(\mathbb{F}_p(X))$ modulo the two sided ideal generated by the element $*$, where $*$ is the base point of X. If $p=2$, the algebra $A^p(V)$ is a non-commutative view of exterior algebras.*

Proposition 8.2. *Let X be a pointed siplicial set. Then there is an isomorphism of simplicial algebras*

$$A^p(X)\xrightarrow{\cong}\mathbb{F}_p\left(F^{\mathbb{Z}/p}(X)\right).$$

Proof: Notice that the simplicial group $F^{\mathbb{Z}/p}(X)$ is the quotient simplicial group of $F(X)$ modulo the normal subsimplicial group generated by x^p for $x \in X$. Consider the simplicial map $i : X \to \mathbb{F}_p\left(F^{\mathbb{Z}/p}(X)\right) i(x) = x - 1$ for $x \in X$. Let $\tilde{\imath} : T(\mathbb{F}_p(X)) \to \mathbb{F}_p\left(F^{\mathbb{Z}/p}(X)\right)$ be a simplicial homomorphism of simplicial algebras which is a (unique) extension of i. Notice that we have the following equations: (1). $\tilde{\imath}(x^p) = (x-1)^p = x^p - 1 = 0$ and (2). $\tilde{\imath}(*) = * - 1 = 0$.

Thus there is a (unique) simplicial homomorphism

$$\phi : A^p(X) \to \mathbb{F}_p\left(F^{\mathbb{Z}/p}(X)\right)$$

of simplicial algebras which is a (unique) extension of i. Conversely, consider the simplicial map $j : X \to A^P(X)$ given by $j(x) = x + 1$ for $x \in X$. Similarly, there is a (unique) simplicial homomorphism

$$\psi : \mathbb{F}_p\left(F^{\mathbb{Z}/p}(X)\right) \to A^p(X)$$

of simplicial algebras which is a (unique) extension of the map j. It is easy to check that $\varphi \circ \psi = 1_{\mathbb{F}_p\left(F^{\mathbb{Z}/p}(X)\right)}$ and $\varphi \circ \psi = 1_{A^p(X)}$, which is the assertion.

Corollary 8.3. *Let X be a pointed simplicial set. Then there is an isomorphism of algebras*

$$\pi_*\left(A^p(X)\right) \cong H_*\left(\Omega\left(K(\mathbb{Z}/p, 1) \wedge X\right); \mathbb{F}_p\right).$$

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MSRI, 1000 Centennial Drive, Berkeley, CA 94720, USA