HOMOTOPY THEORY OF THE SUSPENSIONS OF THE PROJECTIVE PLANE

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ABSTRACT. The homotopy theory of the suspensions of the real projective plane is largely investigated. The homotopy groups are computed up to certain range. The decompositions of the self smashes and the loop spaces are studied with some applications to the Stiefel manifolds.

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1. Introduction

Let $\mathbb{R}\mathrm{P}^2$ be the projective plane. The n-dimensional mod 2 Moore space $P^n(2)$ is defined by $P^n(2) = \Sigma^{n-2}\mathbb{R}\mathrm{P}^2$ for $n \geq 2$. As a space, $P^n(2)$ is the homotopy cofibre of the degree 2 map $[2]\colon S^{n-1} \to S^{n-1}$. In other words, $P^n(2)$ is the cell complex obtained by attaching an n-cell to S^{n-1} where the attaching map is of degree 2. In geometry, $P^{2n}(2)$ is the $2n$-skeleton of the Stiefel manifold $V_{2n+1,2}$ (or the associated spherical bundle $\tau(S^{2n})$ of the tangent bundle of S^{2n}). More precisely, let x be any point in $V_{2n+1,2}$. Then the open manifold $V_{2n+1,2} \setminus \{x\}$ is homotopy equivalent to $P^{2n}(2)$. Also an even dimensional projective plane admits a cellular decomposition in terms of mod 2 Moore spaces.

In this paper, we study homotopy theoretic aspects of mod 2 Moore spaces. Particularly we will compute the homotopy groups up to $8+$dimension. By knowing these homotopy groups, we obtain some information on three-cell complexes. For instance, by using the fact that $\pi_{10}(P^6(2)) = \mathbb{Z}/8$, we show that if a simply connected complex X has the same mod 2 cohomology ring as that of $\tau(S^6)$, then X is homotopy equivalent to $\tau(S^6)$. (See Proposition 5.12.) In particular, if X is a simply connected closed 11-manifold such that its first nontrivial integral homology group is $H_5(X;\mathbb{Z}) = \mathbb{Z}/2$, then X is homotopy equivalent to $\tau(S^6)$. This statement is not true for other Stiefel manifolds in general. For instance, by considering $\pi_6(P^4(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2$, there is a simply connected complex X with property that the mod 2 cohomology ring of X is isomorphic to that of $\tau(S^4)$ but X is not homotopy equivalent to $\tau(S^4)$. The homotopy groups $\pi_*(P^n(2))$ also helps to classify the complexes X which has the cell structure of the form $P^n(2) \cup e^{n+r}$. For example, $\pi_4(P^3(2)) = \mathbb{Z}/4$ shows that if X has the cell structure $P^3(2) \cup e^5$, then X is homotopy equivalent to one of the following three spaces:

1) the wedge $P^3(2) \vee S^5$;
2) the homogeneous space $SU(3)/SO(3)$;
3) $P^3(2) \cup_{2\lambda_3} e^5$, where $\lambda_3\colon S^4 \to P^3(2)$ represents the generator of $\pi_4(P^3(2))$.

If in addition X is a closed 5-manifold, then X is homotopy equivalent to $SU(3)/SO(3)$. (See Example 6.15.)

Our methods for computing the homotopy groups are: (1). to use the results on the homotopy groups of spheres, (2). to investigate various fibre sequences and (3). to study splittings of spaces.

The homotopy groups of spheres have been much studied. Up to the range what we need, we use Toda's book [42] as the main reference for the homotopy groups of spheres and so we directly use the notations for the elements in the $\pi_*(S^n)$ given in [42] without further explanation. Relations between the homotopy groups of spheres and that of a space X can be set up by constructing a map from a sphere or its loops to X or its loops, or vice versa. For instance, by knowing a spherical class in the homology of the loop space of X, we obtain a map from a sphere to the loop space of X. This map then induces a group homomorphism from the homotopy groups of a sphere to that of X. In our case, the mod 2 homology of the loop space $\Omega P^n(2)$ is the tensor algebra on two generators for $n \geq 3$. Many spherical classes can be obtained by considering the Samelson products. We should point out that the existence of certain spherical classes in the mod 2 homology $H_*(\Omega P^n(2))$ is equivalent to the strong form of Kervaire invariant one conjecture. (See Section 2.4 for details.)

The canonical method to study the homotopy groups of a space X is to consider certain fibration resolution of X or its loop spaces. The resolution produces a spectral sequence in general. Depending on the type of fibration resolution considered, there are several types of the Adams spectral sequences using different homology theories. In this paper, roughly speaking, we consider the spherical fibration resolutions of $P^n(2)$. These resolutions will be considered case by case depending on special properties of the considered mod 2 Moore space. For instance, there are several ways to produce fibration resolutions for $P^4(2)$: (1). Consider the pinch map $P^4(2) \to S^4$ and then do analysis on the homotopy fibre $F^4\{2\}$. (2). Consider the canonical inclusion $P^4(2) \to \tau(S^4)$ and study its homotopy fibre. (3). Consider the map $\phi\colon P^4(2) \to \mathbb{H}\mathrm{P}^\infty$ given by the composition of the pinch map $P^4(2) \to S^4$ and the inclusion $S^4 \to \mathbb{H}\mathrm{P}^\infty$. In the last case, the homotopy fibre of the map ϕ is homotopy equivalent to the 7-skeleton of the double loop space $\Omega^2 S^5$. Since our purpose is to compute low homotopy groups, we are not going to study the "entire fibration resolutions" and their associated spectral sequences in this paper. On the other hand, we freely consider different fibrations until the homotopy groups are determined, where the different fibrations will be used particularly for determining the group structure of the homotopy groups.

Although the space $P^n(2)$ itself is indecomposable up to homotopy, its loop space $\Omega P^n(2)$ admits various product decompositions for $n \geq 3$. By applying the functor π_* to decompositions of loop spaces, we obtain decompositions of the homotopy groups and then the homotopy groups can be obtained by studying the factors in the decompositions. For instance, the homotopy group $\pi_{11}(P^3(2)) = \mathbb{Z}/2^{\oplus 14} \oplus \mathbb{Z}/4^{\oplus 2}$ is obtained by taking the direct sum of the corresponding homotopy groups of many factors in a decomposition of the triple loop space of $P^3(2)$.

Our decompositions of $\Omega P^n(2)$ for $n \geq 3$ are the special case of general results in [36, 37]. Let X be a path connected p-local space. Then there is a homotopy decomposition

$$\Omega\Sigma X \simeq A^{\min}(X) \times \Omega(\bigvee_{k=2}^{\infty} Q_k^{\max}(X)),$$

where $A^{\min}$ and $Q_k^{\max}$ are functors and $Q_k^{\max}(X)$ is a functorial retract of the suspension of the k-fold self smash product $X^{(k)}$. By applying the Hilton-Milnor Theorem to the second factor, one gets further decompositions of $\Omega\Sigma X$. The study of this decomposition for the special case where $X = P^{n-1}(2)$ is given in chapter 4.

Since the space $Q_k^{\max}(X)$ is a retract of $\Sigma X^{(k)}$, the study of splittings of self smash products of $P^n(2)$ is important for our purpose to compute the homotopy groups. In chapter 3, we study splittings of the self smash products of $\mathbb{R}\mathrm{P}^2$. The homology of indecomposable factors in $(\mathbb{R}\mathrm{P}^2)^{(k)}$ has been determined in [38]. The cell structure of the indecomposable factors is the main issue that we are going to study in this paper. One of the interesting points is that these indecomposable factors admits smash product decomposition. For instance, the smallest retract of $(\mathbb{R}\mathrm{P}^2)^{(7)}$ which contains the bottom cell is homotopy equivalent $\mathbb{H}\mathrm{P}^2\wedge\mathbb{C}\mathrm{P}^2\wedge\mathbb{R}\mathrm{P}^2$. (See Corollary 3.11.) This gives a relation among different projective planes. We should point out that the splittings of self smash products of more general spaces play a role in Morava K-theory, see [32]. Furthermore, the problem on determining functorial splittings of self smashes of suspensions is equivalent to the fundamental problem in the modular representation theory of the symmetric groups, see [38]. This problem remains open in both homotopy theory and the representation theory in general. However much progress has been achieved recently in both areas. In homotopy theory, the homology of functorial indecomposable retracts of self smashes of a two-cell suspension has been determined in [38] with applications given in [10]. The corresponding results in the representation theory was obtained in [13, 14].

The EHP sequence is one of the main tools for computing the homotopy groups of spheres. In Section 4.4, we study an analogue for mod 2 Moore spaces. More precisely, we consider the reduced James-Hopf map $\bar{H}_2\colon \Omega P^{n+1}(2)\to\Omega P^{2n+1}(2)$ for $n\geq 2$ which is given by the second James-Hopf map $H_2\colon \Omega P^{n+1}(2)\to\Omega\Sigma(P^n(2))^{(2)}$ composing with the loop of the pinch map $\Sigma(P^n(2))^{(2)}\to P^{2n+1}(2)$. We show that the atomic retract of the homotopy fibre of the map $\bar{H}_2$ is homotopy equivalent to $S^n\{2\}$, the homotopy fibre of the degree 2 map $[2]\colon S^n\to S^n$. (See Theorem 4.21.) Also a product decomposition of the homotopy fibre of the map $\bar{H}_2$ is given in Section 4.4 up to certain dimension.

The space $P^3(2)=\Sigma\mathbb{R}\mathrm{P}^2$ will be treated as a special case. There is a decomposition of the triple loop space of $P^3(2)$ which is obtained as follows. Let $\gamma_2\colon\Sigma\mathbb{R}\mathrm{P}^2\to BSO(3)$ be the adjoint map of the canonical inclusion $\mathbb{R}\mathrm{P}^2\to SO(3)$. Then the homotopy fibre of γ_2 is homotopy equivalent to $\Sigma\mathbb{R}\mathrm{P}^4_2\vee P^6(2)$, where $\mathbb{R}\mathrm{P}^b_a=\mathbb{R}\mathrm{P}^b/\mathbb{R}\mathrm{P}^{a-1}$ is a truncated projective space. Let Ω_0X denote the path-connected component of ΩX which contains the base-point and let $X\langle n\rangle$ denote the n-connected cover of X. Then we have a decomposition

$$\Omega^3_0P^3(2)\simeq\Omega^3_0(\Sigma\mathbb{R}\mathrm{P}^4_2\vee P^6(2))\times\Omega^2(S^3\langle 3\rangle),$$

see Theorem 6.7. This induces a decomposition of $\pi_r(P^3(2))$ for $r\geq 4$. By applying the Hilton-Milnor theorem to $\Omega(\Sigma\mathbb{R}\mathrm{P}^4_2\vee P^6(2))$, we are able to compute $\pi_r(P^3(2))$ for $r\leq 11$.

The main part of this paper, particularly the calculation of the homotopy groups, is from my Ph.D. thesis [45] of the University of Rochester under the supervise of Professor Frederick Cohen. The table of the homotopy groups computed in this paper have been announced without proofs in [8, pp.1202]. Some of the homotopy groups of the mod 2 Moore spaces have been computed by Mukai and others with applications [27, 28, 29] and so there are some overlap between our work and theirs. An application to construct non-suspension co-H-spaces is given in [48].

Some terminologies and notations used in this paper are as follows. Every space is localized at 2. We always assume that a space is a path-connected CW-complex with a non-degenerate base point $*$ and any map is a pointed map. Let $f\colon X\to Y$ be a map. We write F_f for the homotopy fibre of f and C_f is the homotopy cofibre of f.

The homotopy fibre of the pinch map $P^n(2) \to S^n$ is denoted by $F^n\{2\}$. The mod 2 homology of a space X will be denoted by $H_*(X)$. On the other hand, the integral homology is denoted by $H_*(X;\mathbb{Z})$ to avoid confusion. The n-fold self smash product of X is denoted by $X^{(n)}$. Let X be a co-H-space. A map of *degree* k, denoted by $[k]\colon X \to X$, means the composite $X \xrightarrow{\mu'_k} \bigvee_{j=1}^{k} X \xrightarrow{\text{fold}} X$, where μ'_k is a choice of k-fold comultiplication. (**Note.** If X is not homotopy associative, the map $[k]$ depends on the choice of a k-fold comultiplication and so may not be unique in general.) The homotopy fibre of $[k]\colon S^n \to S^n$ is denoted by $S^n\{k\}$. The homotopy cofibre of $[k]\colon S^n \to S^n$ is the Moore space $P^{n+1}(k)$. For an H-space X, a *power map of degree* k, denoted by $k\colon X \to X$, means the composite $X \xrightarrow{\Delta_k} X^k \xrightarrow{\mu_k} X$, where μ_k is a choice of k-fold multiplication. Let X be a complex. We write $\mathrm{sk}_n(X)$ for the n-skeleton of X. The n-connected cover of X is denoted by $X\langle n\rangle$. The base-point path-connected component of the loop space of X is denoted by $\Omega_0 X$. In addition, we write $\mathbb{R}\mathrm{P}^b_a$ for $\mathbb{R}\mathrm{P}^b/\mathbb{R}\mathrm{P}^{a-1}$. Furthermore we directly use notations for the elements of the homotopy groups of spheres in [42]. Let x be a homogeneous element in a graded module V. We write $|x|$ for the degree of x. The notation $y_n \in V$ indicates that y is a homogeneous element in V of degree n, that is, $y \in V_n$. Let α be an element in the set of homotopy classes $[X, Y]$. We write $\alpha\colon X \to Y$ for a map which represents the homotopy class α.

The article is organized as follows. In chapter 2, we give some preliminaries and discuss some classical problems related to mod 2 Moore spaces. The study of self smashes of the projective plane $\mathbb{R}\mathrm{P}^2$ is given in chapter 3. In chapter 4, we study the loop space of mod 2 Moore spaces. The computation of the homotopy groups of $P^n(2)$ for $n \geq 4$ is given in chapter 5. The homotopy theory of $P^3(2)$ is studied in chapter 6. The computation of the homotopy groups of $P^3(2)$ is given in the last section of this chapter.

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2. Preliminary and the Classical Homotopy Theory

2.1. **Action of the Steenrod Algebra.** Recall the Steenrod algebra $\mathcal{A}$ can be described as all linear natural transformations of the mod 2 cohomology functor. We refer the reader to [40] for several basic properties of $\mathcal{A}$. The algebra $\mathcal{A}$ is generated by symbols Sq^n, $n \geq 1$, $|Sq^n| = n$, subject to the "Adem relations". That is, if one takes the tensor algebra on these symbols and quotients by the two-sided ideal generated by the Adem relations one obtains $\mathcal{A}$. The fundamental properties of Sq are as follows

1) For all integers $i \geq 0$ and $n \geq 0$, there is a linear natural transformation

$$Sq^i\colon H^n(X,A) \longrightarrow H^{n+i}(X,A).$$

2) $Sq^0 = 1$ and Sq^1 is the Bockstein homomorphism.
3) $Sq^n x = x^2$ if $|x| = n > 0$.
4) $Sq^i x = 0$ if $|x| < n$.
5) The Cartan formula

$$Sq^n(x \otimes y) = \sum_{i=0}^{n} Sq^i x \otimes Sq^{n-i} y.$$

6) The Adem relations

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

for $0 < a < 2b$, where $[a/2] = \max\{k \in \mathbb{Z} | k \leq a/2\}$.

We write $Sq^n_* \colon H_q(X) \to H_{q-n}(X)$ for the dual-action of Sq^n on the homology functor. By the duality, the Cartan formula and the Adem relations are given by

$$Sq^n_*(x \otimes y) = \sum_{i=0}^{n} Sq^i_*(x) \otimes Sq^{n-i}_*(y)$$

$$Sq^b_* Sq^a_* = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^j_* Sq^{a+b-j}_*$$

for $0 < a < 2b$, respectively.

Recall that the cohomology ring $H^*(\mathbb{R}\mathrm{P}^2)$ is the quotient of the polynomial algebra on the symbol x, $|x| = 1$, by the ideal generated by x^3. Thus $H^*(\mathbb{R}\mathrm{P}^2)$ has a basis $\{1, x, x^2 = Sq^1 x\}$ and so the reduced homology $\bar{H}_*(\mathbb{R}\mathrm{P}^2)$ has a basis $\{u, v\}$ with $|v| = 2$ and $Sq^1_* v = u$. It follows that $\bar{H}_*(P^n(2))$ has a basis $\{u, v\}$ with $|v| = n$ and $Sq^1_* v = u$. Let $[\mathcal{A}, \mathcal{A}]$ be the two-sided ideal generated by the commutator $[\alpha, \beta] = \alpha\beta - \beta\alpha$ for $\alpha, \beta \in \mathcal{A}$. Recall that $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ is a divided polynomial algebra on one generator, that is

$$Sq^i Sq^j = \binom{i+j}{i} Sq^{i+j} \mod [\mathcal{A}, \mathcal{A}],$$

see [40, pp.25]. Let $\Gamma(x_1, \ldots, x_k)$ be the divided polynomial algebra generated one $x_1, \ldots, x_k$. We have $\mathcal{A}/[\mathcal{A}, \mathcal{A}] \cong \Gamma(Sq^1)$.

Lemma 2.1. *Let X be a path-connected space. Suppose that the commutator $[Sq^a_*, Sq^b_*]$ acts trivially on $H_*(X)$ for $a, b \geq 1$. Then the action of $\mathcal{A}$ on*

(1). $H_*(\Omega\Sigma X)$ and
(2). $\bar{H}_*\left((X)^{(n)}\right)$ for $n \geq 1$

factors through the quotient $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$.

Proof. It suffices to show that $[Sq^a_*, Sq^b_*]$ acts trivially on $H_*(\Omega\Sigma X)$. Let $V = \bar{H}_*(X)$. Recall that, as an algebra, $H_*(\Omega\Sigma X)$ is isomorphic to the tensor algebra $T(V)$. Thus the action of the Steenrod algebra on $H_*(\Omega\Sigma X)$ is uniquely determined by its action on V and the Cartan formula. By using the fact that $Sq^i_*Sq^j_*x = Sq^j_*Sq^i_*x$ for $x \in V$, we have

$$Sq^b_*Sq^a_*(x_1x_2\cdots x_k) = \sum_{\substack{i_1+i_2+\ldots+i_k=a\\ j_1+j_2+\ldots+j_k=b}} Sq^{j_1}_*Sq^{i_1}_*x_1Sq^{j_2}_*Sq^{i_2}_*x_2\cdots Sq^{j_k}_*Sq^{i_k}_*x_k$$

$$= \sum_{\substack{i_1+i_2+\ldots+i_k=a\\ j_1+j_2+\ldots+j_k=b}} Sq^{i_1}_*Sq^{j_1}_*x_1Sq^{i_2}_*Sq^{j_2}_*x_2\cdots Sq^{i_k}_*Sq^{j_k}_*x_k = Sq^a_*Sq^b_*(x_1x_2\cdots x_k)$$

for $x_i \in V$ and hence the result. □

Corollary 2.2. *The action of $\mathcal{A}$ on $H_*(\Omega P^n(2))$ and $\bar{H}_*\left((P^n(2))^{(m)}\right)$ factors through $\mathcal{A}/[\mathcal{A},\mathcal{A}] \cong \Gamma(Sq^1)$.*

Lemma 2.3. *Let $f\colon A \to B$ be a map and let $i\colon B \to C_f$ be the inclusion. Suppose that there exist positive integers r and k such that*

(1). *$H_j(B) = 0$ for $j \geq r$ and $H_r(C_f) \neq 0$.*
(2). *there is an element $v \in H_r(C_f)$ with the property that $Sq^k_*v \neq 0$ and Sq^j_*v lies in the image of $i_*\colon H_*(B) \to H_*(C_f)$ for $j \geq k$.*

Then the map $f \wedge X\colon A \wedge X \longrightarrow B \wedge X$ is essential for any finite dimensional complex X with $\bar{H}_*(X) \neq 0$.

Proof. Suppose that $f \wedge X\colon A \wedge X \to B \wedge X$ were null homotopic. Then there is a map $\phi\colon C_f \wedge X \to B \wedge X$ such that $\phi \circ (i \wedge X)$ is homotopic to the identity map of $B \wedge X$. Let y be a top cell of X, that is, $H_j(X) = 0$ for $j > |y|$ and y is not zero in $H_{|y|}(X)$. There exists an integer $t \geq 0$ such that $Sq^j_*y = 0$ for $j > t$ and $Sq^t_*y \neq 0$. By the assumption (1), $H_{r+|y|}(B \wedge X) = 0$ and so $\phi_*(v \otimes y) = 0$. For each $j \geq k$, let u_j be an element in $H_{r-j}(B)$ such that $Sq^j_*v = i_*(u_j)$. It follows that

$$0 = Sq^{k+t}_*\phi_*(v \otimes y) = \phi_*(Sq^{k+t}_*(v \otimes y)) = \sum_{j \geq k} \phi_*(Sq^j_*v \otimes Sq^{k+t-j}_*y)$$

$$= u_k \otimes Sq^t_*y + \sum_{j>k} u_j \otimes Sq^{k+t-j}_*y \neq 0$$

because $\bar{H}_{k+t}(B \wedge X) = \bigoplus_j \bar{H}_j(B) \otimes \bar{H}_{k+t-j}X$. We obtain a contradiction and hence the result. □

Corollary 2.4. *Let X be any finite dimensional complex with $\bar{H}_*(X) \neq 0$. Then the degree map $[2]\colon \Sigma X \to \Sigma X$ is essential.*

Proposition 2.5. *Let $n \geq 3$. Then the degree map $[2]\colon P^n(2) \to P^n(2)$ is homotopic to the composite*

$$P^n(2) \xrightarrow{\text{pinch}} S^n \xrightarrow{\eta} S^{n-1} \hookrightarrow P^n(2),$$

where η is a suspension of the Hopf map.

Proof. It suffices to show that the statement holds for $n = 3$. Clearly the map $[2]$ restricted to the bottom cell S^2 is null homotopic and so there is a map $\alpha\colon S^3 \to P^3(2)$ such that the diagram

$$\begin{array}{ccc} P^3(2) & \xrightarrow{\text{pinch}} & S^3 \\ \Big\downarrow{\scriptstyle [2]} & & \Big\downarrow{\scriptstyle \alpha} \\ P^3(2) & = & P^3(2) \end{array}$$

commutes up to homotopy. Since $P^3(2)$ is a torsion space, the composite

$$S^3 \xrightarrow{\alpha} P^3(2) \xrightarrow{\text{pinch}} S^3$$

is null homotopic and so α lifts to the homotopy fibre $F^3\{2\}$ of the pinch map $P^3(2) \to S^3$. By computing the low homology of $F^3\{2\}$, the bottom cell S^2 is the 3-skeleton of $F^3\{2\}$ and so there is a map $\alpha'\colon S^3 \to S^2$ such that the map α is homotopic to the composite $S^3 \xrightarrow{\alpha'} S^2 \hookrightarrow P^3(2)$. By Corollary 2.4, the map $[2]\colon P^3(2) \to P^3(2)$ is essential and so is the composite $P^3(2) \xrightarrow{\text{pinch}} S^3 \xrightarrow{\alpha'} S^2$. It follows that $[\alpha']$ is a generator of $\pi_3(S^2) = \mathbb{Z}$ or $\alpha' \simeq \pm\eta$. The assertion follows. ☐

Proposition 2.6. *Let $n \geq 3$. Then the group $[P^n(2), P^n(2)]$ is isomorphic to $\mathbb{Z}/4$.*

Proof. We only need to compute $[P^3(2), P^3(2)]$. By the cofibre sequence

$$S^2 \xrightarrow{[2]} S^2 \longrightarrow P^3(2),$$

there is an exact sequence

$$[S^3, P^3(2)] \xrightarrow{2} [S^3, P^3(2)] \longrightarrow [P^3(2), P^3(2)] \longrightarrow [S^2, P^3(2)] \xrightarrow{2} [S^2, P^3(2)].$$

Thus there is a short exact sequence

$$0 \longrightarrow \pi_3(P^3(2))/2 \longrightarrow [P^3(2), P^3(2)] \longrightarrow \pi_2(P^3(2)) = \mathbb{Z}/2 \longrightarrow 0.$$

By using the fact that S^2 is the 3-skeleton of $F^3\{2\}$, the map $\pi_3(S^2) \longrightarrow \pi_3(P^3(2))$ is an epimorphism and so $\pi_3(P^3(2))$ is a cyclic group $\mathbb{Z}/k$ for some integer k. It follows that $[P^3(2), P^3(2)]$ is isomorphic to $\mathbb{Z}/4$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/2$. By Corollary 2.4, we have

$$2 \cdot [P^3(2), P^3(2)] \neq 0$$

and hence $[P^3(2), P^3(2)] = \mathbb{Z}/4$, which is the result. ☐

Corollary 2.7. *Let f be a self homotopy equivalence of $P^n(2)$ with $n \geq 3$. Then f is either homotopic to the identity map or the degree map $[-1]\colon P^n(2) \to P^n(2)$.*

Note. Propositions 2.5 and 2.6 can be found in [43, Corollary, p.307] and [26, Lemma 1.1, p.271].

2.2. **The James Construction and the Cohen Groups.** Let X be a pointed space with the base point $*$. Recall that the James construction $J(X)$ is the free monoid on X subject to the single relation $* = 1$, see for instance [18, 44]. The James filtration $\{J_n(X)\}_{n\geq 0}$ with $J_0X = *$ and $J_1X = X$ is induced by the word-length filtration. Note that $J_n(X)/J_{n-1}(X) \cong X^{(n)}$. Let $q_n\colon X^n \to J_n(X)$ be the quotient map. The fundamental properties of $J(X)$ are as follows [5]:

1) If X is path-connected, then $J(X)$ is (weak) homotopy equivalent to $\Omega\Sigma X$.
2) The quotient $\Sigma q_n\colon X^n \to \Sigma J_n(X)$ has a functorial cross-section.
3) The inclusion $\Sigma J_{n-1}(X) \to \Sigma J_n(X)$ has a functorial retraction.
4) There is a functorial decomposition
$$\Sigma J_n(X) \simeq \bigvee_{j=1}^{n} X^{(n)}$$
for $0 \leq n \leq \infty$.
5) The James-Hopf map $H_n\colon J(X) \longrightarrow J(X^{(n)})$ is defined combinatorially by
$$H_n(x_1x_2\cdots x_q) = \prod_{1\leq i_1<i_2<\ldots<i_n\leq q} (x_{i_1}\wedge x_{i_2}\wedge\ldots\wedge x_{i_n}),$$
where the product of the right side is ordered lexicographically from right.
6) There is a fibre sequence, which is called EHP sequence,
$$S^n \longrightarrow J(S^n) \simeq \Omega S^{n+1} \xrightarrow{H_2} J(S^{2n}) \simeq \Omega S^{2n+1}$$
localized at 2.

Below we go over some ideas and results from [7]. Let $f\colon X \to \Omega Y$ be a map. The group $K_n(f)$ is defined to be the subgroup of $[X^n, \Omega Y]$ generated by $x_i(f)$ represented by the composite
$$X^n \xrightarrow{\pi_i} X \xrightarrow{f} \Omega Y$$
for $1 \leq i \leq n$, where π_i is the i-th coordinate projection. Let F_n be the free group of rank n generated by letters $x_1, \ldots, x_n$ and let
$$e_f\colon F_n \longrightarrow [X^n, \Omega Y]$$
be the group homomorphism such that $e_f(x_i) = x_i(f)$ for $1 \leq i \leq n$. If $g\colon Y \to Y'$ and $h\colon X' \to X$ be maps, then there is a commutative diagram
$$\begin{array}{ccc} F_n & \xrightarrow{e_f} & [X^n, \Omega Y] \\ \Vert & & \downarrow{\scriptstyle (\Omega g)_*^{h^n}} \\ F_n & \xrightarrow{e_{\Omega g\circ f\circ h}} & [X'^n, \Omega Y']. \end{array}$$
Let $\mathcal{C}$ be a class (or set) of maps $f\colon X \to \Omega Y$, that is, X and Y run over certain class of spaces and f runs over certain maps from X to ΩY. The *Cohen group* $K_n(\mathcal{C})$ is the quotient group of F_n such that

1) If $f\colon X \to \Omega Y$ lies in $\mathcal{C}$, then the homomorphism $e_f\colon F_n \to [X^n, \Omega Y]$ factors through $K_n(\mathcal{C})$.

2) There exists an object $f\colon X\to \Omega Y$ in $\mathcal{C}$ such that the resulting homomorphism $e_f\colon K_n(\mathcal{C})\to [X^n,\Omega Y]$ is a monomorphism.

Note.

1) If $f\colon X\to\Omega Y$ is map between p-local (or p-complete) spaces, then the map e_f extends to the p-localization (or p-completion) of F_n. Thus one can obtain the p-local (or p-complete) version of the Cohen groups along the same lines.
2) There is a short exact sequence

$$1\longrightarrow \bigcap_{f\in\mathcal{C}}\mathrm{Ker}(e_f)\longrightarrow F_n\longrightarrow K_n(\mathcal{C})\longrightarrow 1.$$

Define the group homomorphisms $d_i\colon F_n\to F_{n-1}$ and $s_i\colon F_{n-1}\to F_n$ by

$$d_i(x_j)=\begin{cases} x_j & \text{for } j<i\\ 1 & \text{for } j=i\\ x_{j-1} & \text{for } j>i\end{cases}\qquad s_i(x_j)=\begin{cases} x_j & \text{for } j<i\\ x_{j+1} & \text{for } j\geq i.\end{cases}$$

Let

$$d^i\colon X^{n-1}\to X^n,\quad (z_1,z_2,\ldots,z_{n-1})\mapsto(z_1,\ldots,z_{i-1},*,z_i,\ldots,z_{n-1})$$

$$\tilde{\pi}_i\colon X^n\to X^{n-1},\quad (z_1,z_2,\ldots,z_n)\mapsto(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n)$$

be the maps for $1\leq i\leq n$. Then there are commutative diagrams

$$\begin{array}{ccc} F_n & \xrightarrow{e_f} & [X^n,\Omega Y]\\ \downarrow{d_i} & & \downarrow{d^{i*}}\\ F_{n-1} & \xrightarrow{e_f} & [X^{n-1},\Omega Y]\end{array}\qquad \begin{array}{ccc} F_{n-1} & \xrightarrow{e_f} & [X^{n-1},\Omega Y]\\ \downarrow{s_i} & & \downarrow{\tilde{\pi}_i^*}\\ F_n & \xrightarrow{e_f} & [X^n,\Omega Y]\end{array}$$

for $1\leq i\leq n$. It follows that the composite $F_n\xrightarrow{d_i}F_{n-1}\twoheadrightarrow K_{n-1}(\mathcal{C})$ factors through $K_n(\mathcal{C})$ and the composite $F_{n-1}\xrightarrow{s_i}F_n\twoheadrightarrow K_n(\mathcal{C})$ factors through $K_{n-1}(\mathcal{C})$. Let $d_i\colon K_n(\mathcal{C})\to K_{n-1}(\mathcal{C})$ and $s_i\colon K_{n-1}(\mathcal{C})\to K_n(\mathcal{C})$ denote the resulting homomorphisms. Then there are commutative diagrams

$$\begin{array}{ccc} K_n(\mathcal{C}) & \xrightarrow{e_f} & [X^n,\Omega Y]\\ \downarrow{d_i} & & \downarrow{d^{i*}}\\ K_{n-1}(\mathcal{C}) & \xrightarrow{e_f} & [X^{n-1},\Omega Y]\end{array}\qquad \begin{array}{ccc} K_{n-1}(\mathcal{C}) & \xrightarrow{e_f} & [X^{n-1},\Omega Y]\\ \downarrow{s_i} & & \downarrow{\tilde{\pi}_i^*}\\ K_n(\mathcal{C}) & \xrightarrow{e_f} & [X^n,\Omega Y]\end{array}$$

for $f\in\mathcal{C}$ and $1\leq i\leq n$.

By direct calculation, we have

Lemma 2.8. *In the sequence of the groups $\{K_n(\mathcal{C})\}_{n\geq 1}$, the following identities hold*

1) $d_jd_i=d_id_{j+1}$ *for* $j\geq i$.
2) $s_js_i=s_{i+1}s_j$ *for* $j\leq i$.

3)
$$d_j s_i = \begin{cases} s_{i-1} d_j & \text{for} \quad j < i \\ \text{id} & \text{for} \quad j = i \\ s_i d_{j-1} & \text{for} \quad j > i. \end{cases}$$

Note. The third identity is different from the simplicial identity.

Lemma 2.9. *The $[J_n(X), \Omega Y]$ is the equalizer of the homomorphisms*
$$d^{i*}\colon [X^n, \Omega Y] \to [X^{n-1}, \Omega Y]$$
for $1 \le i \le n$.

Proof. The proof is given by induction on n and the assertion holds for $n = 1$. Suppose that the assertion holds for $n-1$ with $n > 1$. By the suspension splitting theorem, $q_n^*\colon [J_n(X), \Omega Y] \to [X^n, \Omega Y]$ is a monomorphism and so $[J_n(X), \Omega Y]$ can be regarded as a subgroup of $[X^n, \Omega Y]$. Let G_n be the equalizer of the homomorphisms $d_i\colon [X^n, \Omega Y] \to [X^{n-1}, \Omega Y]$ for $1 \le i \le n$. By the definition of $J_n(X)$, we have
$$[J_n(X), \Omega Y] \subseteq G_n.$$
Conversely let $\alpha \in G_n$, that is, $d_1\alpha = d_2\alpha = \ldots = d_n\alpha$. Then $d_1\alpha \in G_{n-1}$ and so $d_1\alpha \in [J_{n-1}(X), \Omega Y]$ by induction. Consider the commutative diagram
$$\begin{array}{ccc} [J_n(X), \Omega Y] & \hookrightarrow & [X^n, \Omega Y] \\ \downarrow & & \downarrow d_1 \\ [J_{n-1}(X), \Omega Y] & \hookrightarrow & [X^{n-1}, \Omega Y]. \end{array}$$
There exists an element $\beta \in [J_n(X), \Omega Y]$ such that $d_1\beta = d_1\alpha$. It follows that $d_i(\alpha\beta^{-1}) = 1$ for all $1 \le i \le n$. Let $g\colon X^n \to \Omega Y$ be a map which represents $\alpha\beta^{-1}$ and let $g'\colon \Sigma X^n \to Y$ be the adjoint of g. Then the composite
$$\Sigma \bigvee_{i=1}^{n} X^{n-1} \xrightarrow{\Sigma(d^1, \ldots, d^n)} \Sigma X^n \xrightarrow{g'} Y$$
is null homotopic. let
$$(X|*)^n = \{(z_1, \ldots, z_n) \in X^n | z_i = * \text{ for some } i\}.$$
Recall that the map
$$\Sigma \bigvee_{i=1}^{n} X^{n-1} \xrightarrow{\Sigma(d^1, d^2, \ldots, d^n)} \Sigma(X|*)^n$$
has a cross-section. Thus the composite
$$\Sigma(X|*)^n \hookrightarrow \Sigma X^n \xrightarrow{g'} Y$$
is null homotopic and so g' factors through $\Sigma X^{(n)}$ up to homotopy or
$$\alpha\beta^{-1} \in [X^{(n)}, \Omega Y] \subseteq [J_n(X), \Omega Y] \subseteq [X^n, \Omega Y].$$
Thus $\alpha \in [J_n(X), \Omega Y]$ and hence the result. □

The *Cohen group* $\mathcal{H}_n(\mathcal{C})$ is defined to the equalizer of the group homomorphisms $d_i\colon K_n(\mathcal{C}) \to K_{n-1}(\mathcal{C})$ for $1 \leq i \leq n$. By Lemma 2.9, there is a homomorphism $e_f\colon \mathcal{H}_n(\mathcal{C}) \to [J_n(X), \Omega Y]$ such that the diagram

$$\begin{array}{ccc} \mathcal{H}_n(\mathcal{C}) & \xrightarrow{e_f} & [J_n(X), \Omega Y] \\ \downarrow & & \downarrow \\ K_n(\mathcal{C}) & \xrightarrow{e_f} & [X^n, \Omega Y] \end{array}$$

commutes for $f \in \mathcal{C}$. By the definition, there is a homomorphism $p_n\colon \mathcal{H}_n(\mathcal{C}) \to \mathcal{H}_{n-1}(\mathcal{C})$ such that the diagram

$$\begin{array}{ccc} \mathcal{H}_n(\mathcal{C}) & \hookrightarrow & K_n(\mathcal{C}) \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle d_i} \\ \mathcal{H}_{n-1}(\mathcal{C}) & \hookrightarrow & K_{n-1}(\mathcal{C}) \end{array}$$

commutes for each $1 \leq i \leq n$.

Lemma 2.10. *The map $p_n\colon \mathcal{H}_n(\mathcal{C}) \to \mathcal{H}_{n-1}(\mathcal{C})$ is an epimorphism for each n.*

Proof. We show by induction that $p_{k,n} = p_{k+1} \circ \ldots \circ p_n\colon \mathcal{H}_n(\mathcal{C}) \to \mathcal{H}_k(\mathcal{C})$ is an epimorphism for $k \leq n$. Clearly $p_{1,n}$ is an epimorphism. Suppose that $p_{k-1,n}$ is an epimorphism with $k > 1$ and let $\alpha \in \mathcal{H}_k(\mathcal{C})$. Since $p_{k-1,n}\colon \mathcal{H}_n(\mathcal{C}) \to \mathcal{H}_{k-1}(\mathcal{C})$ is onto, we may assume that α lies in the kernel of $p_k\colon \mathcal{H}_k(\mathcal{C}) \to \mathcal{H}_{k-1}(\mathcal{C})$. Let

$$\alpha_{k,n} = \prod_{1 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n} s_{i_{n-k}} s_{i_{n-k-1}} \ldots s_{i_1} \alpha \in K_n(\mathcal{C})$$

with lexicographic order from right. Then it is a routine work to check that $\alpha_{k,n} \in \mathcal{H}_n(\mathcal{C})$ with $p_{k,n}(\alpha_{k,n}) = \alpha$ and hence the result. $\square$

Thus one gets the progroup

$$\mathcal{H}(\mathcal{C}) = \lim_{p_n} \mathcal{H}_n(\mathcal{C})$$

together with the representation $e_f\colon \mathcal{H}(\mathcal{C}) \to [J(X), \Omega Y]$ for $f \in \mathcal{C}$. The group $\mathcal{H}(\mathcal{C})$ catches most interesting elements in $[J(X), \Omega Y]$ which are related to the Hopf invariants and the Whitehead products.

Let G be a group. We write $[a, b] = a^{-1}b^{-1}ab$ for $a, b \in G$. Let $\{\Gamma_n G\}$ be the descending central series of G starting with $\Gamma_1 G = G$. Let $L_n(G) = \Gamma_n G/\Gamma_{n+1} G$

Example 2.11. *Let $\mathcal{C}$ consist of the single object $f\colon S^{m-1} \hookrightarrow \Omega S^m$ localized at 2 with $m \geq 1$. We determine $\mathcal{H}(\mathcal{C})$.*

Solution. Consider $e_f\colon F_n \to [X^n, \Omega S^m]$, where $X = S^{m-1}$. Recall that the 3-fold Samelson product $W_3\colon S^{3(m-1)} \to \Omega S^m$ is null homotopic localized at 2, see for instance [5]. Thus

$$e_f([[x_i, x_j], x_k]) = 1$$

for any i, j, k and so e_f factors through $F_n/\Gamma_3 F_n$. By direct calculation, we have $\mathcal{H}_n(\mathcal{C}) \cong \mathcal{H}_2(\mathcal{C})$ for $n \geq 2$ and so $\mathcal{H}(\mathcal{C}) \cong \mathcal{H}_2(\mathcal{C})$. Now we consider several cases:
Case I. The Whitehead square $\omega_m = 0$, that is, $m = 1, 3$ or 7. Then $e_f([x_i, x_j]) = 1$ for any i and j. Thus e_f factors through $F_n/\Gamma_2 F_n = L_1(F_n)$. It follows that $K_n(\mathcal{C})$ is a free abelian group of rank n and so $\mathcal{H}(\mathcal{C}) \cong \mathcal{H}_1(\mathcal{C}) \cong \mathbb{Z}$.
Case II. The Whitehead square $\omega_m \neq 0$ with m odd. Since $2\omega_m = 0$, we have $e_f([x_i, x_j]^2) = 1$. It follows that there is a short exact sequence

$$1 \longrightarrow L_2(F_n)/2L_2(F_n) \longrightarrow K_n(\mathcal{C}) \longrightarrow L_1(F_n) \longrightarrow 1.$$

By direct calculation, $\mathcal{H}_1(\mathcal{C}) \cong \mathbb{Z}$ and $\mathcal{H}_n(\mathcal{C}) \cong \mathcal{H}_2(\mathcal{C})$ for $n \geq 2$ with a short sequence

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathcal{H}_2(\mathcal{C}) \longrightarrow \mathbb{Z} \longrightarrow 1.$$

Case III. m is even. In this case, $k\omega_m \neq 0$ for any non-zero integer k and so $e_f([x_i, x_j]^k) \neq 1$ for $k \neq 0$. Then there is a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_2(\mathcal{C}) \longrightarrow \mathbb{Z} \longrightarrow 1.$$

□

The following theorem is a well-known result in classical homotopy theory [5].

Theorem 2.12. *Let $m > 1$. Then $\Omega[2] \simeq 2 \cdot (H_2 \circ \Omega\omega_m) \colon \Omega S^m \to \Omega S^m$.*

Proof. Let $\mathcal{C}$ consist of the single object $S^{m-1} \hookrightarrow \Omega S^m$. Observe that the maps $2|_{J_n(S^{m-1})}$ and $\Omega[2]|_{J_n(S^{m-1})} \colon J_n(S^{m-1}) \to \Omega S^m$ represent the element $e_f(x_1 \cdots x_n)^2$ and $e_f(x_1^2 x_2^2 \cdots x_n^2)$, respectively. By commutator calculus,

$$x_1^2 x_2^2 \cdots x_n^2 \equiv (x_1 x_2 \cdots x_n)^2 \prod_{1 \leq i < j \leq n} [x_i, x_j] \mod \Gamma_3(F_n).$$

The assertion follows from Example 2.11. □

A space X is called *conilpotent* of class $\leq k$ if the $(k+1)$-fold reduced diagonal $X \longrightarrow \bigwedge_{i=1}^{k+1} X$ is null homotopic. Clearly a space of LS-category $\leq k$ is conilpotent of class $\leq k$. A *coabelian* space means a conilpotent space of class ≤ 1. For instance, a co-H-space is coabelian.

Proposition 2.13. *Let X be a conilpotent space of class $\leq k$ and let $f \colon X \to \Omega Y$ be a map. Then the following relation holds in the group $K_n(f)$:*

The (iterated) commutator on generators

$$[[x_{i_1}, x_{i_2}], \ldots, x_{i_t}] = 1$$

if one of the generators $x_1, \ldots, x_n$ occurs at least $k+1$ times in the commutator bracket.

Proof. Let $\{j_1, j_2, \ldots, j_s\} = \{i_1, i_2, \ldots, i_t\}$ with $j_1 < j_2 < \ldots < j_s$. Let a_i be the occurrence of x_{j_i} in the commutator bracket. Then

$$e_f([[x_{i_1}, x_{i_2}], \ldots, x_{i_t}]) \in [X^n, \Omega Y]$$

is represented by the composite

$$X^n \xrightarrow{\pi_{j_1, j_2, \ldots, j_s}} X^s \xrightarrow{\text{pinch}} X^{(s)} \xrightarrow{\bar{\Delta}} X^{(a_1)} \wedge \ldots \wedge X^{(a_s)} = X^{(t)} \xrightarrow{\sigma} X^{(t)} \xrightarrow{W_t} J(X) \xrightarrow{Jf} \Omega Y,$$

where Jf is the loop map induced by f, σ is a permutation of coordinates, and W_t is the t-fold Samelson product. By the assumption, one of a_i is $\geq k+1$ and hence the result. □

Corollary 2.14. *Let $f\colon X \to \Omega Y$. Suppose that X is a conilpotent space of class $< \infty$. Then $K_n(f)$ is a nilpotent group for each n. In particular, if X is a path-connected finite complex, then $K_n(f)$ is nilpotent for each n.*

Let R be a ring and let $\bar{V}$ be a free R module of rank n with a basis $\{x_1, x_2, \ldots, x_n\}$. Recall that the module Lie(n) over R is the submodule of $\bar{V}^{\otimes n}$ spanned by the Lie elements $\{[[x_{\sigma(1)}, x_{\sigma(2)}], \ldots, x_{\sigma(n)})] | \sigma \in S_n\}$. The symmetric group S_n acts on Lie(n) by permuting letters $x_1, \ldots, x_n$. The following theorem is due to Cohen.

Theorem 2.15. [7] *Let $\mathcal{E}$ be the class of maps $f\colon X \to \Omega Y$ with the condition that X is coabelian and let $\mathcal{E}^{p^r}$ be the subclass of $\mathcal{E}$ consisting of $f\colon X \to \Omega Y$ with the additional condition that $p^r[f] = 1$ in the group $[X, \Omega Y]$. Then*

1) *$K_n(\mathcal{E})$ is the quotient group of $F_n = F(x_1, \ldots, x_n)$ subject to the relations for the iterated commutators on generators*
$$[[x_{i_1}, x_{i_2}], \ldots, x_{i_t}] = 1$$
with $i_a = i_b$ for some $1 \leq a < b \leq t$.
2) *The kernel of the epimorphism $\mathcal{H}_n(\mathcal{E}) \to \mathcal{H}_{n-1}(\mathcal{E})$ is isomorphic to* Lie(n) *over $\mathbb{Z}$.*
3) *$K_n(\mathcal{E}^{p^r})$ is the quotient group of $K_n(\mathcal{E})$ subject to the additional relations*
$$x_i^{p^r} = 1$$
for $1 \leq i \leq n$.
4) *The kernel of the epimorphism $\mathcal{H}_n(\mathcal{E}^{p^r}) \to \mathcal{H}_{n-1}(\mathcal{E}^{p^r})$ is isomorphic to* Lie(n) *over $\mathbb{Z}/p^r$.*

The following proposition is useful.

Proposition 2.16. [7] *In the group $K_n(\mathcal{E})$, the following identity*
$$[[x_{i_1}^{n_1}, x_{i_2}^{n_2}], \ldots, x_{i_t}^{n_t}] = [[x_{i_1}, x_{i_2}], \ldots, x_{i_t}]^{n_1 n_2 \cdots n_t}$$
holds for the generators x_i.

2.3. **An Application to the Stiefel Manifolds.** Recall that the Stiefel manifold $V_{m+n,m}$ is the space of all m-frames in R^{m+n} or, equivalently,
$$V_{m+n,m} = O(m+n)/O(n) = SO(m+n)/SO(n)$$
the quotient homogeneous space of $O(m+n)$ by the sub Lie group $O(n)$. In particular,
$$V_{2n+1,2} = \{(x,y) \in \mathbb{R}^{2n+1} \times R^{2n+1} | \langle x, y\rangle = 0, \|x\| = 1, \|y\| = 1)\}$$
is the spherical tangent bundle $\tau(S^{2n})$ of S^{2n}. Let M be an orientable m-manifold and let $f\colon M \to M$ be a map. The degree of f, denoted by $\deg(f)$, is defined to the degree of the homomorphism $f_*\colon H_m(M, \mathbb{Z}) = \mathbb{Z} \to H_m(M, \mathbb{Z}) = \mathbb{Z}$. A question arising from geometry is whether there is a self homeomorphism f of M which changes the orientation, that is, $\deg(f) = -1$. In the case where $M = V_{2n+1,2}$ with $n > 1$, we will show that for any self homotopy equivalence of preserves the orientation and so the answer of the question above is negative when $M = V_{2n+1,2}$ with $n > 1$.

Recall that $P^{2n}(2)$ is the $(4n-2)$-skeleton of $V_{2n+1,2}$ and so there is a cofibre sequence

$$S^{4n-2} \xrightarrow{\lambda_{2n}} P^{2n}(2) \longrightarrow V_{2n+1,2}.$$

The following lemma can be found in [31], [25, Lemma 12, p.185] or [26, Proposition 2.1, p.274].

Lemma 2.17. *The order of the homotopy class* $[\lambda_{2n}]$ *in* $\pi_{4n-2}(P^{2n}(2))$ *is* 4 *if* $n \equiv 0 \mod 2$ *and* 8 *if* $n \equiv 1 \mod 2$ *and* $n > 1$.

Let $\lambda'_{2n}\colon S^{4n-3} \to \Omega P^{2n}(2)$ be the adjoint map of λ_{2n}.

Lemma 2.18. *Let* $n > 1$. *Then*

$$[2]\circ\lambda_{2n} \simeq * \quad\text{and}\quad [-1]\circ\lambda_{2n} \simeq \lambda_{2n}\circ[-3]\colon S^{4n-2} \to P^{2n}(2)$$

for $n > 1$.

Proof. Observe that $J_2(P^{2n-1}(2))$ is the $(6n-7)$-skeleton of $J(P^{2n-1}(2)) \simeq \Omega P^{2n}(2)$. It suffices to show that the homotopy class

$$[\Omega[-1]|_{J_2(P^{2n-1})}\circ\lambda'_{2n}] = -3[\lambda'_{2n}]$$

in $\pi_{4n-3}(\Omega P^{2n}(2))$. Since the pinch map $P^{2n}(2) \to S^{2n}$ factors through $V_{2n+1,2}$, the composite $[2]\circ\lambda_{2n}$ is null homotopic by Proposition 2.5. Consider the Cohen representation

$$e_f\colon \mathcal{H}_2(\mathcal{E}) \to [J_2(P^{2n-1}(2)), \Omega P^{2n}(2)],$$

where f is the inclusion $P^{2n-1} \hookrightarrow \Omega P^{2n}(2)$. The maps

$$\Omega[-1]|_{J_2(P^{2n-1}(2))}, \quad \Omega[2]|_{J_2(P^{2n-1}(2))} \quad\text{and}\quad k|_{J_2(P^{2n-1}(2))}$$

represent elements $x_1^{-1}x_2^{-1}$, $x_1^2x_2^2$ and $(x_1x_2)^k$, respectively. Since

$$x_1^2x_2^2 = (x_1x_2)^2\cdot[x_1,x_2] = (x_1x_2)^2\cdot(x_1^{-1}x_2^{-1})\cdot(x_1x_2)$$

in $\mathcal{H}_2(\mathcal{E})$, we have

$$0 = [\Omega[2]|_{J_2(P^{2n-1}(2))}\circ\lambda'_{2n}] = 2[\lambda'_{2n}] + [\Omega[-1]|_{J_2(P^{2n-1}(2))}\circ\lambda'_{2n}] + [\lambda'_{2n}]$$

in $\pi_{4n-3}(\Omega P^{2n}(2))$ and hence the result. □

Theorem 2.19. *Let* $n > 1$ *and let* $f\colon V_{2n+1,2} \to V_{2n+1,2}$ *be a map. Let*

$$g = f|_{P^{2n}(2)}\colon P^{2n}(2) \to P^{2n}(2)$$

be the restriction of f. *Then*

$$\deg(f) \equiv \begin{cases} 1 \mod 4 & \text{if} \quad g\simeq \mathrm{id} \quad\text{and}\quad n\equiv 0 \mod 2\\ 1 \mod 8 & \text{if} \quad g\simeq \mathrm{id} \quad\text{and}\quad n\equiv 1 \mod 2\\ 1 \mod 4 & \text{if} \quad g\simeq [-1] \quad\text{and}\quad n\equiv 0 \mod 2\\ 5 \mod 8 & \text{if} \quad g\simeq [-1] \quad\text{and}\quad n\equiv 1 \mod 2\\ 0 \mod 4 & \text{if} \quad g\simeq [2] \quad\text{and}\quad n\equiv 0 \mod 2\\ 0 \mod 8 & \text{if} \quad g\simeq [2] \quad\text{and}\quad n\equiv 1 \mod 2\\ 0 \mod 4 & \text{if} \quad g\simeq * \quad\text{and}\quad n\equiv 0 \mod 2\\ 0 \mod 8 & \text{if} \quad g\simeq * \quad\text{and}\quad n\equiv 1 \mod 2 \end{cases}$$

Proof. There is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega S^{4n-1} & \xrightarrow{\partial} & F_q & \longrightarrow & V_{2n+1,2} & \xrightarrow{q} & S^{4n-1} \\
\downarrow{\scriptstyle \Omega[\deg(f)]} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle [\deg(f)]} \\
\Omega S^{4n-1} & \xrightarrow{\partial} & F_q & \longrightarrow & V_{2n+1,2} & \xrightarrow{q} & S^{4n-1},
\end{array}$$

where q is the pinch map. By computing low-dimensional homology of F_q, $P^{2n}(2)$ is the $(6n-4)$-skeleton of F_q and so there is a homotopy commutative diagram

$$\begin{array}{ccc}
S^{4n-2} & \xrightarrow{\lambda_{2n}} & P^{2n}(2) \\
\downarrow{\scriptstyle [\deg(f)]} & & \simeq \downarrow{\scriptstyle g} \\
S^{4n-2} & \xrightarrow{\lambda_{2n}} & P^{2n}(2).
\end{array}$$

The assertion follows from Lemmas 2.17 and 2.18. □

Corollary 2.20. *Let $n > 1$. Then*

(1). *Any self homotopy equivalence of $V_{2n+1,2}$ preserves the orientation. In particular, there is no self homeomorphism of $V_{2n+1,2}$ which changes the orientation.*
(2). *If f is a 2-local homotopy equivalence of $V_{2n+1,2}$, then $\deg(f) \equiv 1 \mod 4$.*
(3). *If $n \equiv 0 \mod 2$, then there is no self map of $V_{2n+1,2}$ of degree 2 or 3 $\mod 4$.*
(4). *If $n \equiv 1 \mod 2$, then there is no self map of $V_{2n+1,2}$ is of degree $2, 3, 4, 6$ or $7 \mod 8$.*
(5). *If $n \equiv 1 \mod 2$ and f is a self homotopy equivalence of $V_{2n+1,2}$, then f restricted to $P^{2n}(2)$ is homotopic to the identity map of $P^{2n}(2)$.*
(6). *If $n \equiv 0 \mod 2$, then there is a self homotopy equivalence f of $V_{2n+1,2}$ such that $f|_{P^{2n}(2)}\colon P^{2n}(2) \to P^{2n}(2)$ is homotopic to $[-1]$.*

2.4. **The Strong Form of the Kervaire Invariant One Problem.** Consider the Whitehead square ω_n in $\pi_{2n-1}(S^n)$. That ω_n is zero precisely when $n = 1, 3, 7$ is equivalent to the classical problem of the existence of elements of Hopf invariant one which was solved completely in [1]. Restrict attention to odd integers n which are not equal to $1, 3$ or 7. In these cases ω_n is non-zero and is of order 2. There is a short exact sequence of groups localized at 2

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{2n-1}(S^n) \longrightarrow \pi^s_{2n-1}(S^n) \longrightarrow 0.$$

Hence ω_n is divisible by 2 in $\pi_{2n-1}(S^n)$ if and only if this sequence fails to split. It is well known that ω_n is *not* divisible by 2 if $n \neq 2^k - 1$ for some k. A proof can be found in [5]. Of the remaining cases given by $n = 2^k - 1$, it is known that ω_n is divisible by 2 if $n = 1, 3, 7, 15, 31$ or 63, see [3, 20, 34]. The cases for which $n = 2^k - 1 > 63$ remain open.

Note. For even integers n, it is a routine exercise to show that ω_n is *not* divisible by 2 if $n \neq 2, 4$ or 8. According to [42], ω_2 is divisible by 2 but ω_4 and ω_8 are *not*.

The Strong form of the Kervaire invariant one conjecture is that ω_n is divisible by 2 when $n = 2^k - 1$. There are many reformations of this conjecture, see for instance [5, 8, 34]. Several reformulations in terms of mod 2 Moore spaces and the Stiefel manifolds are discussed below.

2.4.1. *Spherical Classes.* Let X be a space. An element $a \in H_*(X)$ is called *spherical* if there is an element b in $\pi_*(X)$ with Hurewicz image in mod 2 homology given by a. Recall that $H_*(\Omega P^{n+1}(2)) = T(u, v)$, where $|v| = n$ and $Sq^1_* v = u$.

Proposition 2.21. *The Whitehead square ω_{2n-1} is divisible by 2 in $\pi_{4n-3}S^{2n-1}$ if and only if u^2 is spherical in $H_*(\Omega P^{2n+1}(2))$.*

Proof. First suppose that $\omega_{2n-1} : S^{4n-3} \to S^{2n-1}$ is divisible by 2. Let $f\colon S^{4n-3} \to S^{2n-1}$ be any map and let $f'\colon S^{4n-4} \to \Omega S^{2n-1}$ be the adjoint map of f. Then the composite $S^{4n-4} \xrightarrow{f'} \Omega S^{2n-1} \xrightarrow{H_2} \Omega S^{4n-3}$ is null homotopic and so, by Theorem 2.12, there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{[2]} & S^{2n-1} \\ \uparrow f & & \uparrow f \\ S^{4n-3} & \xrightarrow{[2]} & S^{4n-3}. \end{array}$$

Thus the composite $S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1} \longrightarrow P^{2n}(2)$ is null homotopic. Consider the homotopy commutative diagram

$$\begin{array}{ccc} S^{4n-3} & \xrightarrow{\omega_{2n-1}} & S^{2n-1} \\ \cap\downarrow & & \cap\downarrow \\ \Sigma P^{2n-1}(2) \wedge P^{2n-1}(2) & \xrightarrow{W_2} & P^{2n}(2), \end{array}$$

where $W_2 : \Sigma X \wedge X \to \Sigma X$ is Whitehead product. It follows that there is a homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccccc} S^{4n-3} & \longrightarrow & * & \longrightarrow & S^{4n-2} & = & S^{4n-2} \\ \cap\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma P^{2n-1}(2) \wedge P^{2n-1}(2) & \xrightarrow{W_2} & P^{2n}(2) & \hookrightarrow & J_2(P^{2n}(2)) & \xrightarrow{\text{pinch}} & P^{2n}(2) \wedge P^{2n}(2) \end{array}$$

and so u^2 is spherical.

Conversely suppose that u^2 is spherical in $H_*(\Omega P^{2n+1}(2))$. We may assume that $n > 1$. Since $J_2(P^{2n}(2))$ is the (6n-4)-skeleton of $\Omega P^{2n+1}(2) \simeq J(P^{2n}(2))$, the element u^2 is spherical in $H_* J_2(P^{2n}(2))$. Thus there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega(P^{2n}(2)\wedge P^{2n}(2)) & \longrightarrow & F_q & \longrightarrow & J_2(P^{2n}(2)) & \xrightarrow{q} & P^{2n}(2)\wedge P^{2n}(2) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow i \\
\Omega S^{4n-2} & \longrightarrow & * & \longrightarrow & S^{4n-2} & = & S^{4n-2}
\end{array}$$

where i is the canonical inclusion and p is the pinch map. By inspecting the Serre spectral sequence for the fibre sequence

$$F \longrightarrow J_2(P^{2n}(2)) \longrightarrow P^{2n}(2)\wedge P^{2n}(2),$$

the space $P^{2n}(2)$ is the $(6n-5)$-skeleton of F. Thus the composite

$$S^{4n-3} \longrightarrow \Sigma P^{2n-1}(2)\wedge P^{2n-1}(2) \longrightarrow P^{2n}(2)$$

is null homotopic or the composite

$$S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1} \longrightarrow P^{2n}(2)$$

is null homotopic. It follows that $\omega_{2n-1}: S^{4n-3}\to S^{2n-1}$ lifts to the homotopy fibre E of the injection $S^{2n-1}\to P^{2n}(2)$. Consider the fibre sequence

$$\Omega S^{2n-1} \xrightarrow{\Omega j} \Omega P^{2n}(2) \longrightarrow E \xrightarrow{f} S^{2n-1} \xrightarrow{j} P^{2n}(2).$$

Since $\Omega j_*: H_*(\Omega S^{2n-1}) \to H_*(\Omega P^{2n}(2))$ is a monomorphism, the Serre spectral sequence for the fibre sequence $\Omega S^{2n-1}\to \Omega P^{2n}(2)\to E$ collapses. By [12, Lemma 2.1], the class $[u,v]$ is spherical in $H_*\Omega P^{2n}(2)$ and so is in $H_*(E)$. Hence the $(4n-3)$-skeleton $\mathrm{sk}_{4n-3}(E)$ of E is homotopy equivalent to $S^{2n-1}\vee S^{4n-3}$ and $f|_{S^{4n-3}}$ is null. Now let $\theta: S^{4n-3}\longrightarrow E$ be a lifting of ω_{2n-1}. Let θ_1 and θ_2 denote the composites

$$S^{4n-3} \xrightarrow{\theta} \mathrm{sk}_{4n-3}(E) \xrightarrow{\text{proj.}} S^{2n-1} \hookrightarrow E \quad \text{and}$$

$$S^{4n-3} \xrightarrow{\theta} \mathrm{sk}_{4n-3}(E) \xrightarrow{proj.} S^{4n-3} \hookrightarrow E,$$

respectively. Because $f|_{S^{2n-1}}$ is of degree 2, we have

$$\omega_{2n-1} \simeq f\circ\theta \simeq f\circ\theta_1 \simeq [2]\circ\theta_1.$$

The assertion follows from the fact that

$$[2]_* = 2 : \pi_{4n-3}(S^{2n-1}) \to \pi_{4n-3}(S^{2n-1}).$$

□

Recall that $H_*(S^n\{2\}) \cong P(u)\otimes E(v)$ as $P(u)$-module with $|v|=n$ and $Sq^1_* v = u$.

Corollary 2.22. *The Whitehead square ω_{2n-1} is divisible by 2 if and only if u^2 is spherical in $H_*(S^{2n}\{2\})$.*

Proof. The homotopy commutative diagram

$$\begin{array}{ccccccc} \Omega S^{2n+1} & \longrightarrow & F^{2n+1}\{2\} & \longrightarrow & P^{2n+1}(2) & \xrightarrow{\text{pinch}} & S^{2n+1} \\ \uparrow\cup & & \uparrow & & & & \\ S^{2n} & \xrightarrow{[2]} & S^{2n} & & & & \end{array}$$

induces a canonical inclusion $j\colon S^{2n}\{2\} \to \Omega P^{2n+1}(2)$ so that

$$j_*\colon H_r(S^{2n}\{2\}) \longrightarrow H_r(\Omega P^{2n+1}(2))$$

is one to one for all r and onto for $r \leq 4n-2$ and hence the result. □

2.4.2. *Exponents of Homotopy Groups.*

Proposition 2.23. *Let $n \geq 2$. Suppose that u^2 is spherical in $H_{4n-2}(S^{2n}\{2\})$. Let $x \in \pi_{4n-2}(S^{2n}\{2\})$ be any element with Hurewicz image in mod 2 homology given by u^2. Then x generates a $\mathbb{Z}/8$-summand in $\pi_{4n-2}(S^{2n}\{2\})$.*

Proof. Since u^2 is spherical, we have

$$\mathrm{sk}_{4n-2}(S^{2n}\{2\}) \simeq P^{2n}(2) \vee S^{4n-2}$$

and so there is a cofibre sequence

$$S^{4n-2} \xrightarrow{\phi} P^{2n}(2) \vee S^{4n-2} \overset{j}{\hookrightarrow} \mathrm{sk}_{4n-1}(S^{2n}\{2\}) \longrightarrow S^{4n-1}.$$

Let ϕ_1 and ϕ_2 denote the composites

$$S^{4n-2} \xrightarrow{\phi} P^{2n}(2) \vee S^{4n-2} \xrightarrow{\text{proj.}} P^{2n}(2) \quad \text{and} \quad S^{4n-2} \xrightarrow{\phi} P^{2n}(2) \vee S^{4n-2} \xrightarrow{\text{proj.}} S^{4n-2},$$

respectively. Since $\bar{H}_*(\mathrm{sk}_{4n-1}(S^{2n}\{2\}))$ has a basis u, v, u^2 and uv with $Sq^1_* v = u$ and $Sq^1_*(uv) = u^2$, the map $\phi_2\colon S^{4n-2} \to S^{4n-2}$ is of degree 2. Consider the homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccccc} S^{4n-2} & \xrightarrow{\phi} & P^{2n}(2) \vee S^{4n-2} & \longrightarrow & \mathrm{sk}_{4n-1}(S^{2n}\{2\}) & \longrightarrow & S^{4n-1} \\ \| & & \downarrow & & \downarrow q & & \| \\ S^{4n-2} & \xrightarrow{\phi_1} & P^{2n}(2) & \xrightarrow{f} & X^{2n} & \longrightarrow & S^{4n-1}, \end{array}$$

where $X^{2n} = C_{\phi_1}$. Since $q_*\colon H_*(\mathrm{sk}_{4n-1}(S^{2n}\{2\})) \to H_*(X^{2n})$ is onto, we have

$$H_*(X^{2n}) \cong E(\bar{u}) \otimes E(\bar{v})$$

as coalgebras, where $\bar{x} = q_*(x)$. By inspecting the Eilenberg-Moore spectral sequence for $\Omega X^{2n} \to * \to X^{2n}$, we have

$$H_*(\Omega X^{2n}) \cong P(s^{-1}\bar{u}, s^{-1}\bar{v})$$

the polynomial on $s^{-1}\bar{u}$ and $s^{-1}\bar{v}$. Thus

$$\Omega f_*\colon H_*(\Omega P^{2n}(2)) \cong T(s^{-1}\bar{u}, s^{-1}\bar{v}) \to H_*(\Omega X^{2n}) \cong P(s^{-1}\bar{u}, s^{-1}\bar{v})$$

is onto and so there is a short exact sequence of Hopf algebras

$$H_*(\Omega F_f) \longrightarrow H_*(\Omega P^{2n}(2)) \longrightarrow H_*(\Omega X^{2n}).$$

Observe that $\phi_1 : S^{4n-2} \to P^{2n}(2)$ lifts to F_f with $\phi_{1*} : H_{4n-2}(S^{4n-2}) \cong H_{4n-2}(F_f)$ and $\bar{H}_r(F_f) = 0$ for $r < 4n-2$. Let $\phi_1' : S^{4n-3} \to \Omega P^{2n}(2)$ the adjoint map of ϕ_1. Then $\phi'_{1*}(\iota_{4n-3}) = [s^{-1}\bar{u}, s^{-1}\bar{v}]$ which is the only primitive element in $H_{4n-3}(\Omega P^{2n}(2))$. Since $2n$ should be a power of 2, the homotopy class $[\phi_1]$ of ϕ_1 is of order 4 in $\pi_{4n-2}(P^{2n}(2))$ by [12, Theorem 2.2]. Now, by the cofibre sequence

$$S^{4n-2} \xrightarrow{\phi} P^{2n}(2) \vee S^{4n-2} \xrightarrow{j} \mathrm{sk}_{4n-1}(S^{2n}\{2\}) \longrightarrow S^{4n-1},$$

there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} = \pi_{4n-2}(S^{4n-2}) \xrightarrow{\phi_*} \pi_{4n-2}(P^{2n}(2)) \oplus \mathbb{Z} \xrightarrow{j_*} \pi_{4n-2}(S^{2n}\{2\}) \longrightarrow 0$$

with $\phi_*(\iota_{4n-2}) = [\phi_1] + 2[\iota_{4n-2}]$. Thus

$$8x = 8j_*([\iota_{4n-2}]) = 4j_*(2[\iota_{4n-2}]) = 4j_*([\phi_1]) = 0$$

and

$$4x = 4j_*([\iota_{4n-2}]) = j_*(2[\phi_1]) \neq 0$$

since $2[\phi_1] \notin \mathrm{Im}(\phi_*)$. Observe that x is not divisible by 2. Thus x generates a $\mathbb{Z}/8$-summand and hence the result. ☐

Proposition 2.24. *If ω_{2n-1} is not divisible by 2, then $4\pi_r(S^{2n}\{2\}) = 0$ for $r \leq 4n-2$.*

Proof. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccccc}
\Omega^2 S^{4n-1} & \xrightarrow{\Omega^2[4]} & \Omega^2 S^{4n-1} & \longrightarrow & \Omega(S^{4n-1}\{4\}) & \longrightarrow & \Omega S^{4n-1} & \xrightarrow{\Omega[4]} & \Omega S^{4n-1} \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \bar{H}_2} & & \uparrow{\scriptstyle H_2} & & \uparrow{\scriptstyle H_2} \\
\Omega^2 S^{2n} & \xrightarrow{\Omega^2[2]} & \Omega^2 S^{2n} & \xrightarrow{\Omega f} & \Omega(S^{2n}\{2\}) & \longrightarrow & \Omega S^{2n} & \xrightarrow{\Omega[2]} & \Omega S^{2n} \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \bar{E}} & & \uparrow{\scriptstyle E} & & \uparrow{\scriptstyle E} \\
\Omega S^{2n-1} & \xrightarrow{\Omega[2]} & \Omega S^{2n-1} & \longrightarrow & S^{2n-1}\{2\} & \longrightarrow & S^{2n-1} & \xrightarrow{[2]} & S^{2n-1}.
\end{array}$$

The map

$$\bar{E}_* \colon \pi_r(S^{2n-1}\{2\}) \longrightarrow \pi_r(\Omega(S^{2n}\{2\}))$$

is onto for $r \leq 4n-4$ and there is an exact sequence

$$\pi_{4n-3}(S^{2n-1}\{2\}) \longrightarrow \pi_{4n-3}(\Omega(S^{2n}\{2\})) \longrightarrow \pi_{4n-3}(\Omega(S^{4n-1}\{4\})).$$

Since u^2 is not spherical, $\pi_{4n-3}(\Omega(S^{2n}\{2\})) \to \pi_{4n-3}(\Omega(S^{4n-1}\{4\})) = \mathbb{Z}/4$ is not an epimorphism. Recall that

$$[\Omega H_2 \circ \omega'_{2n}] = 2[\iota_{4n-3}]$$

in $\pi_{4n-3}(\Omega^2 S^{4n-1})$, where $\omega'_{2n} \colon S^{4n-3} \to \Omega^2 S^{2n}$ is the adjoint map of Whitehead square ω_{2n}. Thus

$$\mathrm{Im}(\bar{H}_{2*} : \pi_{4n-3}(\Omega(S^{2n}\{2\})) \longrightarrow \pi_{4n-3}(\Omega(S^{4n-1}\{4\}))) = \mathbb{Z}/2$$

which is generated by $[\bar{H}_2 \circ \Omega f \circ \omega'_{2n}]$. By Theorem 2.12, we have

$$\Omega^2[2] \circ \omega'_{2n} = 2\omega'_{2n} + \Omega^2\omega_{2n} \circ \Omega H_2 \circ \omega'_{2n} = 4\omega'_{2n}.$$

Thus $4\Omega f_*(\omega'_{2n}) = 0$ and so

$$\pi_{4n-3}(S^{2n-1}\{2\}) \oplus \mathbb{Z}/4 \to \pi_{4n-3}(\Omega(S^{2n}\{2\}))$$

is onto. According to [12], $4\pi_r(S^{2n-1}\{2\}) = 0$ for $r \leq 4n-3$ and hence the result. □

Corollary 2.25. *Let $n \geq 2$. Then the Whitehead square ω_{2n-1} is not divisible by 2 if and only if $4\pi_r(S^{2n}\{2\}) = 0$ for $r \leq 4n-2$.*

Proposition 2.26. *Let $n > 1$. Suppose that u^2 is spherical in $H_*(\Omega P^{2n+1}(2))$. Let x be any element in $\pi_{4n-2}(\Omega P^{2n+1}(2))$ with the Hurewicz image in mod 2 homology given by u^2. Then x is of order 4.*

Proof. Suppose that $2x = 0$. Then there is a map $\phi : P^{4n+1}(2) \to \Omega P^{2n+1}(2)$ such that $\phi|_{S^{4n-2}}$ is a representative of x in $\pi_{4n-2}(\Omega P^{2n+1}(2))$. Let $\{U, V\}$ be a basis for $\bar{H}_*(P^{4n-1}(2))$ with $Sq^1_*V = U$. Since $\phi_*(U) = u^2$, we have $Sq^1_*\phi_*(V) = \phi_*(U) = u^2$ and so $\phi_*(V) = uv$ or vu. This is a contradiction because uv and vu are *not* primitive. Thus $2x \neq 0$.

Now we show that $4\pi_{4n-2}(\Omega P^{2n+1}(2)) = 0$ from which we obtain $4x = 0$. In the Cohen group $K_2(\mathcal{E}^{\mathbb{Z}/4})$, we have

$$(x_1x_2)^4 = [x_1, x_2]^{-2}$$

and so

$$4y = -2(\Omega W_2 \circ H_2)_*(y)$$

for any $y \in \pi_r\Omega P^{2n+1}(2)$ with $r \leq 6n-4$, where $W_2 : \Sigma(P^{2n}(2))^{(2)} \to P^{2n+1}(2)$ is the Whitehead product. Since $2 \cdot \pi_{4n-2}(\Omega\Sigma(P^{2n}(2))^{(2)}) = 0$, we have $4y = 0$ and hence the result. □

2.4.3. *Partial H-spaces.* A map $f\colon X \to Y$ is called a *partial H-space* if there is a map $\tilde{f}\colon J_2(X) \to Y$ such that $\tilde{f}|_X = f$, or equivalently there is a map $\mu\colon X \times X \to Y$ such that $\mu|_{X\times *} = \mu|_{*\times X} = f$. The identity map $f = \mathrm{id}_X\colon X \to X$ is a partial H-space if and only if X is an H-space. If X is an H-space, then any map from $X \to Y$ is a partial H-space.

It is well known that S^{2n-1} is an H-space if and only if ω_{2n-1} is null homotopic (and so if and only if $2n-1 = 1, 3$ or 7). Consider the spherical fibration

$$S^{2n-1} \xrightarrow{i} V_{2n+1,2} \longrightarrow S^{2n}.$$

One may ask whether the inclusion $i\colon S^{2n-1} \to V_{2n+1,2}$ is a partial H-space. This question is equivalent to the strong form of the Kervaire invariant one problem.

Proposition 2.27. *The following are equivalent.*

(1). *The Whitehead square ω_{2n-1} is divisible by 2.*
(2). *the inclusion $i\colon S^{2n-1} \to P^{2n}(2)$ is a partial H-space.*
(3). *the inclusion $i\colon S^{2n-1} \to V_{2n+1,2}$ is a partial H-space.*

Proof. (2) ⇒ (3) is obvious. (1) ⇒ (2). Since ω_{2n-1} is divisible by 2, the composite

$$S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1} \xhookrightarrow{i} P^{2n}(2)$$

is null homotopic and so the inclusion $i\colon S^{2n-1} \to P^{2n}(2)$ extends to $J_2(S^{2n-1})$ or i is a partial H-space.

(3) ⇒ (1). Since the inclusion $i\colon S^{2n-1} \to V_{2n+1,2}$ extends to $J_2(S^{2n-1})$, the composite

$$S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1} \xhookrightarrow{i} V_{2n+1,2}$$

is null homotopic and so there is a homotopy commutative diagram

$$\begin{array}{ccccccc}
S^{4n-3} & \xrightarrow{\omega} & S^{2n-1} & & & & \\
\downarrow{\scriptstyle g} & & \| & & & & \\
\Omega S^{2n} & \xrightarrow{\partial} & S^{2n-1} & \xrightarrow{i} & V_{2n+1,2} & \longrightarrow & S^{2n} \\
\uparrow{\scriptstyle E} & & \| & & & & \\
S^{2n-1} & \xrightarrow{[2]} & S^{2n-1}, & & & &
\end{array}$$

where the middle row is the fibre sequence. Recall that $E_*\colon \pi_r(S^{2n-1}) \to \pi_r(\Omega S^{2n})$ is onto for $r \leq 4n-3$. Thus there is a map $\tilde{g}\colon S^{4n-3} \to S^{2n-1}$ such that $\omega_{2n-1} \simeq [2] \circ \tilde{g}$ and hence the result. □

The question whether the inclusion $i\colon S^{2n-1} \to V_{2n+1,2}$ is a partial H-space admits some geometric means, namely whether there is a map

$$\phi\colon S^{2n-1} \times S^{2n-1} \xrightarrow{(\phi_0,\theta_0,\phi_1,\theta_1)} \mathbb{R}^{2n} \times \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}$$

such that

1) $\phi_0(x,*) = \phi_0(*,x) = x$;
2) $\phi_1(x,*) = \phi_1(*,x) = 0$
3) $\theta_0(x,*) = \theta_0(*,x) = 0$;
4) $\theta_1(x,*) = \theta_1(*,x) = 1$;
5) (ϕ_0,θ_0) and (ϕ_1,θ_1) are nowhere zero;
6) (ϕ_1,θ_1) is orthogonal to (ϕ_0,θ_0).

The map ϕ_0 looks like a multiplication on $\mathbb{R}^{2n}$ with possible zero divisors, where $2n$ is assumed to be a power of 2. The Cayley-Dickson multiplication might be a candidate for the map ϕ_0. Condition (5) requires that $\theta_0(x,y) \neq 0$ whenever $\phi_0(x,y) = 0$ for $x,y \in S^{2n-1}$. The map (ϕ_1,θ_1) looks like certain *'derivative'* of (ϕ_0,θ_0).

2.4.4. *Spherical Fibrations.*

Proposition 2.28. *Suppose that the Whitehead square ω_{2n-1} is divisible by 2. Then there is a spherical fibration*

$$S^{2n-1} \longrightarrow E^{4n-1} \longrightarrow P^{2n}(2)$$

such that the boundary map $\partial\colon \Omega P^{2n}(2) \to S^{2n-1}$ is onto in mod 2 homology.

Proof. When $2n-1 = 1, 3$ or 7, the space E^{4n-1} can be chosen as the homotopy pull back of the diagram $S^{4n-1} \xrightarrow{q} S^{2n} \xleftarrow{\text{pinch}} P^{2n}(2)$ and so we may assume that $n \neq 1, 3, 7$.

Let $q\colon P^{2n-1}(2) \to S^{2n-1}$ be the pinch map. Consider the homotopy commutative diagram

$$\begin{array}{ccccc}
S^{4n-5} & \hookrightarrow & P^{2n-1}(2) \wedge P^{2n-2}(2) & \xrightarrow{\text{pinch}} & P^{2n-1}(2) \wedge P^{2n-2}(2)/S^{4n-5} \\
\downarrow{\scriptstyle \omega_{2n-2}} & & \downarrow{\scriptstyle W_2} & & \downarrow{\scriptstyle \bar{W}_2} \\
S^{2n-2} & \hookrightarrow & P^{2n-1}(2) & \xrightarrow{q} & S^{2n-1} \\
\cap\downarrow & & \downarrow & & \cap\downarrow \\
J_2(S^{2n-2}) & \longrightarrow & J_2(P^{2n-1}(2)) & \longrightarrow & J_2(P^{2n-1}(2))/J_2(S^{2n-2}),
\end{array}$$

where $\bar{W}_2$ is the composite

$$P^{2n-1}(2) \wedge P^{2n-2}(2)/S^{4n-5} \xrightarrow{\text{pinch}} P^{4n-3}(2) \xrightarrow{\text{pinch}} S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1}.$$

Since ω_{2n-1} is divisible by 2, the map $\bar{W}_2$ is null homotopic and so there is a retraction $r\colon J_2(P^{2n-1}(2))/J_2(S^{2n-2}) \to S^{2n-1}$. Let $\bar{q}\colon J_2(P^{2n-1}(2)) \to S^{2n-1}$ be the composite

$$J_2(P^{2n-1}(2)) \xrightarrow{\text{pinch}} J_2(P^{2n-1}(2))/J_2(S^{2n-2}) \xrightarrow{r} S^{2n-1}.$$

Then $\bar{q}|_{P^{2n-1}(2)} = q$. Let $\{u, v\}$ be a basis for $\bar{H}_*(P^{2n-1}(2))$ with $Sq^1_* v = u$. Consider the Serre spectral sequence for the fibre sequence

$$\Omega S^{2n-1} \longrightarrow F_{\bar{q}} \longrightarrow J_2(P^{2n-1}(2)) \xrightarrow{\bar{q}} S^{2n-1}.$$

Then

$$H_r(F_{\bar{q}}) \to H_r(J_2(P^{2n-1}(2)))$$

is a monomorphism for $r \leq 6n-7$. A basis for $H_r(F_{\bar{q}})$ for $r \leq 6n-7$ is given by $\{u, u^2, [u, v], v^2]\}$. Since $\bar{q}|_{J_2(S^{2n-2})}$ is null homotopic. The inclusion

$$J_2(S^{2n-2}) \to J_2(P^{2n-1}(2))$$

lifts to $F_{\bar{q}}$ and so $\mathrm{sk}_{4n-4}(F_{\bar{q}}) \simeq J_2(S^{2n-2})$ with a cofibre sequence

$$P^{4n-3}(2) \longrightarrow J_2(S^{2n-2}) \longrightarrow \mathrm{sk}_{4n-2}(F_{\bar{q}}).$$

Let $s\colon \Sigma X^{(2)} \to \Sigma J_2(X)$ be a functorial cross-section of the pinch map $\Sigma J_2(X) \to \Sigma X^{(2)}$ and let E^{4n-1} be the homotopy cofibre of the composite

$$S^{4n-3} \xrightarrow{s} \Sigma J_2(S^{2n-2}) \hookrightarrow \Sigma\, \mathrm{sk}_{4n-2} F_{\bar{q}}.$$

Then there is a homotopy commutative diagram

$$\begin{array}{ccc}
S^{4n-3} & \hookrightarrow & P^{2n-1}\wedge P^{2n}(2) \\
\downarrow & & \downarrow{\scriptstyle s} \\
\Sigma\,\mathrm{sk}_{4n-2}(F_{\bar{q}}) & \hookrightarrow & \Sigma J_2(P^{2n-1}(2)) \\
\downarrow{\scriptstyle \phi} & & \downarrow \\
E^{4n-1} & \xrightarrow{\theta} & P^{2n}(2).
\end{array}$$

We show that the homotopy fibre of the map θ is S^{2n-1}. By considering the commutative diagram

$$\begin{array}{ccc}
H_*(\Omega\Sigma\,\mathrm{sk}_{4n-2}(\bar{F}_{\bar{q}})) & \longrightarrow & H_*(\Omega\Sigma J_2(P^{2n-1}(2))) \\
\downarrow & & \downarrow \\
H_*(\Omega E^{4n-1}) & \xrightarrow{\theta_*} & H_*(\Omega P^{2n}(2)),
\end{array}$$

the elements $u, [u,v]$ and v^2 of $H_*(\Omega P^{2n}(2)) = T(u,v)$ lie in the image of θ_* and so the Poincaré series

$$\chi(H_*(\Omega E^{4n-1})) \geq \chi(T(u,[u,v],v^2)).$$

By considering the Eilenberg-Moore spectral sequence for $\Omega E^{4n-1} \to * \to E^{4n-1}$, we have

$$\chi(H_*(\Omega E^{4n-1})) \leq \chi(T(u,[u,v],v^2)).$$

Thus $\chi(H_*(\Omega E^{4n-1})) = \chi(T(u,[u,v],v^2))$ or $H_*(\Omega E^{4n-1}) \cong T(u,[u,v],v^2)$ and therefore $\theta_*\colon H_*(\Omega E^{4n-1}) \to H_*(\Omega P^{2n}(2))$ is a monomorphism. Hence the Serre spectral sequence for the fibre sequence $\Omega E^{4n-1} \longrightarrow \Omega P^{2n}(2) \xrightarrow{\partial} F_\theta$ collapses and so

$$\chi(H_*(F_\theta)) = \frac{\chi(T(u,v))}{\chi(T(u,[u,v],v^2))} = 1 + t^{2n-1}$$

It follows that F_θ is homotopy equivalent to S^{2n-1} and hence the result. □

Proposition 2.29. *Let $n \geq 2$. Suppose that there is a spherical fibration*

$$S^{2n-1} \longrightarrow E^{4n-1} \longrightarrow P^{2n}(2)$$

such that the boundary map $\partial\colon \Omega P^{2n}(2) \to S^{2n-1}$ is onto in mod 2 homology. Then

1) *the homotopy cofibre of the map $E^{4n-1} \to P^{2n}(2)$ has a non-trivial cup product.*
2) *The Whitehead square ω_{2n-1} is divisible by 2.*

Proof. Consider the Serre spectral sequence for $S^{2n-1} \to E^{4n-1} \to P^{2n}(2)$. The homology $\bar{H}_*(E^{4n-1})$ has a basis x_{2n-1}, x_{4n-2} and x_{4n-1}, where $|x_i| = i$. Since $\partial_*\colon H_*(\Omega P^{2n}(2)) \to S^{2n-1}$ is onto, the Serre spectral sequence for

$$\Omega E^{4n-1} \to \Omega P^{2n}(2) \to S^{2n-1}$$

collapses and so $H_*(\Omega E^{4n-1}) \cong T(u, [u, v], v^2)$. By expecting the Serre spectral sequence for $\Omega E^{4n-1} \to * \to E^{4n-1}$, the homology suspension

$$\sigma\colon Q(H_*(\Omega E^{4n-1})) \to \bar{H}_*(E^{4n-1})$$

is an isomorphism, where $Q(A)$ is the set of indecomposable elements of an algebra A. It follows that $Sq^1_* x_{4n-1} = x_{4n-2}$.

Let X be the homotopy fibre of $E^{4n-1} \to P^{2n}(2)$. Then $\bar{H}_*(X)$ has a basis y_{2n}, y_{4n-1} and y_{4n} with $Sq^1_* y_{4n} = y_{4n-1}$. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc}
\Omega P^{2n}(2) & \longrightarrow & S^{2n-1} & \longrightarrow & E^{4n-1} & \longrightarrow & P^{2n}(2) \\
\downarrow{\scriptstyle \Omega f} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\
\Omega X & = & \Omega X & \longrightarrow & * & \longrightarrow & X.
\end{array}$$

Let τ denote the transgression. We have

$$\tau(y_{2n})^2 = \Omega f_*(v^2) = 0.$$

By expecting the Serre spectral sequence for $\Omega X \to * \to X$, we have

$$y_{2n}^{*2} = y_{4n}^*$$

in the cohomology $H^*(X)$ and hence assertion (1). By [5, Theorem 9.1], assertion (2) follows. This finishes the proof. $\square$

2.5. **The Exponent Problem.**

2.5.1. *The Exponent Problems in the Homotopy Theory.* Let p be a prime integer and let G be an abelian group. We write $\mathrm{Tor}_p(G)$ for the p-torsion component of G. We call p^r an *exponent* of G if $p^r \cdot \mathrm{Tor}_p(G) = 0$. Let X be a space. The integer p^r is called an exponent of $\pi_*(X)$ if $p^r \cdot \mathrm{Tor}_p(\pi_n(X)) = 0$ for all $n \geq 2$. If X is a simply connected CW of finite type, then each $\pi_n(X)$ has a bounded exponent which depends on n in general. For instance $\pi_*(S^2 \vee S^2)$ does not have a bounded exponent. On the other hand, one can check that the homotopy groups of the mapping space from a simply connected finite torsion space to a space has a bounded exponent.

It was first known by James [19] that $\pi_*(S^{2n+1})$ has an exponent bounded by 2^{2n}. The improvements given in [5, 35] state that the exponent of $\pi_*(S^{2n+1})$ is bounded by $2^{2n-[n/2]}$, where $[a]$ is the maximal integer $\leq a$. In the cases where $p > 2$, Toda [41] showed that $\pi_*(S^{2n+1})$ has an exponent bounded by p^{2n}. Selick [33] then showed that $\pi_*(S^3)$ has an exponent bounded by p for $p > 2$. Later Cohen-Moore-Neisendorfer [9] proved that $\pi_*(S^{2n+1})$ has an exponent bounded by p^n for $p > 2$. This is the best exponent because a theorem of Gray [15] gives that there are $\mathbb{Z}/p^n$-summands in $\pi_*(S^{2n+1})$ for $p > 2$. The best 2-exponent of $\pi_*(S^{2n+1})$ remains open.

The Moore conjecture says that the homotopy groups of a simply connected finite complex X has a bounded p-exponent if the rational cohomology ring $H^*(\Omega X; \mathbb{Q})$ is finitely generated. For any given simply connected finite complex X, it was known by McGibbon and Wilkerson [21] that the Moore conjecture holds when the prime p is sufficiently large. By the arguments above, the Moore conjecture holds for spheres (for all primes p) and so for any space which admits a finite fibration resolution in terms of spheres, for instance the classical Lie groups.

The Barratt conjecture states that if the degree map $[p^r]\colon \Sigma X \to \Sigma X$ is null homotopic, then $p^{r+1}\pi_*(\Sigma X) = 0$. It was known by Neisendorfer [30] that the Barratt conjecture holds for the Moore spaces $P^n(p^r)$ with $p > 2$. By [9], there is a family of $\mathbb{Z}/p^{r+1}$-summands in $\pi_*(P^n(p^r))$ and so p^{r+1} is the best exponent.

In [6], Cohen showed that $\pi_*(P^n(2^r))$ has a bounded exponent when $r \geq 2$ and so the Moore conjecture holds for these spaces. It is unknown whether the Barratt conjecture holds for $P^n(2^r)$ with $r \geq 2$. But there is a family of $\mathbb{Z}/2^{r+1}$-summands in $\pi_*(P^n(2^r))$, see [4], and so 2^{r+1} will be the best exponent if the Barratt conjecture holds.

It is still unknown whether $\pi_*(P^n(2))$ has a bounded exponent. In other words, the conjectures of Moore and Barratt both remain open for mod 2 Moore spaces. In [12], we showed that there is a family of $\mathbb{Z}/8$-summands in $\pi_*(P^n(2))$ and so 8 will be the best exponent for $\pi_*(P^n(2))$ if the Barratt conjecture holds.

2.5.2. *Exponents of* $\pi_*(S^n\{2\})$. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \longrightarrow & \Omega(S^{2n+1}\{4\}) & \longrightarrow & \Omega S^{2n+1} & \xrightarrow{\Omega[4]} & \Omega S^{2n+1} \\
\uparrow & & \uparrow \bar{H}_2 & & \uparrow H_2 & & \uparrow H_2 \\
\Omega^2 S^{n+1} & \longrightarrow & \Omega(S^{n+1}\{2\}) & \longrightarrow & \Omega S^{n+1} & \xrightarrow{\Omega[2]} & \Omega S^{n+1} \\
\uparrow & & \uparrow \bar{E} & & \uparrow E & & \uparrow E \\
\Omega S^n & \longrightarrow & S^n\{2\} & \longrightarrow & S^n & \xrightarrow{[2]} & S^n.
\end{array}$$

We obtain the fibre sequence

$$S^n\{2\} \xrightarrow{\bar{E}} \Omega(S^{n+1}\{2\}) \xrightarrow{\bar{H}_2} \Omega(S^{2n+1}\{4\}).$$

Recall that $S^n\{2\}$ is an $\Omega S^n = J(S^{2n-1})$-space. By the above diagram, the map $\bar{E}$ can be chosen as an ΩS^n-map.

Lemma 2.30. *Let $n \geq 3$. Then the map*

$$4 \cdot \bar{E} : S^n\{2\} \longrightarrow \Omega(S^{n+1}\{2\})$$

is null homotopic.

Proof. Filter $S^n\{2\}$ by setting $F_k S^n\{2\}$ to be the $((n-1)k+1)$-skeleton. Let $\mu\colon J(S^{n-1}) \times P^n(2) \to S^n\{2\}$ be the $J(S^{n-1})$-action on $S^n\{2\}$. Then $\mu_k = \mu|_{J_k(S^{n-1}) \times P^n(2)}$

lifts to $F_{k+1}S^n\{2\}$. Recall that

$$\Sigma J_k S^{n-1}\times P^n(2)\simeq \Sigma\bigvee_{j=1}^{k} S^{(n-1)j}\wedge P^n(2)\vee\Sigma\bigvee_{j=1}^{k} S^{(n-1)j}\vee P^n(2)\quad\text{and}$$

$$\Sigma F_{k+1}S^n\{2\}\simeq\Sigma\bigvee_{j=0}^{k} S^{(n-1)j}\wedge P^n(2).$$

The map $\Sigma\mu_k$ is a retraction for $0\le k\le+\infty$. Let

$$q_k\colon (S^{n-1})^k\times P^n(2)\longrightarrow J_k(S^{n-1})\times P^n(2)$$

denote the projection. Then Σq_k is a retraction and therefore

$$q_k^*\circ\mu_k^*\colon [F_{k+1}S^n\{2\},\Omega Y]\to[(S^{n-1})^k\times P^n(2),\Omega Y]$$

is a monomorphism for any Y. Since $\bar{E}$ is an ΩS^n-map, there is a homotopy commutative diagram

$$\begin{array}{ccc}
(S^{n-1})^k\times P^n(2) & \xrightarrow{(\bar{E}|_{S^{n-1}})^k\times\bar{E}|_{P^n(2)}} & (\Omega(S^{n+1}\{2\}))^{k+1}\\
\downarrow{\scriptstyle q_k} & & \downarrow{\scriptstyle \mu\times\mathrm{id}}\\
J_kS^{n-1}\times P^n(2) & \xrightarrow{\bar{E}|_{J_k(S^{n-1})}\times\bar{E}|_{P^n(2)}} & \Omega(S^{n+1}\{2\})\times\Omega(S^{n+1}\{2\})\\
\downarrow{\scriptstyle \mu_k} & & \downarrow{\scriptstyle \mu}\\
F_{k+1}S^n\{2\} & \xrightarrow{\bar{E}} & \Omega(S^{n+1}\{2\}).
\end{array}$$

Let x_i and y denote the homotopy classes of

$$(S^{n-1})^k\times P^n(2)\xrightarrow{\pi_i} S^{n-1}\xrightarrow{\bar{E}|_{S^{n-1}}}\Omega(S^{n+1}\{2\})\quad\text{and}$$

$$(S^{n-1})^k\times P^n(2)\xrightarrow{\pi_{k+1}} P^n(2)\xrightarrow{\bar{E}|_{P^n(2)}}\Omega(S^{n+1}\{2\}),$$

respectively, where π_i the i-th coordinate projection. Then

$$q_k^*\circ\mu_k^*([\bar{E}|_{F_{k+1}S^n\{2\}}])=x_1x_2\cdots x_ky.$$

Now let G_k be the subgroup of $[(S^{n-1})^k\times P^n(2),\Omega(S^{n+1}\{2\})]$ generated by x_i, $1\le i\le k$, and y. Then the following identities hold in G_k:

1) $x_i^2=1$ for $1\le i\le k$;
2) $y^4=1$;
3) the commutators $[x_i,x_j]=1$;
4) The (iterated) commutators

$$[[a_1,a_2],\dots,a_s]=1$$

if $a_i=a_j$ for some $i<j$, where $a_i=y$ or x_j for some j.

Identities (1) and (2) are obvious. Identity (4) follows from Theorem 2.15 and (3) follows from the homotopy commutative diagram

$$\begin{array}{ccc} \Omega^2 S^{n+1} & \longrightarrow & \Omega(S^{2n+1}\{2\}) \\ \Big\uparrow{\scriptstyle \Omega E} & & \Big\uparrow{\scriptstyle \bar{E}} \\ \Omega S^n & \longrightarrow & S^n\{2\}. \end{array}$$

Now, by using identities (1) to (4),

$$(x_1 \cdots x_k y)^2 = x_1 \cdots x_k y \cdot x_1 \cdots x_k y = (x_1 \cdots x_k)^2 \cdot y^2 \cdot \delta = y^2 \cdot \delta$$

for some $\delta \in \overline{<y>}$, where $\overline{<y>}$ is the subgroup of G_k generated by $[y, \alpha]$ for $\alpha \in G_k$. By identity (4), $\overline{<y>}$ is abelian. Thus $(x_1 \cdots x_k y)^4 = y^4 \cdot \delta^2 = 1$ and so $4 \cdot \bar{E}|_{F_{k+1} S^n\{2\}}$ is null homotopic for each k. Since

$$[S^n\{2\}, \Omega Y] \cong \prod_{j=0}^{\infty} [P^{n+(n-1)j}(2), \Omega Y]$$

as sets for any Y, the map $4 \cdot \bar{E}$ is null homotopic and hence the result. □

An H-space X is said to have an *exponent* p^r if the power map $p^r \colon X \to X$ is null homotopic.

Proposition 2.31. *Let $n \geq 3$. Then the space $\Omega^2(S^n\{2\})$ has an exponent* 16.

Proof. By Lemma 2.30, the power map $4 : \Omega(S^n\{2\}) \to \Omega(S^n\{2\})$ lifts to the fibre $\Omega^3(S^{2n+1}\{4\})$. Since $\Omega^2[4] \simeq 4 : \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$, $\Omega^3(S^{2n+1}\{4\})$ is homotopy equivalent to the homotopy fibre $(\Omega^2 S^{2n+1})\{4\}$ of the power map $4 \colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$. Thus $\Omega^4(S^{2n+1}\{4\})$ has exponent 4 and so $\Omega^2(S^n\{2\})$ has exponent 16, which is the assertion. □

Corollary 2.32. *Let $n \geq 3$. Then $16 \cdot \pi_*(S^n\{2\}) = 0$.*

For the case $n = 2$, we have

Proposition 2.33. *There is a homotopy decomposition*

$$\Omega^2(S^2\{2\}\langle 3\rangle) \simeq \Omega^2(S^3\langle 3\rangle) \times \Omega^3_0 S^3.$$

In particular, $4 \cdot (\pi_*(S^2\{2\})) = 0$.

Proof. By the homotopy commutative diagram

$$\begin{array}{ccc} \Omega S^3 & \xrightarrow{\Omega[4] \simeq 4} & \Omega S^3 \\ \Big\uparrow{\scriptstyle H_2} & & \Big\uparrow{\scriptstyle H_2} \\ \Omega S^2 & \xrightarrow{\Omega[2]} & \Omega S^2, \end{array}$$

we have $\Omega_0(S^2\{2\}) \simeq (\Omega S^3)\{4\}$. According to [4], $\Omega^2(S^3\langle 3\rangle)$ has an exponent 4 and hence the result. □

By Proposition 2.23, there are $\mathbb{Z}/8$-summands in $\pi_*(S^n\{2\})$ for $n = 4, 8, 16, 32$ or 64. We do not know any element in $\pi_*(S^n\{2\})$ which order is 16 and so we wonder whether $\pi_*(S^n\{2\})$ has an exponent bounded by 8. This is true when n is odd.

Proposition 2.34. *The space* $\Omega^4(S^{2n+1}\{2\})$ *has an exponent* 8.

Proof. By using the fact that $2\Omega[2] \simeq 4 : \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$, there is a homotopy commutative diagram of fibre sequences

$$
\begin{array}{ccccc}
(\Omega^3 S^{2n+1})\{2\} & \xrightarrow{\delta} & \Omega^3 S^{2n+1} & \xrightarrow{2} & \Omega^3 S^{2n+1} \\
\| & & \downarrow{\scriptstyle k} & & \downarrow \\
(\Omega^3 S^{2n+1})\{2\} & \xrightarrow{g} & \Omega^2(S^{2n+1}\{2\}) & \xrightarrow{f} & (\Omega^2 S^{2n+1})\{4\} \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow{\scriptstyle \Omega^2[2]} & & \downarrow{\scriptstyle 4} \\
(\Omega^2 S^{2n+1})\{2\} & \longrightarrow & \Omega^2 S^{2n+1} & \xrightarrow{2} & \Omega^2 S^{2n+1}.
\end{array}
$$

Observe that

$$2\Omega g \simeq 2\Omega k \circ \Omega\delta \simeq \Omega k \circ 2 \circ \Omega\delta \simeq *.$$

There is a homotopy commutative diagram of fibre sequences

$$
\begin{array}{ccccccc}
\Omega^4(S^{2n+1}\{2\}) & \longrightarrow & (\Omega^4 S^{2n+1})\{4\} & \longrightarrow & (\Omega^4 S^{2n+1})\{2\} & \xrightarrow{\Omega g} & \Omega^3(S^{2n+1}\{2\}) \\
\downarrow{\scriptstyle 2} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 2} \\
\Omega^4(S^{2n+1}\{2\}) & = & \Omega^4(S^{2n+1}\{2\}) & \longrightarrow & * & \longrightarrow & \Omega^3(S^{2n+1}\{2\})
\end{array}
$$

and so a homotopy commutative diagram

$$
\begin{array}{ccc}
\Omega^4(S^{2n+1}\{2\}) & \longrightarrow & (\Omega^4 S^{2n+1})\{4\} \\
\downarrow{\scriptstyle 4} & & \downarrow{\scriptstyle 4 \simeq *} \\
\Omega^4(S^{2n+1}\{2\}) & \longrightarrow & (\Omega^4 S^{2n+1})\{4\} \\
\downarrow{\scriptstyle 2} & & \downarrow \\
\Omega^4(S^{2n+1}\{2\}) & = & \Omega^4(S^{2n+1}\{2\})
\end{array}
$$

Thus $8\colon \Omega^4(S^{2n+1}\{2\}) \to \Omega^4(S^{2n+1}\{2\})$ is null homotopic and hence the result. □

Corollary 2.35. *The homotopy groups $\pi_*(S^{2n+1}\{2\}) = 0$ have an exponent bounded by* 8*.*

So far we do not find any element in $\pi_*(S^{2n+1}\{2\})$ which order is 8. Below is a conjecture due to F. R. Cohen.

Conjecture 2.36 (Cohen)**.** *The homotopy groups $\pi_*(S^{2n+1}\{2\})$ have an exponent bounded by* 4*.*

2.6. **Some Lemmas on Fibrations.** We refer the reader to [22] for basic properties of Hopf algebras. A *Hopf algebra* means an (graded) algebra A together with a comultiplication $\psi\colon A \to A \otimes A$ such that A is a coalgebra under ψ and ψ is an algebraic map. Let M be an augmented module with the augmentation $\epsilon\colon M \to \mathbf{k}$, where $\mathbf{k}$ is the ground field. We write IM for the kernel of ϵ. Let A be a Hopf algebra. Let I^nA denote the n-fold self product of IA in A and let $Q(A) = IA/I^2A$ be the set of indecomposable elements.

In the case where A is a tensor algebra (as an algebra), the indecomposable elements of subHopf algebras of A can be understood in the following sense.

Lemma 2.37. *Let A be a connected Hopf algebra of finite type and let B be a subHopf algebra of A. Suppose that A is a tensor algebra as an algebra. Then there is a short exact sequence*

$$0 \longrightarrow Q(B) \longrightarrow (k \otimes_B A) \otimes Q(A) \longrightarrow I(k \otimes_B A) \longrightarrow 0.$$

Proof. Let $C = k \otimes_B A$. Then C is a right A-module. Let V be a submodule of IA such that $V \cong Q(A)$ and let J be the left ideal of A generated by IB. Consider the commutative diagram

$$\begin{array}{ccccc}
J \otimes V & = & J \otimes V & & \\
\big\downarrow{\scriptstyle \text{mult}} & & \big\downarrow & & \\
J & \hookrightarrow & A \otimes V \cong IA & \twoheadrightarrow & IC \\
\big\downarrow & & \big\downarrow & & \big\| \\
J/(J \cdot V) & \hookrightarrow & C \otimes V & \twoheadrightarrow & IC.
\end{array}$$

we obtain the short exact sequence

$$0 \longrightarrow J/(J \cdot V) \longrightarrow C \otimes V \twoheadrightarrow IC.$$

Now the composite

$$IB \hookrightarrow J \longrightarrow J/(J \cdot V) = J/(J \cdot IA)$$

is epimorphism and it factors through $Q(B) = IB/I^2B$. Let $\phi\colon Q(B) \to J/(J \cdot V)$ be the resulting map. Since ϕ is an epimorphism, the Poincaré series

$$\chi(Q(B)) \geq \chi(J/(J \cdot V)) = \chi(C \otimes V) - \chi(IC) = \chi(C)\chi(V) - \chi(C) + 1.$$

Recall that any sub algebra of tensor algebra is free. By [22, Theorem 4.4], there is an isomorphism of B-modules

$$A \cong B \otimes C.$$

In particular, we have

$$\frac{1}{1-\chi(V)} = \frac{1}{1-\chi(Q(B))} \cdot \chi(C)$$

or $\chi(Q(B)) = \chi(C)\chi(V) - \chi(C) + 1$. Thus the map $\phi\colon Q(B) \to J/(J \cdot V)$ is an isomorphism and hence the result. □

This lemma helps to understand the homology of certain spaces.

Lemma 2.38. *Let $F \xrightarrow{i} E \xrightarrow{q} \Sigma X$ be a fibration. Suppose that*

1) *X is path-connected;*
2) *F is simply connected and*
3) *the boundary $H_*(\Omega\Sigma X) \to H_*(F)$ is an epimorphism.*

Then

1) *There is a short exact sequence*

$$0 \longrightarrow Q(H_*(\Omega E)) \longrightarrow H_*(F) \otimes \bar{H}_*(X) \longrightarrow \bar{H}_*(F) \longrightarrow 0.$$

2) *The homology suspension*

$$\sigma\colon Q(H_*(\Omega E)) \to \bar{H}_{*+1}(E)$$

is an isomorphism.

Proof. Since $H_*(\Omega\Sigma X) \to H_*(F)$ is an epimorphism, the Serre spectral sequence for the fibre sequence $\Omega E \longrightarrow \Omega X \longrightarrow F$ collapses and so assertion (1) follows from Lemma 2.37.

Consider the cofibre sequence

$$F \xrightarrow{i} E \xrightarrow{\theta} \Sigma X \times F/(* \times F).$$

Since $H_*(\Omega\Sigma X) \to H_*(F)$ is onto, the map $i_*\colon \bar{H}_*(F) \to \bar{H}_*(E)$ is zero or

$$\theta_*\colon \bar{H}_*(E) \to \bar{H}_*(\Sigma X \times F/(* \times F))$$

is a monomorphism. It follows that

$$\chi(s^{-1}\bar{H}_*(E)) = \chi(Q(H_*(\Omega E))),$$

where s^{-1} is the desuspension of graded modules. Since

$$\chi(H_*(\Omega E)) = \frac{1}{1-\chi(s^{-1}\bar{H}_*(E))},$$

the cobar spectral sequence for E collapses and hence assertion (2). We finish the proof. □

Lemma 2.39. *Let $F \xrightarrow{i} E \xrightarrow{q} \Sigma X$ be a fibration and let $f\colon X \to F$ be the composite $X \hookrightarrow \Omega\Sigma X \xrightarrow{\partial} F$. Then there is a cofibre sequence*

$$C_f \longrightarrow E \longrightarrow \Sigma X \wedge F.$$

Proof. By [44, Theorem 1.1, chapter VII], there is a cofibre sequence

$$F \xrightarrow{i} E \longrightarrow (\Sigma X \times F)/(* \times F).$$

Since $i \circ f \colon X \to E$ is null homotopic, there is a homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccc}
F & \longrightarrow & C_f & \longrightarrow & \Sigma X \\
\| & & \downarrow & & \downarrow \\
F & \xrightarrow{i} & E & \longrightarrow & (\Sigma X \times F)/(* \times F) \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \Sigma X \wedge F & = & \Sigma X \wedge F
\end{array}$$

and hence the result. □

3. Decompositions of Self Smash Products

In this chapter, we investigate splittings of self smash products of the projective plane $\mathbb{R}P^2$. An instructional example is the 3-fold self smashes $(\mathbb{R}P^2)^{(3)}$. First we can decompose this space as wedge of 3 factors

$$(\mathbb{R}P^2)^{(3)} \simeq X_1 \vee X_2 \vee X_3,$$

where X_i is an indecomposable space. Since there is only one bottom cell in $(\mathbb{R}P^2)^{(3)}$, we assume that X_1 contains the bottom cell, that is, $H_3(X_1) = \mathbb{Z}/2$. Then we show that $X_2 \simeq X_3 \simeq P^5(2)$. By analysis on the cell structure, we show that $X_1 \simeq \mathbb{C}P^2 \wedge \mathbb{R}P^2$. In other words, the atomic space X_1 admits a smash product decomposition. This gives an interesting relation between different projective planes. We will largely investigate this kind of decompositions in this chapter.

There is a connection between decompositions of self smashes of $\mathbb{R}P^2$ and the modular representation theory. The homology of the (functorial) indecomposable factors of self smashes of a two-cell suspension has been determined recently in [38]. We will go over these results and connections in the first section. In the second section, we focus on decompositions of self smash products of $\mathbb{R}P^2$. Since $\mathbb{R}P^2$ is not a suspension, we first desuspension the decomposition formulas. Then we study the smash product decompositions of indecomposable factors.

3.1. **Functorial Decompositions of Self Smashes of Two-cell Complexes.**

Let X be a path-connected p-local CW-complex. We start with some observations. Consider the n-fold self smash product $X^{(n)}$. Let the symmetric group S_n act on $X^{(n)}$ by permuting positions. Thus, for each $\sigma \in S_n$, we have a map $\sigma \colon X^{(n)} \to X^{(n)}$. Let $Z_{(p)}$ denote p-local integers and let $\alpha = \sum_{\sigma \in S_n} k_\sigma \sigma$ be an element in the group algebra

$\mathbb{Z}_{(p)}(S_n)$. By using the group structure in $[\Sigma X^{(n)}, \Sigma X^{(n)}]$, we obtain a map

$$\alpha = \sum_{\sigma \in S_n} k_\sigma \sigma \colon \Sigma X^{(n)} \to \Sigma X^{(n)}.$$

(**Note.** $[\Sigma X^{(n)}, \Sigma X^{(n)}]$ is not abelian in general and the product above is given by a fixed choice of order.) The map α has the following properties:

1) The map α is functorial with respect to X.
2) Let $S_*(Y)$ be the singular chain complex of Y with coefficients in $\mathbb{Z}_{(p)}$. Then the induced chain map

$$\alpha_* \colon S_*(\Sigma X^{(n)}) \simeq \Sigma S_*(X)^{\otimes n} \to S_*(\Sigma X^{(n)}) \simeq \Sigma S_*(X)^{\otimes n}$$

is homotopic to the linear combination $\sum_{\sigma \in S_n} k_\sigma \Sigma\sigma$, where S_n acts on $S_*(X)^{\otimes n}$ by permuting factors in graded sense. In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_{(p)}(S_n) & \xrightarrow{\theta_1} & [\Sigma S_*(X)^{\otimes n}, \Sigma S_*(X)^{\otimes n}] \\ \| & & \uparrow \\ \mathbb{Z}_{(p)}(S_n) & \xrightarrow{\theta_2} & [\Sigma X^{(n)}, \Sigma X^{(n)}], \end{array}$$

where θ_1 is a map of rings and θ_2 is *only* a function.

For each $\alpha \in \mathbb{Z}_{(p)}(S_n)$, let $\alpha(X) = \operatorname{hocolim}_\alpha \Sigma X^{(n)}$ be the homotopy colimit of the sequence

$$\Sigma X^{(n)} \xrightarrow{\alpha} \Sigma X^{(n)} \xrightarrow{\alpha} \Sigma X^{(n)} \xrightarrow{\alpha} \cdots .$$

Now let

(1)
$$1 = \sum_\alpha e_\alpha$$

be an orthogonal decomposition of the identity in $\mathbb{Z}_{(p)}(S_n)$ in terms of primitive idempotents. For each e_α, we have a $e_\alpha(X) = \operatorname{hocolim}_{e_\alpha} \Sigma X^{(n)}$. The composite

(2)
$$\Sigma X^{(n)} \xrightarrow{\text{comult}} \bigvee_\alpha \Sigma X^{(n)} \longrightarrow \bigvee_\alpha e_\alpha(X)$$

is a homotopy equivalence because its induced map on the singular chains over p-local integers is a homotopy equivalence. Furthermore, this decomposition is functorial with respect to X. In other words, each e_α is a functor from path-connected spaces to co-H-spaces. (**Note:** for each α, we can fix a choice of the representative map e_α. This gives a particular choice of space $e_\alpha(X)$ such that $e_\alpha(X)$ is strictly functorial with respect to Y. Furthermore, there is a choice of the map

$$\phi \colon \Sigma X^{(n)} \to \bigvee_\alpha e_\alpha(X)$$

such that ϕ is strictly functorial on X and is a homotopy equivalence for each X.)

Recall that primitive idempotents, up to conjugates, in $\mathbb{Z}_{(p)}(S_n)$ are one-to-one corresponds to p-regular partitions of n. A *(proper) partition* of n is a sequence of positive integers $\lambda = (\lambda_1, \cdots, \lambda_s)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \text{ and } \sum_{j=1}^{s} = n.$$

s is called the length of λ, which is denoted by $\operatorname{len}(\lambda)$. A partition $\lambda = (\lambda_1, \cdots, \lambda_s)$ is called *p-regular* if there is NO subscript i with $1 \leq i \leq s$ such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1}$. For each p-regular partition λ, there is an idempotent $Q^\lambda \in \mathbb{Z}_{(p)}(S_n)$ such that the (left) ideal generated by Q^λ is a projective cover of the Specht module corresponding to λ (See [18] for details). Any primitive idempotent in $\mathbb{Z}_{(p)}(S_n)$ is conjugate to one and only one Q^λ. Let d_λ be the number of idempotents in decomposition 1 which are conjugate to Q^λ. We called d_λ the *multiplicity* of primitive idempotent Q^λ in $\mathbb{Z}_{(p)}(S_n)$. Let $Q^\lambda(X)$ be the homotopy colimit

$$Q^\lambda(X) = \operatorname{hocolim}_{e^\lambda} \Sigma X^{(n)}.$$

Then decomposition 2 can be rewritten as

$$\Sigma X^{(n)} \simeq \bigvee_{\lambda} \bigvee^{d_\lambda} Q^\lambda(X), \tag{3}$$

where λ runs over all p-regular partitions of n.

Let $H_*(X)$ denote the mod p homology of X.

Theorem 3.1. [38, Theorem 1.1] *Let $X = \Sigma Y$ be a p-local suspension. Suppose that $p = 2$ or $\bar{H}_*(X)^{\mathrm{odd}} = 0$. Then $Q^\lambda(X)$ is contractible if and only if* $\operatorname{len}(\lambda) > \dim \bar{H}_*(X)$.

Note: If $p > 2$ and $\bar{H}_*(X)^{\mathrm{odd}} \neq 0$, then $Q^\lambda(X)$ may not be contractible even if $\operatorname{len}(\lambda) > \dim \bar{H}_*(X)$. For instance, let X be an odd sphere, then $Q^{(1,1)}(X) \simeq \Sigma X^{(2)}$.

Let X be a path-connected two-cell complex localized at 2. By Theorem 3.1, we have

$$\Sigma X^{(n)} \simeq \bigvee_{n/2<a\leq n} \bigvee^{c_{(a,n-a)}} Q^{(a,n-a)}(X). \tag{4}$$

The homology of $Q^{(a,n-a)}(Y)$ for each $n/2 < a \leq n$ and the multiplicity $d_{(a,n-a)}$ can be determined.

Let u, v be a basis for $V = \bar{H}_*(X)$ with $|u| \leq |v|$. Let $S(V)$ be the free commutative algebra generated by V and let $P(V)$ be the set of primitive elements in $S(V)$. Then

$$P(V) = \bigoplus_{k=0}^{\infty} P_{2^k}(V)$$

and $P_{2^k}(V)$ has a basis u^{2^k}, v^{2^k}. Since V is a module over the Steenrod algebra, $P_{2^k}(V)$ is a module over the Steenrod algebra in the sense that $P_{2^k}(V)$ is a submodule of

$S_{2^k}(V)$ over the Steenrod algebra. Let $s(n)$ be the integer such that $2^{s(n)}-1 \leq n \leq 2^{s(n)+1}-2$ and let

$$n - 2^{s(n)} + 1 = 2^{a(1;n)} + 2^{a(2;n)} + \cdots + 2^{a(l(n);n)}$$

with $0 \leq a(1;n) < a(2;n) < \cdots < a(l(n);n)$.

Theorem 3.2. [38, Theorem 1.2] *Let X be a path connected two-cell complex localized at 2 and let $V = \bar{H}_*(X)$ with a basis $|u|, |v|$. Then*

1) *there is an isomorphism of modules over the Steenrod algebra*

$$\Sigma^{-1}\bar{H}_*(Q^{(n)}(X)) \cong \bigotimes_{j=0}^{s(n)-1} P_{2^j}(V) \otimes \bigotimes_{j=1}^{l(n)} P_{2^{a(j;n)}}(V);$$

2) *there is a homotopy equivalence*

$$Q^{(a,n-a)}(X) \simeq \Sigma^{(|u|+|v|)(n-a)} Q^{(n-2(n-a))}(X)$$

for $n/2 < a \leq n$.

3) *the* (mod 2) *homology of the decomposition* 4 *is a complete decomposition over the Steenrod algebra if $\bar{H}_*(X)$ has a non-trivial Steenrod operation.*

Assertion 3 says that any idempotent $\phi\colon \bar{H}_*(X^{(n)}) \to \bar{H}_*(X^{(n)})$ over the Steenrod algebra can be geometrically realizable by an idempotent $f\colon \Sigma X^{(n)} \to \Sigma X^{(n)}$, that is $f_* = \phi$. Furthermore the map f can be chosen to be a functorial map

$$f = \sum_{\sigma \in S_n} k_\sigma \sigma\colon \Sigma X^{(n)} \to \Sigma X^{(n)}$$

for some $k_\sigma \in \mathbb{Z}_{(2)}$.

For each a with $n/2 < a \leq n$, we have

$$2^{s(2a-n)} - 1 \leq 2a - n \leq 2^{s(2a-n)+1} - 2.$$

Let

$$2a - n - 2^{s(2a-n)} + 1 = \eta_0 + \eta_1 2 + \cdots + \eta_{s(2a-n)-1} 2^{s(2a-n)-1)}$$

be the 2-adic resolution, where $\eta_i = 0, 1$. Let

$$\omega(a) = (\eta_0 + 1, \eta_1 + 1, \cdots, \eta_{s(2a-n)-1} + 1).$$

Let m be a non-negative integer. Define

$$\epsilon(m) = \begin{cases} 2 & \text{if } m > 0 \text{ and m is even,} \\ 1 & \text{if } \quad \text{m is odd,} \\ 0 & \text{if } \quad m = 0. \end{cases} \tag{5}$$

Let $I = (i_1, \cdots, i_s)$ be a sequence of non-negative integers. Let

$$\epsilon(I) = (\epsilon(i_1), \cdots, \epsilon(i_s)).$$

Theorem 3.3. [38, Theorem 1.3] *For each $n/2 < a \le n$, the multiplicity $d_{(a,n-a)}$ of $Q^{(a,n-a)}$ is given by the formula*

$$d_{(a,n-a)} = \sum_{\substack{I=(i_0,\cdots,i_s),\, i_0=n \\ \epsilon(I)=\omega(a) \\ 0\le i_l \le \frac{i_{l-1}-\epsilon(i_{l-1})}{2} \\ i_{l-1} > \epsilon(i_{l-1}) \text{ for } 0\le l\le s \\ 0\le i_s\le \epsilon(i_s)}} k_I, \tag{6}$$

where

$$k_I = \binom{\frac{i_0-\epsilon(i_0)}{2}}{i_1}\cdot\binom{\frac{i_1-\epsilon(i_1)}{2}}{i_2}\cdots\binom{\frac{i_{s-1}-\epsilon(i_{s-1})}{2}}{i_s} 2^{(n-(\epsilon(i_0)+i_1+\epsilon(i_1)+\cdots+i_{s-1}+\epsilon(i_{s-1})+i_s))/2}.$$

3.2. **Decompositions of Self Smashes of $\mathbb{R}\mathrm{P}^2$.** When X is a projective plane, the decompositions in Theorem 3.2 are desuspensionable.

Proposition 3.4. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{K}$ and let $X = \mathbb{F}\mathrm{P}^2$. Then there is a sequence of spaces $\mathbb{F}\mathrm{Q}^k$ for $k \ge 1$ such that*

1) $X^{(n)} \simeq \mathbb{F}\mathrm{Q}^n \vee \bigvee_{i=1}^{[\frac{n-1}{2}]} \bigvee_{j=1}^{c_i} \Sigma^{(|u|+|v|)i}\mathbb{F}\mathrm{Q}^{n-2i}$.
2) $\Sigma\mathbb{F}\mathrm{Q}^n \simeq Q^{(n)}(X)$.
3) $\mathbb{F}\mathrm{Q}^{2n} \simeq \mathbb{F}\mathrm{Q}^{2n-1} \wedge \mathbb{F}\mathrm{P}^2$.

Proof. Suppose that there is a sequence of space $\mathbb{F}\mathrm{Q}^n$ such that assertions (1) and (2) hold. Then the composite

$$\mathbb{F}\mathrm{Q}^{2n-1} \wedge \mathbb{F}\mathrm{P}^2 \hookrightarrow (\mathbb{F}\mathrm{P}^2)^{(2n)} \longrightarrow \mathbb{F}\mathrm{Q}^{2n}$$

is a homotopy equivalence by checking the homology. Now we show that there exists a sequence of spaces $\mathbb{F}\mathrm{Q}^n$ such that assertions (1) and (2) hold by induction on n. Let $\mathbb{F}\mathrm{Q}^1 = \mathbb{F}\mathrm{P}^2$. Suppose that the spaces $\mathbb{F}\mathrm{Q}^k$ defined for $k < n$ with $n > 1$ such that assertions (1) and (2) holds for all $k < n$. Let $d = \dim \mathbb{F}$ over $\mathbb{R}$. Then $Q^{(n)}(X)$ is (dn)-connected $(2dn+1)$-dimensional complex and so there is a space Y such that $\Sigma Y \simeq Q^{(n)}(X)$. Since $X^{(n)}$ is $(dn-1)$-connected $(2dn)$-dimensional complex, there is a decomposition

$$\mathrm{sk}_{2dn-1}(X^{(n)}) \simeq \mathrm{sk}_{2dn-1}(Y) \vee B,$$

where

$$B = \bigvee_{i=1}^{[\frac{n-1}{2}]} \bigvee_{j=1}^{c_i} \Sigma^{(|u|+|v|)i}\mathbb{F}\mathrm{Q}^{n-2i}(X).$$

Consider the cofibre sequence

$$S^{2dn-1} \xrightarrow{\partial} \mathrm{sk}_{2dn-1}(Y) \vee B \longrightarrow X^{(n)}.$$

Since $\Sigma X^{(n)} \simeq \Sigma Y \vee \Sigma B$ and B is (dn)-connected, the composite

$$S^{2dn-1} \xrightarrow{\partial} \mathrm{sk}_{2dn-1}(Y) \vee B \xrightarrow{\text{proj.}} B$$

is null homotopic. Observe that $\mathrm{sk}_{2dn-1}(Y)$ is the $(2dn-1)$-skeleton of the homotopy fibre of the projection map $\mathrm{sk}_{2dn-1}(Y) \vee B \to B$. There is a homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccc}
S^{2dn-1} & \longrightarrow & \mathrm{sk}_{2dn-1}(Y) & \longrightarrow & \mathbb{F}\mathrm{Q}^n \\
\Vert & & \cap\downarrow & & \downarrow \\
S^{2dn-1} & \xrightarrow{\partial} & \mathrm{sk}_{2dn-1}(Y) \vee B & \longrightarrow & X^{(n)}.
\end{array}$$

By expecting the homology, the map $\mathbb{F}\mathrm{Q}^n \vee B \to X^{(n)}$ is a homotopy equivalence, which is assertion (1). By assertion 3 of Theorem 3.2, $\Sigma\mathbb{F}\mathrm{Q}^n \simeq Q^{(n)}(X)$. The induction is finished and hence the result. □

Now we investigate the atomic spaces $\mathbb{R}\mathrm{Q}^n$ for low n. By expecting the homology, $\mathbb{R}\mathrm{Q}^1 = \mathbb{R}\mathrm{P}^2$ and $\mathbb{R}\mathrm{Q}^2 = \mathbb{R}\mathrm{P}^2 \wedge \mathbb{F}\mathrm{P}^2$. The first proper factor is $\mathbb{R}\mathrm{Q}^3$. By expecting the homology, this is a four cell complex. This factor of $X^{(3)}$ can obtained in an explicit way after suspension.

Lemma 3.5. *Let X be a 2-local path-connected space and let $\sigma = (123)$ be the cyclic element in S_3 of order 3. Then*

$$Q^{(3)}(X) = \mathrm{hocolim}_{\frac{1}{3}(\sigma+\sigma^2+1)} \Sigma X^{(3)}.$$

The proof follows immediately by observing that $\dfrac{1}{3}(\sigma + \sigma^2 + 1)$ is a primitive idempotent in $\mathbb{Z}/2(S_3)$ which acts on $x \otimes x \otimes x$ trivially.

Proposition 3.6. *The space $\mathbb{R}\mathrm{Q}^3$ is homotopy equivalent to $\mathbb{R}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2$.*

Proof. By assertion 1 of Theorem 3.2, $\bar{H}_*(\mathbb{R}\mathrm{Q}^3)$ has a basis x_3, x_4, x_5 and x_6 with the nontrivial Steenrod operations $Sq^1_* x_6 = x_5$, $Sq^2_* x_6 = x_4$, $Sq^3_* x_6 = x_3$, $Sq^2_* x_5 = x_3$ and $Sq^1_* x_4 = x_3$. Clearly $\mathrm{sk}_4(\mathbb{R}\mathrm{Q}^3) \simeq \mathbb{R}\mathrm{P}^4_3 = P^4(2)$. Since $\pi_4(P^4(2)) = \mathbb{Z}/2$, we have $\mathrm{sk}_5(\mathbb{R}\mathrm{Q}^3) \simeq \mathbb{R}\mathrm{P}^5_3$. Now we compute $\pi_5(\mathbb{R}\mathrm{P}^5_3)$. Since $P^4(2)$ is the 6-skeleton of the homotopy fibre of the pinch map $q\colon \mathbb{R}\mathrm{P}^5_3 \to S^5$, there is an exact sequence

$$\pi_5(S^4) = \mathbb{Z}/2 \longrightarrow \pi_5(P^4(2)) = \mathbb{Z}/4 \longrightarrow \pi_5(\mathbb{R}\mathrm{P}^5_3) \longrightarrow \pi_5(S^5) \longrightarrow \pi_4(P^4(2)) \longrightarrow 0$$

and so a short exact sequence

$$0 \longrightarrow \pi_5(P^4)/2 \longrightarrow \pi_5(\mathbb{R}\mathrm{P}^5_3) \xrightarrow{q_*} 2 \cdot \pi_5(S^5).$$

Let $\alpha \in \pi_5(\mathbb{R}\mathrm{P}^5_3)$ such that $q_*(\alpha) = 2\iota_5$ and let δ be the nontrivial element of $\mathrm{Im}(\pi_5(P^4(2)) \to \pi_5(\mathbb{R}\mathrm{P}^5_3)) = \mathbb{Z}/2$. Let β be any element in $\pi_5(\mathbb{R}\mathrm{P}^5_3)$ such that $q_*(\beta) = \pm 2 \cdot \iota_5$. Then β is one of the four elements $\{\pm\alpha, \pm(\alpha + \delta)\}$. It follows that there are exactly two different, up to homotopy, 4-cell complexes Y such that $\mathrm{sk}_5(Y) \simeq \mathbb{R}\mathrm{P}^5_3$ and $Sq^1_*\colon H_6(Y) \to H_5(Y)$ is an isomorphism. Observe that $\mathbb{R}\mathrm{P}^6_3$ and $\mathbb{R}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2$ are the complexes which satisfy these conditions with different Steenrod operations. By expecting the Steenrod operations, $\mathbb{R}\mathrm{Q}^3$ is homotopy equivalent to $\mathbb{R}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2$ and hence the result. □

Corollary 3.7. *There is homotopy decomposition*

$$(\mathbb{R}\mathrm{P}^2)^{(3)} \simeq \mathbb{R}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \vee P^5(2) \vee P^5(2).$$

By using this corollary, we can directly work out decompositions of $(\mathbb{R}\mathrm{P})^{(n)}$ for $n < 6$.

Example 3.8. . When $n = 4$, then

$$(\mathbb{R}\mathrm{P}^2)^{(4)} \simeq (\mathbb{R}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \vee P^5(2) \vee P^5(2)) \wedge \mathbb{R}\mathrm{P}^2 \simeq (\mathbb{R}\mathrm{P}^2)^{(2)} \vee P^5(2) \wedge \mathbb{R}\mathrm{P}^2 \vee P^5(2) \wedge \mathbb{R}\mathrm{P}^2$$

is a complete decomposition by considering the homology and so $\mathbb{R}\mathrm{Q}^4 \simeq (\mathbb{R}\mathrm{P}^2)^{(2)} \wedge \mathbb{C}\mathrm{P}^2$. When $n = 5$, then

$$\begin{aligned}(\mathbb{R}\mathrm{P}^2)^{(5)} &\simeq (\mathbb{R}\mathrm{P}^2)^{(3)} \wedge \mathbb{C}\mathrm{P}^2 \vee \Sigma^3(\mathbb{R}\mathrm{P}^2)^{(3)} \vee \Sigma^3(\mathbb{R}\mathrm{P}^2)^{(3)} \\ &\simeq \mathbb{R}\mathrm{P}^2 \wedge (\mathbb{C}\mathrm{P}^2)^{(2)} \vee \bigvee_{j=1}^{4} P^5(2) \wedge \mathbb{C}\mathrm{P}^2 \vee \bigvee_{j=1}^{4} P^8(2).\end{aligned}$$

Thus $\mathbb{R}\mathrm{Q}^5 \simeq \mathbb{R}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2$. □

The next example is $\mathbb{R}\mathrm{Q}^7$. Observe that $\mathbb{R}\mathrm{Q}^7$ is a retract of $\mathbb{R}\mathrm{Q}^6 \wedge \mathbb{R}\mathrm{P}^2$ and so, by Corollary 3.7, $\mathbb{R}\mathrm{Q}^7$ is a retract of $\mathbb{R}\mathrm{P}^2 \wedge (\mathbb{C}\mathrm{P}^2)^{(3)}$. The space $(\mathbb{C}\mathrm{P}^2)^{(3)}$ has further decomposition by Proposition 3.4.

Let $f, g \in \pi_n(Y)$. We write $f \sim g$ if there is a homotopy commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ S^n & \xrightarrow{g} & Y \end{array}$$

for some self homotopy equivalences ϕ_1 and ϕ_2. Clearly, if $f \sim g$, then the homotopy cofibre $C_f \simeq C_g$.

Lemma 3.9. *Let $Z = S^4 \cup e^8$ be a complex such that $Sq^4 \colon H^4(Z) \to H^8(Z)$ is an isomorphism. Then Z is homotopy equivalent to $\mathbb{H}\mathrm{P}^2$ localized at 2.*

Proof. Let $f \colon S^7 \to S^4$ be the attaching map for Z. Recall that $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/4$ with generators ν_4 and $E(\nu')$, see [42]. Since $Sq^4 \colon H^4(Z) \to H^8(Z)$ is an isomorphism, we may assume that $f = \nu_4 + aE(\nu')$, where $a = 0, \pm 1$ or 2. Now we apply the Cohen group to show that $\nu_4 + aE(\nu') \sim \nu_4$ for each a. The result will follow from this statement.

Observe that $\Omega[k] \colon \Omega S^4 \to \Omega S^4$ restricted to $J_2(S^3)$ is represented by the element $x_1^k x_2^k$ in the Cohen group for the inclusion $E \colon S^3 \hookrightarrow J(S^3)$. Let α_k denote the element $x_1^k x_2^k (x_1 x_2)^{-k}$. Then $\alpha_k \in \Gamma_2 K_2(E)$. By Example 2.11, we have $\Gamma_3 K_2(E) = 1$ and so

$$\alpha_{k+1} = x_1^{k+1} x_2^{k+1} (x_1 x_2)^{-k-1} = x_1 \cdot x_1^k \cdot x_2 \cdot x_2^k \cdot (x_1 x_2)^k \cdot (x_1 x_2)^{-1}$$

$$= x_1 [x_1^k, x_2] x_2 (x_1^k \cdot x_2^k \cdot (x_1 x_2)^{-k}) \cdot (x_1 x_2)^{-1} = [x_1, x_2]^k \alpha_k \cdot x_1 \cdot x_2 \cdot (x_1 x_2)^{-1} = [x_1, x_2]^k \alpha_k.$$

Since $\alpha_1 = 1$, we have

$$\alpha_k = [x_1, x_2]^{(k-1)+(k-2)+\ldots+1} = [x_1, x_2]^{k(k-1)/2}.$$

Note that $[x_1, x_2]$ is represented by the composite

$$J_2(S^3) \xrightarrow{\text{pinch}} S^6 \xrightarrow{\omega_4} J(S^3) \simeq \Omega S^4.$$

Thus $[x_1, x_2]_*(\nu_4) = \omega_4 = 2\nu_4 - E(\nu')$ and so

$$\Omega([k])_*(\nu_4) = k\nu_4 + [x_1, x_2]_*^{k(k-1)/2}(\nu_4) = k\nu_4 + \frac{k(k-1)}{2}(2\nu_4 - E(\nu')) = k^2\nu_4 - \frac{k(k-1)}{2}E(\nu').$$

By taking $k = 3$, we have

$$\nu_4 \sim 9\nu_4 - 3E(\nu') \sim \nu_4 - \frac{1}{3}E(\nu') = \nu_4 + E(\nu')$$

and

$$\nu_4 - E(\nu') \sim 9\nu_4 - 3E(\nu') - 3E(\nu') \sim \nu_4 + 2E(\nu').$$

By taking $k = 5$, we have

$$\nu_4 \sim 25\nu_4 - 10E(\nu') \sim \nu_4 - \frac{10}{25}E(\nu') = \nu_4 + 2E(\nu')$$

and hence the result. □

A homology class $x \in H_n(Y; \mathbb{Z}/2)$ is called (mod 2) *cospherical* if there is a map $f\colon Y \to S^n$ such that $f_*(x) = \iota_n \in H_n(S^n; \mathbb{Z}/2)$

Lemma 3.10. *There is a homotopy decomposition*

$$(\mathbb{C}\mathrm{P}^2)^{(3)} \simeq \mathbb{H}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \vee \Sigma^6\mathbb{C}\mathrm{P}^2 \vee \Sigma^6\mathbb{C}\mathrm{P}^2$$

localized at 2 *and so* $\mathbb{C}\mathrm{Q}^3 \simeq \mathbb{H}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2$.

Proof. Observe that the homology class $[u, v] = u \otimes v - v \otimes u$ is spherical in $\bar{H}_*(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2)$. There is a cofibre sequence

$$S^6 \xrightarrow{f} \mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \longrightarrow C_f,$$

where $f_*(\iota_6) = [u, v]$. Now $\bar{H}_*(C_f)$ has a basis $\{x_4, x_6, x_8\}$ with the nontrivial Steenrod operations $Sq^2_* x_6 = x_4$ and $Sq^4_* x_8 = x_4$. Since $Sq^2_* x_8 = 0$, the homology class x_6 is cospherical. Let $g\colon C_f \to S^6$ such that $g_*(x_6) = \iota_6$. Let Z be the 8-skeleton of the homotopy fibre of $g\colon C_f \to S^6$. By expecting the Serre spectral sequence for the fibre sequence $F_g \to C_f \to S^6$, the map $Z \to C_f$ induces a monomorphism in homology and so there is a cofibre sequence

$$Z \longrightarrow C_f \xrightarrow{g} S^6.$$

Since Z is a two-cell complex with nontrivial Sq^4, we have $Z \simeq \mathbb{H}\mathrm{P}^2$ by Lemma 3.9. By Proposition 3.4, we have

$$(\mathbb{C}\mathrm{P}^2)^{(3)} \simeq S^6 \wedge \mathbb{C}\mathrm{P}^2 \vee C_f \wedge \mathbb{C}\mathrm{P}^2 \simeq S^6 \wedge \mathbb{C}\mathrm{P}^2 \vee S^6 \wedge \mathbb{C}\mathrm{P}^6 \vee \mathbb{H}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2$$

and hence the result. □

Corollary 3.11. *There is a homotopy decomposition*

$$(\mathbb{R}\mathrm{P}^2)^{(7)} \simeq \mathbb{H}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge \mathbb{R}\mathrm{P}^2 \vee \bigvee_{j=1}^{6} \Sigma^3(\mathbb{C}\mathrm{P}^2)^{(2)} \wedge \mathbb{R}\mathrm{P}^2 \vee \bigvee_{j=1}^{14} \Sigma^6\mathbb{C}\mathrm{P}^2 \wedge \mathbb{R}\mathrm{P}^2 \vee \bigvee_{j=1}^{8} P^{11}(2).$$

and so $\mathbb{R}\mathrm{Q}^7 \simeq \mathbb{H}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge \mathbb{R}\mathrm{P}^2$.

In general, we have the following theorem for $\mathbb{R}\mathrm{Q}^n$ with $n \geq 7$.

Theorem 3.12. *Let* $k \geq 1$*. Then*

1) $\mathbb{R}Q^{8k-1} \simeq \mathbb{H}Q^{2k-1} \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2$;
2) $\mathbb{R}Q^{8k} \simeq \mathbb{H}Q^{2k-1} \wedge \mathbb{C}P^2 \wedge (\mathbb{R}P^2)^{(2)}$;
3) $\mathbb{R}Q^{8k+1} \simeq \mathbb{H}Q^{2k-1} \wedge (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2$;
4) $\mathbb{R}Q^{8k+2} \simeq \mathbb{H}Q^{2k-1} \wedge (\mathbb{C}P^2)^{(2)} \wedge (\mathbb{R}P^2)^{(2)}$;
5) $\mathbb{R}Q^{8k+3} \simeq \mathbb{H}Q^{2k-1} \wedge \mathbb{H}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2$;
6) $\mathbb{R}Q^{8k+4} \simeq \mathbb{H}Q^{2k-1} \wedge \mathbb{H}P^2 \wedge \mathbb{C}P^2 \wedge (\mathbb{R}P^2)^{(2)}$;
7) $\mathbb{R}Q^{8k+5} \simeq \mathbb{H}Q^{2k-1} \wedge \mathbb{H}P^2 \wedge (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2$;
8) $\mathbb{R}Q^{8k+6} \simeq \mathbb{H}Q^{2k-1} \wedge \mathbb{H}P^2 \wedge (\mathbb{C}P^2)^{(2)} \wedge (\mathbb{R}P^2)^{(2)}$;

The proof follows by induction and by using Theorem 3.2, Propositions 3.4 and 3.6 and Lemma 3.9.

Example 3.13. The spaces $\mathbb{R}Q^{2n+1}$ for $9 \leq n \leq 23$ admit the following smash product decompositions:

$$\mathbb{R}Q^9 \simeq \mathbb{H}P^2 \wedge (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2; \quad \mathbb{R}Q^{11} \simeq (\mathbb{H}P^2)^{(2)} \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2;$$

$$\mathbb{R}Q^{13} \simeq (\mathbb{H}P^2)^{(2)} \wedge (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2; \quad \mathbb{R}Q^{15} \simeq \mathbb{H}Q^3 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2;$$

$$\mathbb{R}Q^{17} \simeq \mathbb{H}Q^3 \wedge (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2; \quad \mathbb{R}Q^{19} \simeq \mathbb{H}Q^3 \wedge \mathbb{H}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2;$$

$$\mathbb{R}Q^{21} \simeq \mathbb{H}Q^3 \wedge \mathbb{H}P^2 \wedge (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2; \quad \mathbb{R}Q^{23} \simeq \mathbb{H}Q^5 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2.$$

One may wonder whether we can get the geometric analogue of the algebraic theorem 3.2, namely, whether $\mathbb{H}Q^n$ admits further smash product decompositions. The first case is $\mathbb{H}Q^3$ which appears in $\mathbb{R}Q^{15}$. The space $\mathbb{H}Q^3$ has the same homology as $\mathbb{K}P^2 \wedge \mathbb{H}P^2$. However, by analysis on the cell structure, $\mathbb{H}Q^3$ is *not* homotopy equivalent to $\mathbb{K}P^2 \wedge \mathbb{H}P^2$. Below we are going to determine the cell structure of $Q^{(3,0)}(X)$ for a general two-cell complex in the stable category. In particular, this will give the stable cell structure of $\mathbb{H}Q^3$.

For convenience, we write $\bar{Q}^n(X)$ for $\Sigma^{-1}Q^{(n)}(X)$ in the stable category. Thus $\bar{Q}^n(X)$ is the *functorially* atomic factor of $X^{(n)}$ which contains the bottom cell. Let $\{u, v\}$ be a basis for $\bar{H}_*(X)$. We assume that $|u| = 0$ and $|v| > 1$. (So mod 2 Moore space is excluded.) As a stable complex, $X = S^0 \cup_f e^{|v|}$, where f runs over all elements in the 2-torsion of the stable homotopy groups $\pi_*^s(S^0)$. Observe that the integral homology of X is torsion free in these cases. We write ab for $a \otimes b$ in a tensor product $V \otimes W$ and write $s^n x$ for the n-fold suspension of an element x in a graded module.

Lemma 3.14. *Let $X = S^0 \cup_f e^{|v|}$ be a 2-cell complex with $|v| > 1$. Suppose that $|v| \equiv 0$ mod 2. Then there exists a map $\phi_n \colon \Sigma^{(n-1)|v|}X \to X^{(n)}/\operatorname{sk}_{(n-2)|v|}(X^{(n)})$ such that in the integral homology $\phi_{n*}(s^{(n-1)|v|}v) = v^n$ and $\phi_{n*}(s^{(n-1)|v|}u) = \sum_{i+j=n-1} v^i u v^j$.*

Proof. The proof is given by induction on n. The statement is obvious for $n = 1$. Now we show that the statement holds for $n = 2$. Consider the cofibre sequence

$$\Sigma^{-1}S^{2|v|} \xrightarrow{\partial} \operatorname{sk}_{|v|}(X^{(2)}) \longrightarrow X^{(2)} \xrightarrow{\text{pinch}} S^{2|v|} \xrightarrow{-\partial} \Sigma \operatorname{sk}_{|v|}(X^{(2)}).$$

Let $g_1, g_2 \colon S^{|v|} \to \operatorname{sk}_{|v|}(X^{(2)})/S^0$ be the maps which sends $\iota_{|v|}$ to uv and vu, respectively. By using the cofibre sequence

$$X \wedge S^0 \longrightarrow X^{(2)} \longrightarrow X \wedge S^{|v|} \xrightarrow{-\operatorname{id}_X \wedge (\Sigma f)} X \wedge S^1,$$

we obtain the commutative diagram

$$\begin{array}{ccccc}
\Sigma^{-1}S^{2|v|} & \xrightarrow{\partial} & \mathrm{sk}_{|v|}(X^{(2)})/S^0 & \longrightarrow & X^{(2)}/S^0 \\
\| & & \downarrow{p_1} & & \downarrow \\
\Sigma^{-1}S^{2|v|} & \xrightarrow{\Sigma^{-1}(\Sigma f\wedge \mathrm{id}_{S^{|v|}})} & S^0\wedge S^{|v|} & \hookrightarrow & X\wedge S^{|v|}.
\end{array}$$

By computing the homology, we have

$$(p_1\circ g_1)_*(\iota_{|v|})=\iota_{|v|}\quad\text{and}\quad(p_1\circ g_2)_*(\iota_{|v|})=0.$$

Similarly, we have a map $p_2\colon \mathrm{sk}_{|v|}(X^{(2)})/S^0\to S^{|v|}$ such that

$$p_2\circ\partial=\Sigma^{-1}(\mathrm{id}_{S^{|v|}}\wedge\Sigma f)\colon \Sigma^{-1}S^{2|v|}\longrightarrow S^{|v|},$$

$$(p_2\circ g_1)_*(\iota_{|v|})=0,\quad\text{and}\quad(p_2\circ g_2)_*(\iota_{|v|})=\iota_{|v|}.$$

Since $\Sigma^{-1}(\mathrm{id}_{S^{|v|}}\wedge\Sigma f)\simeq(-1)^{|u||v|}\Sigma^{-1}(\Sigma f\wedge\mathrm{id}_{S^{|v|}})\colon S^{2|v|-1}\longrightarrow S^{|v|}$, there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\Sigma^{-1}S^{2|v|} & \xrightarrow{\Sigma^{-1}(\Sigma f\wedge \mathrm{id}_{S^{|v|}})} & S^{|v|} & \longrightarrow & \Sigma^{|v|}X \\
\| & & \downarrow{g} & & \downarrow{\phi_2} \\
\Sigma^{-1}S^{2|v|} & \xrightarrow{\partial} & \mathrm{sk}_{|v|}(X^{(2)})/S^0 & \longrightarrow & X^{(2)}/S^0,
\end{array}$$

where in the integral homology g_* sends $\iota_{|v|}$ to $uv+(-1)^{|u||v|}vu=uv+vu$. Thus the statement holds for $n=2$.

Suppose that the statement holds for $n-1$ with $n>2$. There is a homotopy commutative diagram

$$\begin{array}{ccc}
\Sigma^{(n-2)|v|}X\wedge X & \xrightarrow{\phi_{n-1}\wedge\mathrm{id}_X} & (X^{(n-1)}/\,\mathrm{sk}_{(n-3)|v|}(X^{(n-1)}))\wedge X \\
\downarrow{\text{pinch}} & & \downarrow{\text{pinch}} \\
\Sigma^{(n-2)|v|}X^{(2)}/S^0 & \xrightarrow{\tilde\phi} & X^{(n)}/\,\mathrm{sk}_{(n-2)|v|}(X^{(n)}).
\end{array}$$

Let $\phi_n\colon \Sigma^{(n-1)|v|}X\to X^{(n)}/\,\mathrm{sk}_{(n-2)|v|)}(X^{(n)})$ be the map defined by the composite

$$\Sigma^{(n-1)|v|}X\xrightarrow{\Sigma^{(n-2)|v|}\phi_2}\Sigma^{(n-2)|v|}X^{(2)}/S^0\xrightarrow{\tilde\phi}X^{(n)}/\,\mathrm{sk}_{(n-2)|v|}(X^{(n)}).$$

By induction, we have $\phi_{n*}(s^{(n-1)|v|}v)=v^n$ and

$$\phi_{n*}(s^{(n-1)|v|}u)=\sum_{i+j=n-2}v^iuv^jv+v^{n-1}u=\sum_{i+j=n-1}v^iuv^j.$$

Hence the result. □

Theorem 3.15. *Let $X = S^0 \cup_f e^{|v|}$ be a stable 2-local two-cell complex with $|v| > 0$ and $|v| \equiv 0 \mod 2$. Then the cell structure of $\bar{Q}^{2^k-1}(X)$ is given by*

$$\bar{Q}^{2^k-1}(X) = S^0 \cup_f e^{|v|} \cup_{2f} e^{2|v|} \cup_{3f} e^{3|v|} \cup \cdots \cup_{(2^k-1)f} e^{(2^k-1)|v|}$$

for each $k \geq 0$.

Proof. Let $V = \bar{H}_*(X)$ and let $q\colon X^{(2^k-1)} \to \bar{Q}^{2^k-1}(X)$ be the projection map. By Theorem 3.2, $\bar{H}_*(Q^{2^k-1}(X))$ is the symmetric quotient $S_{2^k-1}(V)$ of $V^{\otimes 2^k-1}$. Let

$$j_n\colon X^{(n)} = (S^0)^{(2^k-1-n)} \wedge X^{(n)} \hookrightarrow X^{(2^k-1)}$$

be the injection for $n \leq 2^k - 1$. Let θ be the composite

$$\Sigma^{(n-1)|v|}X \xrightarrow{\phi_n} X^{(n)}/\operatorname{sk}_{(n-2)|v|}(X^{(n)}) \xrightarrow{j_n} \operatorname{sk}_{n|v|}(\bar{Q}^{2^k-1}(X))/\operatorname{sk}_{(n-2)|v|}(\bar{Q}^{2^k-1}(X)),$$

where ϕ_n is given in Lemma 3.14. Then

$$\theta_*\colon H_j(\Sigma^{(n-1)|v|}X) \longrightarrow H_j(\operatorname{sk}_{n|v|}(Q^{2^k-1}(X))/\operatorname{sk}_{(n-2)|v|}(X))$$

is an isomorphism for $j \neq (n-1)|v|$ and of degree n for $j = (n-1)|v|$. Since $\Sigma^{(n-1)|v|}X$ is attached by f, the attaching map for the two-complex

$$\operatorname{sk}_{n|v|}(Q^{2^k-1}(X))/\operatorname{sk}_{(n-2)|v|}(Q^{2^k-1}(X))$$

is nf and hence the result. □

Corollary 3.16. *The space $\mathbb{H}\mathrm{Q}^3$ has the stable cell structure*

$$S^{12} \cup_\nu e^{16} \cup_{2\nu} e^{20} \cup_{3\nu} e^{24}.$$

Thus $\mathbb{H}\mathrm{Q}^3$ is not homotopy equivalent to $\mathbb{K}\mathrm{P}^2 \wedge \mathbb{H}\mathrm{P}^2$.

Proposition 3.17. *Let $X = S^0 \cup_f e^{|v|}$ be a stable two-cell complex with $|v| > 0$ and $|v| \equiv 0 \mod 2$. Then*

$$\operatorname{sk}_n(\bar{Q}^{2^k-1}(X)) \simeq \operatorname{sk}_n(\bar{Q}^{2^t-1}(X))$$

for $n \leq \min\{2^k - 1, 2^t - 1\}$.

Proof. We may assume that $2^k - 1 \leq 2^t - 1$. Then the composite

$$\bar{Q}^{2^k-1}(X) \hookrightarrow X^{(2^k-1)} \xrightarrow{j_{2^k-1}} X^{(2^t-1)} \xrightarrow{\text{pinch}} \bar{Q}^{2^t-1}(X)$$

induces a homotopy equivalence from $\bar{Q}^{2^k-1}(X)$ to $\operatorname{sk}_{2^k-1}(\bar{Q}^{2^t-1}(X))$. □

Let $\bar{J}_n(X)$ denote the $n|v|$-skeleton of $\bar{Q}^{2^k-1}(X)$ for some k such that $n \leq 2^k - 1$. By the above proposition, the homotopy type of $\bar{J}_n(X)$ is independent on the choice of $\bar{Q}^{2^k-1}(X)$. By Theorem 3.15, the cell structure of $\bar{J}_n(X)$ is given by

$$\bar{J}_n(X) = S^0 \cup_f e^{|v|} \cup_{2f} e^{2|v|} \cup_{3f} e^{2|v|} \cup \cdots \cup_{nf} e^{n|v|}.$$

The projection $q\colon X^{(2^k-1)} \to \bar{Q}^{2^k-1}(X)$ induces a map

$$q_n\colon X^{(n)} = \operatorname{sk}_n(X^{(2^k-1)}) \xrightarrow{q|_{X^{(n)}}} \bar{J}_n(X) = \operatorname{sk}_n(\bar{Q}^{2^k-1}(X)).$$

Let D denote the Spanier-Whitehead dual and let c_n be the composite

$$\Sigma^{2n|v|}D(\bar{J}_n(X)) \xrightarrow{Dq_n} \Sigma^{2n|v|}D(X^{(n)}) \simeq X^{(n)} \xrightarrow{q_n} \bar{J}_n(X).$$

The homology information on the map c_n is given as follows.

Proposition 3.18. *Let* $X = S^0 \cup_f e^{|v|}$ *with* $|v| > 0$ *and* $|v| \equiv 0 \mod 2$. *Then*

$$c_{n*}\colon \bar{H}_{j|v|}(\Sigma^{2n|v|} D(\bar{J}_n(X)); \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)} \to \bar{H}_{j|v|}(\bar{J}_n(X); \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}$$

is of degree $\binom{n}{j}$.

Proof. Let $V = \bar{H}_*(X)$. Observe that

$$q_{n*}\colon \bar{H}_*(X^{(n)}) = V^{\otimes n} \longrightarrow \bar{H}_*(\bar{J}_n(X)) = S_n(V)$$

is the canonical quotient from the self tensor product to symmetric product. Thus

$$Dq_{n*}\colon \bar{H}_*(\Sigma^{2n|v|} D(\bar{J}_n(X))) = S_n(V)^* \longrightarrow \bar{H}_*(X^{(n)}) = V^{\otimes n}$$

is the canonical inclusion of the n-th homogeneous of the commutative divided algebra into n-fold self tensor product. It follows that

$$c_{n*}\colon \bar{H}_{j|v|}(\Sigma^{2n|v|} D(\bar{J}_n(X))) \to \bar{H}_{j|v|}(\bar{J}_n(X))$$

is of degree $\dim(V_j^{\otimes n}) = \binom{n}{j}$, which is the assertion. □

Corollary 3.19. *Let* $X = S^0 \cup_f e^{|v|}$ *with* $|v| > 0$ *and* $|v| \equiv 0 \mod 2$. *Then* $\bar{Q}^{2^k - 1}(X)$ *is self-dual for each* $k \geq 0$.

As an application, we give a new proof of the classical result that the Whitehead square ω_{15} is divisible by 2.

Theorem 3.20 (Toda)**.** *The Whitehead square* $\omega_{15} \in \pi_{29}(S^{15})$ *is divisible by* 2.

Proof. Let $X = \Sigma^{-8}\mathbb{K}\mathrm{P}^2$ and let f be the composite

$$\mathbb{K}\mathrm{P}^2 \simeq \Sigma^{32} D(\bar{J}_2(X))/S^0 \xrightarrow{c_2} \bar{J}_2(X)/S^0 \hookrightarrow \bar{J}_3(X)/S^0.$$

Then $f_*\colon H_*(\mathbb{K}\mathrm{P}^2; \mathbb{Z}_{(2)}) \to H_*(\bar{J}_3(X)/S^0; \mathbb{Z}_{(2)})$ is of degree 2 and 1 in dimensions 8 and 16, respectively. Thus $\bar{H}_*(C_f)$ has a basis $\{x_8, x_9, x_{24}\}$ with $Sq^1_* x_9 = x_8$ and $Sq^{16}_* x_{24} = x_8$. It is well-known that the existence of such a complex if and only if the Whitehead square ω_{15} is divisible by 2 and hence the result. □

Example. Let $X = \Sigma^{-8}\mathbb{K}\mathrm{P}^2$ and let g be th composite

$$\Sigma^{16}\bar{J}_3(X) \simeq (\Sigma^{80} D(\bar{J}_5(X)))/\operatorname{sk}_8(\Sigma^{80} D(\bar{J}_5(X))) \xrightarrow{c_5} \bar{J}_5(X)/X \hookrightarrow \bar{J}_6(X)/X.$$

Then in $\mathbb{Z}_{(2)}$-homology, g_* is of degree $1, 5, 10$ and 10 in dimensions $40, 32, 24$ and 16, respectively. Thus $\bar{H}_*(C_g)$ has a basis $\{x_{16}, x_{17}, x_{24}, x_{25}, x_{48}\}$ with $Sq^1_* x_{17} = x_{16}$, $Sq^1_* x_{25} = x_{24}$, $Sq^8_* x_{24} = x_{16}$, $Sq^8_* x_{25} = x_{17}$ and $Sq^{32}_* x_{48} = x_{16}$. Clearly the attaching map between the 48-cell and 25-cell is trivial and so we obtain a stable complex Y such that $\bar{H}_*(Y)$ has a basis $\{x_{16}, x_{17}, x_{24}, x_{48}\}$ with $Sq^1_* x_{17} = x_{16}$, $Sq^8_* x_{24} = x_{16}$ and $Sq^{32}_* x_{48} = x_{16}$.

4. Decompositions of the Loop Spaces

This chapter is the preliminary for computing the homotopy groups of mod 2 Moore spaces. In the first section, we briefly go over some results in [12, 36, 37, 39]. In section 2, we will apply these general results to our special space and determine the factors of $\Omega P^n(2)$ up to certain dimension. This will help computing the homotopy groups up to certain range. We give some properties on the spaces $F^n\{2\}$ in section 3. In section 4, we study the EHP sequences for mod 2 Moore spaces.

4.1. **Functorial Decompositions of Loop Suspensions.** We start with some algebraic results. Let V be a vector space over a field $\mathbf{k}$ and let $T(V)$ be the tensor algebra generated by V. $T(V)$ is a Hopf algebra by saying V primitive. Consider T as a functor from (ungraded) modules to coalgebras. Let $A^{\min}$ is a smallest retract of T which contains V in the following sense

1) $A^{\min}(V)$ is a functorial coalgebra retract of $T(V)$;
2) $V \subseteq A^{\min}(V)$;
3) If $A(V)$ is any functorial coalgebra retract of $T(V)$ with $V \subseteq A(V)$, then $A^{\min}(V)$ is a functorial coalgebra retract of $A(V)$.

It was shown in [36] that the functor $A^{\min}$ exists and unique up to natural equivalence. The coalgebra $A^{\min}(V)$ can described as follows.

Recall that $\mathrm{Lie}(n)$ is module over the symmetric group $\mathbf{k}(S_n)$, see 2.15. Let $\mathrm{Lie}^{\max}(n)$ be the maximal projective $\mathbf{k}(S_n)$-submodule of $\mathrm{Lie}(n)$. (If $\mathrm{Lie}(n)$ itself is projective, then $\mathrm{Lie}^{\max}(n) = \mathrm{Lie}(n)$.) For any $\mathbf{k}$-module V, define the functor

$$L_n^{\max}(V) = V^{\otimes n} \otimes_{\mathbf{k}(S_n)} \mathrm{Lie}^{\max}(n).$$

Then $L_n^{\max}(V)$ is a functorial submodule of $L_n(V)$. Let $B^{\max}(V)$ be the subHopf algebra of $T(V)$ generated by $L_n^{\max}(V)$ for $n \geq 2$. Let $Q_n^{\max}(V)$ denote the set of indecomposable elements of $B^{\max}(V)$ of tensor length n.

Theorem 4.1. [36, Theorem 6.5, Corollary 8.9] *There are functorial coalgebra decompositions*

$$T(V) \cong B^{\max}(V) \otimes A^{\min}(V) \cong T(\bigoplus_{n=2}^{\infty} Q_n^{\max}(V)) \otimes A^{\min}(V).$$

Let p be the characteristic of $\mathbf{k}$.

Theorem 4.2. [36, Theorem 1.5] *If n is not a power of p, then $L_n(V) \subseteq B^{\max}(V)$.*

Let $A_n^{\min}(V)$ be the submodule of $A^{\min}(V)$ consisting of homogeneous elements of tensor length n.

Proposition 4.3. [36, Proposition 8.10] *There is a functorial isomorphism of* $\mathbf{k}$-*modules*

$$A_{n-1}^{\min}(V) \otimes V \cong Q_n^{\max}(V) \oplus A_n^{\min}(V)$$

for each $n \geq 1$.

These algebraic results admit *geometric realization* in the following sense. A *homotopy functor* from CW-complexes to spaces means a functor from the homotopy category of CW-complexes to the homotopy category of spaces.

Theorem 4.4. [37, Theorem 1.2] *Let X be any path-connected p-local CW complex. Then there are homotopy functors $\bar{Q}_n^{\max}$, $n \geq 2$, and $A^{\min}$ from path-connected p-local CW complexes to spaces with the following properties*

1) *$\bar{Q}_n^{\max}(X)$ is a functorial retract of $\Sigma X^{(n)}$;*
2) *there is a functorial fibre sequence*
$$A^{\min}(X) \xrightarrow{j_X} \bigvee_{n=2}^{\infty} \bar{Q}_n^{\max}(X) \xrightarrow{\pi_X} \Sigma X$$
with $j_X \simeq *$ functorially;
3) *there is a functorial decomposition*
$$\Omega\Sigma X \simeq A^{\min}(X) \times \Omega(\bigvee_{n=2}^{\infty} \bar{Q}_n^{\max}(X));$$
4) *there is a functorial coalgebra filtration on the mod p homology $H_*(A^{\min}(X))$ such that there is a functorial isomorphism of coalgebras*
$$\mathrm{Gr}.H_*(A^{\min}(X)) \cong A^{\min}(V),$$
where $V = \bar{H}_*(X)$ by forgetting the comultiplication.

The following lemmas will be useful.

Lemma 4.5. [12, Lemma 3.3] *The symmetric group module* $\mathrm{Lie}(n)$ *is projective if and only if $n \not\equiv 0 \mod p$. Thus* $\mathrm{Lie}^{\max}(n) = \mathrm{Lie}(n)$ *if and only if $n \not\equiv 0 \mod p$.*

Recall that the set of finitely generated indecomposable projective modules of the symmetric group $\mathbf{k}(S_n)$ is one-to-one correspondent to the set of p-regular partitions of n. The indecomposable projective $\mathbf{k}(S_n)$-module corresponding to λ is denoted by Q^λ.

Lemma 4.6. [39] *The symmetric group module* $\mathrm{Lie}^{\max}(8)$ *has the decomposition*
$$\mathrm{Lie}^{\max}(8) \cong [L_3, L_5] + Q^{(6,2)} + 4Q^{(5,2,1)} + Q^{(4,3,1)} \cong 2Q^{(6,2)} + Q^{(5,3)} + 8Q^{(5,2,1)} + 4Q^{(4,3,1)},$$
where kM is the k-fold self direct sum of a module M and $M + N = M \oplus N$.

The following lemma is well-known.

Lemma 4.7. *Let G be a finite group and let $\mathrm{Syl}_p(G)$ be the Sylow p-subgroup. Suppose that P is a finitely generated projective $\mathbf{k}(G)$-module. Then P is a free $\mathbf{k}(\mathrm{Syl}_p(G))$-module, where p is the characteristic of $\mathbf{k}$. In particular, the dimension of P is divisible by the order of the Sylow p-subgroup of G.*

4.2. **Determination of $\bar{Q}_k^{\max}(P^n(2))$ for $k \leq 9$.** In this section, we determine $\bar{Q}_k^{\max}(P^n(2))$ for low k. Since $\bar{Q}_k^{\max}(P^n(2))$ is a retract of $\Sigma(P^n(2))^{(k)}$, there is a corresponding retract $Q_k^{\max}(P^n(2))$ of $(P^n(2))^{(k)}$ such that $\bar{Q}_k^{\max}(P^n(2)) \simeq \Sigma Q_k^{\max}(P^n(2))$ by Proposition 3.4. We will determine $Q_k^{\max}(P^n(2))$ for $n \geq 2$ and $k \leq 9$. We assume that the ground field $\mathbf{k}$ is of characteristic 2. Let V be a two-dimensional module with a basis $\{u, v\}$ and $|v| = |u| + 1 = n$. Let $\mathrm{ad}(v)(u) = [u, v]$ and $\mathrm{ad}^n(v)(u) = [\mathrm{ad}^{n-1}(v)(u), v]$ for $n \geq 2$. For a 2-regular partition λ of k. Let $Q^\lambda(V) = V^{\otimes k} \otimes_{\mathbf{k}(S_k)} Q^\lambda$. By Theorem 3.1, we have $Q^\lambda(V) = 0$ if $\mathrm{len}(\lambda) > 2$. By Proposition 3.4, each module $Q^\lambda(V)$ is geometric realizable by a space $Q^\lambda(P^n(2))$

which is a retract of $(P^n(2))^{(k)}$. For simplicity, we write kX for the k-fold self wedge of X.

Our algorithm for determining decompositions of $Q_k^{\max}(P^n(2))$ for low k is as follows:

1) Determine the module $Q_k^{\max}(V)$.
2) Count the dimension $t_1 = \dim Q_k^{\max}(V)_l$, where $Q_k^{\max}(V)_j = 0$ for $j > l$ and $Q_k^{\max}(V)_l \neq 0$. In other words, count the dimension of the top degree submodule of $Q_k^{\max}(V)$. By Theorem 3.2, there is one and only one indecomposable factor $Q^{(a_1,k-a_1)}(P^n(2))$ of $(P^n(2))^{(k)}$ such that (1) $Q^{(a_1,k-a_1)}(P^n(2))$ has exactly one top cell in dimension l and (2) the t_1-fold self wedge of $Q^{(a_1,k-a_1)}(P^n(2))$ is a retract $Q_k^{\max}(P^n(2))$. In this step, we obtain the factor $Q^{(a_1,k-a_1)}(P^n(2))$ with the multiplicity t_1.
3) By subtracting the factor $t_1Q^{(a_1,k-a_1)}(V)$ from $Q_k^{\max}(V)$, we obtain the proper summand $Q_k^{\max}(V)/(t_1Q^{(a,k-a)}(V)$ of $Q_k^{\max}(V)$. By repeating step 2, we obtain a factor $Q^{(a_2,k-a_2)}(P^n(2))$ of $Q_k^{\max}(P^n(2))$ with a multiplicity t_2.
4) After a finite steps, we obtain a (finite) sequence of indecomposable factors of $Q_k^{\max}(P^n(2))$ together with their multiplicities. This will give a complete decomposition of $Q_k^{\max}(P^n(2))$ in terms of indecomposable factors.

4.2.1. $Q_k^{\max}(P^n(2))$ *for* $k \leq 4$. Since $\dim \mathrm{Lie}(2) = 1$ and $\dim \mathrm{Lie}(4) = 6$, we obtain that $\mathrm{Lie}^{\max}(2) = \mathrm{Lie}^{\max}(4) = 0$ by Lemma 4.7. It follows that

$$Q_2^{\max}(V) = L_2^{\max}(V) = Q_4^{\max}(V) = L_4^{\max}(V) = 0$$

and so

$$Q_2^{\max}(P^n(2)) \simeq Q_4^{\max}(P^n(2)) \simeq *.$$

By Proposition 4.5, $\mathrm{Lie}^{\max}(3) = \mathrm{Lie}(3)$ and so

$$Q_3^{\max}(V) = L_3^{\max}(V) = L_3(V).$$

This is a two-dimensional module with a basis $\{[[u,v],v],[[u,v],u]\}$ and so

Proposition 4.8. *There is a homotopy equivalence*

$$Q_3^{\max}(P^n(2)) \simeq P^{3n-1}(2).$$

4.2.2. $Q_5^{\max}(P^n(2))$. By Proposition 4.5, $\mathrm{Lie}^{\max}(5) = \mathrm{Lie}(5)$ and so $L_5^{\max}(V) = L_5(V)$. Observe that $L_5^{\max}(V) \to Q(B^{\max}(V))$ is a monomorphism. We have

$$Q_5^{\max}(V) = L_5(V).$$

This is 6-dimensional module.

Observe that the only top dimensional element in $L_5(V)$ is given by $[[u,v],v,v,v]$. Thus $Q^{(4,1)}(V)$ is a summand of $L_5(V)$ with the multiplicity one. By Example 3.8, $Q^{(4,1)}(P^n(2)) \simeq \mathbb{C}\mathrm{P}^2 \wedge P^{5n-5}(2)$.

Now $L_5(V)/Q^{(4,1)}(V)$ is a two-dimensional module. Thus we have

Proposition 4.9. *There is a homotopy decomposition*

$$Q_5^{\max}(P^n(2)) \simeq \mathbf{C}P^2 \wedge P^{5n-5}(2) \vee P^{5n-2}(2).$$

4.2.3. $Q_6^{\max}(P^n(2))$. By Theorem 4.2, we have

$$L_6(V) \subseteq B^{\max}(V) \cong T(Q_3^{\max}(V) \oplus Q_5^{\max}(V) \oplus Q_6^{\max}(V) \oplus \ldots)$$

and so $Q_6^{\max}(V) = L_6(V)/[L_3(V), L_3(V)]$. Observe that the only top dimensional element in $L_6(V)/[L_3(V), L_3(V)]$ is represented by $\mathrm{ad}^5(v)(u)$. Thus $Q^{(5,1)}(V)$ is a summand of $Q_6^{\max}(V)$. Since

$$\dim L_6(V)/[L_3(V), L_3(V)] = \frac{1}{6}(2^6 - 2^3 - 2^2 + 2) - 1 = 8 = \dim Q^{(5,1)}(V)$$

by the Witt formula and Example 3.8, we have

Proposition 4.10. *There is a homotopy equivalence*

$$Q_6^{\max}(P^n(2)) \simeq \mathbb{C}\mathrm{P}^2 \wedge \mathbb{R}\mathrm{P}^2 \wedge P^{6n-7}(2).$$

4.2.4. $Q_7^{\max}(P^n(2))$. By Theorem 4.2, we have $Q_7^{\max}(V) = L_7(V)$. Since the top dimensional element of $L_7(V)$ is $\mathrm{ad}^6(v)(u)$, $Q^{(6,1)}(V)$ is a summand of $L_7(V)$. By expecting the next top dimensional elements in $L_7(V)$ and $Q^{(6,1)}(V)$, $Q^{(5,2)}(V)$ is a summand of V with multiplicity 2. By extracting these three factors from $L_7(V)$, we obtain a 2-dimensional factor which is $Q^{(4,3)}(V)$. By Corollary 3.11, we have

Proposition 4.11. *There is a homotopy decomposition*

$$Q_7^{\max}(P^n(2)) \simeq (\mathbb{C}\mathrm{P}^2)^{(2)} \wedge P^{7n-9}(2) \vee 2(\mathbb{C}\mathrm{P}^2 \wedge P^{7n-6}(2)) \vee P^{7n-3}(2).$$

4.2.5. $Q_8^{\max}(V)$. Since $\dim V = 2$, we have

$$L_8^{\max}(V) \cong [L_3(V), L_5(V)] \oplus Q^{(6,2)}(V) \cong Q^{(6,2)}(V) \oplus Q^{(6,2)}(V) \oplus Q^{(5,3)}(V)$$

by Lemma 4.6. Observe that the factor $[L_3(V), L_5(V)]$ lies in the set of decomposable elements of $B^{\max}(V)$ and in dimension 8, there are no more possible decomposable elements. We obtain

$$Q_8^{\max}(V) \cong Q^{(6,2)}(V).$$

Thus we have

Proposition 4.12. *There is a homotopy equivalence*

$$Q_8^{\max}(P^n(2)) \simeq \mathbb{C}\mathrm{P}^2 \wedge \mathbb{R}\mathrm{P}^2 \wedge P^{8n-8}(2).$$

This proposition also gives information for primitive elements of tensor length 8 in $A^{\min}(V)$. Let $L_n^{\min}(V) = L_n(V) \cap A^{\min}(V)$. Observe that there is a decomposition

$$L_8(V) \cong [L_3(V), L_5(V)] \oplus Q_8^{\max}(V) \oplus L_8^{\min}(V).$$

Thus

$$\dim L_8^{\min}(V) = \dim L_8(V) - \dim[L_3(V), L_5(V)] - \dim Q_8^{\max}(V) = \frac{1}{8}(2^8 - 2^4) - 2 \times 6 - 8 = 10.$$

Let L be the sub Lie algebra of $L(V)$ generated by $L_2(V), L_3(V), L_4(V), L_5(V)$ and $L_6(V)$. Then L is a free Lie algebra generated by

$$\{L_2(V), L_3(V), L_4(V)/L_2(L_2(V)) = L_4(V), L_5(V), L_6'(V)\},$$

where $L_6'(V) \cong L_6(V)/(L_3(L_2(V)) \oplus L_2(L_3(V))) \cong L_6(V)/[L_3(V), L_3(V)]$. This gives another (non-functorial) decomposition of $L_8(V)$:

$$L_8(V) \cong L_4(L_2(V)) \oplus [L_4(V), L_4(V)] \oplus [L_2(V), L_6'(V)] \oplus [L_3(V), L_5(V)] \oplus M_8,$$

where $\dim M_8 = 7$. According to [39], $Q_8^{\max}(V)$ is a summand of $[L_2(V), L_6(V)]$ and so $Q_8^{\max}(V) \cong [L_2(V), L_6'(V)]$. Observe that $L_4(L_2(V)) = 0$. Thus

$$L_8^{\min}(V) \cong [L_4(V), L_4(V)] \oplus M_8$$

as modules over the Steenrod algebra. A basis for M_8 is given by $\{Sq_*^i v^8 \mid 1 \le i \le 7\}$ and $\dim[L_4(V), L_4(V)]$ is 3.

Note. If $\dim W \ge 3$, then $[L_4(W), L_4(W)]$ contains factors which lie in $B^{\max}(V)$. These factors come from some of the factors Q^λ with $\mathrm{len}(\lambda) \ge 3$ in Lemma 4.6.

4.2.6. $Q_9^{\max}(P^n(2))$. By expecting the Lie elements of tensor length 9 in the free Lie algebra

$$L(L_3(V), L_5(V), Q_6^{\max}(V), L_7(V), Q_8^{\max}(V)),$$

we have the short exact sequence

$$0 \longrightarrow L_3(L_3(V)) \oplus [L_3(V), Q_6^{\max}(V)] \longrightarrow L_9(V) \longrightarrow Q_9^{\max}(V) \longrightarrow 0.$$

Thus

$$\dim Q_9^{\max}(V) = \frac{1}{9}(2^9 - 3) - 2 - 2 \times 8 = 38.$$

The top element in $Q_9^{\max}(V)$ is $\mathrm{ad}^8(v)(u)$ which occurs once. Thus

$$Q^{(8,1)}(V) \cong L_2(V) \otimes Q^{(7)}(V)$$

is a summand of $Q_9^{\max}(V)$ with multiplicity one. Now consider the second top cells, that is, in the submodule of $V^{\otimes 9}$ spanned by monomials in which u occurs twice and v occurs 7 times. The modules $L_9(V)$, $L_3(L_3(V))$ and $[L_3, Q_6^{\max}(V)]$ restricted to this submodule are of dimension 4, 0 and 1, respectively, by the Witt formula. Thus $Q_9^{\max}(V)$ restricted to this submodule is of dimension 3. By Theorem 3.2, $Q^{(8,1)}(V)$ restricted to this submodule is of dimension 1. Thus $Q^{(7,2)}(V)$ is a summand of $Q_9^{\max}(V)$ with multiplicity 2. Similarly, by expecting the submodule of $V^{\otimes 9}$ spanned by monomials in which u occurs 3 times and v occurs 6 times, $Q^{(6,3)}(V)$ is a summand of $Q_9^{\max}(V)$ with multiplicity 3. By subtracting the factors obtained, there is a two-dimensional submodule of $Q_9^{\max}(V)$ left. Thus $Q^{(5,4)}(V)$ is a summand of $Q_9^{\max}(V)$ with multiplicity 1 and so we have

Proposition 4.13. *There is a homotopy decomposition*

$$Q_9^{\max}(P^n(2)) \simeq \mathbb{H}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^{9n-13}(2) \vee 2(\mathbb{C}\mathrm{P}^2)^{(2)} \wedge P^{9n-10}(2) \vee 3\mathbb{C}\mathrm{P}^2 \wedge P^{9n-7}(2) \vee P^{9n-4}(2).$$

Note. By using this algorithm, one may determine $Q_k^{\max}(V)$ for $k \le 15$ with the help of computer program.

4.3. **The Fibre of the Pinch Map.** Let $F^{n+1}\{2\}$ be the homotopy fibre of the pinch map $P^{n+1}(2) \longrightarrow S^{n+1}$. There is a principal fibre sequence

$$\Omega S^{n+1} \xrightarrow{s_{n+1}} F^{n+1}\{2\} \longrightarrow P^{n+1}(2)$$

and so ΩS^{n+1} acts on $F^{n+1}\{2\}$.

Lemma 4.14. [9, Proposition 8.1, Corollary 8.2] *If $n \ge 2$, then*

1) *the reduced integral homology $\bar{H}_*(F^{n+1}\{2\}; \mathbb{Z})$ is a free $\mathbb{Z}$-module on generators y_k with $k \ge 1$ and degree $|y_k| = kn$;*
2) *The map $s_{(n+1)*}\colon H_{kn}(\Omega S^{n+1}; \mathbb{Z}) = \mathbb{Z} \to H_{kn}(F^{n+1}\{2\}; \mathbb{Z}) = \mathbb{Z}$ is of degree 2;*

3) *$\bar{H}_*(F^{n+1}\{2\})$ is a free $H_*(\Omega S^{n+1})$ module on a single generator y_1.*

The following proposition can be found in [16, Corollary 5.8. p.513] and [27, Lemma 1.4].

Proposition 4.15. *The $(2n)$-skeleton $\mathrm{sk}_{2n}(F^{n+1}\{2\})$ of $F^{n+1}\{2\}$ is the homotopy cofibre of $2\omega_n\colon S^{2n-1} \to S^n$. Thus $\mathrm{sk}_{4n-2}(F^{2n}\{2\}) \simeq S^{2n-1} \vee S^{4n-2}$.*

Proof. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega S^{n+1} & \xrightarrow{\Omega[2]} & \Omega S^{n+1} & \longrightarrow & S^{n+1}\{2\} & \longrightarrow & S^{n+1} \\
\| & & \uparrow g & & \uparrow & & \| \\
\Omega S^{n+1} & \xrightarrow{s_{n+1}} & F^{n+1}\{2\} & \longrightarrow & P^{n+1}(2) & \longrightarrow & S^{n+1}.
\end{array}$$

By assertion 2 of Lemma 4.14, we have

$$2^k \iota_n^k = \Omega[2]_*(\iota_n^k) = g_*(s_{(n+1)*}(\iota_n^k)) = 2g_*(y_k)$$

and so $g_*(y_k) = 2^{k-1}\iota_n^k$

$$g_*\colon H_{kn}(F^{n+1}\{2\};\mathbb{Z}) \longrightarrow H_{kn}(\Omega S^{n+1};\mathbb{Z})$$

is of degree 2^{k-1} for $k \geq 1$. Thus there is a homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccccc}
S^{2n-1} & \xrightarrow{\partial} & S^n & \hookrightarrow & \mathrm{sk}_{2n}(F^{n+1}\{2\}) & \xrightarrow{\text{pinch}} & S^{2n} \\
\downarrow [2] & & \| & & \downarrow g & & \downarrow [2] \\
S^{2n-1} & \xrightarrow{\omega_n} & S^n & \hookrightarrow & J_2(S^n) & \xrightarrow{\text{pinch}} & S^{2n}
\end{array}$$

and hence the result. □

By assertion 2 of Theorem 4.4, there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
A^{\min}(P^n(2)) & \xrightarrow{j_{P^n(2)}} & \bigvee_{k=2}^{\infty} \bar{Q}_k^{\max}(P^n(2)) & \xrightarrow{\pi_{P^n(2)}} & P^{n+1}(2) \\
\downarrow & & \downarrow & & \downarrow \\
A^{\min}(S^n) & \xrightarrow{j_{S^n}} & \bigvee_{k=2}^{\infty} \bar{Q}_k^{\max}(S^n) \simeq * & \xrightarrow{\pi_{S^n}} & S^{n+1}.
\end{array}$$

Thus the map $\bigvee_{k=1}^{\infty} Q_n^{\max}(P^n(2)) \to P^{n+1}(2)$ lifts to $F^{n+1}\{2\}$ and so there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
\tilde{A}^{n+1} & = & \tilde{A}^{n+1} & \longrightarrow & * \\
\downarrow & & \downarrow{\scriptstyle \tilde{j}} & & \downarrow \\
A^{\min}(P^n(2)) & \xrightarrow{j_{P^n(2)}} & \Sigma \bigvee_{k=2}^{\infty} Q_k^{\max}(P^n(2)) & \xrightarrow{\pi_{P^n(2)}} & P^{n+1}(2) \\
\downarrow & & \downarrow & & \| \\
\Omega S^{n+1} & \longrightarrow & F^{n+1}\{2\} & \longrightarrow & P^{n+1}(2).
\end{array}$$

Since $j_{P^n(2)}$ is null homotopic, so is $\tilde{j}$. Thus we have

Proposition 4.16. *If $n \geq 2$, then there is a homotopy decomposition*

$$\Omega F^{n+1}\{2\} \simeq \tilde{A}^{n+1} \times \Omega\Sigma(\bigvee_{k=2}^{\infty} Q_k^{\max}(P^n(2)))$$

with a fibre sequence $\tilde{A}^{n+1} \longrightarrow A^{\min}(P^n(2)) \longrightarrow \Omega S^{n+1}$.

Let $\tilde{B}^{n+1}$ be the pull back of the diagram $S^n \hookrightarrow F^{n+1}\{2\} \longleftarrow \Sigma \bigvee_{k=2}^{\infty} Q_k^{\max}(P^n(2))$. There is a principal fibre sequence $\Omega S^n \longrightarrow \tilde{A}^{n+1} \longrightarrow \tilde{B}^{n+1}$. When n is odd, this fibre sequence splits off.

Lemma 4.17. *There is a homotopy decomposition*

$$\tilde{A}^{2n} \simeq \Omega S^{2n-1} \times \tilde{B}^{2n}.$$

for $n > 1$.

Proof. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
F & \longrightarrow & F^{2n}\{2\} & \longrightarrow & S^{2n-1} \\
\| & & \downarrow & & \downarrow \\
F & \longrightarrow & P^{2n}(2) & \hookrightarrow & \tau(S^{2n}) \\
\downarrow & & \downarrow{\scriptstyle \text{pinch}} & & \downarrow \\
* & \longrightarrow & S^{2n} & = & S^{2n}.
\end{array}$$

The bottom cell S^{2n-1} is a retract of $F^{2n}\{2\}$ and so ΩS^{2n-1} is a retract $\Omega F^{2n}\{2\}$. It follows that ΩS^{2n-1} is a retract of $\tilde{A}^{2n}$ and hence the result. $\square$

4.4. **The EHP Sequences for mod 2 Moore Spaces.** Let $\bar{H}_2$ denote the composite

$$\bar{H}_2\colon \Omega P^{n+1} \simeq J(P^n(2)) \xrightarrow{H_2} J(P^n(2)\wedge P^n(2)) \xrightarrow{J(q)} J(P^{2n}(2)) \simeq \Omega P^{2n+1}(2),$$

where q is the pinch map $(P^n(2))^{(2)} \longrightarrow (P^n(2))^{(2)}/(S^{n-1}\wedge P^n(2)) = P^{2n}(2)$. There is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{P} & S^n & \xrightarrow{E} & \Omega S^{n+1} & \xrightarrow{H_2} & \Omega S^{2n+1}\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\Omega^2 P^{2n+1}(2) & \xrightarrow{P} & T^{n+1} & \xrightarrow{E} & \Omega P^{n+1}(2) & \xrightarrow{\bar{H}_2} & \Omega P^{2n+1}(2).
\end{array}$$

We will determine the atomic factor of T^{n+1} which contains the bottom cell. We need some terminologies on simplicial sets. Let X and Y be pointed simplicial sets and let $f\colon X\to Y$ be a simplicial map. Let $F(Y)$ be the Milnor construction on Y. Define the simplicial set E^f to be the pull-back of the diagram

$$J(X) \xrightarrow{J(f)} J(Y) \subseteq F(Y) \longleftarrow PF(Y),$$

where PZ is the Moore path-space of a fibrant simplicial set Z. As a space, E^f is homotopy equivalent to the loop space of the homotopy fibre of $\Sigma f\colon \Sigma X\to\Sigma Y$. Consider the strictly commutative diagram

$$\begin{array}{ccccccc}
J(X) & \xrightarrow{J(f)} & J(Y) & \hookrightarrow & F(Y) & \longleftarrow & PF(Y)\\
\downarrow{\scriptstyle H_2} & & \downarrow{\scriptstyle H_2} & & \downarrow{\scriptstyle \tilde{H}_2} & & \downarrow{\scriptstyle P\tilde{H}_2}\\
J(X\wedge X) & \xrightarrow{J(f\wedge f)} & J(Y\wedge Y) & \hookrightarrow & F(Y\wedge Y) & \longleftarrow & PF(Y\wedge Y),
\end{array}$$

where $\tilde{H}_2\colon F(Y)\to F(Y\wedge Y)$ is a functorial extension of the map H_2. There is a unique simplicial map

$$H_2\colon E^f \longrightarrow E^{f\wedge f}$$

such that the diagram

$$\begin{array}{ccc}
E^f & \hookrightarrow & J(X)\times PF(Y)\\
\downarrow{\scriptstyle H_2} & & \downarrow{\scriptstyle H_2\times P\tilde{H}_2}\\
E^{f\wedge f} & \hookrightarrow & J(X\wedge X)\times PF(Y\wedge Y)
\end{array}$$

commutes strictly.

Lemma 4.18. *Let $f\colon X\to Y$ be a simplicial map and let Z be a simplicial subset of X such that $f|_Z\colon Z\to Y$ is the trivial map. Then there exist simplicial homomorphisms $\phi_1\colon J(Z)\to E^f$ and $\phi_2\colon J(Z\wedge X)\to E^{f\wedge f}$ such that*

1) *ϕ_1 is a lifting of $J(Z)\hookrightarrow J(X)$;*

2) *ϕ_2 is a lifting of $J(X\wedge Z)\hookrightarrow J(X\wedge X)$ and*
3) *the diagram*

$$\begin{array}{ccc}
J(Z) & \xrightarrow{\ \phi_1\ } & E^f \\
\Big\downarrow{\scriptstyle H_2} & & \Big\downarrow{\scriptstyle H_2} \\
J(Z\wedge Z) & \xrightarrow{\ \phi_2|_{J(Z\wedge Z)}\ } & E^{f\wedge f}
\end{array}$$

commutes strictly.

Proof. Since $J(f)|_{J(Z)}$ and $J(f\wedge f)|_{J(Z\wedge X)}$ are trivial maps, the simplicial submonoids $J(Z)$ of $J(X)$ and $J(X\wedge Z)$ of $J(X\wedge X)$ admits the cross-sections, $x\mapsto(x,1)$, into the pull-back E^f and $E^{f\wedge f}$, respectively. The assertions follow immediately. □

Lemma 4.19. *There exists a map $\phi\colon T^{n+1}\to S^n\{2\}$ such that*

$$\phi_*\colon H_{n-1}(T^{n+1})\longrightarrow H_{n-1}(S^n\{2\})$$

is an isomorphism.

Proof. Let $f\colon X\to Y$ be a fibration with the fibre F such that

1) $X\simeq Y\simeq S^n$;
2) f is a map of degree 2.

Then there is a map $g\colon P^n(2)\to F$ such that the composite $P^n(2)\to F\to X$ is homotopic to the pinch map $P^n(2)\to S^n$. By Lemma 4.18, there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
T^{n+1} & \longrightarrow & \Omega P^{n+1}(2) & \xrightarrow{\ \bar{H}_2\ } & \Omega\Sigma(P^n(2)\wedge S^n) \\
\Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_2} \\
S^n\{2\} & \longrightarrow & \Omega S^{n+1}\{2\} & \xrightarrow{\ H_2\ } & \Omega S^{2n+1}\{4\}.
\end{array}$$

Clearly ϕ induces an isomorphism on H_{n-1} and hence the result. □

Our next step is to construct a cross-section map for $T^{n+1}\to S^n\{2\}$. Let

$$A\overset{i}{\hookrightarrow} X\longrightarrow X/A$$

be a cofibration such that $A\simeq X\simeq S^{n-1}$ and i is of degree 2. Then $\Omega P^{n+1}(2)\simeq F(X/A)$ is the quotient simplicial group of $F(X)$ by the normal subgroup generated by $F(A)$. Clearly the simplicial coset of $F(X)$ by the action of the simplicial subgroup $F(A)$ is homotopy equivalent to $S^n\{2\}$. Thus there is a map of $F(X)$-spaces

$$s\colon S^n\{2\}\to\Omega P^{n+1}(2)$$

such that the diagram

$$\begin{array}{ccc} F(X) & = & F(X) \\ \downarrow & & \downarrow \\ S^n\{2\} & \xrightarrow{s} & \Omega P^{n+1}(2) \end{array}$$

commutes. Let $c\colon J(X)\wedge J(Y) \longrightarrow J(X\wedge Y)$ be the map defined by

$$c((x_1x_2\cdots x_n)\wedge(y_1y_2\cdots y_m)) =$$
$$(x_1\wedge y_1)\cdot(x_2\wedge y_1)\cdots(x_n\wedge y_1)\cdot(x_1\wedge y_2)\cdots(x_n\wedge y_2)\cdots(x_1\wedge y_m)\cdots(x_n\wedge y_m)$$

for $x_i\in X$ and $y_j\in Y$, see [47]. Let $\bar{K}_{k,l}$ be the subgroup of

$$[(S^{n-1})^k\times(P^n(2))^l,\Omega P^{n+1}(2)]$$

generated by the elements x_i, $1\le i\le k$ and y_j, $1\le j\le l$, represented by the maps:

$$x_i\colon (S^{n-1})^k\times(P^n(2))^l \xrightarrow{\pi_i} S^{n-1} \hookrightarrow \Omega P^{n+1}(2),$$
$$y_j\colon (S^{n-1})^k\times(P^n(2))^l \xrightarrow{\pi_{k+j}} P^n(2) \hookrightarrow \Omega P^{n+1}(2),$$

respectively, where π_i the i-th coordinate projection. By the proof of Lemma 2.30, we have

Lemma 4.20. *If $n\ge 3$, then the following identities hold in the group $\bar{K}_{k,l}$:*
1) $x_i^2=1$ *for* $1\le i\le k$;
2) $y_j^4=1$;
3) *The (iterated) commutators*

$$[[a_1,a_2],\ldots,a_s]=1$$

if $a_i=a_j$ for some $i<j$, where $a_i=y_s$ or x_t for some s,t.

Theorem 4.21. *If $n\ge 3$, then the map $\phi\colon T^{n+1}\to S^n\{2\}$ has a cross-section and so $S^n\{2\}$ is a retract of T^{n+1}.*

Proof. Since $S^n\{2\}$ is atomic, it suffices to show that the composite

$$S^n\{2\}\xrightarrow{s}\Omega P^{n+1}(2)\xrightarrow{\bar{H}_2}\Omega P^{2n+1}(2)$$

is null homotopic. We use the filtration $\{F_kS^n\{2\}\}$ is the proof of Lemma 2.30. Observe that the product element $x_1x_2\cdots x_ky_1\in\bar{K}_{k,1}$ is represented by the composite

$$(S^{n-1})^k\times P^n(2)\longrightarrow F_{k+1}S^n\{2\}\xrightarrow{s}\Omega P^{n+1}(2).$$

Consider the function

$$H_{2*}\colon[(S^{n-1})^k\times P^n(2),\Omega P^{n+1}(2)]\longrightarrow[(S^{n-1})^k\times P^n(2),\Omega\Sigma(P^n(2))^{(2)}].$$

According to [47, Lemma 3.4], The element $H_{2*}(x_1x_2\cdots x_ky)$ is represented by the product of the maps α and β given by

$$\alpha\colon (S^{n-1})^k\times P^n(2)\xrightarrow{\text{proj.}}(S^{n-1})^k\xrightarrow{\text{proj.}}J_k(S^{n-1})\xrightarrow{H_2}J(S^{2n-2})\hookrightarrow J(P^n(2)\wedge P^n(2)),$$
$$\beta\colon (S^{n-1})^k\times P^n(2)\xrightarrow{\text{proj.}}J_k(S^{n-1})\wedge P^n(2)\xrightarrow{c}J(S^{n-1}\wedge P^n(2))\hookrightarrow J(P^n(2)\wedge P^n(2)).$$

Since both α and β are null homotopic after composing with the map

$$J((P^n(2))^{(2)}) \longrightarrow J((P^n(2))^{(2)}/(S^{n-1}\wedge P^n(2))) = J(P^{2n}(2)),$$

the element

$$\bar{H}_2(x_1\cdots x_k y_1) = 1$$

in $[(S^{n-1})^k \times P^n(2), \Omega P^{2n+1}(2)]$ and hence the result. □

The homology of the space T^{n+1} can be described as follows.

Proposition 4.22. *If $n\geq 3$, then the Serre spectral sequence for the fibre sequence*

$$T^{n+1} \longrightarrow \Omega P^{n+1}(2) \xrightarrow{\bar{H}_2} \Omega P^{2n+1}(2)$$

collapses. Furthermore, there is a submodule W of the free Lie algebra $L(u,v)$ such that $H_*(T^{n+1})$ is isomorphic to $T(W)\otimes H_*(S^n\{2\})$ as coalgebras.

Proof. Let $\tilde{V} = \bar{H}_*(P^{2n}(2))$ with a basis $\{\tilde{u},\tilde{v}\}$ and $Sq^1_*\tilde{v} = \tilde{u}$. Consider

$$\bar{H}_{2*}\colon H_*(\Omega P^{n+1}(2)) = T(V) \longrightarrow H_*(\Omega P^{2n+1}(2)) = T(\tilde{V}).$$

Let A be the cotensor product $\mathbf{k}\square_{E(u,v)} T(u,v)$. Then A is a subHopf algebra of $T(u,v)$ generated by $\{u^2, [u,v], v^2, [[u,v],u], [[u,v],v]\}$. Let $j\colon A\to T(u,v)$ be the inclusion map and let $s\colon E(u,v)\to T(u,v)$ be a $\mathbf{k}$-linear map defined by $s(u)=u, s(v)=v$ and $s(uv)=uv$. Then s is a coalgebra map and the composite

$$A\otimes E(u,v) \xrightarrow{j\otimes s} T(u,v)\otimes T(u,v) \xrightarrow{\mu} T(u,v)$$

is an isomorphism of coalgebras over A. According to [11, Proposition 5.1] and [47, Proposition 3.12], the map

$$H_{2*}\colon T(V)\to T(V\otimes V)$$

is a morphism of coalgebras over A and so is $\bar{H}_{2*}$, where the map $\bar{H}_{2*}$ restricted to A is a morphism of Hopf algebras given by $\bar{H}_{2*}([u,v]) = \tilde{u}$, $\bar{H}_{2*}(v^2) = \tilde{v}$ and

$$\bar{H}_{2*}(u^2) = \bar{H}_{2*}([[u,v],u]) = \bar{H}_2([[u,v],v]) = 0.$$

Let $B = \mathbf{k}\square_{T(\tilde{V})} A$. By [22, Theorem 4.4], there is an isomorphism of coalgebras

$$A \cong B\otimes T(\tilde{V}),$$

where the coaction of $T(\tilde{V})$ on B is trivial. Observe that the composite

$$E(u,v) \xrightarrow{s} T(u,v) \xrightarrow{\bar{H}_{2*}} T(\tilde{V})$$

is trivial map. Thus

$$T(V) \cong E(u,v)\otimes B\otimes T(\tilde{V})$$

as comodules over $T(\tilde{V})$, where the coactions of $T(\tilde{V})$ on the factors $E(u,v)$ and B are trivial. Thus $H^*(\Omega P^{n+1}(2))$ is a free $H^*(\Omega P^{2n+1}(2))$-module. By expecting the Eilenberg-Moore spectral sequence for the fibre sequence

$$T^{n+1} \longrightarrow \Omega P^{n+1}(2) \longrightarrow \Omega P^{2n+1}(2),$$

$H_*(T^{n+1})$ is a sub-quotient of $E(u,v)\otimes B$. In particular, the Poincaré series

$$\chi(H_*(T^{n+1})) \leq \chi(E(u,v))\chi(B).$$

By expecting the Serre spectral sequence for the fibre sequence above, we have

$$\chi(E(u,v))\chi(B)\chi(T(\tilde{V})) = \chi(T(V)) \leq \chi(H_*(T^{n+1}))\chi(T(\tilde{V})).$$

Thus

$$\chi(H_*(T^{n+1})) = \chi(E(u,v))\chi(B)$$

and so the Serre spectral sequence collapses. Furthermore, there is an isomorphism of coalgebra

$$H_*(T^{n+1}) \cong E(u,v) \otimes B.$$

Consider the commutative diagrams of short exact sequences of Hopf algebras

$$\begin{array}{ccccc}
B\square_{P(u^2)} & \hookrightarrow & B & \twoheadrightarrow & P(u^2) \\
\downarrow & & \downarrow & & \| \\
A\square_{P(u^2)} & \hookrightarrow & A & \twoheadrightarrow & P(u^2) \\
\downarrow & & \downarrow & & \\
T(\tilde{V}) & = & T(\tilde{V}). & &
\end{array}$$

There is a submodule W of $L(V)$ such that $B\square_{P(u^2)} \cong T(W)$. It follows that

$$H_*(T^{n+1}) \cong E(u,v) \otimes P(u^2) \otimes T(W) \cong H_*(S^n\{2\}) \otimes T(W)$$

as coalgebras. The proof is finished. $\square$

Now we are going to give a product decomposition of T^{n+1} up to certain dimension. We need to do some lemmas on the James-Hopf maps. Let $\mathcal{E}$ be the collection of spaces consisting of all suspensions. For each suspension X, let $K_n(k)(X)$ be the subgroup of $[X^n, J(X^{(k)})]$ generated by the elements labelled by $\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}$, $1 \leq i_1, i_2, \ldots, i_s \leq n$, represented by the map

$$X^n \xrightarrow{\pi_{i_1,i_2,\ldots,i_k}} X^k \xrightarrow{\text{pinch}} X^{(k)} \hookrightarrow J(X^{(k)}),$$

where π_I is the coordinate projection. By letting X run over all objects in $\mathcal{E}$, one gets a universal group $K_n(k)$. The generators for $K_n(k)$ are labelled by $\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}$ subject to the relations given in [47, Lemma 2.2]. Observe that $K_n(1) = K_n(\mathcal{E})$ is the Cohen group given in Section 2.2. The James-Hopf map $H_k\colon J(X) \to J(X^{(k)})$ induces a function

$$H_{k*}\colon [X^n, J(X)] \longrightarrow [X^n, J(X^{(k)})].$$

When X runs over all objects in $\mathcal{E}$, this function induces a unique function $H_k\colon K_n(1)\to K_n(k)$ such that the diagram

$$\begin{array}{ccc} K_n(1) & \xrightarrow{e_X} & [X^n, J(X)] \\ \big\downarrow H_k & & \big\downarrow H_{k*} \\ K_n(k) & \xrightarrow{e_X} & [X^n, J(X^{(k)})] \end{array}$$

commutes for all $X\in\mathcal{E}$. Let $\beta_n\colon X^{(n)}\to J(X)$ be the n-fold Samelson product. Observe that the element $e_X([[x_1,x_2],\cdots,x_n])$ in $[X^n,J(X)]$ is represented by the composite $X^n\xrightarrow{\text{pinch}}X^{(n)}\xrightarrow{W_n}J(X)$. Thus the element

$$e_X(H_k([[x_1,x_2],\cdots,x_n]))$$

in $[X^n,J(X^{(k)})]$ is represented by composite

$$X^n\xrightarrow{\text{pinch}}X^{(n)}\xrightarrow{\beta_n}J(X)\xrightarrow{H_k}J(X^{(k)}).$$

Lemma 4.23. *If n is odd, then the composite*

$$X^{(n)}\xrightarrow{\beta_n}J(X)\xrightarrow{H_2}J(X^{(2)})$$

is null homotopic for any suspension X.

Proof. The homotopy commutative diagram

$$\begin{array}{ccc} J(X) & \xrightarrow{J([-1])} & J(X) \\ \big\downarrow H_2 & & \big\downarrow H_2 \\ J(X^{(2)}) & \xrightarrow{\text{id}} & J(X^{(2)}) \end{array}$$

induces a group homomorphism $J([-1])\colon K_n(1)\to K_n(1)$ such that the diagram

$$\begin{array}{ccc} K_n(1) & \xrightarrow{J([-1])} & K_n(1) \\ \big\downarrow H_2 & & \big\downarrow H_2 \\ K_n(2) & = & K_n(2) \end{array}$$

commutes. Observe that $J([-1])(x_i)=-x_i$. Thus we have

$$H_2([[x_1,x_2],\cdots,x_n])=H_2\circ J([-1])([[x_1,x_2],\cdots,x_n])=H_2([[x_1,x_2],\cdots,x_n]^{(-1)^n}).$$

By [47, Theorem 3.8], H_2 restricted to the commutator subgroup $\Gamma^2K_n(2)$ is a group homomorphism. Thus the element

$$(H_2([[x_1,x_2],\cdots,x_n]))^2=1$$

in the group $K_n(2)$. Since $K_n(2)$ is a torsion free group, the element

$$H_2([[x_1,x_2],\cdots,x_n])=1$$

and so the composite

$$X^n \xrightarrow{q} X^{(n)} \xrightarrow{\beta_n} J(X) \xrightarrow{H_2} J(X^{(2)})$$

is null homotopic. The assertion follows from the fact that $q^*\colon [X^{(n)},\Omega Y]\to[X^n,\Omega Y]$ is a monomorphism for any Y. □

Note. This lemma can be generalized as follows: The composite

$$X^{(l)} \xrightarrow{W_l} J(X) \xrightarrow{H_k} J(X^{(k)})$$

is null homotopic if $l\not\equiv 0 \mod k$.

Corollary 4.24. *If $n\geq 3$ and k is odd, then the composite*

$$(P^n(2))^{(k)} \xrightarrow{\beta_k} \Omega P^{n+1} \xrightarrow{\bar H_2} \Omega P^{2n+1}(2)$$

is null homotopic.

Lemma 4.25. *In the group $K_4(2)$, the element $H_2([[x_1,x_2],x_3,x_4])$ is equal to the product*

$$[\{x_4|x_3\},\{x_1|x_2\}]\cdot[\{x_4|x_1\},\{x_3|x_2\}]\cdot[\{x_3|x_1\},\{x_4|x_2\}]\cdot[\{x_2|x_1\},\{x_4|x_3\}].$$

The proof follows from a direct calculation by using the methods in [47].

Lemma 4.26. *If $n\geq 3$, then the composite*

$$(P^n(2))^{(3)}\wedge S^{n-1} \hookrightarrow (P^n(2))^{(4)} \xrightarrow{\beta_4} \Omega P^{n+1}(2) \xrightarrow{\bar H_2} \Omega P^{2n+1}(2)$$

is null homotopic.

Proof. Let

$$q\colon (P^n(2))^{(2)} \longrightarrow P^{2n}(2)=(P^n(2))^{(2)}/S^{n-1}\wedge P^n(2)$$

be the pinch map. Observe that the element $\{x_4|x_i\}$ restricted to $(P^n(2))^3\times S^{n-1}$ is represented by the composite

$$(P^n(2))^3\times S^{n-1} \xrightarrow{\pi_{4,i}} S^{n-1}\wedge P^n(2) \hookrightarrow J((P^n(2))^{(2)}).$$

Thus $J(q)_*(\{x_4|x_i\})=1$ for $1\leq i\leq 3$. The assertion follows from Lemma 4.25. □

Let $f\colon X\to J(Y)$ be a map. We write $J(f)\colon J(X)\to J(Y)$ be the homomorphism of monoids induced by f.

Lemma 4.27. *Let $\phi\colon Y\to (P^n(2))^{(k)}$ be a map such that the composite*

$$Y \xrightarrow{\phi} (P^n(2))^{(k)} \xrightarrow{\beta_k} \Omega P^{n+1}(2) \xrightarrow{\bar H_2} \Omega P^{2n+1}(2)$$

is null homotopic. Suppose that $n\geq 3$ and $k\geq 2$. Then there is a map

$$\mu_\phi\colon \Omega\Sigma Y\times T^{n+1}\to T^{n+1}$$

such that the diagram

$$\begin{array}{ccc} \Omega\Sigma Y \times T^{n+1} & \xrightarrow{\mu_\phi} & T^{n+1} \\ \downarrow{\scriptstyle \Omega\Sigma\phi\times E} & & \downarrow{\scriptstyle E} \\ \Omega\Sigma(P^n(2))^{(k)} \times \Omega P^{n+1} & \xrightarrow{J(\beta_k)\cdot \mathrm{id}_{\Omega P^{n+1}(2)}} & \Omega P^{n+1}(2) \end{array}$$

commutes up to homotopy.

Proof. By Theorem [47, Theorem 3.10], there is a map $\beta_{k,2}\colon \Sigma X^{(k)} \longrightarrow \Sigma X^{(2)}$ such that $\Omega\beta_{k,2}$ is homotopic to the composite

$$\Omega\Sigma X^{(k)} \xrightarrow{J(\beta_k)} \Omega\Sigma X \xrightarrow{H_2} \Omega\Sigma X^{(2)}$$

for any suspension X and any integer $k \geq 2$. According to [47, Proposition 3.12], there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega\Sigma X^{(k)} \times \Omega\Sigma X & \xrightarrow{J(\beta_k)\cdot \mathrm{id}_{\Omega\Sigma X}} & \Omega\Sigma X \\ \downarrow{\scriptstyle \Omega\beta_{k,2}\times H_2} & & \downarrow{\scriptstyle H_2} \\ \Omega\Sigma X^{(2)} \times \Omega\Sigma X^{(2)} & \xrightarrow{\mu} & \Omega\Sigma X^{(2)} \end{array}$$

for any suspension X and any integer $k \geq 3$. It follows that the composite

$$\Omega\Sigma Y \times T^{n+1} \xrightarrow{\Omega\Sigma\phi\times E} \Omega\Sigma(P^n(2))^{(k)} \times \Omega P^{n+1}(2) \xrightarrow{J(\beta_k)\cdot \mathrm{id}_{\Omega P^{n+1}(2)}} \Omega P^{n+1}(2)$$

lifts to the fibre T^{n+1} and hence the result. □

Theorem 4.28. *If $n \geq 3$, there is a map*

$$\theta\colon S^n\{2\} \times \Omega P^{3n}(2) \times \Omega P^{4n-1}(2) \times \Omega(\mathbb{C}\mathrm{P}^2 \wedge P^{5n-4}(2) \vee P^{5n-1}(2)) \longrightarrow T^{n+1}$$

such that $H_j(\theta)$ is an isomorphism for $j \leq 6n-6$.

Proof. Let $s\colon S^n\{2\} \to T^{n+1}$ be a cross-section of the map ϕ defined in Lemma 4.19. By Lemma 4.26 and Corollary 4.24, the maps

$$\beta_3\colon Q_3^{\max}(P^n(2)) = P^{3n-1}(2) \longrightarrow \Omega P^{n+1}(2),$$

$$\beta_5\colon Q_5^{\max}(P^n(2)) = \mathbb{C}\mathrm{P}^2 \wedge P^{5n-5}(2) \vee P^{5n-2}(2) \longrightarrow \Omega P^{n+1}(2),$$

$$\bar\beta_4\colon Q_3^{\max}(P^n(2)) \wedge S^{n-1} = P^{4n-2}(2) \hookrightarrow Q_3^{\max}(P^n(2)) \wedge P^n(2) \xrightarrow{\beta_4} \Omega P^{n+1}(2)$$

are null homotopic after composing with $\bar H_2$. The assertion follows from Lemma 4.27 and by expecting the homology of T^{n+1} described in Proposition 4.22. □

5. The Homotopy Groups $\pi_{n+r}(\Sigma^n\mathbb{R}\mathrm{P}^2)$ for $n \geq 2$ and $r \leq 8$

In this chapter, we compute specific homotopy groups. Thus this chapter will be divided into parts **A:** $\pi_*(P^4(2))$, **B:** $\pi_*(P^5(2))$, **C:** $\pi_*(P^6(2))$, **D:** $\pi_*(P^7(2))$, **E:** $\pi_*(P^8(2))$, **F:** $\pi_*(P^9(2))$ and **G:** $\pi_*(P^n(2))$ for $n \geq 10$.

5.1. **Preliminary and Notations.**

Proposition 5.1. *The first three homotopy groups are as follows.*

(1). $\pi_{n-1}(P^n(2))=\mathbb{Z}/2\cong\pi_1^s(\mathbb{R}\mathrm{P}^2)$ *for* $n\geq 2$*;*
(2). $\pi_n(P^n(2))=\mathbb{Z}/2\cong\pi_2^s(\mathbb{R}\mathrm{P}^2)$ *for* $n\geq 4$*;*
(3). $\pi_{n+1}(P^n(2))=\mathbb{Z}/4\cong\pi_3^s(\mathbb{R}\mathrm{P}^2)$ *for* $n\geq 4$*.*

The proof is immediate.

Recall that there is a decomposition

$$\Omega P^{n+1}(2)\simeq\Omega\Sigma(\bigvee_{k=3}^{\infty}Q_k^{\max}(P^n(2)))\times A^{\min}(P^n(2)).$$

By applying the Hilton-Milnor theorem, we have a decomposition

$$\Omega\Sigma(\bigvee_{k=3}^{\infty}Q_k^{\max}(P^n(2)))\simeq\prod_{k=3}^{\infty}\Omega\Sigma L_k.$$

Let $V=\bar{H}_*(P^n(2))$ with a basis $\{u,v\}$ with $Sq^1_*v=u$. When k is odd, We write L_k for $L_k(P^n(2))$ as a geometric realization of module $L_k(V)$. When $k=6$, then L_k is a geometric realization of $L_6(V)/[L_3(V),L_3(V)]=Q_6^{\max}(V)$. Observe that L_k is $(kn-k)$-connected. For computing $\pi_{n+r}(\Omega P^{n+1}(2))$ with $n\geq 3$ and $r\leq 8$, we only need factors $L_3\simeq P^{3n-1}(2)$ and $L_5\simeq\mathbb{C}\mathrm{P}^2\wedge P^{5n-5}\vee P^{5n-2}(2)$. We rewrite the decomposition

$$\Omega P^{n+1}(2)\simeq\Omega L_3\times\Omega L_5\times Y_{n+1},$$

where $\pi_{n+r}(Y_{n+1})=\pi_{n+r}(A^{\min}(P^n(2)))$ for $n\geq 3$ and $r\leq 8$. The decomposition in Proposition 4.16 is rewritten as

$$\Omega F^{n+1}\{2\}\simeq\Omega L_3\times\Omega L_5\times A_{n+1},$$

where $\pi_{n+r}(A_{n+1})=\pi_{n+r}(\tilde{A}^{n+1})$ for $n\geq 3$ and $r\leq 8$. Recall that there is a fibre sequence

$$A_{n+1}\longrightarrow Y_{n+1}\longrightarrow\Omega S^{n+1}.$$

By Lemma 4.17, A_{2n} has a further decomposition

$$A_{2n}\simeq\Omega S^{2n-1}\times B_{2n}.$$

We will need some information on the stable homotopy groups of $\mathbb{R}\mathrm{P}^2$.

Lemma 5.2. *The stable homotopy groups* $\pi_r^s(\mathbb{R}\mathrm{P}^2)$ *for* $4\leq r\leq 10$ *are as follows*

(1). $\pi_4^s(\mathbb{R}\mathrm{P}^2)=\mathbb{Z}/2\oplus\mathbb{Z}/2$,
(2). $\pi_5^s(\mathbb{R}\mathrm{P}^2)=\mathbb{Z}/2$,
(3). $\pi_6^s(\mathbb{R}\mathrm{P}^2)=0$,
(4). $\pi_7^s(\mathbb{R}\mathrm{P}^2)=\mathbb{Z}/2$,
(5). $\pi_8^s(\mathbb{R}\mathrm{P}^2)=\mathbb{Z}/2\oplus\mathbb{Z}/2$,
(6). $\pi_9^s(\mathbb{R}\mathrm{P}^2)=\mathbb{Z}/2^{\oplus 3}$,
(7). $\pi_{10}^s(\mathbb{R}\mathrm{P}^2)=\mathbb{Z}/2\oplus\mathbb{Z}/4\oplus\mathbb{Z}/4$.

Proof. The proof follows from the long exact sequence of stable homotopy groups associated with the cofibre sequence

$$S^1\xrightarrow{[2]}S^1\longrightarrow\mathbb{R}\mathrm{P}^2$$

together with calculations from [42] for the homotopy groups of spheres and the fact that the degree 2 map $[2] : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$, in the stable category, is the composite

$$\mathbb{R}\mathrm{P}^2 \xrightarrow{\text{pinch}} S^2 \xrightarrow{\eta} S^1 \hookrightarrow \mathbb{R}\mathrm{P}^2.$$

□

5.2. **The Homotopy Groups $\pi_*(P^4(2))$.** Now we start to compute $\pi_* P^4(2)$ for $* \leq 12$.

5.2.1. *The Space B_4.*

Lemma 5.3. *There exists a map $\phi : \Omega S^6 \times P^{10}(2) \to B_4$ so that*

$$\phi_* : H_*(\Omega S^6 \times P^{10}(2)) \to H_* B_4$$

is an isomorphism for $* \leq 10$.

Proof. The primitive elements $PH_r(B_4)$ for $r \leq 10$ has a basis

$$\{[u,v], [u,v]^2, [[[u,v],u],u], [[[u,v],v],u]\}.$$

Let $\phi_1 : \Omega S^6 \to \Omega F^4\{2\}$ be a map so that $(\phi_1)_*(\iota_5) = [u,v]$. Since

$$L_3(P^n(2)) = P^{3n-1}(2),$$

there exists a map $\psi \colon P^8(2) \to \Omega F^4\{2\}$ such that

$$\psi_*(v_8) = [[u,v],v] \quad \text{and} \quad \psi_*(u_7) = [[u,v],u].$$

Let $j \colon S^2 \to \Omega F^4\{2\}$ be the injection so that $j_*(\iota_2) = u$. Define ϕ_2 to be Samelson product $[\psi, j] : P^{10}(2) = P^8(2) \wedge S^2 \to \Omega F^4\{2\}$. Let ϕ be the composite

$$\Omega S^6 \times P^{10}(2) \xrightarrow{\phi_1 \times \phi_2} \Omega F^4\{2\} \times \Omega F^4\{2\} \longrightarrow \Omega F^4\{2\} \longrightarrow B_4.$$

Then the map ϕ has the desired property and hence the result. □

Lemma 5.4. *There is a cofibre sequence*

$$S^{10} \xrightarrow{\psi} \mathrm{sk}_{15}(\Omega S^6 \times P^{10}(2)) \longrightarrow \mathrm{sk}_{15}(B_4),$$

which satisfies:

1) *the composite*

$$S^{10} \xrightarrow{\psi} \Omega S^6 \times P^{10}(2) \xrightarrow{\text{proj.}} P^{10}(2)$$

is $\bar{\eta}$ which is a representative of the generator for $\pi_{10} P^{10}(2) = \mathbb{Z}/2$ and

2) *the composite*

$$S^{10} \xrightarrow{\psi} \Omega S^6 \times P^{10}(2) \longrightarrow \Omega S^6$$

is $2\omega_6$.

Proof. Note that $PH_r(B_4)$ for $r \leq 16$ has a basis

$$[u,v], [[u,v,u^2], [u,v]^2, [[u,v,v,u], [[u,v,v,v]$$

with $Sq^1_*[[u,v,v,u] = [[u,v,u^2]$ and $Sq^2_*[[u,v,v,v] = [[u,v,u^2]$. By [9, Lemma 10.4], $\beta_2[[u,v,v,v] = [u,v]^2$. Thus $\mathrm{sk}_{15}(B_4)/\,\mathrm{sk}_{15}(\Omega S^6 \times P^{10}(2)) = S^{11}$ and there is a cofibre sequence

$$S^{10} \xrightarrow{\psi} \mathrm{sk}_{15}(\Omega S^6 \times P^{10}(2)) \longrightarrow \mathrm{sk}_{15}(B_4).$$

Since $Sq^2_*[[u,v,v,v] = [[u,v,u^2] \neq 0$, $S^{10} \xrightarrow{\psi} \Omega S^6 \times P^{10}(2) \longrightarrow P^{10}(2)$ is essential and hence condition (1). Let ψ_2 be the composite $S^{10} \xrightarrow{\psi} \Omega S^6 \times P^{10}(2) \longrightarrow \Omega S^6$. There is a homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccccc} S^{10} & \xrightarrow{\psi} & \mathrm{sk}_{15}(\Omega S^6 \times P^{10}(2)) & \longrightarrow & \mathrm{sk}_{15}(B_4) & \longrightarrow & S^{11} \\ \| & & \downarrow & & \downarrow & & \| \\ S^{10} & \xrightarrow{\psi_2} & J_3(S^5) & \longrightarrow & X & \longrightarrow & S^{11}. \end{array}$$

Since $H_*(\mathrm{sk}_{15}(B_4)) \to H_*X$ is onto, the Bockstein $\beta_2 : H_{11}X \to H_{10}X$ is an isomorphism. Observe that $\phi_2 \colon S^{10} \to \Omega S^6$ maps into the 10-skeleton $J_2(S^5)$ up to homotopy. The composite

$$S^{10} \xrightarrow{\psi_2} J_2(S^5) \xrightarrow{\text{pinch}} S^{10},$$

is of degree 4. Since $\pi_{11}(S^6) = \mathbb{Z}$ which is generated by ω_6, we have $\psi_2 = 2\omega_6$ and hence the result. □

5.2.2. *The Homotopy Groups $\pi_*(F^4\{2\})$.*

Lemma 5.5. *The homotopy groups $\pi_k(B_4)$ for $5 \leq k \leq 12$ are as follows:*

$$\pi_k(B_4) = \begin{cases} \mathbb{Z} & \text{for} \quad k=5 \\ \mathbb{Z}/2 & \text{for} \quad k=6 \\ \mathbb{Z}/2 & \text{for} \quad k=7 \\ \mathbb{Z}/8 & \text{for} \quad k=8 \\ \mathbb{Z}/2 & \text{for} \quad k=9 \\ \mathbb{Z}/4 & \text{for} \quad k=10 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for} \quad k=11 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for} \quad k=12. \end{cases}$$

Proof. By Lemma 5.4, $\pi_r(B_4) \cong \pi_r(\Omega S^6)$ for $r \leq 8$ and $\pi_9(B_4) = \pi_9(\Omega S^6) \oplus \pi_9(P^{10}(2)) = \mathbb{Z}/2$.

1. $\pi_{10}(B_4)$: There is an exact sequence

$$\pi_{10}(S^{10}) \xrightarrow{\psi_*} \pi_{10}(\Omega S^6) \oplus \pi_{10}(P^{10}(2)) \longrightarrow \pi_{10}B_4 \longrightarrow 0 \quad \text{or}$$

$$\mathbb{Z} \xrightarrow{(2,\eta_*)} \mathbb{Z} \oplus \mathbb{Z}/2 \longrightarrow \pi_{10}(B_4) \longrightarrow 0.$$

Thus $\pi_{10}(\Omega S^6) \longrightarrow \pi_{10}(B_4)$ is onto and $\pi_{10}(B_4) = \mathbb{Z}/4$.

2. $\pi_{11}(B_4)$: There is an exact sequence

$$\pi_{11}(S^{10}) \xrightarrow{\psi_*} \pi_{11}(\Omega S^6) \oplus \pi_{11}(P^{10}(2)) \longrightarrow \pi_{11}(B_4) \longrightarrow 0 \quad \text{or}$$

$$\mathbb{Z}/2 \xrightarrow{\psi_*} \mathbb{Z}/2 \oplus \mathbb{Z}/4 \longrightarrow \pi_{11}(B_4) \longrightarrow 0.$$

Observe that

$$\psi_*(\eta) = 2\omega_6 \circ \eta + \bar{\eta}^2 = \bar{\eta} \circ \eta = 2\alpha,$$

where α is the generator of $\pi_{11}(P^{10}(2)) = \mathbb{Z}/4$. Thus

$$\pi_{11}(B_4) \cong \pi_{11}(\Omega S^6) \oplus \pi_{11}(P^{10}(2))/2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

3. $\pi_{12}(B_4)$: Consider the exact sequence

$$\pi_{12}(S^{10}) \xrightarrow{\psi_*} \pi_{12}(\Omega S^6) \oplus \pi_{12}(P^{10}(2)) \longrightarrow \pi_{12}(B_4) \longrightarrow 0.$$

Note that

$$\psi_*(\eta^2) = 2\omega_6 \circ \eta^2 + \bar{\eta} \circ \eta^2 = 0.$$

Thus

$$\pi_{12}(B_4) \cong \pi_{12}(\Omega S^6) \oplus \pi_{12}(P^{10}(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

The calculation is finished now. □

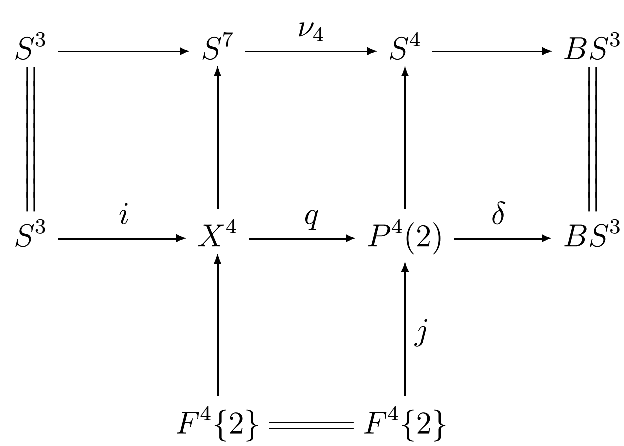
By Lemmas 5.2 and 5.5, we have

Proposition 5.6. *The homotopy groups $\pi_r = \pi_r(F^4\{2\})$ for $r \le 13$ are as follows.*

$\pi_r =$	$\pi_r(S^3)\oplus$	$\pi_{r-1}(B_4)\oplus$	$\pi_r(P^9(2))\oplus$	$\pi_r(X)\oplus$	$\pi_r(P^{14}(2))^{\oplus 2}$
r=3	$\mathbb{Z}$				
4	$\mathbb{Z}/2$				
5	$\mathbb{Z}/2$				
6	$\mathbb{Z}/4$	$\mathbb{Z}$			
7	$\mathbb{Z}/2$	$\mathbb{Z}/2$			
8	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		
9	0	$\mathbb{Z}/8$	$\mathbb{Z}/2$		
10	0	$\mathbb{Z}/2$	$\mathbb{Z}/4$		
11	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 2}$		
12	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	
13	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 2} \oplus \mathbb{Z}/4$	0	0	$(\mathbb{Z}/2)^{\oplus 2}$

where $X = \mathbb{C}\mathrm{P}^2 \wedge P^{11}(2)$, one copy of $P^{14}(2)$ comes from ΣL_5 and another copy of $P^{14}(2)$ is the bottom two-cells for ΣL_6.

5.2.3. *The Fibre of $P^4(2) \to BS^3$.* Now consider the homotopy commutative diagram of fibre sequences



where $\nu_4\colon S^7 \to S^4$ is the Hopf fibration.

Lemma 5.7. *There is a cofibre sequence*

$$S^6 \xrightarrow{(\nu',2)} S^3 \vee S^6 \longrightarrow X^4$$

Proof. By Lemma 2.38, the mod 2 homology $\bar{H}_r(X^4) = \mathbb{Z}/2$ for $r = 3, 6, 7$ and 0 otherwise.

Consider the Serre spectral sequence for the fibre sequence $F^4\{2\} \to X^4 \to S^7$. There is a cofibre sequence

$$\mathrm{sk}_6(F^4\{2\}) \longrightarrow X^4 \longrightarrow S^7.$$

Since $F^4\{2\}_{(6)} = S^3 \vee S^6$, there is a cofibre sequence

$$S^6 \xrightarrow{\theta} S^3 \vee S^6 \xrightarrow{f} X^4.$$

Now consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} \Omega S^7 & \longrightarrow & F^4\{2\} & \longrightarrow & X^4 \\ \downarrow & & \| & & \downarrow \\ \Omega S^4 & \longrightarrow & F^4\{2\} & \longrightarrow & P^4(2). \end{array}$$

Observe that $H_6(\Omega S^4; \mathbb{Z}) \to H_6(F^4\{2\}; \mathbb{Z})$ is of degree 2. The composite

$$S^6 \xrightarrow{\theta} S^3 \vee S^6 \xrightarrow{\text{proj.}} S^6$$

is of degree 2. Recall that $\pi_6(S^3) = \mathbb{Z}/4$ which is generated by ν'. The composite $S^6 \xrightarrow{\theta} S^3 \vee S^6 \xrightarrow{\text{proj.}} S^3$ is $k\nu'$. We claim that $k = \pm 1$.

Suppose that $k \equiv 0 \mod 2$. Let $\tilde{\lambda}_4$ denote the composite

$$S^6 \hookrightarrow S^3 \vee S^6 = \mathrm{sk}_6(F^4\{2\}) \longrightarrow X^4.$$

Then, in $\pi_6(X^4)$, $2[\tilde{\lambda}_4] = -kf_*(\nu')$. Since $i : S^3 \to X^4$ is of degree 2 and S^3 is an H-space, we have

$$i_*(k/2\nu') = f_*([2]_*(-k/2\nu')) = f_*(-k\nu') = 2[\tilde{\lambda}_4].$$

Hence $2q_*[\tilde{\lambda}_4] = 0$ in $\pi_6(P^4(2))$. Let $\lambda_4 \colon S^5 \to \Omega P^4(2)$ denote the adjoint map of $q \circ \tilde{\lambda}_4 \colon S^6 \to P^4(2)$. Then $(\lambda_4)_*(\iota_5) = [u, v]$ in $H_5(\Omega P^4(2))$. By [12, Theorem 2.2],

$$2q_*[\tilde{\lambda}_4] = 2[\lambda_4] \neq 0.$$

This is a contradiction. Thus $k = \pm 1$ and hence the result. □

5.2.4. *The Homotopy Groups $\pi_*(P^4(2))$.*

Theorem 5.8. *The homotopy groups $\pi_r(P^4(2))$ for $6 \le r \le 12$ are as follows.*

$\pi_r(P^4(2))$	$\pi_{r-1}(Y_4)\oplus$	$\pi_r(P^9(2))\oplus$	$\pi_r(\mathbb{C}\mathrm{P}^2 \wedge P^{11}(2))$
$r = 6$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$		
7	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$		
8	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	
9	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	
10	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/4$	
11	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	
12	$\mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

Proof. **1.** $\pi_6(P^4(2))$: By expecting the exact sequence

$$\pi_6(S^6) \xrightarrow{(\nu',2)} \pi_6(S^3) \oplus \pi_6(S^6) \longrightarrow \pi_6(X^4) \longrightarrow 0,$$

we have $\pi_6(X^4) = \mathbb{Z}/8$ generated by $[\tilde{\lambda}_4]$. Since

$$i_*(\nu') = 2\nu' = 4[\tilde{\lambda}_4],$$

there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \pi_6(P^4(2)) \longrightarrow \pi_5(S^3) \longrightarrow 0.$$

Since $q_*([\tilde{\lambda}_4]) = [\lambda_4]$ in $\pi_6(P^4(2)) \cong \pi_5(\Omega P^4(2))$ and $h([\lambda_4]) = [u,v]$ in $H_*\Omega P^4(2)$, the element $[\lambda_4]$ is not divisible by 2. Thus the above short exact sequence splits and so $\pi_6(P^4(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2$.

2. $\pi_7(P^4(2))$: Consider the fibre sequence

$$\Omega^2 S^7 \xrightarrow{\delta} \Omega S^3 \times B_4 \times Q \longrightarrow \Omega X^4 \to \Omega S^7,$$

where $Q = \prod_{k\geq 3} \Omega\Sigma L_k$. Since $\Omega P^4(2) \simeq Y_4 \times Q$, the composite

$$\Omega^2 S^7 \xrightarrow{\delta} \Omega S^3 \times B_4 \times Q \xrightarrow{\text{proj.}} Q$$

is null homotopic. By Lemma 5.7, the diagram

$$\begin{array}{ccccc} \Omega S^6 & \xrightarrow{E} & \Omega^2 S^7 & \xrightarrow{\delta} & \Omega S^3 \times B_4 \times Q \\ \downarrow{\scriptstyle \Omega[2]} & & & & \downarrow{\scriptstyle \text{proj.}} \\ \Omega S^6 & & \xrightarrow{\phi_1} & & B_4 \end{array}$$

commutes up to homotopy commutes and the composite

$$\Omega S^6 \xrightarrow{E} \Omega^2 S^7 \xrightarrow{\delta} \Omega S^3 \times B_4 \times Q \xrightarrow{\text{proj.}} \Omega S^3$$

is $\Omega\nu'$. Consider the exact sequence

$$\pi_6(\Omega^2 S^7) \xrightarrow{\delta_*} \pi_6(\Omega S^3) \oplus \pi_6(B_4) \oplus \pi_6(Q) \longrightarrow \pi_6(\Omega X^4) \longrightarrow 0.$$

Observe that

$$\delta_*(\eta) = \nu' \circ \eta + (\phi_1)_* \circ \Omega[2]_*(\eta) = \nu' \circ \eta.$$

Thus $\pi_6(\Omega X^4) \cong \pi_6(B_4) \oplus \pi_6(Q) = \mathbb{Z}/2$ and $\delta_* : \pi_6(\Omega^2 S^7) \to \pi_6(\Omega F^4\{2\})$ is 1-1. Since $i_*(\nu' \circ \eta) = f_*(2\nu' \circ \eta) = f_*(0) = 0$, there is a short exact sequence

$$0 \longrightarrow \pi_7(X^4) \xrightarrow{q_*} \pi_7(P^4(2)) \longrightarrow 2 \cdot \pi_6(S^3) \longrightarrow 0 \quad \text{or}$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_7(P^4(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and therefore $(\lambda_4)_* : \pi_6(S^5) \to \pi_7(P^4(2))$ is a monomorphism. Consider the second reduced James-Hopf map $\bar{H}_2 : \Omega P^4(2) \longrightarrow \Omega P^7(2)$. Then

$$(\bar{H}_2 \circ \lambda_4)_*(\iota_5) = (\bar{H}_2)_*([u,v]) \neq 0$$

in $H_5(\Omega P^7(2))$ and so $\bar{H}_2\circ\lambda_4 : S^5 \to \Omega P^7(2)$ is the inclusion of the bottom cell. Since $\bar{H}_2 \circ \lambda_4$ induces an isomorphism on π_6, the subgroup $\pi_6(S^5) = \mathbb{Z}/2$ is a summand of $\pi_6(\Omega P^4(2))$ or the above short exact sequence for $\pi_7(P^4(2))$ splits. Thus

$$\pi_7(P^4(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

generated by $\lambda_4 \circ \eta$ and $\overline{2\nu'}$.

3. $\pi_8(P^4(2))$: By the commutative diagram

$$\begin{array}{ccccccc}
 & & \pi_8(S^7) & \longrightarrow & \pi_8(S^4) & & \\
 & & \uparrow{\scriptstyle 0} & & \uparrow & & \\
\pi_8(S^3) & \xrightarrow{i_*} & \pi_8(X^4) & \xrightarrow{q_*} & \pi_8(P^4(2)) & \xrightarrow{\delta'_*} & \pi_7(S^3) \\
 & & \uparrow & & \uparrow & & \\
 & & \pi_7(\Omega S^3) \oplus \pi_7(B_4) \oplus \pi_7(Q) & = & \pi_8(F^4\{2\}) & & \\
 & & \uparrow{\scriptstyle \delta_*} & & \uparrow & & \\
 & & \pi_9(S^7) & \longrightarrow & \pi_9(S^4), & &
\end{array}$$

we have

$$\delta_*(\eta^2) = \nu' \circ \eta^2 + (\phi_1)_* \circ [2]_*(\eta^2) = \nu' \circ \eta^2 \neq 0.$$

Thus

$$\pi_8(X^4) \cong \pi_7(B_4) \oplus \pi_7(Q) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

and there is an exact sequence

$$\pi_8(S^3) \xrightarrow{i_*} \pi_7(B_4) \xrightarrow{q_*} \pi_7(Y_4) \xrightarrow{\delta'_*} \pi_7(S^3) \longrightarrow 0.$$

Since $i_*(\nu' \circ \eta^2) = f_*(2(\nu' \circ \eta^2)) = 0$, there is a short exact sequence

$$0 \longrightarrow \pi_7(B_4) \xrightarrow{q_*} \pi_7(Y_4) \xrightarrow{\delta'_*} \pi_7(S^3) \longrightarrow 0.$$

Recall that $(\lambda_4)_*(\iota_5) = [u, v]$. There is a commutative diagram

$$\begin{array}{ccccc}
\pi_7(B_4) & \xrightarrow{q_*} & \pi_7(Y_4) & \longrightarrow & \pi_8(P^4(2)) \\
\uparrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle H_{2*}} \\
\mathbb{Z}/2 = \pi_7(S^5) & \xrightarrow{\mu_*} & \pi_7(\Omega(P^6(2) \vee S^6)) & \xrightarrow{\Omega g_*} & \pi_7(\Omega\Sigma(P^3(2) \wedge P^3(2)),
\end{array}$$

where $\mu : S^5 \to \Omega(P^6(2) \vee S^6)$ is the composite $S^5 \xrightarrow{E} \Omega S^6 \hookrightarrow \Omega(P^6(2) \vee S^6)$ and

$$P^6(2) \vee S^6 \xrightarrow{g} \Sigma P^3(2) \wedge P^3(2) \longrightarrow S^7$$

is a cofibre sequence. Observe that

$$g_*\colon \pi_8(P^6(2)\vee S^6)=\mathbb{Z}/4\oplus\mathbb{Z}/2 \longrightarrow \pi_8(P^3(2)\wedge P^4(2))=\mathbb{Z}/4\oplus\mathbb{Z}/2$$

is an isomorphism. There exists a homomorphism $\psi\colon \pi_7(\Omega\Sigma(P^3(2)\wedge P^3(2)))\to\mathbb{Z}/2$ so that

$$\pi_7(Y_4)\cong\pi_7(B_4)\oplus\pi_7(S^3)$$

generated by $\lambda_4\circ\eta^2$ and $\overline{\nu'\circ\eta}$. Thus

$$\pi_8(P^4(2))=\pi_7(Y_4)\oplus\pi_7(Q)=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2.$$

4. $\pi_9(P^4(2))$: Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & \pi_9(S^7) & \longrightarrow & \pi_9(S^4) & & & & \\
 & & \uparrow{\scriptstyle 0} & & \uparrow & & & & \\
\pi_9(S^3)=0 & \xrightarrow{i_*} & \pi_9(X^4) & \xrightarrow{q_*} & \pi_9(P^4(2)) & \xrightarrow{\delta'_*} & \pi_8(S^3) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \pi_8(\Omega S^3)\oplus\pi_8(B_4)\oplus\pi_8(Q) & = & \pi_9(F^4\{2\}) & & & & \\
 & & \uparrow{\scriptstyle \delta_*} & & \uparrow & & & & \\
 & & \pi_{10}(S^7) & \longrightarrow & \pi_{10}(S^4) & & & & .
\end{array}$$

Since $\pi_8(B_4)\cong\pi_8(\Omega S^6)=\mathbb{Z}/8$, we have $\delta_*(\nu_7)=2\nu_6$ and so

$$\pi_9(X^4)\cong\pi_8(B_4)/2\oplus\pi_8(Q)=\mathbb{Z}/2\oplus\mathbf{Z}/2$$

with a short exact sequence

$$0\longrightarrow\pi_8(\Omega S^6)/2\longrightarrow\pi_8(Y_4)\longrightarrow\pi_8(S^3)\longrightarrow 0.$$

Now consider the commutative diagram

$$\begin{array}{ccccc}
\pi_8(\Omega S^6)/2 & \hookrightarrow & \pi_8(Y_4) & \longrightarrow & \pi_8(\Omega P^4(2)) \\
\uparrow & & & & \downarrow{\scriptstyle \bar{H}_{2*}} \\
\pi_8(\Omega S^6) & & \xrightarrow{\Omega g_*} & & \pi_8(\Omega P^7(2))=\mathbb{Z}/2\oplus\mathbb{Z}/2.
\end{array}$$

Since $\Omega g_*(\nu_6)\neq 0$, the above short exact sequence for $\pi_8(Y_4)$ splits and therefore

$$\pi_8(Y_4)\cong\pi_8\Omega S^6/2\oplus\pi_8 S^3.$$

Thus $\pi_9(P^4(2))\cong\pi_8(\Omega S^6)/2\oplus\pi_8(S^3)\oplus\pi_8(Q)=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2.$

5. $\pi_{10}(P^4(2))$: Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & \pi_{10}(S^7) & \longrightarrow & \pi_{10}(S^4) & & \\
 & & \uparrow & & \uparrow & & \\
\pi_{10}(S^3)=0 & \xrightarrow{i_*} & \pi_{10}(X^4) & \xrightarrow{q_*} & \pi_{10}(P^4(2)) & \xrightarrow{\delta'_*} & \pi_9(S^3)=0 \\
 & & \uparrow & & \uparrow & & \\
 & & \pi_9(\Omega S^3)\oplus\pi_9(B_4)\oplus\pi_9(Q) & = & \pi_{10}(F^4\{2\}) & & \\
 & & \uparrow{\scriptstyle \delta_*} & & \uparrow & & \\
 & & \pi_{11}(S^7)=0 & \longrightarrow & \pi_{11}(S^4). & &
\end{array}$$

Since $\mathrm{Ker}(\delta_*\colon \pi_{10}(S^7)\to\pi_9(F^4\{2\})) = 4\cdot\pi_{10}(S^7)$, we have $\pi_{10}(X^4)\cong\pi_{10}(P^4(2))$ with short exact sequences

$$0\longrightarrow \pi_9(B_4)\oplus\pi_9(Q)\longrightarrow \pi_{10}(P^4(2))\longrightarrow 4\cdot\pi_{10}(S^7)\longrightarrow 0 \quad\text{and}$$

$$0\longrightarrow \pi_9(B_4)\longrightarrow \pi_9(Y_4)\longrightarrow 4\cdot\pi_{10}(S^7)\longrightarrow 0.$$

Recall that $\pi_9(B_4)\cong\pi_9(P^{10}(2))=\mathbb{Z}/2$. Let α be the non-zero element in $\pi_9(B_4)$. Then the Hurewicz image $H(\alpha)=[[u,v],u],u]$ in $H_*(B_4)$ and so in $H_*\Omega P^4(2)$. Thus α is not divisible by 2 in $\pi_{10}(P^4(2))$ and

$$\pi_9(Y_4)\cong\pi_9(B_4)\oplus 4\cdot\pi_{10}(S^7)$$

generated by Samelson product $\langle\langle\lambda_4,u\rangle,u\rangle$ and $\overline{4\nu_4^2}$. Hence

$$\pi_{10}(P^4(2))\cong\pi_9(B_4)\oplus 4\cdot\pi_{10}(S^7)\oplus\pi_9(Q)=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4.$$

6. $\pi_{11}(P^4(2))$: Consider the commutative diagram of exact sequences

$$
\begin{array}{ccccccc}
 & & \pi_{11}(S^7)=0 & \longrightarrow & \pi_{11}(S^4) & & \\
 & & \uparrow & & \uparrow & & \\
\pi_{11}(S^3)=\mathbb{Z}/2 & \xrightarrow{i_*} & \pi_{11}(X^4) & \xrightarrow{q_*} & \pi_{11}(P^4(2)) & \xrightarrow{\delta'_*} & \pi_{10}(S^3)=0 \\
 & & \uparrow & & \uparrow & & \\
 & & \pi_{10}(\Omega S^3)\oplus\pi_{10}(B_4)\oplus\pi_{10}(Q) & = & \pi_{11}(F^4\{2\}) & & \\
 & & \uparrow{\scriptstyle \delta_*} & & \uparrow & & \\
 & & \pi_{12}(S^7)=0 & \longrightarrow & \pi_{12}(S^4). & &
\end{array}
$$

Recall that $\pi_{11}(S^3)=\mathbb{Z}/2$ generated by ϵ_3 and $i_*(\epsilon_3)=f_*\circ[2]_*(\epsilon_3)=f_*(2\epsilon_3)=0$. Thus

$$\pi_{11}(P^4(2))\cong\pi_{11}(X^4)\cong\pi_{11}(F^4\{2\})=\mathbb{Z}/2\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2.$$

7. $\pi_{12}(P^4(2))$: Consider the commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
 & & \pi_{12}(S^7)=0 & \longrightarrow & \pi_{12}(S^4) & \xrightarrow{\cong} & \pi_{11}(S^3) & & \\
 & & \uparrow & & \uparrow{\scriptstyle p_*} & & \| & & \\
\pi_{12}(S^3) & \xrightarrow{i_*} & \pi_{12}(X^4) & \xrightarrow{q_*} & \pi_{12}(P^4(2)) & \xrightarrow{\delta'_*} & \pi_{11}(S^3) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \pi_{11}(\Omega S^3)\oplus\pi_{11}(B_4)\oplus\pi_{11}(Q) & = & \pi_{12}(F^4\{2\}) & & & & \\
 & & \uparrow{\scriptstyle \delta_*} & & \uparrow & & & & \\
 & & \pi_{13}(S^7)=\mathbb{Z}/2 & \longrightarrow & \pi_{13}(S^4). & & & &
\end{array}
$$

Observe that

$$\delta_*(\nu^2)=\nu'\circ\nu_6^2+(\phi_1)_*\Omega[2]_*(\nu_6^2)=\nu'\circ\nu_6^2=0$$

and $2\pi_{12}S^3=0$. Thus

$$\pi_{12}(X^4)\cong\pi_{11}(\Omega S^3)\oplus\pi_{11}(B_4)\oplus\pi_{11}(Q)$$

with a short exact sequence

$$0 \longrightarrow \pi_{11}(\Omega S^3) \oplus \pi_{11}(B_4) \longrightarrow \pi_{11}(Y_4) \xrightarrow{\delta'_*} \pi_{11}(S^3) = \mathbb{Z}/2 \longrightarrow 0.$$

Now let $\alpha \in \pi_{11}(Y_4) \subseteq \pi_{11}(\Omega P^4(2))$ so that $\delta'_*(\alpha) = \epsilon_3$. Then $p_*(\alpha) = \epsilon_4 \in \pi_{11}(\Omega S^4)$. Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_{11}(\Omega F^4\{2\}) & \longrightarrow & \pi_{11}(\Omega P^4(2)) & \xrightarrow{p_*} & \pi_{11}(\Omega S^4) \\
\downarrow & & \downarrow E^\infty & & \downarrow E^\infty \\
\pi_{11}(Q(S^2)) & \longrightarrow & \pi_{11}(Q(P^3(2))) & \xrightarrow{p_*} & \pi_{11}(Q(S^3))
\end{array}$$

Then $\delta'_*(E^\infty\alpha) = E^\infty\epsilon_3 = \epsilon$ and $E^\infty(2\alpha) = 2E^\infty\alpha = \epsilon \circ \eta \neq 0$ in $\pi_{11}Q(P^3(2))$. It follows that $2\alpha \neq 0$. Since $2\pi_{11}\Omega S^3 = 2\pi_{11}B_4 = 0$, we have

$$\pi_{11}(Y_4) \cong \frac{1}{2}\pi_{11}(S^3) \oplus \pi_{11}(B_4) \oplus \pi_{11}(\Omega S^3)/(\epsilon_3 \circ \eta) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

generated by $1/2\epsilon_3, \mu_3, \lambda_4 \circ \nu_6^2, \beta$, where β is the image of the generator for $\pi_{11}(P^{10}(2))$ with $2\beta = 0$ in $\pi_{11}(\Omega P^4(2))$. Hence

$$\pi_{12}(P^4(2)) \cong \pi_{11}(Y_4) \oplus \pi_{12}(P^9(2)) \oplus \pi_{12}(\mathbb{C}\mathrm{P}^2 \wedge P^{11}(2)) \cong (\mathbb{Z}/2)^{\oplus 5} \oplus \mathbb{Z}/4.$$

We finish the calculation. □

5.3. **The Homotopy Groups $\pi_*(P^5(2))$.** Now we compute $\pi_*P^5(2)$. By Proposition 4.15, there is a cofibre sequence

$$S^7 \xrightarrow{2\omega_4} S^4 \longrightarrow \mathrm{sk}_8(F^5\{2\})$$

and so there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega F^5\{\} & = & \Omega F^5\{2\} & \longrightarrow & * & \longrightarrow & F^5\{2\} \\
\uparrow \Omega j & & \uparrow \phi_1 & & \uparrow j & & \\
\Omega S^4 & \longrightarrow & X^5 & \xrightarrow{2\omega_4} & S^4. & &
\end{array}$$

Let $\phi_2 = [S_3, j] : L_3(P^4(2)) \wedge S^3 = P^{14}(2) \to \Omega F^5\{2\}$ be Samelson product and let ϕ denote the composite

$$X^5 \times P^{14}(2) \xrightarrow{\phi_1 \times \phi_2} \Omega F^5\{2\} \times \Omega F^5\{2\} \xrightarrow{\text{multi.}} \Omega F^5\{2\} \longrightarrow A_5.$$

Lemma 5.9. *The map $\phi_* : H_r(X^5 \times P^{14}(2)) \to H_r(A_5)$ is an isomorphism for $r \leq 13$ and therefore $\phi_* : \pi_r(X^5 \times P^{14}(2)) \to \pi_r(A_5)$ is an isomorphism for $r \leq 12$.*

Proof. The proof follows by observing that the primitive elements $P\bar{H}_r(A_5)$ for $r \leq 14$ has a basis $u, u^2, [u,v], u^4, [[[u,v],u],u], [u,v]^2, [[[u,v,],v,],u]$ with $\beta_2[u,v] = u^2$ and $Sq^1_*[[[u,v],v],u] = [[[u,v],u],u]$. □

There is an overlap between the following theorem and [28, Theorem 1.2, p.64].

Theorem 5.10. *The homotopy groups $\pi_r(P^5(2))$ for $7 \le r \le 13$ are as follows.*

$\pi_r(P^5(2))$	$\pi_{r-1}(Y_5)\oplus$	$\pi_r(P^{12}(2))$
$r = 7$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	
8	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	
9	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	
10	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	
11	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$
12	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$
13	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/4$

Proof. **1.** $\pi_7(P^5(2))$: Consider the exact sequence

$$\pi_7(S^7) = \mathbb{Z} \xrightarrow{(2\omega_4)_*} \pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}/4 \xrightarrow{j_*} \pi_6(X^5) \longrightarrow 0.$$

Since

$$(2\omega_4)_*(\iota_7) = \omega_4 \circ (2\iota_7) = 2(2\nu_4 - E\nu') = 4\nu_4 - 2E\nu',$$

we have

$$\pi_6(X^5) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$$

generated by $j_*(\nu_4)$ and $j_*(\omega_4)$. Now consider the exact sequence

$$\pi_8(S^5) \xrightarrow{\delta_*} \pi_7(F^5\{2\}) \cong \pi_6(X^5) \xrightarrow{i_*} \pi_7(P^5(2)) \xrightarrow{p_*} \pi_7(S^5) \longrightarrow 0$$

and the commutative diagram

$$\begin{array}{ccc} \pi_8(S^5) & \xrightarrow{\delta_*} & \pi_7(F^5\{2\}) \\ \uparrow{\scriptstyle E} & & \uparrow{\scriptstyle j_*} \\ \pi_7(S^4) & \xrightarrow{[2]_*} & \pi_7(S^4). \end{array}$$

Since $[2]_*(v_4) = 2\nu_4 + \omega_4$, we have

$$\delta_*(\nu_5) = \delta_* \circ E(\nu_4) = 2j_*(\nu_4) + j_*(\omega_4).$$

Thus there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/4(i_*j_*(\nu_4)) \longrightarrow \pi_7(P^5(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

The homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} F^5\{2\} & \longrightarrow & P^5(2) & \longrightarrow & S^5 \\ \downarrow & & \downarrow & & \downarrow \\ Q(S^4) & \longrightarrow & Q(P^5(2)) & \longrightarrow & Q(S^5) \end{array}$$

induces a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \pi_7(P^5(2)) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow E^{\infty} & & \| & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \pi_7(Q(P^5(2))) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0.
\end{array}$$

Since $E^{\infty}(i_*j_*(\nu_4)) \neq 0$ and $\pi_7(Q(P^5(2))) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, the element $i_*j_*(\nu_4)$ is not divisible by 2 and so

$$\pi_7(P^5(2)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$$

with the epimorphism $E^{\infty}\colon \pi_7(P^5(2)) \longrightarrow \pi_7^s(P^5(2))$.

2. $\pi_8(P^5(2))$: By expecting the exact sequence

$$\pi_8(S^7) \xrightarrow{(2\omega_4)_*=0} \pi_8(S^4) \longrightarrow \pi_7(X^5) \longrightarrow 0,$$

we have

$$\pi_8(S^4) \cong \pi_7(X^5) \cong \pi_7(A_5) \cong \pi_8(F^5\{2\}).$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc}
\pi_9(S^5) & \xrightarrow{\delta_*} & \pi_8(F^5\{2\}) & \longrightarrow & \pi_8(P^5(2)) & \longrightarrow & 4\cdot\pi_8(S^5) & \longrightarrow & 0 \\
\uparrow E & & \uparrow \cong & & & & & & \\
\pi_8(S^4) & \xrightarrow{[2]_*} & \pi_8(S^4), & & & & & &
\end{array}$$

where the top arrow is exact. Since $[2]_*(\nu_4 \circ \eta) = 2\nu_4 \circ \eta + \omega_4 \circ \eta = -E\nu' \circ \eta$ and $\pi_8(S^4) = \mathbb{Z}/2(\nu_4 \circ \eta) \oplus \mathbb{Z}/2(E\nu' \circ \eta)$, there is a short exact sequence

$$\mathbb{Z}/2(\nu_4 \circ \eta) \longrightarrow \pi_8(P^5(2)) \longrightarrow 4\cdot\pi_8(S^5) \longrightarrow 0$$

with the monomorphism $\delta_* : \pi_9(S^5) \longrightarrow \pi_8(F^5\{2\})$. By the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
F^5\{2\} & \longrightarrow & P^5(2) & \longrightarrow & S^5 \\
\downarrow & & \downarrow E & & \downarrow E \\
\Omega F^6\{2\} & \longrightarrow & \Omega P^6(2) & \longrightarrow & \Omega S^6,
\end{array}$$

there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\pi_9(S^5) & \xrightarrow{\delta_*} & \pi_8(F^5\{2\}) & \longrightarrow & \pi_8(P^5(2)) & \longrightarrow & \pi_8 S^5 & \longrightarrow & \pi_7(F^5\{2\}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
0 = \pi_{10}(S^6) & \xrightarrow{\delta_*} & \pi_9(F^6\{2\}) & \longrightarrow & \pi_9(P^6(2)) & \longrightarrow & \pi_9(S^6) & \longrightarrow & \pi_8(F^6\{2\}).
\end{array}$$

By Proposition 4.16 and Lemma 4.17, there is decomposition

$$\Omega F^6\{2\} \simeq \Omega S^5 \times B_6 \times Q_6,$$

where both B_6 and Q_6 are 8-connected. Thus $\pi_r(S^5) \cong \pi_r(F^6\{2\})$ for $r \leq 9$ and so there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2(\nu_4 \circ \eta) & \longrightarrow & \pi_8(P^5(2)) & \longrightarrow & 4 \cdot \pi_8(S^5) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow E & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathbb{Z}/2(\nu_5 \circ \eta) & \longrightarrow & \pi_9(P^6(2)) & \longrightarrow & 4 \cdot \pi_9(S^6) & \longrightarrow & 0. \end{array}$$

Hence $\pi_8(P^5(2)) \cong \pi_9(P^6(2))$. We show that $2 \cdot \pi_9(P^6(2)) = 0$. Recall that the degree 2 map $[2] : P^6(2) \to P^6(2)$ is the composite

$$P^6(2) \xrightarrow{\text{pinch}} S^6 \xrightarrow{\eta_5} S^5 \hookrightarrow P^6(2)$$

and $\eta_5 \circ \nu_6 = 0$ (see [42, pp.44]). Thus $[2]_* : \pi_9(P^6(2)) \to \pi_9(P^6(2))$ is zero. By the distributivity law [2],

$$\Omega[2]_* = 2 + (\Omega W_2 \circ H_2)_* : \pi_8(\Omega P^6(2)) \longrightarrow \pi_8(\Omega P^6(2)).$$

(Note the dimension restrictions in this last equation.) Since the composite

$$P^5(2) \xrightarrow{E} \Omega P^6(2) \xrightarrow{H_2} \Omega\Sigma(P^5(2) \wedge P^5(2)) \xrightarrow{\Omega W_2} \Omega P^6(2)$$

is null homotopic, the composite

$$\pi_8(P^5(2)) \underset{\cong}{\xrightarrow{E}} \pi_8(\Omega P^6(2)) \xrightarrow{(\Omega W_2 \circ H_2)_*} \pi_8(\Omega P^6(2))$$

is zero and so is $(\Omega W_2 \circ H_2)_* : \pi_8(\Omega P^6(2)) \to \pi_8(\Omega P^6(2))$. Thus

$$2 \cdot \pi_8(\Omega P^6(2)) = 0$$

and hence $\pi_8(P^5(2)) \cong \pi_9(P^6(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

3. $\pi_9(P^5(2))$: From the exact sequence

$$\pi_9(S^7) \xrightarrow{(2\omega_4)_*} \pi_9(S^4) \longrightarrow \pi_8(X^5) \longrightarrow \pi_8(S^7) \longrightarrow 0,$$

we obtain a short exact sequence

$$0 \longrightarrow \pi_9(S^4) \longrightarrow \pi_8(X^5) \longrightarrow \pi_8(S^7) \longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} \pi_{10}(S^5) & & \xrightarrow{\qquad\delta_*\qquad} & & \pi_9(F^5\{2\}) & \longrightarrow & \pi_9(P^5(2)) & \longrightarrow & 0 \\ \uparrow E & & & & \uparrow \cong & & & & \\ \pi_9(S^4) & \xrightarrow{[2]_*} & \pi_9(S^4) & \underset{j_*}{\longrightarrow} & \pi_8(X^5) & & & & \end{array}$$

Since $[2]_*(\nu_4 \circ \eta^2) = 2\nu_4 \circ \eta^2 + \omega_4 \circ \eta^2$, we have $\delta_*(\nu_5 \circ \eta^2) = j_*(\omega_4 \circ \eta^2) \neq 0$ and so there is a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_9(S^4)/(\omega_4 \circ \eta^2) & \longrightarrow & \pi_8(X^5)/j_*(\omega_4 \circ \eta^2) & \longrightarrow & \pi_8(S^7) & \longrightarrow & 0 \\ & & \| & & \downarrow \cong & & & & \\ & & \mathbb{Z}/2(\nu_4 \circ \eta^2) & & \pi_9(P^5(2)) & & & & \end{array}$$

The commutative diagram of fibre sequences

$$\begin{array}{ccccc} \Omega S^5 & \longrightarrow & S^5\{2\} & \longrightarrow & S^5 \\ \uparrow & & \uparrow & & \| \\ F^5\{2\} & \longrightarrow & P^5(2) & \longrightarrow & S^5 \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} \pi_9(S^4) & \longrightarrow & \pi_9(F^5\{2\}) \\ \downarrow E & & \downarrow \\ \mathbb{Z}/2 = \pi_9(\Omega S^5) & = & \pi_9(\Omega S^5), \end{array}$$

where E is onto. Thus $j_*(\nu_4 \circ \eta^2)$ is not divisible by 2 in $\pi_9(F^5\{2\})$ and therefore in $\pi_9(P^5(2))$. Hence $\pi_9(P^5(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

4. $\pi_{10}(P^5(2))$: Consider the exact sequence

$$\pi_{10}(S^7) \xrightarrow{(2\omega_4)_*} \pi_{10}(S^4) \longrightarrow \pi_9(X^5) \longrightarrow \pi_9(S^7) \longrightarrow 0.$$

By the EHP sequence $\pi_{12}(S^5) \xrightarrow{H} \pi_{12}(S^9) \xrightarrow{P} \pi_{10}(S^4) \xrightarrow{E} \pi_{11}(S^5) \longrightarrow 0$, we have $P(\nu_9) = 2\nu_4^2$ and so $(2\omega_4)_*(\nu_7) = 4\nu_4^2$. By the commutative diagram

$$\begin{array}{ccc} \pi_{10}(S^4) = \mathbb{Z}/8 & \longrightarrow & \pi_9(X^5) \cong \pi_{10}(F^5\{2\}) \\ \downarrow E & & \downarrow \\ \mathbb{Z}/2 = \pi_{11}(S^5) & = & \pi_{11}(S^5), \end{array}$$

the element ν_4^2 is not divisible by 2 in $\pi_9(X^5)$ and so

$$\pi_9(X^5) \cong \pi_{10}(S^4)/4 \oplus \pi_9(S^7) = \mathbb{Z}/4 \oplus \mathbb{Z}/2.$$

Consider the exact sequence

$$\begin{array}{ccccccc}
\pi_{11}(S^5) & \xrightarrow{\delta_*} & \pi_{10}(F^5\{2\}) & \longrightarrow & \pi_{10}(P^5(2)) & \longrightarrow & 0 \\
\uparrow{\scriptstyle E} & & \uparrow & & & & \\
\pi_{10}(S^4) & \xrightarrow{[2]_*} & \pi_{10}(S^4). & & & &
\end{array}$$

Since $[2]_*(\nu_4^2) = 2\nu_4^2 + \omega_4 \circ \nu_7 = 4\nu_4^2 \equiv 0$ in $\pi_{10}(F^5\{2\})$, we have $\delta_*(\nu_5^2) = 0$ and so $\pi_{10}(F^5\{2\}) \cong \pi_{10}(P^5(2))$. Thus $\pi_{10}(P^5(2)) \cong \pi_{10}(S^4)/4 \oplus \pi_9(S^7) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$.

5 $\pi_{11}(P^5(2))$: Consider the exact sequences

$$\pi_{12}(S^5) \xrightarrow{\delta_*} \pi_{11}(F^5\{2\}) \longrightarrow \pi_{11}(P^5(2)) \longrightarrow \pi_{11}(S^5) \xrightarrow{0} \pi_{10}(F^5\{2\})$$

$$\pi_{11}(S^4) = 0 \longrightarrow \pi_{10}(X^5) \longrightarrow \pi_{10}(S^7) \xrightarrow{(2\omega_4)_*} \pi_{10}(S^4).$$

Since $(2\omega_4)_*(\nu_7) = 4\nu_4^2$, we have $\pi_{10}(X^5) \cong 2 \cdot \pi_{10}(S^7) \cong \mathbb{Z}/4$. By Lemma 5.9, $\pi_{10}(X^5) \cong \pi_{10}(A_5)$ and

$$\pi_{11}(F^5\{2\}) \cong \pi_{10}(A_5) \oplus \pi_{11}(P^{12}(2)) \cong \pi_{10}(X^5) \oplus \pi_{11}(P^{12}(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2,$$

where $\Omega P^{12}(2) = \Omega\Sigma L_3$ is a common factor of $\Omega F^5\{2\}$ and $\Omega P^5(2)$. We need to determine the boundary $\delta_* \colon \pi_{12}(S^5) \to \pi_{11}(F^5\{2\})$. Consider the homotopy commutative diagram of EHP sequences

$$\begin{array}{ccccccc}
\Omega^2 S^9 & \xrightarrow{P} & S^4 & \xrightarrow{E} & \Omega S^5 & \xrightarrow{H} & \Omega S^9 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^2 P^9(2) & \xrightarrow{P} & T^5 & \xrightarrow{E} & \Omega P^5(2) & \xrightarrow{\bar{H}_2} & \Omega P^9(2) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^2 F^9\{2\} & \xrightarrow{P} & E_5 & \xrightarrow{\bar{E}} & \Omega F^5\{2\} & \xrightarrow{\bar{H}_2} & \Omega F^9\{2\}.
\end{array}$$

Observe that the inclusion $\Omega S^4 \hookrightarrow \Omega F^5\{2\}$ lifts to E_5 and the composite

$$L_3(P^4(2)) \longrightarrow \Omega F^5\{2\} \xrightarrow{\bar{H}_2} \Omega F^9\{2\}$$

is null homotopic. Thus $\mathrm{sk}_{12}(E_5) \simeq \mathrm{sk}_{12}(\Omega S^4 \times \Omega P^{12}(2))$ and so

$$\pi_{10}(E_5) = \pi_{11}(S^4) \oplus \pi_{11}(P^{12}(2)) = 0 \oplus \mathbb{Z}/2 = \mathbb{Z}/2.$$

It follows that the composite

$$\mathbb{Z}/4 = \pi_{10}(X^5) \hookrightarrow \pi_{10}(\Omega F^5\{2\}) \longrightarrow \pi_{10}(\Omega F^9\{2\}) \cong \pi_{11}(S^8) = \mathbb{Z}/8$$

is a monomorphism. Now consider the commutative diagram of exact sequences

$$\begin{array}{ccccc} \pi_{12}(S^5) & \xrightarrow{\delta_*} & \pi_{10}(\Omega F^5\{2\}) & \longrightarrow & \pi_{10}(\Omega P^5(2)) \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \bar{H}_{2*}} & & \downarrow{\scriptstyle \bar{H}_{2*}} \\ \pi_{12}(S^9) & \xrightarrow{\delta'} & \pi_{10}(\Omega F^9\{2\}) & \longrightarrow & \pi_{10}(\Omega P^9\{2\}). \end{array}$$

Recall that $\pi_{12}(S^5)$ is generated by σ''' with $H(\sigma''') = 4\nu_9$, see [42, Lemma 5.13]. We obtain

$$\bar{H}_{2*} \circ \delta_*(\sigma''') = \delta'_* \circ H(\sigma''') = \delta'_*(4\nu_9) = 8\nu_7 = 0$$

in $\pi_{10}(S^7) \cong \pi_{10}(\Omega F^9\{2\})$. Thus $\delta_*(\sigma''') = 0$ and so there are short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{10}(\Omega F^5\{2\}) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \longrightarrow & \pi_{10}(\Omega P^5(2)) & \longrightarrow & \pi_{11}(S^5) = \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \uparrow\cup & & \uparrow\cup & & \| & & \\ 0 & \longrightarrow & \pi_{10}(X^5) = \mathbb{Z}/4 & \longrightarrow & \pi_{10}(Y_5) & \longrightarrow & \pi_{11}(S^5) = \mathbb{Z}/2 & \longrightarrow & 0. \end{array}$$

Now we need to solve the extension problem. By the distributivity law [2],

$$\Omega[2]_* = 2 + (\Omega W_2)_* \circ (\bar{H}_2)_* + \sum_{\sigma\in\Sigma_3} n_\sigma \Omega W_{3*} \circ \Omega\sigma_* \circ H_{3*} : \pi_{10}(\Omega P^5(2)) \longrightarrow \pi_{10}(\Omega P^5(2))$$

for some $n_\sigma \in \mathbf{Z}$. Note that the diagram

$$\begin{array}{ccc} \Omega\Sigma(P^4(2))^{(3)} & \longrightarrow & \Omega\Sigma L_3 = \Omega P^{12}(2) \\ \downarrow{\scriptstyle \Omega W_3} & & \downarrow \\ \Omega P^5(2) & = & \Omega P^5(2) \end{array}$$

commutes up to homotopy and $\pi_{10}(\Omega P^{12}(2)) = \mathbb{Z}/2$. We have

$$2\Omega W_{3*} \circ \Omega\sigma_* \circ H_{3*}(\alpha) = 0$$

for any $\alpha \in \pi_{10}(\Omega P^5(2))$. Recall that $\Omega[2] : \Omega P^5(2) \to \Omega P^5(2)$ is the composite

$$\Omega P^5(2) \longrightarrow \Omega S^5 \longrightarrow \Omega S^4 \hookrightarrow \Omega P^5(2).$$

Since $\pi_{10}(\Omega S^4) = 0$, $\Omega[2]_* : \pi_{10}(\Omega P^5(2)) \longrightarrow \pi_{10}(\Omega P^5(2))$ is zero. Since

$$\pi_{11}(P^5(2) \wedge P^4(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

we have $2(\Omega W_2 \circ H_2)_*(\alpha) = 0$ for any $\alpha \in \pi_{11}(P^5(2))$. Thus

$$4\alpha = 0$$

for all $\alpha \in \pi_{11}P^5(2)$ and so $\pi_{10}(Y_5) = \mathbb{Z}/4 \oplus \mathbb{Z}/2$. Hence

$$\pi_{11}(P^5(2)) = \pi_{10}(Y_5) \oplus \mathbb{Z}/2 = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

6. $\pi_{12}(P^5(2))$: Consider the exact sequence

$$\pi_{12}(S^7) = 0 \longrightarrow \pi_{12}(S^4) \longrightarrow \pi_{11}(X^5) \longrightarrow 0.$$

We have $\pi_{12}(S^4) \cong \pi_{11}(X^5) \cong \pi_{11}(A_5)$. Now consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\pi_{13}(S^5) & \xrightarrow{\delta_*} & \pi_{11}(A_5) & \xrightarrow{j_*} & \pi_{11}(Y_5) & \longrightarrow & \pi_{12}(S^5) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle E^\infty} & & \downarrow{\scriptstyle E^\infty} & & \downarrow{\scriptstyle E^\infty} & & \\
\pi^s_{13}(S^5) & \xrightarrow{2} & \pi^s_{12}(S^4) & \xrightarrow{j_*} & \pi^s_{12}(P^5(2)) & \longrightarrow & \pi^s_{12}(S^5), & &
\end{array}$$

where

$$E^\infty\colon \pi_{11}(A_5) \cong \pi_{12}(S^4) \longrightarrow \pi^s_{13}S^5 \quad \text{and} \quad E^\infty\colon \pi_{12}(S^5) \longrightarrow \pi^s_{12}(S^5)$$

are monomorphisms, see [42, Proposition 5.15, Theorem 7.1]. Observe that

$$E^\infty j_*(\epsilon_4) = j_*(\epsilon) \neq 0$$

in $\pi^s_{12}(P^5(2)) = \mathbb{Z}/2^{\oplus 3}$. Thus there is a short exact sequence

$$0 \longrightarrow \pi_{11}(A_5) \longrightarrow \pi_{11}(Y_5) \longrightarrow \pi_{12}(S^5) \longrightarrow 0.$$

Since $E^\infty \pi_{11}(A_5)$ is a summand of $\pi^s_{12}(P^5(2))$, we have

$$\pi_{11}(Y_5) = \pi_{11}(A_5) \oplus \pi_{12}(S^5)$$

and $E^\infty\colon \pi_{11}(Y_5) \longrightarrow \pi^s_{12}(P^5(2))$ is a monomorphism. Thus $\pi_{12}(P^5(2)) = \pi_{11}(Y_5) \oplus \pi_{12}(P^{12}(2)) = \mathbb{Z}/2^{\oplus 3}$.

7. $\pi_{13}(P^5(2))$: Consider the exact sequence

$$\pi_{13}(S^7) \xrightarrow{(2\omega_4)_*} \pi_{13}(S^4) \longrightarrow \pi_{12}(X^5) \longrightarrow \pi_{12}(S^7) = 0.$$

Since $2 \cdot \pi_{13}(S^7) = 0$, $(2\omega_2)_*\colon \pi_{13}(S^7) \longrightarrow \pi_{13}(S^4)$ is zero and so

$$\pi_{13}(S^4) \cong \pi_{12}(X^5) \cong \pi_{12}(A_5).$$

Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\pi_{14}(S^5) & \xrightarrow{\delta_*} & \pi_{12}(A_5) & \longrightarrow & \pi_{12}(Y_5) & \longrightarrow & \pi_{13}(S^5) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cap\downarrow & & \\
\pi^s_{14}(S^5) & \xrightarrow{2} & \pi^s_{13}(S^4) & \longrightarrow & \pi^s_{13}(P^5(2)) & \longrightarrow & \pi^s_{13}(S^5) & \longrightarrow & 0
\end{array}$$

By Theorem [42, Theorem 7.2], $\pi_{13}(S^4) \cong \pi^s_{13}(S^4) \cong \mathbb{Z}/^{\oplus 3}$ generated by ν_4^3, μ_4 and $\eta_4 \circ \epsilon_5$. There is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{13}(S^4) & \longrightarrow & \pi_{12}(Y_5) & \longrightarrow & \pi_{13}(S^5) = \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \cap\downarrow & & \\
0 & \longrightarrow & \pi^s_{13}(S^4) & \longrightarrow & \pi^s_{13}(P^5(2)) & \longrightarrow & \pi^s_{13}(S^5) & \longrightarrow & 0
\end{array}$$

Thus $\pi_{12}(Y_5) \to \pi^s_{13}(P^5(2))$ is a monomorphism and

$$\pi_{12}(Y_5) \cong \frac{1}{2}\pi_{13}(S^5) \oplus \pi_{13}(S^4)/(\eta_4 \circ \epsilon_5) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Thus $\pi_{13}(P^5(2)) = \pi_{12}(Y_5) \oplus \pi_{13}(P^{12}(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$. This completes the calculation. □

5.4. **The Homotopy Groups $\pi_*(P^6(2))$.**

Theorem 5.11. *The homotopy groups $\pi_r(P^6(2))$ for $r \leq 14$ are as follows.*

1) $\pi_r(P^6(2)) \cong \pi^s_r(P^6(2))$ *for* $r \leq 8$.
2) $\pi_9(P^6(2)) \cong \pi_8(P^5(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *maps onto* $\pi^s_9(P^6(2))$.
3) $\pi_{10}(P^6(2)) = \mathbb{Z}/8$ *maps onto* $\pi^s_{10}(P^6(2)) = 0$.
4) $\pi_{11}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *maps onto* $\pi^s_{11}(P^6(2)) = \mathbb{Z}/2$.
5) $\pi_{12}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *with the stable image* $\mathbb{Z}/2$ *in* $\pi^s_{12}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
6) $\pi_{13}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *with the stable image* $\mathbb{Z}/2^{\oplus 2}$ *in* $\pi^s_{13}(P^6(2)) = \mathbb{Z}/2^{\oplus 3}$.
7) $\pi_{14}(P^6(2)) = \mathbb{Z}/2^{\oplus 4} \oplus \mathbb{Z}/4$ *with the stable image* $\mathbb{Z}/2^{\oplus 2} \oplus \mathbb{Z}/4$ *in* $\pi^s_{14}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.

An application is as follows.

Proposition 5.12. *Let X be a simply connected CW-complex. Suppose that the integral homotopy $H_5(X;\mathbb{Z}) = \mathbb{Z}/2$ and the mod 2 cohomology $H^*(X)$ is an exterior algebra $E(u,v)$ with $|u| = 5$ and $|v| = 6$. Then X is homotopy equivalent to $\tau(S^6)$. In other words, the Stiefel manifold $\tau(S^6)$ is uniquely determined, up to homotopy, by its fundamental group and its cohomology ring.*

Proof. By the assumption, $X = P^6(2) \cup_f e^{11}$. Since $H^*(X) \cong E(u,v)$, the homology $H_*(\Omega X)$ is isomorphic to the polynomial algebra $S(u,v)$. Let $f' \colon S^9 \to \Omega P^6(2)$ be the adjoint of the attaching map $f \colon S^{10} \to P^6(2)$. Then $f_*(\iota_9) = [u_4, v_5]$ and the homotopy class $[f]$ is not divisible by 2 in $\pi_{10}(P^6(2))$. Let $\lambda_6 \colon S^{10} \to P^6(2)$ be the attaching map for $\tau(S^6)$. By assertion 3, $f \simeq k\lambda_6$ for $k = \pm 1$ or ± 3. Clearly $X \simeq \tau(S^6)$ if $k = \pm 1$. By Lemma 2.18, there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{10} & \xrightarrow{\lambda_6} & P^6(2) \\ \downarrow{\scriptstyle [-3]} & & \downarrow{\scriptstyle [-1]} \\ S^{10} & \xrightarrow{\lambda_6} & P^6(2). \end{array}$$

Thus $X \simeq \tau(P^6(2))$ if $k = -3$. If $k = 3$, then $X \simeq P^6(2) \cup_{-f} e^{11} \simeq \tau(S^6)$ and hence the result. □

Proof of Theorem 5.11. **1.** $\pi_r P^6(2)$ for $r \leq 9$: Since S^5 is the 9-skeleton of $F^6\{2\}$, assertion (1) follows. Assertion (2) was proved in the Step 2 of Theorem 5.10.
2. $\pi_{10}(P^6(2))$: By Lemma 4.17, $A_6 \simeq \Omega S^5 \times B_6$. Since S^9 is the 16-skeleton of B_6, $\pi_r(\Omega S^5 \times S^9) \cong \pi_r(A_6)$ for $r \leq 15$. Observe that $\Sigma L_3(P^5(2)) = P^{15}(2)$. The

homotopy groups $\pi_r(P^6(2)) \cong \pi_{r-1}(Y_6)$ for $r \leq 13$ and $\pi_{14}(P^6(2)) = \pi_{13}(Y_4) \oplus \mathbb{Z}/2$. Now consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\pi_{11}(S^6) & \xrightarrow{\delta_*} & \pi_{10}(F^6\{2\}) & \longrightarrow & \pi_{10}(P^6(2)) & \longrightarrow & \pi_{10}(S^6) = 0 \\
\downarrow H & & \downarrow \bar{H}_{2*} & & \downarrow \bar{H}_{2*} & & \downarrow H \\
\pi_{11}(S^{11}) & \xrightarrow{2} & \pi_{10}(F^{11}\{2\}) & \longrightarrow & \pi_{10}(P^{11}(2)) & \longrightarrow & \pi_{10}(S^{11}) = 0.
\end{array}$$

Since $H\colon \pi_{11}(S^6) \to \pi_{11}(S^{11})$ is of degree 2, the composite

$$\pi_{11}(S^6) \xrightarrow{\delta_*} \pi_{10}(F^6\{2\}) \cong \pi_9(\Omega S^5 \times S^9) \xrightarrow{\text{proj.}} \pi_9(S^9)$$

is of degree 4. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta_*} \mathbb{Z}/2 \oplus \mathbb{Z} \longrightarrow \pi_{10}(P^6(2)) \longrightarrow 0,$$

where $\delta_*(\omega_6) = k\nu_5 \circ \eta^2 + 4\iota_9$ for some $k = 0, 1$. By [12, Theorem 2.2], there is a $\mathbb{Z}/8$-summand in $\pi_{10}(P^6(2))$. Thus $k = 1$ and $\pi_{10}(P^6(2)) = \mathbb{Z}/8$.

3. $\pi_{11}(P^6(2))$: Consider the commutative diagram

$$\begin{array}{ccccccc}
\pi_{12}(S^6) & \longrightarrow & \pi_{11}(F^6\{2\}) & \longrightarrow & \pi_{11}(P^6(2)) & \longrightarrow & 0 \\
\uparrow \cong & & \uparrow & & & & \\
\pi_{11}(S^5) & \xrightarrow{[2]_*} & \pi_{11}(S^5) & & & &
\end{array}$$

Recall that $\Omega[2] \simeq 2 + \Omega\omega_5 \circ H_2\colon \Omega S^5 \to \Omega S^5$. Thus $[2]_*(\nu_5^2) = 2\nu_5^2 = 0$ and

$$\pi_{11}(P^6(2)) \cong \pi_{11}(F^6\{2\}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Since $\pi_{11}(S^5) \cong \pi_{11}^s(S^5)$, the map $\pi_{11}(P^6(2)) \to \pi_{11}^s(P^6(2))$ is onto.

4. $\pi_{12}(P^6(2))$: Consider the commutative diagram

$$\begin{array}{ccccccccc}
\mathbb{Z}/4 = \pi_{13}(S^6) & \xrightarrow{\delta_*} & \pi_{12}(F^6\{2\}) = \pi_{12}(S^5) \oplus \pi_{11}(S^9) & \longrightarrow & \pi_{12}(P^6(2)) & \longrightarrow & \pi_{12}(S^6) & \longrightarrow & 0 \\
\downarrow & & \downarrow E^\infty & & \downarrow & & \downarrow \cong & & \\
\pi_{12}^s S^5 = \mathbb{Z}/16 & \xrightarrow{2} & \pi_{12}^s S^5 = \mathbb{Z}/16 & \longrightarrow & \pi_{12}^s(P^6(2)) & \longrightarrow & \pi_{12}^s(S^6) & \longrightarrow & 0
\end{array}$$

where $\pi_{13}(S^6) \longrightarrow \pi_{12}^s(S^5)$ is a monomorphism by [42, Proposition 5.15]. Thus

$$E^\infty \circ \delta_*(\sigma'') = 8\sigma \neq 0$$

and so there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{12}(P^6(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\mathbb{Z}/2 & \longrightarrow & \pi_{12}(F^6\{2\}) = \mathbb{Z}/2^{\oplus 2} & \longrightarrow & \pi_{12}(P^6(2)) & \longrightarrow & \pi_{12}(S^6) & \longrightarrow & 0 \\
 & & \downarrow \bar{H}_{2*} & & \downarrow \bar{H}_{2*} & & 0 \downarrow H & & \\
0 & \longrightarrow & \pi_{12}(F^{11}\{2\}) = \mathbb{Z}/2 & \longrightarrow & \pi_{12}(P^{11}(2)) = \mathbb{Z}/4 & \longrightarrow & \pi_{12}(S^{11}) = \mathbb{Z}/2 & \longrightarrow & 0.
\end{array}$$

We have $\pi_{12}(P^6(2)) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. By [12, Lemma 2.1], the composite

$$S^9 \xrightarrow{\lambda_6} \Omega P^6(2) \longrightarrow Q(P^5(2))$$

is null homotopic and so there is an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{12}(P^6(2)) \xrightarrow{E^\infty} \pi^s_{12}(P^6(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

5. $\pi_{13}(P^6(2))$: Consider the commutative diagram

$$\begin{array}{ccccccccc}
\mathbb{Z}/2 = \pi_{13}(S^5) & \xrightarrow{[2]_*} & \pi_{13}(S^5) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\mathbb{Z}/2 \oplus \mathbb{Z}/8 = \pi_{14}(S^6) & \xrightarrow{\delta_*} & \pi_{13}(F^6\{2\}) & \longrightarrow & \pi_{13}(P^6(2)) & \longrightarrow & 2 \cdot \pi_{13}(S^6) & \longrightarrow & 0 \\
\downarrow & & \downarrow H_{2*} & & \downarrow & & \downarrow 0 & & \\
\mathbb{Z}/8 = \pi_{14}(S^{11}) & \xrightarrow{2} & \mathbb{Z}/8 = \pi_{13}(F^{11}\{2\}) & \longrightarrow & \pi_{13}(P^{11}(2)) & \longrightarrow & \pi_{13}(S^{11}) & \longrightarrow & 0,
\end{array}$$

where $\pi_{14}(S^6) \to \pi_{14}(S^{11})$ and $\pi_{13}(F^6\{2\}) \to \pi_{13}(F^{11}\{2\})$ are epimorphisms and $E\colon \pi_{13}(S^5) \to \pi_{14}(S^6)$ is a monomorphism. Observe that

$$\pi_{13}(F^6\{2\}) \cong \pi_{12}(\Omega S^5 \times S^9) = \mathbb{Z}/2 \oplus \mathbb{Z}/8 \quad \text{and} \quad [2]_*(\epsilon_5) = 2\epsilon_5 + (\Omega\omega_5 \circ H_2)_*(\epsilon_5) = 0.$$

Thus $\delta_*(\bar{\nu}_6) = 2\lambda_6 \circ \nu_9 + k\epsilon_5$ for some $k = 0, 1$ and $\delta(\epsilon_6) = 0$. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_{14}(S^6) & \xrightarrow{\delta_*} & \pi_{13}(F^6\{2\}) \\
\downarrow & & \downarrow E^\infty \\
\pi^s_{13}(S^5) & \xrightarrow{2} & \pi^s_{13}(S^5)
\end{array}$$

Since $E^\infty(\lambda_6) = 0$, we have $E^\infty(\lambda_6 \circ \nu_9) = 0$ and so

$$E^\infty \circ \delta_*(\bar{\nu}_6) = E^\infty(k\epsilon_5) = k\epsilon = 2E^\infty(\bar{\nu}_6) = 2\bar{\nu} = 0$$

Thus $k = 0$ and $\delta_*(\bar{\nu}_6) = 2\lambda_6 \circ \nu_9$ with a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \pi_{13}(P^6(2)) \longrightarrow 2 \cdot \pi_{13}S^6 \to 0.$$

Now consider the commutative diagram

$$\begin{array}{ccc}
\pi_{13}(P^6(2)) & \twoheadrightarrow & 2\cdot\pi_{13}S^6 \\
\uparrow & & \cong\uparrow E \\
\pi_{12}(P^5(2)) & \longrightarrow & \pi_{12}(S^5)
\end{array}$$

By Step 6 of Theorem 5.10, $\pi_{12}(P^5(2)) \to \pi_{12}S^5$ is split onto and so

$$\pi_{13}(P^6(2)) \cong 2\cdot\pi_{13}(S^6) \oplus \pi_{13}(S^5) \oplus \pi_{12}(S^9)/2 \cong \mathbb{Z}/2^{\oplus 3}$$

with an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{13}(P^6(2)) \longrightarrow \pi^s_{13}(P^6(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

6. $\pi_{14}(P^6(2))$: Consider the exact sequence

$$\pi_{15}(S^6) \xrightarrow{\delta_*} \pi_{13}(A_6) \longrightarrow \pi_{13}(Y_6) \xrightarrow{p_*} \pi_{14}(S^6) \xrightarrow{\delta_*} \pi_{13}(F^6\{2\}),$$

where

$$\mathrm{Ker}(\delta_*\colon \pi_{14}(S^6) \to \pi_{13}(F^6\{2\})) = \{4\bar\nu_6\} \oplus \{\epsilon_5\} = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Since $\pi_{13}(S^9) = 0$, we have $\pi_{13}(A_6) \cong \pi_{14}(S^5)$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & \pi_{14}(S^5) = \mathbb{Z}/2^{\oplus 3} & \xrightarrow{j_*} & \pi_{13}(Y_6) & \xrightarrow{p_*} & \{4\bar\nu_6\} \oplus \{\epsilon_6\} & \longrightarrow & 0 \\
 & & \cong\downarrow E^\infty & & \downarrow E^\infty & & \downarrow & & \\
0 & \longrightarrow & \pi^s_{14}(S^5) & \xrightarrow{j_*} & \pi^s_{14}(P^6(2)) & \xrightarrow{p_*} & \pi^s_{14}(S^6) & \longrightarrow & 0.
\end{array}$$

Thus $\pi_{14}(S^5) \longrightarrow \pi_{13}(Y_6)$ is a monomorphism. Let $\alpha_1 \in \pi_{13}(Y_6)$ so that $p_*(\alpha_1) = \epsilon_6$. Then $E^\infty(2\alpha_1) = \eta \circ \epsilon_6 = j_* \circ E^\infty(\eta_5 \circ \epsilon_6) \neq 0$ and so α_1 is of order 4. Now let $\alpha_2 \in \pi_{13}(Y_6)$ so that $p_*(\alpha_2) = 4\bar\nu_6$. Then there exists $\beta \in \pi_{14}(S^5)$ so that $j_*(\beta) = 2\alpha_2$. Since $p_* \circ E^\infty(\alpha_2) = E^\infty 4\bar\nu_6 = 0$, we have $E^\infty(\alpha_2) \in \pi^s_{14}(S^5)$ and so

$$2E^\infty(\alpha_2) = E^\infty(2\alpha_2) = E^\infty j_*(\beta) = j_* E^\infty(\beta) = 0.$$

Thus $\beta = 0$ or $2\alpha_2 = 0$. Hence

$$\pi_{13}(Y_6) \cong \{4\bar\nu_6\} \oplus 1/2\{\epsilon_5\} \oplus \pi_{14}(S^5)/\{\eta_5 \circ \epsilon_6\} \cong \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4 \quad \text{and}$$

$$\pi_{14}(P^6(2)) \cong \pi_{13}(Y_6) \oplus \pi_{14}(P^{15}(2)) \cong \mathbb{Z}/2^{\oplus 4} \oplus \mathbb{Z}/4$$

with an exact sequence $0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \pi_{14}(P^6(2)) \longrightarrow \pi^s_{14}(P^6(2))$. We finish the proof. $\square$

5.5. The Homotopy Groups $\pi_*(P^7(2))$.

Theorem 5.13. *The homotopy groups $\pi_r(P^7(2))$ for $r \leq 15$ are as follows.*

1) $\pi_r(P^7(2)) \cong \pi_r^s P^7(2)$ *for* $r \leq 10$.
2) $\pi_{11}(P^7(2)) = \mathbb{Z}/2$.
3) $\pi_{12}(P^7(2)) = \mathbb{Z}/2$.
4) $\pi_{13}(P^7(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
5) $\pi_{14}(P^7(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
6) $\pi_{15}(P^7(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8$.

Proof. **1.** $\pi_r(P^7(2))$ for $r \leq 10$: Since S^6 is the 11-skeleton of $F^7\{2\}$, assertion (1) follows.

2. $\pi_{11}(P^7(2))$: By Proposition 4.15, the 17-skeleton $\mathrm{sk}_{17}(F^7\{2\})$ of $F^7\{2\}$ is the homotopy cofibre of $2\omega_6\colon S^{11} \to S^6$. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega F^7\{2\} & = & \Omega F^7\{2\} & \longrightarrow & * & \longrightarrow & F^7\{2\} \\
\uparrow{\scriptstyle \Omega j} & & \uparrow{\scriptstyle \phi} & & \uparrow & & \uparrow{\scriptstyle j} \\
\Omega S^6 & \longrightarrow & X^7 & \longrightarrow & S^{11} & \xrightarrow{2\omega_6} & S^6.
\end{array}$$

Let ψ be the composite

$$X^7 \times \Omega\Sigma L_3(P^6(2)) \xrightarrow{\phi \times J(\beta_3)} \Omega F^7\{2\} \times \Omega F^7\{2\} \xrightarrow{\text{multi.}} \Omega F^7\{2\}.$$

Then

$$\psi_* : H_r(X^7 \times \Omega\Sigma L_3(P^6(2)) \to H_r(\Omega F^7\{2\})$$

is an isomorphism for $r \leq 19$. Since $\Sigma L_3(P^6(2)) = P^{18}(2)$, the map

$$\psi_*\colon \pi_r(X^7) \longrightarrow \pi_r(\Omega F^7\{2\})$$

is an isomorphism for $r \leq 15$. By expecting the exact sequence

$$\pi_{11}(S^{11}) \xrightarrow{(2\omega_6)_*} \pi_{11}(S^6) \longrightarrow \pi_{10}(X^7) \longrightarrow 0,$$

we have $\pi_{10}(X^7) = \mathbb{Z}/2$. From the exact sequence

$$\pi_{12}(S^7) = 0 \longrightarrow \pi_{10}(X^7) \longrightarrow \pi_{11}(P^7(2)) \longrightarrow \pi_{11}(S^7) = 0,$$

we have $\pi_{11}(P^7(2)) \cong \pi_{10}(X^7) = \mathbb{Z}/2$.

3. $\pi_{12}(P^7(2))$: Consider the exact sequence

$$\pi_{12}(S^{11}) \xrightarrow{(2\omega_{6*}} \pi_{12}(S^6) \longrightarrow \pi_{11}(X^7) \longrightarrow 0.$$

Since $\pi_{12}(S^{11}) = \mathbb{Z}/2$, $(2\omega_6)_* : \pi_{12}(S^{11}) \to \pi_{12}(S^6)$ is zero and so $\pi_{12}(S^6) \cong \pi_{11}(X^7)$. Now consider the commutative diagram

$$\begin{array}{ccccccc}
\pi_{13}(S^7) & \xrightarrow{\delta_*} & \pi_{11}(X^7) & \longrightarrow & \pi_{12}(P^7(2)) & \longrightarrow & 0 \\
\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} & & & & \\
\pi_{12}(S^6) & \xrightarrow{[2]_*} & \pi_{12}(S^6). & & & &
\end{array}$$

Recall that $\Omega[2]_* = 2 + (\Omega\omega_6 \circ H_2)_*$ and $(H_2)_*(\nu_6^2) = 0$. Thus $[2]_*(\nu_6^2) = 2\nu_6^2 = 0$ and so $\delta_*(\nu_7^2) = 0$. Hence $\pi_{12}(P^7(2)) \cong \pi_{11}(X^7) \cong \mathbb{Z}/2$.

4. $\pi_{13}(P^7(2))$: Consider the exact sequence

$$\pi_{13}(S^{11}) \xrightarrow{(2\omega_6)_*} \pi_{13}(S^6) \xrightarrow{j_*} \pi_{12}(X^7) \longrightarrow \pi_{12}(S^{11}) \longrightarrow 0.$$

Since $(2\omega_6)_* : \pi_{13}(S^{11}) \to \pi_{13}(S^6)$ is zero, there is a short exact sequence

$$0 \longrightarrow \pi_{13}(S^6) \longrightarrow \pi_{12}(X^7) \longrightarrow \pi_{12}(S^{11}) \longrightarrow 0.$$

From the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\pi_{13}(\Omega S^7) & \xrightarrow{\Omega[2]_* = 2} & \pi_{13}(\Omega S^7) & \longrightarrow & \pi_{13}(S^7)\{2\} & \longrightarrow & \pi_{13}(S^7) \\
\| & & \uparrow f_{1*} & & \uparrow f_{2*} & & \| \\
\pi_{13}(\Omega S^7) & \xrightarrow{\delta_*} & \pi_{12}(X^7) & \longrightarrow & \pi_{13}(P^7(2)) & \longrightarrow & \pi_{13}(S^7),
\end{array}$$

we have $f_{1*} \circ \delta_*(\sigma') = 2\sigma'$. Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\pi_{14}(S^7) & \xrightarrow{\delta_*} & \pi_{12}(X^7) & \longrightarrow & \pi_{13}(P^7(2)) & \longrightarrow & \pi_{13}(S^7) \\
\downarrow & & \downarrow & & \downarrow \bar{H}_{2*} & & \downarrow 0 \\
\pi_{14}(S^{13}) & \xrightarrow{\delta'_*} & \pi_{13}(F^{13}\{2\}) \cong \pi_{13}(S^{12}) & \longrightarrow & \pi_{13}(P^{13}(2)) & \longrightarrow & \pi_{13}(S^{13})
\end{array}$$

Since $\delta'_* : \pi_{14}(S^{13}) \to \pi_{13}(F^{13}\{2\})$ is zero, the composite

$$\pi_{14}(S^7) \xrightarrow{\delta_*} \pi_{12}(X^7) \longrightarrow \pi_{12}(S^{11})$$

is zero and so $\delta_*(\sigma') = j_*(k\sigma'')$ for some integer k. Now

$$2\sigma' = (f_1)_* \circ \delta_*(\sigma') = (f_1)_* \circ j_*(k\sigma'') = kE(\sigma'') = 2k\sigma'.$$

Thus $k \equiv 1(4)$ and so $\delta_*(\sigma') = \pm j_*(\sigma'')$. Hence there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{13}(P^7(2)) \longrightarrow \pi_{13}(S^7) \longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_{13}(P^7(2)) & \longrightarrow & \pi_{13}(S^7) & \longrightarrow & 0 \\
\uparrow E & & \cong \uparrow E & & \\
\pi_{12}(P^6(2)) & \longrightarrow & \pi_{12}(S^6) & \longrightarrow & 0
\end{array}$$

By Step 4 of Theorem 5.11, we have $\pi_{12}(P^6(2)) \cong \pi_{12}(S^6) \oplus \mathbb{Z}/2$ and so

$$\pi_{13}(P^7(2)) \cong \pi_{12}(X^7)/(\pi_{13}(S^6)) \oplus \pi_{13}(S^7) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

5. $\pi_{14}(P^7(2))$: Consider the exact sequence

$$\pi_{14}(S^{11}) \xrightarrow{(2\omega_6)_*} \pi_{14}(S^6) \longrightarrow \pi_{13}(X^7) \longrightarrow \pi_{13}(S^{11}) \longrightarrow 0.$$

By [42, Lemma 6.2], $P(\nu_{13}) = \pm 2\bar{\nu}_6$ in the EHP sequence for $S^6 \xrightarrow{E} \Omega S^7 \xrightarrow{H} \Omega S^{13}$. Thus $(2\omega_6)_*(\nu_{11}) = \pm 4\bar{\nu}_6$ and so there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/4 \longrightarrow \pi_{13}(X^7) \xrightarrow{p_*} \pi_{13}(S^{11}) \longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccccccc} \pi_{14}(S^6) & \xrightarrow{j_*} & \pi_{13}(X^7) & \xrightarrow{p_*} & \pi_{13}(S^{11}) & \longrightarrow & 0 \\ \downarrow E & & \downarrow f_{1*} & & & & \\ \pi_{15}(S^7) & = & \pi_{15}(S^7). & & & & \end{array}$$

Let $\alpha \in \pi_{13}(X^7)$ so that $p_*(\alpha) = \eta^2$. Then $2\alpha = j_*(k\bar{\nu}_6 + l\epsilon_6)$ for some integers k, l. Then

$$E(k\bar{\nu}_6 + l\epsilon_6) = (f_1)_* \circ j_*(k\bar{\nu}_6 + l\epsilon_6) = (f_1)_*(2\alpha) = 0 = k\bar{\nu}_7 + l\epsilon_7.$$

It follows that $l = 0$ and $k \equiv 0(2)$. Now $p_*(\alpha - j_*(k/2\bar{\nu}_6)) = \eta^2$ and $2(\alpha - j_*(k/2\bar{\nu}_6)) = 0$. Thus

$$\pi_{13}(X^7) \cong \pi_{13}(S^{11}) \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

Now consider the exact sequence

$$\pi_{15}(S^7) \xrightarrow{\delta_*} \pi_{13}(X^7) \longrightarrow \pi_{14}(P^7(2)) \longrightarrow \pi_{14}(S^7) \xrightarrow{\delta_*} \pi_{12}(X^7).$$

By Step 4, $\delta_*(\sigma') = \pm j_*\sigma''$ and so $\mathrm{Ker}(\delta_* : \pi_{14}(S^7) \to \pi_{12}(X^7)) = 4 \cdot \pi_{14}(S^7)$. By [42, pp. 64 and Theorem 7.1], we have

$$\pi_{15}(S^7) = \{\sigma' \circ \eta_{14}\} \oplus \{\bar{\nu}_7\} \oplus \{\epsilon_7\} = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \text{and} \quad \sigma'' \circ \eta_{13} = 4\bar{\nu}_6.$$

Hence

$$\delta_*(\sigma' \circ \eta_{14}) = j_*(\sigma'' \circ \eta_{13}) = j_*(4\bar{\nu}_6) = 0$$

in $\pi_{13}(X^7)$. Now

$$\delta_*(\bar{\nu}_7) = \delta_* E(\bar{\nu}_6) = j_* \circ [2]_*(\bar{\nu}_6) = j_*(2\bar{\nu}_6 + (\Omega\omega_6 \circ H_2)_*(\bar{\nu}_6))$$

$$= j_*(2\bar{\nu}_6 + (\Omega\omega_6)_*(\nu_{11})) = j_*(2\bar{\nu}_6 + P(\nu_{13})) = j_*(4\bar{\nu}_6) = 0.$$

and

$$\delta_*(\epsilon_7) = j_* \circ [2]_*(\epsilon_6) = j_*(2\epsilon_6) + (\Omega\omega_6 \circ H_2)_*(\epsilon_6)) = 0$$

This shows that the boundary map $\delta_* \colon \pi_{15}(S^7) \longrightarrow \pi_{13}(X^7)$ is zero and so there is a short exact sequence

$$0 \longrightarrow \pi_{13}(X^7) \longrightarrow \pi_{14}(P^7(2)) \longrightarrow 4 \cdot \pi_{14}(S^7) \longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_{14}(P^7(2)) & \longrightarrow & 4\cdot\pi_{14}(S^7) & \longrightarrow & 0\\
\uparrow E & & \cong\uparrow E & & \\
\pi_{13}(P^6(2)) & \longrightarrow & 2\cdot\pi_{13}(S^6) & \longrightarrow & 0\\
\uparrow & & \cong\uparrow E & & \\
\pi_{12}(P^5(2)) & \longrightarrow & \pi_{12}(S^5) & \longrightarrow & 0
\end{array}$$

By Step 6 of Theorem 5.10, $\pi_{12}P^5(2)\to\pi_{12}S^5$ is split onto. Thus

$$\pi_{14}(P^7(2)) \longrightarrow 4\cdot\pi_{14}(S^7)$$

is split onto and so

$$\pi_{14}(P^7(2))\cong 4\cdot\pi_{14}(S^7)\oplus\pi_{13}(X^7)=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4.$$

6. $\pi_{15}(P^7(2))$: Consider the exact sequence

$$\pi_{15}(S^{11})=0\xrightarrow{(2\omega_6)_*}\pi_{15}(S^6)\longrightarrow\pi_{14}(X^7)\longrightarrow\pi_{14}(S^{11})\xrightarrow{(2\omega_6)_*}\pi_{14}(S^6),$$

where $\mathrm{Ker}((2\omega_6)_*:\pi_{14}(S^{11})\to\pi_{14}(S^6))=2\cdot\pi_{14}(S^{11})$. There is a short exact sequence

$$0\longrightarrow\pi_{15}(S^6)\longrightarrow\pi_{14}(X^7)\longrightarrow 2\cdot\pi_{14}(S^{11})\longrightarrow 0.$$

Now consider the commutative diagram

$$\begin{array}{ccc}
\pi_{15}(S^6) & \hookrightarrow & \pi_{14}(X^7)\\
\downarrow E & & \downarrow\\
\pi_{16}(S^7) & = & \pi_{16}(S^7).
\end{array}$$

By [42, Theorem 7.2], we have $\pi_{16}(S^7)\cong\pi_{15}(S^6)\oplus\mathbb{Z}/2$ and so $\pi_{15}(S^6)$ is a summand of $\pi_{14}(X^7)$. Thus

$$\pi_{14}(X^7)\cong\pi_{15}(S^6)\oplus 2\cdot\pi_{14}(S^{11}).$$

Consider the exact sequence

$$\pi_{16}(S^7)\xrightarrow{\delta_*}\pi_{14}(X^7)\longrightarrow\pi_{15}(P^7(2))\longrightarrow\pi_{15}(S^7)\longrightarrow 0.$$

According to [42, Theorem 7.2],

$$\begin{aligned}\pi_{16}S^7&=\{\sigma'\circ\eta_{14}^2\}\oplus\{\nu_7^3\}\oplus\{\mu_7\}\oplus\{\eta_7\circ\epsilon_8\}\\&=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2.\end{aligned}$$

Observe that

$$\delta_*(\sigma'\circ\eta_{14}^2)=\delta_*(\sigma'\circ\eta_{14})\circ\eta_{15}=0$$
$$\delta_*(\nu_7^3)=\delta_*E(\nu_6^3)=j_*\circ[2]_*(\nu_6^3)=j_*(2\nu_6^3+(\Omega\omega_6\circ H_2)_*(\nu_6^3))=0$$
$$\delta_*(\mu_7)=\delta_*E\mu_6=j_*\circ[2]_*(\mu_6)=j_*(2\mu_6+(\Omega\omega_6\circ H_2)_*(\mu_6))=0$$

$\delta_*(\eta_7\circ\epsilon_8)=\delta_*E(\eta_6\circ\epsilon_7)=j_*\circ[2]_*(\eta_6\circ\epsilon_7)=j_*(2\eta_6\circ\epsilon_7+(\Omega\omega_6\circ H_2)_*(\eta_6\circ\epsilon_7))=0.$
Thus $\delta_*:\pi_{16}(S^7)\longrightarrow\pi_{14}(X^7)$ is zero and so there is a short exact sequence

$$0\longrightarrow\pi_{14}(X^7)\longrightarrow\pi_{15}(P^7(2))\longrightarrow\pi_{15}(S^7)\longrightarrow 0.$$

Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{14}(X^7) & \longrightarrow & \pi_{15}(P^7(2)) & \longrightarrow & \pi_{15}(S^7) & \longrightarrow & 0\\
 & & \downarrow E^\infty & & \downarrow E^\infty & & \downarrow E^\infty & & \\
0 & \longrightarrow & \pi^s_{15}(S^6) & \longrightarrow & \pi^s_{15}(P^7(2)) & \longrightarrow & \pi^s_{15}(S^7) & \longrightarrow & 0,
\end{array}$$

where $E^\infty\colon\pi_{14}(X^7)\longrightarrow\pi^s_{15}(S^6)$ and $E^\infty\colon\pi_{15}(S^7)\longrightarrow\pi^s_{15}(S^7)$ are epimorphisms. It follows that

$$E^\infty\colon\pi_{15}(P^7(2))\longrightarrow\pi^s_{15}(P^7(2))$$

is an epimorphism and so there is a commutative diagram of short exact sequences

$$\begin{array}{ccccc}
\mathbb{Z}/4 & \hookrightarrow & G & \xrightarrow{p_*} & \mathbb{Z}/2=\{\sigma'\circ\eta_{14}\}\\
\downarrow & & \downarrow & & \downarrow\\
\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4 & \hookrightarrow & \pi_{15}(P^7(2)) & \xrightarrow{p_*} & \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\\
\downarrow & & \downarrow & & \downarrow\\
\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 & \hookrightarrow & \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4 & \twoheadrightarrow & \mathbb{Z}/2\oplus\mathbb{Z}/2,
\end{array}$$

where the rows and columns are short exact sequence. To solve the group extension problem, we need to compute $\pi_{15}(P^7(2)\wedge P^6(2))$. According to [27, Lemma 1.6(ii), p.518], $\pi_{2n+3}(\Sigma P^n(2)\wedge P^n(2))=\mathbb{Z}/4\oplus\mathbb{Z}/2$ for $n\geq 4$. In particular,

$$\pi_{15}(P^7(2)\wedge P^6(2))=\mathbb{Z}/4\oplus\mathbb{Z}/2.$$

Now let $\alpha\in G$ so that $p_*(\alpha)=\sigma'\circ\eta_{14}$ in $\pi_{15}(S^7)$. From the commutative diagram

$$\begin{array}{ccc}
\pi_{15}(P^7(2)) & \xrightarrow{p_*} & \pi_{15}(S^7)\\
\downarrow H_{2*} & & \downarrow H\\
\pi_{15}(P^7(2)\wedge P^6(2)) & \xrightarrow{p_*} & \pi_{15}(S^{13}),
\end{array}$$

we have

$$p_*\circ H_{2*}(\alpha)=H\circ p_*(\alpha)=H(\sigma'\circ\eta_{14})=\eta_{13}^2.$$

Observe that there is a short exact sequence

$$0\longrightarrow\mathbb{Z}/2\oplus\mathbb{Z}/2=\pi_{15}(P^{12}(2))\oplus\pi_{15}(S^{12})/2\longrightarrow\pi_{15}(P^7(2)\wedge P^6(2))\xrightarrow{p_*}\pi_{15}(S^{13})\longrightarrow 0,$$

Thus $2H_{2*}(\alpha) \neq 0$ and so $2\alpha \neq 0$. It follows that the short exact sequence

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

does not split and so $G = \mathbb{Z}/8$. By Barratt's exponent theorem [2, 7], we have $8\pi_{15}(P^7(2)) = 0$ and so

$$\pi_{15}(P^7(2)) = \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4.$$

This completes the calculation. □

Note. It was known in [12] that there is a family of $\mathbb{Z}/8$-summands in $\pi_*(P^n(2))$ for $n \geq 3$ by expecting certain spherical classes in $H_*(\Omega P^n(2))$. The $\mathbb{Z}/8$-summand of $\pi_{15}(P^7(2)) = \pi_{14}(\Omega P^7(2))$ does not come from a spherical class because $H_{14}(\Omega P^7(2)) = 0$. It might be interesting to know whether this summand is detected by other homology theory. According to [27, Proposition 4.2], a generator of $\mathbb{Z}/8$-summand of $\pi_{15}(P^7(2))$ is given as $\tilde{\eta}_6 \circ \sigma_8$, where $\tilde{\eta}_6$ is a generator for $\pi_8(P^7(2)) = \mathbb{Z}/4$. According to the referee's comments, the other generators for $\pi_{15}(P^7(2))$ are the following: $[\bar{\nu}_7]$, $[\epsilon_7]$ and $i_7\nu_6$, where $[\alpha]$ stands for a lifting of α and $[\epsilon_7] = \tilde{\epsilon}_6$ (a coextension of ϵ_6).

5.6. **The Homotopy Groups $\pi_*(P^8(2))$.** Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
S^7 & \longrightarrow & S^{15} & \longrightarrow & S^8 \\
\| & & \uparrow & \text{pull} & \uparrow p \\
S^7 & \xrightarrow{i} & X^8 & \xrightarrow{q} & P^8(2) \\
 & & \uparrow f & & \uparrow \\
 & & F^8\{2\} & = & F^8\{2\}
\end{array}$$

where $S^7 \to S^{15} \to S^8$ is the Hopf fibration.

Lemma 5.14. *The space X^8 is the homotopy cofibre of $S^{14} \overset{(\sigma',[2])}{\to} S^7 \vee S^{14}$ and therefore X^8 is the 20-skeleton of $\Omega^2 S^9$.*

Proof. By Lemma 2.38, $\bar{H}_*(X^8)$ has a basis $\{x_7, x_{14}x_{15}\}$ with $Sq^1_* x_{15} = x_{14}$ Thus X^8 is the homotopy cofibre of $\delta|_{S^{14}} \colon S^{14} \longrightarrow \mathrm{sk}_{14}(F^8\{2\})$, where $\delta : \Omega S^{15} \to F^8\{2\}$ is the boundary. Since $\mathrm{sk}_{14}(F^8\{2\}) \simeq S^7 \vee S^{14}$, there is a cofibre sequence

$$S^{14} \xrightarrow{\psi} S^7 \vee S^{14} \xrightarrow{f} X^8.$$

with the composite $S^{14} \xrightarrow{\psi} S^7 \vee S^{14} \xrightarrow{\text{proj.}} S^{14}$ of degree 2. Let ϕ_1 be the composite $S^{14} \xrightarrow{\psi} S^7 \vee S^{14} \xrightarrow{\text{proj.}} S^7$. Then $\phi_1 \simeq k\sigma'$ for some k. It suffices to show that $k \not\equiv 0$ mod 2. Suppose that k were even. Let $\bar{\lambda}_8 \colon S^{14} \to F^8\{2\}$ be the composite

$$S^{14} \hookrightarrow S^7 \vee S^{14} \simeq \mathrm{sk}_{14}(F^8\{2\}) \hookrightarrow F^8\{2\}.$$

Then $\bar{\lambda}'_{8*}(\iota_{13}) = [u,v]$ in $H_*\Omega F^8\{2\}$ and therefore in $H_*\Omega P^8(2)$, where $\bar{\lambda}'_8$ is the adjoint map of $\bar{\lambda}_8$. By [12, Theorem 2.2], the homotopy class

$$2[q\circ f\circ\bar{\lambda}_8]\neq 0 \quad\text{and}\quad 4[q\circ f\circ\bar{\lambda}_8]=0$$

in $\pi_{14}(P^8(2))$. Since $i\colon S^7\to X^8$ is of degree 2 on the bottom cell, we have

$$0=q_*\circ i_*(\frac{k}{2}\sigma')=q_*\circ f_*(k\sigma')=q_*(-2f_*[\bar{\lambda}_8])=-2[q\circ f\circ\bar{\lambda}]\neq 0$$

This is a contradiction and hence the result. □

Corollary 5.15. *There is an exact sequence*

$$\pi_r(S^7)\xrightarrow{2\cdot E^2}\pi_r(\Omega^2S^9)\longrightarrow\pi_r(P^8(2))\longrightarrow\pi_{r-1}(S^7)\xrightarrow{2\cdot E^2}\pi_{r-1}(\Omega^2S^9)$$

for $r\leq 19$.

Theorem 5.16. *The homotopy groups $\pi_r(P^8(2))$ for $r\leq 16$ are as follows.*

1) $\pi_r(P^8(2))\cong\pi_r^s(P^8(2))$ *for* $r\leq 13$.
2) $\pi_{14}(P^8(2))=\mathbb{Z}/4\oplus\mathbb{Z}/2$.
3) $\pi_{15}(P^8(2))=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$.
4) $\pi_{16}(P^8(2))=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4\oplus\mathbb{Z}/4$.

Proof. **1.** $\pi_r(P^8(2))$ for $r\leq 13$: Note that $F^8\{2\}\simeq S^7\vee S^{14}$ and $\pi_r(S^7)\cong\pi_r^s(S^7)$ for $r\leq 13$. The assertion (1) follows.

2. $\pi_{14}(P^8(2))$: By Corollary 5.15, there is a short exact sequence

$$0\longrightarrow\mathbb{Z}/4=\pi_{14}(\Omega^2S^9)/4\longrightarrow\pi_{14}(P^8(2))\longrightarrow\pi_{13}(S^7)=\mathbb{Z}/2\longrightarrow 0.$$

By [12, Theorem 2.2], there is a $\mathbb{Z}/4$-summand in $\pi_{14}(P^8(2))$ and so

$$\pi_{14}(P^8(2))=\mathbb{Z}/4\oplus\mathbb{Z}/2.$$

3. $\pi_{15}(P^8(2))$: Since $2\pi_{15}(S^7)=0$ and the kernel of

$$2E^2\colon\pi_{14}(S^7)=\mathbb{Z}/8\longrightarrow\pi_{14}(\Omega^2S^9)=\mathbb{Z}/16$$

is $4\pi_{14}(S^7)\cong\mathbb{Z}/2$. There is a short exact sequence

$$0\longrightarrow\pi_{17}(S^9)\longrightarrow\pi_{14}(P^8(2))\longrightarrow 4\pi_{14}(S^7)\longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_{12}(P^5(2)) & \xrightarrow{E^2} & \pi_{14}(P^7(2)) & \xrightarrow{E} & \pi_{15}(P^8(2))\\
\downarrow & & \downarrow & & \downarrow\\
\pi_{12}(S^5) & \underset{\cong}{\xrightarrow{E^2}} & 4\cdot\pi_{14}(S^7) & = & 4\cdot\pi_{14}(S^7).
\end{array}$$

By Step 6 of Theorem 5.10, the homomorphism $\pi_{12}(P^5(2))\to\pi_{12}(S^5)$ is split onto and so is $\pi_{15}(P^8(2))\to 4\cdot\pi_{14}(S^7)$. Thus

$$\pi_{15}(P^8(2))\cong\pi_{17}(S^9)\oplus 4\cdot\pi_{14}(S^7)\cong\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2.$$

4. $\pi_{16}(P^8(2))$: Recall that $\pi_{15}(S^7)=\mathbb{Z}/2^{\oplus 3}$ is generated by $\{\sigma'\circ\eta_{14},\bar{\nu}_7,\epsilon_7\}$ and $\pi_{16}(S^7)=\mathbb{Z}/2^{\oplus 4}$ is generated by $\{\sigma'\circ\eta_{14}^2,\nu_7^3,\mu_7,\eta_7\circ\epsilon_8\}$, see [42, Theorems 7.1 and 7.2]. Since $2\pi_{16}(S^7)=0$, there is a short exact sequence

$$0\longrightarrow\pi_{16}(\Omega^2S^9)\longrightarrow\pi_{16}(P^8(2))\longrightarrow\pi_{15}(S^7)\longrightarrow 0.$$

Consider the commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega X^8 & \xrightarrow{\Omega q} & \Omega P^8(2) & \xrightarrow{\delta} & S^7 & \longrightarrow & X^8\\
\uparrow{\scriptstyle \Omega j} & & \uparrow{\scriptstyle \phi} & & \| & & \uparrow{\scriptstyle j}\\
\Omega S^7 & \xrightarrow{g} & S^7\{2\} & \xrightarrow{p} & S^7 & \xrightarrow{[2]} & S^7.
\end{array}$$

Since S^7 is an H-space, we have $\Omega[2]\simeq 2:\Omega S^7\to\Omega S^7$ and so

$$\Omega^3(S^7\{2\})\simeq(\Omega^3S^7)\{2\}=\mathrm{Map}_*(P^4(2);S^7).$$

Thus the power map $2:\Omega^3(S^7\{2\})\to\Omega^3(S^7\{2\})$ is homotopic to the composite

$$\Omega^3(S^7\{2\})\simeq\mathrm{Map}_*(P^4(2);S^7)\xrightarrow{\mathrm{id}^s}\Omega^3S^7\xrightarrow{\mathrm{id}^\eta}\Omega^4S^7\longrightarrow\Omega^3(S^7\{2\}),$$

where $s\colon S^3\to P^4(2)$ is the canonical inclusion. Since $2\pi_{15}(S^7)=0$, the map $p_*\colon\pi_{15}(S^7\{2\})\to\pi_{15}(S^7)$ is onto. Let $\alpha\in\pi_{15}(S^7\{2\})$ so that $p_*(\alpha)=\sigma'\circ\eta_{14}$. Then $2\alpha=g_*(\sigma'\circ\eta_{14}^2)$ and so

$$2\phi_*(\alpha)=\phi_*(2\alpha)=\phi_*\circ g_*(\sigma'\circ\eta_{14}^2)=\Omega q_*\circ\Omega j_*(\sigma'\circ\eta_{14}^2).$$

Consider the cofibre sequence

$$S^{14}\xrightarrow{(\sigma',[2])}S^7\vee S^{14}\xrightarrow{f}X^8.$$

Then

$$j\circ\sigma'\circ\eta_{14}^2=f|_{S^7}\circ\sigma'\circ\eta_{14}^2\simeq f|_{S^{14}}\circ[-2]\circ\eta_{14}^2\simeq *.$$

Thus $\Omega j_*(\sigma'\circ\eta_{14}^2)=0$ and $2\phi_*(\alpha)=0$,that is, there exists $\alpha_1\in\pi_{16}(P^8(2))$ so that

(1). $\delta_*(\alpha_1)=\sigma'\circ\eta_{14}$ and
(2). $2\alpha_1=0$.

Now consider the commutative diagram

$$\begin{array}{ccc}
\pi_{16}(P^8(2)) & = & \pi_{16}(P^8(2))\\
\uparrow{\scriptstyle E} & & \downarrow{\scriptstyle \delta_*}\\
\pi_{15}(P^7(2)) & \xrightarrow{p'_*} & \pi_{15}(S^7)
\end{array}$$

By Step 6 of Theorem 5.13, the $p'_*\colon\pi_{15}(P^7(2))\to\pi_{15}(S^7)$ is onto. Let α_2,α_3 be elements in $\pi_{15}(P^7(2))$ so that $p'_*(\alpha_2)=\bar{\nu}_7$ and $p'_*(\alpha_3)=\epsilon_7$. Then

$$\{\alpha_2\}\oplus\{\alpha_3\}\cong E^\infty\{\alpha_2\}\oplus E^\infty\{\alpha_3\}$$

is a summand of $\pi^s_{15}(P^7(2))$ and so $E\{\alpha_2\} \oplus E\{\alpha_3\}$ is a summand of $\pi_{16}(P^8(2))$. Thus there is a commutative diagram of short exact sequences

$$\begin{array}{ccccc}
\mathbb{Z}/2\oplus\mathbb{Z}/2 & = & \mathbb{Z}/2\oplus\mathbb{Z}/2 & & \\
\uparrow & & \uparrow & & \\
\mathbb{Z}/2^{\oplus 4} & \longrightarrow & \pi_{16}(P^8(2)) & \longrightarrow & \mathbb{Z}/2^{\oplus 3} \\
\uparrow & & \uparrow & & \| \\
\mathbb{Z}/2\oplus\mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2\oplus\mathbb{Z}/4\oplus\mathbb{Z}/4=\{\alpha_1\}\oplus\{\alpha_2\}\oplus\{\alpha_3\} & \longrightarrow & \mathbb{Z}/2^{\oplus 3}
\end{array}$$

and so $\pi_{16}(P^8(2)) \cong \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4^{\oplus 2}$. We finish the calculation. □

5.7. The Homotopy Groups $\pi_*(P^9(2))$.

Theorem 5.17. *The homotopy groups $\pi_r(P^9(2))$ for $r \leq 17$ are as follows.*

1) $\pi_r(P^9(2)) \cong \pi^s_r(P^9(2))$ *for* $r \leq 14$.
2) $\pi_{15}(P^9(2)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$.
3) $\pi_{16}(P^9(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
4) $\pi_{17}(P^9(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.

Proof. **1.** $\pi_r(P^9(2))$ for $r \leq 14$: Since S^8 is the 15-skeleton of $F^9\{2\}$. The assertion (1) follows.

2. $\pi_{15}(P^9(2))$: Consider the cofibre sequence $S^{15} \xrightarrow{2\omega_8} S^8 \xrightarrow{j} \mathrm{sk}_{23}(F^9\{2\})$. There is an exact sequence

$$\pi_{15}(S^{15}) \xrightarrow{(2\omega_8)_*} \pi_{15}(S^8) \xrightarrow{j_*} \pi_{15}(F^9\{2\}) \longrightarrow 0.$$

Since $(2\omega_8)_*(\iota_{15}) = 2\cdot(2\sigma_8 - E\sigma') = 4\sigma_8 - 2E\sigma'$, we have

$$\pi_{15}(F^9\{2\}) = \mathbb{Z}/16 \oplus \mathbb{Z}/2$$

generated by $j_*(\sigma_8)$ and $j_*(\omega_8)$. Now consider the commutative diagram

$$\begin{array}{ccccccccc}
\pi_{16}(S^9) & \xrightarrow{\delta_*} & \pi_{15}(F^9\{2\}) & \xrightarrow{i_*} & \pi_{15}(P^9(2)) & \xrightarrow{p_*} & \pi_{15}(S^9) & \longrightarrow & 0 \\
\uparrow{\scriptstyle E} & & \uparrow{\scriptstyle j_*} & & & & & & \\
\pi_{15}(S^8) & \xrightarrow{[2]_*} & \pi_{15}(S^8), & & & & & &
\end{array}$$

where E is onto. Since $[2]_*(\sigma_8) = 2\sigma_8 + \omega_8$, we obtain

$$\delta_*(\sigma_9) = \delta_* E\sigma_8 = j_*(2\sigma_8 + \omega_8) = 2j_*(\sigma_8) \oplus j_*(\omega_8),$$

and so there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \pi_{15}(P^9(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Notice that $h(\sigma_8)=\iota_8^2$ in $H_{14}(\Omega S^8)$. Thus $h([1\circ j\circ\sigma_8])=u^2$ in $H_{14}(\Omega P^9(2))$ and $[i\circ j\circ\sigma_8]$ is not divisible by 2. Hence $\pi_{15}(P^9(2))=\mathbb{Z}/4\oplus\mathbb{Z}/2$.

3. $\pi_{16}(P^9(2))$: Consider the exact sequence

$$\pi_{16}(S^{15})\xrightarrow{(2\omega_8)_*}\pi_{16}(S^8)\xrightarrow{j_*}\pi_{16}(F^9\{2\})\longrightarrow 0.$$

Since $2\pi_{16}(S^{15})=0$, $(2\omega_8)_*:\pi_{16}(S^{15})\longrightarrow\pi_{16}(S^8)$ is zero and $\pi_{16}(S^8)\cong\pi_{16}(F^9\{2\})$. Now consider the exact sequence

$$\pi_{17}(S^9)\xrightarrow{\delta_*}\pi_{16}(F^9\{2\})\longrightarrow\pi_{16}(P^9(2))\longrightarrow 8\cdot\pi_{16}(S^9)\longrightarrow 0.$$

Observe that

$$\delta_*(\sigma_9\circ\eta)=\delta_*(\sigma_9)\circ\eta=j_*(\omega_8\circ\eta)=j_*(E\sigma'\circ\eta)$$
$$\delta_*(\bar\nu_9)=\delta_*\circ E^2(\bar\nu_7)=j_*\circ E(2\bar\nu_7)=2j_*(\bar\nu_8)=0$$
$$\delta_*(\epsilon_9)=\delta_*\circ E^2(\epsilon_7)=j_*\circ E(2\epsilon_7)=0$$

Thus there is a short exact sequence

$$0\longrightarrow\pi_{16}(F^9\{2\})/\{E\sigma'\circ\eta\}\longrightarrow\pi_{16}(P^9(2))\longrightarrow 8\cdot\pi_{16}(S^9)\longrightarrow 0\quad\text{or}$$
$$0\longrightarrow\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\longrightarrow\pi_{16}(P^9(2))\longrightarrow\mathbb{Z}/2\longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccc}\pi_{16}(P^9(2)) & \longrightarrow & 8\pi_{16}S^9\\ \uparrow & & \uparrow\cong\\ \pi_{12}(P^5(2)) & \longrightarrow & \pi_{12}(S^5).\end{array}$$

Then $\pi_{16}(P^9(2))\longrightarrow 8\cdot\pi_{16}(S^9)$ is split onto and so $\pi_{16}(P^9(2))=\mathbb{Z}/2^{\oplus 4}$.

4. $\pi_{17}(P^9(2))$: Consider thee exact sequence

$$\pi_{17}(S^{15})\xrightarrow{(2\omega_8)_*}\pi_{17}(S^8)\xrightarrow{j_*}\pi_{17}(F^9\{2\})\longrightarrow\pi_{16}(S^{15})\longrightarrow 0.$$

Since $2\cdot\pi_{17}(S^{15})=0$, $(2\omega_8)_*:\pi_{17}(S^{15})\to\pi_{17}(S^8)$ is zero and so there is a short exact sequence

$$0\longrightarrow\pi_{17}(S^8)\xrightarrow{j_*}\pi_{17}(F^9\{2\})\longrightarrow\pi_{16}(S^{15})\longrightarrow 0.$$

By [42, Theorem 7.2], $\pi_{17}(S^8)/\{E\sigma'\circ\eta_{15}^2\}\cong\pi_{18}(S^9)$. Recall that the suspension $E\colon S^8\to\Omega S^9$ factors through $F^9\{2\}$. Thus $\pi_{17}(S^8)/\{E\sigma'\circ\eta_{15}^2\}$ is a summand of $\pi_{17}(F^9\{2\})/j_*\{E\sigma'\circ\eta_{15}^2\}$ and so

$$\pi_{17}(F^9\{2\})/j_*\{E\sigma'\circ\eta_{15}^2\}\cong\pi_{17}(S^8)/\{E\sigma'\circ\eta_{15}^2\}\oplus\pi_{16}(S^{15}).$$

Now consider the exact sequence

$$\pi_{18}(S^9)\xrightarrow{\delta_*}\pi_{17}(F^9\{2\})\longrightarrow\pi_{17}(P^9(2))\longrightarrow\pi_{17}(S^9)\xrightarrow{\delta_*}\pi_{16}(F^9\{2\}).$$

By Step 3, $\mathrm{Ker}(\delta_*:\pi_{17}(S^9)\to\pi_{16}(F^9\{2\}))=\{\bar\nu\}\oplus\{\epsilon_9\}=\mathbb{Z}/2\oplus\mathbb{Z}/2$. Note that

$$\delta_*(\nu_9\circ\eta_{16}^2)=\delta_*(\nu_9\circ\eta_{16})\circ\eta_{17}=j_*(E\sigma'\circ\eta_{15}^2)$$
$$\delta_*(\nu_9^3)=\delta_*E^2\nu_7^3=j_*E(2\nu_7^3)=0$$
$$\delta_*(\mu_9)=\delta_*E^2\mu_7=j_*E(2\mu_7)=0$$
$$\delta_*(\epsilon_9)=\delta_*E^2\epsilon_7=j_*E(2\epsilon_7)=0.$$

Thus there is a short exact sequence

$$0 \longrightarrow \pi_{17}(S^8)/\{E\sigma' \circ \eta_{15}^2\} \oplus \pi_{16}(S^{15}) \longrightarrow \pi_{17}(P^9(2)) \longrightarrow \{\bar{\nu}_9\} \oplus \{\epsilon_9\} \longrightarrow 0 \quad \text{or}$$

$$0 \longrightarrow \mathbb{Z}/2^{\oplus 5} \longrightarrow \pi_{17}(P^9(2)) \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccccc} \pi_{17}(P^9(2)) & \longrightarrow & \{\bar{\nu}\} \oplus \{\epsilon_9\} & \longrightarrow & 0 \\ \downarrow & & \downarrow \cong & & \\ \pi^s_{17}(P^9(2)) & \longrightarrow & \pi^s_{17}(S^9) & \longrightarrow & 0 \\ \downarrow p & & \| & & \\ \mathbb{Z}/4 \oplus \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & 0, \end{array}$$

where p is the projection. Thus $\pi_{17}(P^9(2)) = \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4^{\oplus 2}$. This completes the calculation. □

5.8. The Homotopy Groups $\pi_*(P^n(2))$ for $n \geq 10$.

Theorem 5.18. *The homotopy groups $\pi_r(P^n(2))$ for $n \geq 10$ and $r \leq n+8$ are as follows.*

1) $\pi_r(P^{10}(2)) \cong \pi^s_r(P^{10}(2))$ *for* $r \leq 16$.
2) $\pi_r(P^{11}(2)) \cong \pi^s_r(P^{11}(2))$ *for* $r \leq 18$.
3) $\pi_r(P^n(2)) \cong \pi^s_r(P^n(2))$ *for* $r \leq n+8$ *and* $n \geq 12$.
4) $\pi_{17}(P^{10}(2)) = \mathbb{Z}/2^{\oplus 4}$.
5) $\pi_{18}(P^{10}(2)) = \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.
6) $\pi_{19}(P^{11}(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.

Proof. Assertions (1), (2) and (3) are obvious.
1. $\pi_{17}(P^{10}(2))$: Consider the exact sequence

$$\pi_{18}(S^{10}) \xrightarrow{\delta_*} \pi_{17}(F^{10}\{2\}) \longrightarrow \pi_{17}(P^{10}(2)) \longrightarrow 8 \cdot \pi_{17}(S^{10}) \longrightarrow 0.$$

Since $F^{10}\{2\} \simeq S^9 \vee S^{18}$, we have $\pi_{17}(S^9) \cong \pi_{17}(F^{10}\{2\})$. Note that

$$\delta_*(\bar{\nu}_{10}) = \delta_* E^3(\bar{\nu}_7) = j_* E^2(2\bar{\nu}_7) = 0$$

$$\delta_*(\epsilon_{10}) = \delta_* E^3(\epsilon_7) = j_* E^2(2\epsilon_7) = 0$$

Thus there is a short exact sequence

$$0 \longrightarrow \pi_{17}(S^9) \longrightarrow \pi_{17}(P^{10}(2)) \longrightarrow 8 \cdot \pi_{17}(S^{10}) \longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccc} \pi_{17}(P^{10}(2)) & \longrightarrow & 8\cdot\pi_{17}S^{10} \\ \uparrow & & \uparrow\cong \\ \pi_{12}(P^5(2)) & \longrightarrow & \pi_{12}(S^5). \end{array}$$

By Step 6 of Theorem 5.10, the map $\pi_{12}(P^5(2)) \to \pi_{12}(S^5)$ is split onto. Thus $\pi_{17}(P^{10}(2)) \cong \mathbb{Z}/2^{\oplus 4}$.

2. $\pi_{18}(P^{10}(2))$: Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} \pi_{19}(S^{10}) & \xrightarrow{\delta_*} & \pi_{18}(F^{10}\{2\}) & \xrightarrow{j_*} & \pi_{18}(P^{10}(2)) & \xrightarrow{p_*} & \pi_{18}(S^{10}) & \longrightarrow & 0 \\ \downarrow H & & \downarrow \bar{H}_{2*} & & \downarrow H_{2*} & & & & \\ \pi_{19}(S^{19}) & \xrightarrow{\delta'_*} & \pi_{18}(F^{19}\{2\}) & \longrightarrow & \pi_{18}(P^{19}(2)). & & & & \end{array}$$

By [42, Theorem 7.2], $\pi_{19}(S^{10}) = \{\Delta(\iota_{21})\}\oplus\{\nu_{10}^3\}\oplus\{\mu_{10}\}\oplus\{\epsilon_{10} \cong \mathbb{Z}\oplus\mathbb{Z}/2^{\oplus 3}$, where $\Delta = P\colon \Omega^2 S^{21} \to S^{10}$. Note that

$$\delta_*(\nu_{10}^3) = \delta_*(\mu_{10}) = \delta_*(\epsilon_{10}) = 0$$

$$\bar{H}_{2*}\circ\delta_*\Delta(\iota_{21}) = \delta'_*\circ H\Delta(\iota_{21}) = \delta'_*(2\iota_{19}) = 4\iota_{18}.$$

Let $\bar{\lambda}_{10} : S^{18} \to F^{10}\{2\}$ be the composite

$$S^{18} \longrightarrow S^9 \vee S^{18} \simeq F^{10}\{2\}_{(18)} \to F^{10}\{2\}.$$

By [12, Theorem 2.2], $4[j\circ\bar{\lambda}_{10}] \neq 0$ and $8[j\circ\bar{\lambda}_{10}] = 0$. Thus

$$\delta_*\Delta(\iota_{21}) = 4[\bar{\lambda}_{10}] + \alpha$$

for some $\alpha \neq 0$ in $\pi_{18}(S^9)$ and so there is a short exact sequence

$$0 \longrightarrow \pi_{18}(S^9)/\{\alpha\}\oplus\mathbb{Z}/8 \longrightarrow \pi_{18}(P^{10}(2)) \longrightarrow \pi_{18}(S^{10}) \longrightarrow 0.$$

From the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z}/4 \oplus \mathbb{Z}/4 & = & \mathbb{Z}/4 \oplus \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & 0 \\
\Big\| & & \Big\downarrow & & \Big\downarrow \cong & & \\
\Big\| & & \pi_{18}(P^{10}(2)) & \longrightarrow & \pi_{18}(S^{10}) & \longrightarrow & 0 \\
\Big\| & & \Big\downarrow & & \Big\downarrow \cong & & \\
\Big\| & & \pi^s_{18}(P^{10}(2)) & \longrightarrow & \pi^s_{18}(S^{10}) & \longrightarrow & 0 \\
\Big\| & & \Big\downarrow & & \Big\downarrow \cong & & \\
\mathbb{Z}/4 \oplus \mathbb{Z}/4 & = & \mathbb{Z}/4 \oplus \mathbb{Z}/4 & \to & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & 0,
\end{array}$$

we obtain $\pi_{18}(P^{10}(2)) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.

3. $\pi_{19}(P^{11}(2))$: From the exact sequence

$$\pi_{19}(S^{19}) \xrightarrow{(2\omega_{10})_*} \pi_{19}(S^{10}) \longrightarrow \pi_{19}(F^{11}\{2\}) \longrightarrow 0,$$

we have $\pi_{19}(F^{11}\{2\}) = \pi_{19}(S^{10})/\{2\Delta(\iota_{21})\}$. Now consider the exact sequence

$$\pi_{20}(S^{11}) \xrightarrow{\delta_*} \pi_{19}(F^{11}\{2\}) \longrightarrow \pi_{19}(P^{11}(2)) \longrightarrow \pi_{19}(S^{11}) \longrightarrow 0.$$

Since $E^4\colon \pi_{16}(S^7) \to \pi_{20}(S^{11})$ is onto, $\delta_* : \pi_{20}(S^{11}) \longrightarrow \pi_{19}(F^{11}\{2\})$ is zero and so there is a short exact sequence

$$0 \longrightarrow \pi_{19}(F^{11}\{2\}) \longrightarrow \pi_{19}(P^{11}(2)) \longrightarrow \pi_{19}(S^{11}) \longrightarrow 0.$$

By [42, Theorem 7.2], we have $\pi_{19}(S^{11}) \cong \pi^s_{19}(S^{11})$. By Lemma 5.2, the group extension for the short exact sequence above is essential and

$$\pi_{19}(P^{11}(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4.$$

This completes the calculation. □

5.9. **Remarks on the generators for the homotopy groups.** In this subsection, we give some remarks along the suggestions given by the referee. Although our paper essentially contains the classical methods completely, in the metastable range the classical method is very useful to settle generators of the group. The classical method is to use the homotopy exact sequence of a pair $(P^n(2), S^{n-1})$ and to determine the relative homotopy group $\pi_*(P^n(2), S^{n-1})$ by using the James exact sequence [17, Theorem 2.1]. By using this method, it seems not hard to determine the group structures and the generators of $\pi_{n+r}(P^n(2))$ for $r \leq 2n-5$ if $\pi_{n+r}(S^m)$ $(m = n-1, n)$ is given by Toda's composition methods. (**Note.** When n is small, for instance $n = 3$ or 4, the range up to $2n-5$ is not so good.) The following example was given by the referee.

Example 5.19. *$\pi_{19}(P^{11}(2)) \cong (\mathbb{Z}/4)^{\oplus 2} \oplus (\mathbb{Z}/2)^{\oplus 2}$, which is generated by the elements $\tilde{\epsilon}_{10}$, $\tilde{\bar{\nu}}_{10}$, $i_{11}\mu_{10}$ and $i_{11} \circ [\iota_{10}, \iota_{10}]$.*

Proof. We consider the homotopy exact sequence of a pair $(P^{11}(2), S^{10})$:

$$\pi_{20}(P^{11}(2), S^{10}) \xrightarrow{\partial} \pi_{19}(S^{10}) \xrightarrow{i_{11*}} \pi_{19}(P^{11}(2)) \xrightarrow{j_*} \pi_{19}(P^{11}(2), S^{10}).$$

By Blakers-Massey's theorem,

$$\pi_{19}(P^{11}(2), S^{10}) \cong \pi_{19}(S^{11}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

So j_* is equivalent to the homomorphism $p_{11*}\colon \pi_{19}(P^{11}(2)) \longrightarrow \pi_{19}(S^{11})$. Since $\pi_{19}(S^{11}) = \Sigma\pi_{18}(S^{10})$ and $\pi_{18}(S^{10}) = \Sigma^4\pi_{14}(S^6)$, there exist coextensions $\tilde{\epsilon}_{10}$ and $\tilde{\bar{\nu}}_{10}$ of ϵ_{10} and $\bar{\nu}_{10}$ satisfying $p_{11*}(\tilde{\epsilon}_{10}) = \epsilon_{11}$ and $\pi_{11*}(\tilde{\bar{\nu}}_{10}) = \bar{\nu}_{11}$ respectively. By [17, Theorem 2.1], we obtain

$$\pi_{20}(P^{11}(2), S^{10}) = \{[\omega, \iota_{10}]\} \cong \mathbb{Z},$$

where $\omega \in \pi_{11}(P^{11}(2), S^{10})$ is the characteristic map of the 11-cell of $P^{11}(2)$ and [,] stands for the relative Whitehead product. We obtain

$$\partial[\omega, \iota_{10}] = -[2\iota_{10}, \iota_{10}].$$

Thus, by the fact that

$$\pi_{19}(S^{10}) \cong \mathbb{Z}\{[\iota, \iota]\} \oplus \mathbb{Z}/2\{\mu_{10}\} \oplus \mathbb{Z}/2\{\nu_{10}^3\} \oplus \mathbb{Z}/2\{\eta_{10}\epsilon_{11}\},$$

we conclude that

$$\pi_{19}(P^{11}(2)) \cong (\mathbb{Z}/4)^{\oplus 2} \oplus (\mathbb{Z}/2)^{\oplus 2},$$

which is generated by elements $\tilde{\epsilon}_{10}$, $\tilde{\bar{\nu}}_{10}$, $i_{11}\mu_{10}$ and $i_{11} \circ [\iota_{10}, \iota_{10}]$. The relations are the following:

$$2\tilde{\epsilon}_{10} = i_{11}\eta_{10}\epsilon_{11} \quad \text{and} \quad 2\tilde{\bar{\nu}}_{10} = i_{11}\nu_{10}^3.$$

Here we note that $\tilde{\epsilon}_{10}$ and $\tilde{\bar{\nu}}_{10}$ are chosen as suspended, so, by the fact $[2] = i_{11}\eta_{10}p_{11}$, we obtain $2\tilde{\epsilon}_{10} = [2]_*\tilde{\epsilon}_{10} = i_{11}\eta_{10}p_{11} \circ \tilde{\epsilon}_{10} = i_{11}\eta_{10}\epsilon_{11}$ and

$$2\tilde{\bar{\nu}}_{10} = [2]_*\tilde{\bar{\nu}}_{10} = i_{11}\eta_{10}p_{11}\tilde{\bar{\nu}}_{10} = i_{11}\eta_{10}\bar{\nu}_{11} = i_{11}\nu_{10}^3.$$

□

Note. $\tilde{\epsilon}_{10} = \Sigma^7\tilde{\epsilon}_3$ and $\tilde{\bar{\nu}}_{10} = \Sigma^4[\bar{\nu}_7]$.

6. The Homotopy Theory of $\Sigma\mathbb{R}P^2$

In this chapter, we study the homotopy theory of the special space $P^3(2) = \Sigma\mathbb{R}P^2$. This chapter will be divided into six sections. In the first section, we consider certain canonical fibrations over $\Sigma\mathbb{R}P^n$. In section 2, we consider the special case where $n = 2$. In this case, we obtain a product decomposition of the triple loop space of $\Sigma\mathbb{R}P^2$. This decomposition theorem will help us to compute the homotopy groups of $\Sigma\mathbb{R}P^2$. In section 3, we prove that the suspension $E\colon \pi_*(\mathbb{R}P^2) \to \pi_{*+1}(\Sigma\mathbb{R}P^2)$ is the trivial homomorphism for $* \geq 3$. This information has been to help us to find certain non-suspension co-H-spaces X which admits the form $P^3(2) \cup_f e^n$. An application of our results to the homogeneous space $SU(3)/SO(3)$ is given in section 4. We will start to compute the homotopy groups from section 5. From our decomposition theorem, the homotopy groups of $\pi_*(\Sigma\mathbb{R}P^4_2)$ for $* \geq 4$ is a summand of $\pi_*(P^3(2))$ and so we first compute $\pi_*(\Sigma\mathbb{R}P^4_2)$ in section 5 and then do $\pi_*(P^3(2))$ in section 6.

6.1. **Fibrations over $\Sigma\mathbb{R}\mathrm{P}^n$.** Let $\gamma_n\colon \Sigma\mathbb{R}\mathrm{P}^n \to BSO(n+1)$ be the adjoint map of the canonical inclusion $\mathbb{R}\mathrm{P}^n \hookrightarrow SO(n+1) = \Omega BSO(n+1)$, where $BSO(n+1)$ is the classifying space of $SO(n+1)$. Let the space Z_n be the pull-back of the diagram

$$\begin{array}{ccc} Z_n & \longrightarrow & ESO(n+1) \\ \big\downarrow & \text{pull} & \big\downarrow q \\ \Sigma\mathbb{R}\mathrm{P}^n & \xrightarrow{\gamma_n} & BSO(n+1), \end{array}$$

where $SO(n+1) \to ESO(n+1) \to BSO(n+1)$ is the universal principle $SO(n+1)$-bundle. We obtain a principal $SO(n+1)$-bundle over $\Sigma\mathbb{R}\mathrm{P}^n$. For instance,

$$Z_1 \longrightarrow \Sigma\mathbb{R}\mathrm{P}^1 = S^2$$

is the classical Hopf fibration. From now on, we assume that $n \geq 2$.

Let F be any space with $SO(n+1)$-action. We have the induced fibre bundle

$$F \longrightarrow Z_n \times_{SO(n+1)} F \longrightarrow \Sigma\mathbb{R}\mathrm{P}^n.$$

Let $Y_n = Z_n \times_{SO(n+1)} S^n$. Then there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc} SO(n+1) & \longrightarrow & S^n & \longrightarrow & BSO(n) & \longrightarrow & BSO(n+1) \\ \big\uparrow & & \| & & \big\uparrow & \text{pull} & \big\uparrow \gamma_n \\ \Omega\Sigma\mathbb{R}\mathrm{P}^n & \xrightarrow{\partial} & S^n & \longrightarrow & Y_n & \xrightarrow{p} & \Sigma\mathbb{R}\mathrm{P}^n \\ \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow \\ \Omega Z_n & \longrightarrow & * & \longrightarrow & Z_n & = & Z_n. \end{array}$$

Observe that the composite $\mathbb{R}\mathrm{P}^n \hookrightarrow \Omega\Sigma\mathbb{R}\mathrm{P}^n \xrightarrow{\partial} S^n$ is the pinch map and so its homotopy cofibre is $\Sigma\mathbb{R}\mathrm{P}^{n-1}$. By Lemma 2.39, there are cofibre sequences

$$SO(n+1)/\mathbb{R}\mathrm{P}^n \longrightarrow Z_n \xrightarrow{\theta_n} \Sigma\mathbb{R}\mathrm{P}^n \wedge SO(n+1) \quad \text{and}$$

$$\Sigma\mathbb{R}\mathrm{P}^{n-1} \hookrightarrow Y_n \xrightarrow{\tilde{\theta}_n} \Sigma\mathbb{R}\mathrm{P}^n \wedge S^n = \Sigma^{n+1}\mathbb{R}\mathrm{P}^n,$$

where θ_n is a monomorphism in mod 2 homology and $\tilde{\theta}_n$ is an epimorphism in mod 2 homology. Furthermore the map $S^n \to Y_n$ is the composite

$$S^n \xrightarrow{\Sigma q} \Sigma\mathbb{R}\mathrm{P}^{n-1} \hookrightarrow Y_n,$$

where $q\colon S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ is the quotient map. Observe that the composite

$$\Sigma\mathbb{R}\mathrm{P}^{n-1} \hookrightarrow Y_n \xrightarrow{p} \Sigma\mathbb{R}\mathrm{P}^n$$

is the canonical inclusion. Recall that the Thom space C_n of the vector bundle

$$Z_n \times_{SO(n+1)} \mathbb{R}^{n+1} \to \Sigma\mathbb{R}\mathrm{P}^n$$

is the homotopy cofibre of the map $p\colon Y_n \to \Sigma\mathbb{R}\mathrm{P}^n$. We have the homotopy commutative diagram of cofibre sequences

$$\begin{CD}
\Sigma\mathbb{R}\mathrm{P}^{n-1} @>>> Y_n @>>> \Sigma^{n+1}\mathbb{R}\mathrm{P}^n \\
@| @VV{p_n}V @VV{\delta_n}V \\
\Sigma\mathbb{R}\mathrm{P}^{n-1} @>>> \Sigma\mathbb{R}\mathrm{P}^n @>>> S^{n+1} \\
@VVV @VVV @VVV \\
* @>>> C_n @= C_n.
\end{CD}$$

We will give some information on $H^*(C_n)$ and some properties of the map

$$\delta_n\colon \Sigma^{n+1}\mathbb{R}\mathrm{P}^n \to S^{n+1}.$$

We need a lemma. A space X is called *stably atomic* if X does not have a nontrivial stable decomposition. The following lemma may be well-known.

Lemma 6.1. *If $n \neq 3$ or 7, then $\mathbb{R}\mathrm{P}^n$ is stably atomic.*

Proof. It is easy to check that $\mathbb{R}\mathrm{P}^{2n}$ is stably atomic by induction and by considering the Steenrod operations. Suppose that $\mathbb{R}\mathrm{P}^{2n+1}$ is *not* stably atomic. Let $q\colon S^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ be the quotient map. Since $\pi_{4n+1}(\Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n}) \cong \pi^s_{4n+1}(\Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n})$, the map $\Sigma^{2n+1}q\colon S^{4n+1} \to \Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n}$ is null homotopic. By [12], there is a homotopy commutative diagram

$$\begin{CD}
\Omega^2 S^{4n+3} @>{P}>> S^{2n+1} @>{E}>> \Omega S^{2n+2} @>{H}>> \Omega S^{4n+3} \\
@AAA @AA{\phi_{2n}}A @AA{\phi_{2n+1}}A @AAA \\
S^{4n+1} @>{\Sigma^{2n+1}q \simeq *}>> \Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n} @>>> \Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n+1} @>{\text{pinch}}>> S^{4n+2},
\end{CD}$$

where the top row is the EHP-sequence, the bottom row is the cofibre sequence and the map $\phi_n\colon \Sigma^{n+1}\mathbb{R}\mathrm{P}^n \to S^{n+1}$ is the adjoint map of the canonical map

$$\mathbb{R}\mathrm{P}^n \hookrightarrow O(n+1) \longrightarrow \Omega^{n+1}S^{n+1}.$$

It follows that the Whitehead square ω_{2n+1} is null homotopic. Thus $2n+1 = 3$ or 7 and hence the result. $\square$

Proposition 6.2. *Let $n \geq 2$. Then*

1) *$Sq^{n+1}\colon H^{n+1}(C_n) \to H^{2n+2}(C_n)$ is an isomorphism.*
2) *The composite*

$$\Sigma\mathbb{R}\mathrm{P}^n \hookrightarrow \Omega\Sigma^{n+1}\mathbb{R}\mathrm{P}^n \xrightarrow{\Omega\delta_n} \Omega S^{n+1} \xrightarrow{H_2} \Omega S^{2n+1}$$

is non-trivial mod 2 homology.

3) *If $n \neq 3$ or 7, then $\delta_{n*}\colon \pi_{n+2}(\Sigma^{n+1}\mathbb{R}\mathrm{P}^n) \to \pi_{n+2}(S^{n+1})$ is onto.*

Proof. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega\Sigma\mathbb{R}\mathrm{P}^n & \longrightarrow & S^n & \xrightarrow{i_n} & Y_n & \xrightarrow{p_n} & \Sigma\mathbb{R}\mathrm{P}^n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega C_n & = & \Omega C_n & \longrightarrow & * & \longrightarrow & C_n.
\end{array}$$

Let x_{n+1} be a generator for $H_{n+1}(C_n)=\mathbb{Z}/2$. Then

$$\tau(x_{n+1})^2=0$$

because $H_{n+1}(\Sigma\mathbb{R}\mathrm{P}^n)\cong H_{n+1}(C_n)$. Thus the dual x^*_{n+1} has a nontrivial self cup product and hence assertion (1).

(2). Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccccc}
\Omega Y_n & \longrightarrow & \Omega\Sigma\mathbb{R}\mathrm{P}^n & \longrightarrow & S^n & \longrightarrow & Y_n & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}^n \\
\downarrow & & \downarrow{\scriptstyle \Omega\phi} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} \\
\Omega\Sigma^{n+1}\mathbb{R}\mathrm{P}^n & \xrightarrow{\Omega\delta_n} & \Omega S^{n+1} & \longrightarrow & F_n & \longrightarrow & \Sigma^{n+1}\mathbb{R}\mathrm{P}^n & \xrightarrow{\delta_n} & S^{n+1}.
\end{array}$$

Let $\gamma_i(u)$ be a basis for $H_i(\mathbb{R}\mathrm{P}^n)$. Then $H_*(\Omega\Sigma\mathbb{R}\mathrm{P}^n)$ is the tensor algebra generated by $\gamma_i(u)$ for $1\le i\le n$. By the decomposition

$$H_*(\Omega\Sigma\mathbb{R}\mathrm{P}^n)=H_*(\Omega Y_n)\oplus H_*(\Omega Y_n)\cdot\gamma_n(u),$$

we have $\gamma_n(u)^2=\alpha+\beta\cdot\gamma_n(u)$ for some $\alpha\in H_{2n}(\Omega Y_n)$ and $\beta\in H_n(\Omega Y_n)$. Note that $\Omega\phi_*$ is a an algebraic map with $\Omega\phi_*(\gamma_i(u))=0$ for $i<n$ and $Q(H_n(\Omega Y_n))=0$. We have $\Omega\phi_*(\beta\cdot\gamma_n(u))=0$ and so

$$\Omega\phi_*(\gamma_n(u)^2)=\Omega\phi_*(\alpha).$$

It follows that the composite

$$H_{2n}(\Sigma\mathbb{R}\mathrm{P}^n)\cong H_{2n}(\Omega\Sigma^{n+1}\mathbb{R}\mathrm{P}^n)\xrightarrow{\Omega\delta_{n*}} H_{2n}(\Omega S^{n+1})$$

is onto and hence assertion (2).

(3). Let f be the composite

$$\Sigma^\infty\mathbb{R}\mathrm{P}^n\longrightarrow\Sigma^\infty\Omega^{n+1}\Sigma^{n+1}\mathbb{R}\mathrm{P}^n\xrightarrow{\Omega^{n+1}\delta_n}\Sigma^\infty\Omega_0^{n+1}S^{n+1}$$

$$\longrightarrow\Sigma^\infty Q_0(S^0)\xrightarrow{H_2}\Sigma^\infty Q(\mathbb{R}\mathrm{P}^\infty)\longrightarrow\Sigma^\infty\mathbb{R}\mathrm{P}^\infty$$

in the stable category. Then there is a self map g of $\Sigma^\infty\mathbb{R}\mathrm{P}^n$ such that $f=j\circ g$, where $j\colon\mathbb{R}\mathrm{P}^n\to\mathbb{R}\mathrm{P}^\infty$ is the inclusion. By assertion (2), $f_*\colon H_n(\mathbb{R}\mathrm{P}^n)\to H_n(\mathbb{R}\mathrm{P}^\infty)$ is an isomorphism and so is $g_*\colon H_n(\mathbb{R}\mathrm{P}^n)\to H_n(\mathbb{R}\mathrm{P}^n)$. By Lemma 6.1, the map g is a homotopy equivalence and hence the result. □

Question 6.3. *Is the map $\delta_n\colon\Sigma^{n+1}\mathbb{R}\mathrm{P}^n\to S^{n+1}$ homotopic to the adjoint map of the canonical map $\mathbb{R}\mathrm{P}^n\hookrightarrow SO(n+1)\longrightarrow\Omega^{n+1}S^{n+1}$?*

6.2. **A Decomposition of $\Omega^3_0P^3(2)$.** Now we study the spaces Z_n and Y_n in the special case where $n=2$. By Lemma 2.38, $Q(H_*(Y_2))$ and $Q(H_*(Z_2))$ have bases $\{u,[u,v],v^2\}$ and $\{u^2,[u,v],v^2,[[u,v],u],[[u,v],v]\}$, respectively, where $Sq^1_*v^2=[u,v]$, $Sq^2_*v^2=u^2$ and $Sq^1_*([[u,v],v])=[[u,v],u]$. Thus $\bar{H}_*(Y_2)$ and $\bar{H}_*(Z_2)$ have bases $\{x_2,x_4,x_5\}$ and $\{y_3,y_4,y_5,z_5,z_6\}$, respectively, with the Steenrod operations $Sq^1_*x_5=x_4$, $Sq^1_*y_5=y_4$, $Sq^2_*y_5=y_3$ and $Sq^1_*z_6=z_5$. The fibre sequence

$$Z_2 \xrightarrow{j} Y_2 \longrightarrow BSO(2)=\mathbb{C}\mathrm{P}^\infty$$

shows that Z_2 is the 2-connected cover of Y_2 with $j_*(y_4)=x_4$ and $j_*(y_5)=x_5$.

Proposition 6.4. *There is a cofibre sequence* $S^4\xrightarrow{(\eta^2,[2])}S^2\vee S^4\longrightarrow Y_2$.

Proof. Consider the fibre sequence $S^2\longrightarrow Y_2\longrightarrow P^3(2)$. The 4-skeleton

$$\mathrm{sk}_4(Y_2)\simeq S^2\cup_{k\eta}e^4$$

for some $k\in\mathbb{Z}$. If $k\neq 0$, then $\pi_3(Y_2)=\mathbb{Z}/k$ and so the boundary map

$$\pi_4(P^3(2))\to k\cdot\pi_3(S^2)$$

is onto. This is a contradiction because $\pi_*(P^3(2))$ is torsion group. Thus $k=0$ and so $\mathrm{sk}_4(Y_2)\simeq S^2\vee S^4$. By the cofibre sequence $S^2\longrightarrow Y_2\longrightarrow P^5(2)$, we have $Y_2=(S^2\vee S^4)\cup_{(\alpha,[2])}e^5$ for some $\alpha\colon S^4\to S^2$. Since $\pi_4(S^2)=\mathbb{Z}/2$, it suffices to show that α is essential. Suppose that α is null homotopic. Then $Y\simeq S^2\vee P^5(2)$ and so the bottom cell S^2 is retract of Y_2. It follows that the bottom cell S^3 is a retract of Z_2. This is impossible because $Sq^2_*y_5=y_3$. Thus α is essential and hence the result. □

We compute the first three homotopy groups of $P^3(2)$.

Proposition 6.5. *The homotopy groups $\pi_n(P^3(2))$ for $n\leq 4$ are as follows.*

1) $\pi_2(P^3(2))\cong\pi^s_2(P^3(2))=\mathbb{Z}/2$.
2) $\pi_3(P^3(2))=\mathbb{Z}/4$ *with a generator represented by the composite*

$$\bar{\eta}\colon S^3\xrightarrow{\eta}S^2\hookrightarrow P^3(2).$$

3) $\pi_4(P^3(2))=\mathbb{Z}/4$ *with a generator represented by the composite*

$$\lambda_3\colon S^4\hookrightarrow S^2\vee S^4=\mathrm{sk}_4(Y_2)\hookrightarrow Y_2\xrightarrow{p}P^3(2).$$

Proof. Assertion (1) is obvious. Consider the fibre sequence $S^2\xrightarrow{i}Y_2\xrightarrow{p}P^3(2)$. We have the exact sequence

$$\pi_4(S^2)\xrightarrow{i_*}\pi_4(Y_2)\xrightarrow{p_*}\pi_4(P^3(2))\xrightarrow{\partial=0}\pi_3(S^2)\longrightarrow\pi_3(Y_2)\longrightarrow\pi_3(P^3(2))\xrightarrow{\partial=0}\pi_2(S^2).$$

Since $i\colon S^2\to Y_2$ is of degree 2 on the bottom cell of Y_2, we have

$$i_*\colon\pi_3(S^2)=\mathbb{Z}\to\pi_3(Y_2)$$

is of degree 4 and $i_*\colon\pi_4(S^2)\to\pi_4(Y_2)$ is zero. Thus

$$\pi_3(P^3(2))=\mathbb{Z}/4\quad\text{and}\quad\pi_4(Y_2)\cong\pi_4(P^3(2)).$$

By the cofibre sequence

$$S^4\xrightarrow{(\eta^2,[2])}S^2\vee S^4\longrightarrow Y_2,$$

we have $\pi_4(Y_2)=\mathbb{Z}/4$ and hence the result. □

By expecting the homology of Z_2, the classes z_5 and z_6 come from a basis for $L_3(V)$ by homology suspension. This information suggests a splitting of Z_2.

Proposition 6.6. *There is a splitting $Z_2 \simeq \Sigma\mathbb{R}\mathrm{P}^4_2 \vee P^6(2)$.*

Proof. Since $\pi_3(Z_2) \cong \pi_3(Y_2) \cong \pi_3(\mathrm{sk}_4(Y_2)) = \pi_3(S^2 \vee S^4) = \mathbb{Z}$, we have

$$\mathrm{sk}_4(Z_2) \simeq S^3 \vee S^4.$$

Observe that $Sq^1_* y_5 = y_4$ and $Sq^2_* y_5 = y_3$. We obtain that $\mathrm{sk}_5(Z_2) \simeq \Sigma\mathbb{R}\mathrm{P}^4_2 \vee S^5$ and so there is a map $\phi\colon \Sigma\mathbb{R}\mathrm{P}^4_2 \to Z_2$ such that the image of ϕ_* has a basis $\{y_3, y_4, y_5\}$. Let $g\colon S^4 \to Z_2$ be a map such that $g_*(\iota_4) = y_4$ and let $g'\colon S^3 \to \Omega Z_2$ be the adjoint map of g. Let $\psi = [g, \mathrm{id}_{\mathbb{R}\mathrm{P}^2}]\colon S^3 \wedge \mathbb{R}\mathrm{P}^2 \to \Omega Z_2$ be the relative Samelson product on the fibre sequence $\Omega Z_2 \to \Omega P^3(2) \to SO(3)$. Then $\psi_*\colon \bar{H}_*(P^3(2)) \to Q(H_*(\Omega Z_2))$ has the image with a $\{[[u,v],u], [[u,v],v]\}$. Let $\psi'\colon P^6(2) \to Z_2$ be the adjoint map of ψ. Then the map

$$\Sigma\mathbb{R}\mathrm{P}^2 \vee P^6(2) \xrightarrow{\phi\vee\psi'} Z_2 \vee Z_2 \longrightarrow Z_2 \xrightarrow{\text{fold}} Z_2$$

is a homotopy equivalence and hence the result. □

Theorem 6.7. [46, Theorem 1.1] *There are homotopy decompositions localized at* 2. $\Omega^3_0(P^3(2)) \simeq \Omega^2(S^3\langle 3\rangle) \times \Omega^3(\Sigma\mathbb{R}\mathrm{P}^4_2\langle 3\rangle \vee P^6(2))$.

Proof. The assertion follows from the facts that

1) The map $i\colon S^2 \to Y_2$ is of degree 2 on the bottom cell.
2) $\Omega^2([2])\colon \Omega^2(S^2\langle 3\rangle) = \Omega^2(S^3\langle 3\rangle) \to \Omega^2(S^2\langle 3\rangle) = \Omega^2(S^3\langle 3\rangle)$ is null homotopic.
3) The 2-connected cover $Z_2 \simeq \Sigma\mathbb{R}\mathrm{P}^4_2 \vee P^6(2)$ by Proposition 6.6

□

There are some applications of this theorem.

Proposition 6.8. *Let X be a simply connected CW-complex and let $E \to X$ be a principle $SO(3)$-bundle. Suppose that the second and the third Whitney classes w_2 and w_3 are not zero in $H^*(X)$. Then*

$$\Omega^3_0 X \simeq \Omega^3_0 E \times \Omega^2(S^3\langle 3\rangle).$$

In particular, the higher homotopy groups of S^3 are summands of $\pi_*(X)$.

Proof. There is a map $f\colon X \to BSO(3)$ such that E is homotopy equivalent to the homotopy fibre of f. By the assumption, the inclusion $P^3(2) \hookrightarrow BSO(3)$ lifts to X and hence the result. □

Corollary 6.9. *Let G be any 2-connected topological group. Suppose that G contains $SO(3)$ as a subgroup. Let $G/SO(3)$ be the obtained homogeneous space. Then $\Omega^3_0(G/SO(3))$ is (weak) homotopy equivalent to the product $\Omega^3_0 G \times \Omega^2(S^3\langle 3\rangle)$.*

Proof. Consider the fibration $SO(3) \to G \to G/SO(3)$. By the assumption, we have $\pi_2(G/SO(3)) \cong \pi_2(BSO(3))$ and hence the result. □

As an example, consider that $SO(3)$ is a canonical subgroup of $SU(3)$ by complexification. We have the decomposition $\Omega^3_0(SU(3)/SO(3)) \simeq \Omega^3_0 SU(3) \times \Omega^2(S^3\langle 3\rangle)$. By using the language in (unstable) K-theory, we have

Corollary 6.10. *Let X be a path-connected space. Then the complexification of any orientable 3-dimensional vector bundle over $\Sigma^3 X$ is a trivial bundle.*

Consider the inclusion $E\colon \mathbb{R}\mathrm{P}^2 \to \Omega P^3(2)$. Let $q\colon S^2 \to \mathbb{R}\mathrm{P}^2$ be the quotient map. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{q} & \mathbb{R}\mathrm{P}^2 \\ \downarrow{[2]} & & \downarrow{E} \\ S^2 & \xrightarrow{\bar\eta} & \Omega P^3(2). \end{array}$$

Observe that

1) $[2]_*\colon \pi_r(S^2) \to \pi_r(S^2)$ is 0 for $r \geq 4$ and of degree 4 for $r = 3$;
2) $q_*\colon \pi_r(S^2) \to \pi_r(\mathbb{R}\mathrm{P}^2)$ is isomorphic and
3) $\pi_4(P^3(2)) = \mathbb{Z}/4$.

We have the following suspension theorem.

Proposition 6.11. *The suspension $E_*\colon \pi_r(\mathbb{R}\mathrm{P}^2) \to \pi_{r+1}(P^3(2))$ is zero for $r \geq 3$.*

Corollary 6.12. *Let X be a complex which admits the form of $P^3(2) \cup_f e^n$ with $n \geq 5$. If the attaching map f is essential, then X is not homotopy equivalent to a suspension.*

This result has been used for finding a family of non-suspension co-H-spaces X which admits the form of $P^3(2) \cup e^n$ in [48].

The map $\lambda_3\colon S^4 \to P^3(2)$, which represents a generator for $\pi_4(P^3(2))$, has some special properties. First the adjoint map $\lambda_3'\colon S^3 \to \Omega P^3(2)$ has the property that $\lambda'_{3*}(\iota_3) = [u, v]$, that is, λ_3 is from the spherical class $[u, v]$. The homotopy cofibre of λ_3 is actually the homogeneous space $SU(3)/SO(3)$.

Proposition 6.13. *There is a cofibre sequence $S^4 \xrightarrow{\lambda_3} P^3(2) \longrightarrow SU(3)/SO(3)$.*

Proof. By [24, Theorem 6.7], the cohomology $H^*(SU(3)/SO(3))$ is the exterior algebra $E(w_2, w_3)$, where w_i is the i-th Whitney class. Thus there is a cofibre sequence

$$S^4 \xrightarrow{f} P^3(2) \longrightarrow SU(3)/SO(3).$$

Consider the homotopy commutative diagram

$$\begin{array}{ccccc} S^4 & \xrightarrow{f} & P^3(2) & \hookrightarrow & SU(3)/SO(3) \\ \downarrow{g} & & \| & & \downarrow \\ Z_2 & \longrightarrow & P^3(2) & \longrightarrow & BSO(3), \end{array}$$

where the top row is a cofibre sequence and the bottom row is a fibre sequence. The map $g\colon S^4 \to Z_2$ is nontrivial in mod 2 homology. Thus g represents a generator for $\pi_4(Z_2)$ and hence the result. □

The next proposition shows that λ_3 represents a generator for the stable homotopy group $\pi_4^s(P^3(2))$.

Proposition 6.14. *There is an isomorphism $\pi_4(P^3(2)) \cong \pi_4^s(P^3(2))$.*

Proof. By Proposition 6.2, $\delta_{2*}\colon \pi_4(P^5(2)) \to \pi_4(S^3)$ is an isomorphism and so the composite $S^4 \xrightarrow{\lambda_3} P^3(2) \xrightarrow{\text{pinch}} S^3$ is homotopic to η. It follows that the homotopy class $[\lambda_3]$ is a generator for $\pi_4^s(P^3(2)) = \mathbb{Z}/4$ or $\pi_4(P^3(2)) = \mathbb{Z}/4 \to \pi_4^s(P^3(2))$ is an isomorphism and hence the result. □

Example 6.15. Since $\pi_4(P^3(2)) = \mathbb{Z}/4$, there are only three different complexes X which admits the form $X = P^3(2) \cup e^5$ given by $P^3(2) \cup_{\lambda_3} e^5 \simeq SU(3)/SO(3)$, $P^3(2) \cup_{2\lambda_3} e^5$ and $P^3(2) \vee S^5$. The space $P^3(2) \cup_{2\lambda_3} e^5$ has the cell structure $S^2 \cup_{\eta^2} e^4 \cup e^5$. This shows that the space $SU(3)/SO(3)$ is uniquely determined, up to homotopy, by its fundamental group and its cohomology ring. More precisely, if X is a simply connected complex such that the cohomology ring $H^*(X)$ is isomorphic to $H^*(SU(3)/SO(3))$, then X is homotopy equivalent to $SU(3)/SO(3)$. This information might be useful in the theory of 5-dimensional manifolds.

Unlike Theorem 2.19 on the Stiefel manifolds, there is a self homotopy equivalence of $SU(3)/SO(3)$ which changes the orientation.

Proposition 6.16. *There is a self homotopy equivalence of $SU(3)/SO(3)$ which changes the orientation.*

Proof. From the commutative diagram

$$\begin{array}{ccc}
\pi_4(P^3(2)) & \xrightarrow[E^\infty]{\cong} & \pi_4^s(P^3(2)) \\
\big\downarrow{\scriptstyle [-1]_*} & & \big\downarrow{\scriptstyle -1} \\
\pi_4(P^3(2)) & \xrightarrow[E^\infty]{\cong} & \pi_4^s(P^3(2)),
\end{array}$$

we have $[-1]_*(\lambda_3) = -\lambda_3$ and so there is a homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccccc}
S^4 & \xrightarrow{\lambda_3} & P^3(2) & \longrightarrow & SU(3)/SO(3) & \xrightarrow{\text{pinch}} & S^5 \\
\big\downarrow{\scriptstyle [-1]} & & \big\downarrow{\scriptstyle [-1]} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle [-1]} \\
S^4 & \xrightarrow{\lambda_3} & P^3(2) & \longrightarrow & SU(3)/SO(3) & \xrightarrow{\text{pinch}} & S^5.
\end{array}$$

By the 5-lemma, ϕ is a homotopy equivalence and hence the result. □

6.3. **The Homotopy Groups $\pi_*(\Sigma\mathbb{R}\mathrm{P}^2)$.** Now we start to compute $\pi_r(P^3(2))$ for $r \leq 11$. The first three homotopy groups have been computed in the previous section. The higher homotopy groups will be computed using Theorem 6.7. Observe that there are two factors in this decomposition theorem. The homotopy groups of the factor $\Omega^2(S^3\langle 3\rangle)$, up to this range, has been computed in [42]. Thus our work is to compute

$\pi_*(\Sigma\mathbb{R}\mathrm{P}^4_2 \vee P^6(2))$. We first compute $\pi_*(\Sigma\mathbb{R}\mathrm{P}^4_2)$, which is the most difficult part, and then, by applying the Hilton-Milnor theorem, determine the homotopy groups of $\Sigma\mathbb{R}\mathrm{P}^4_2 \vee P^6(2)$.

6.3.1. *The Homotopy Groups* $\pi_*(\Sigma\mathbb{R}\mathrm{P}^4_2)$. We need some preliminary lemmas.

Notation 6.17. Let F and B denote the spaces in the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
S^3 & \xrightarrow{i} & SU(3) & \longrightarrow & S^5 \\
\uparrow & & \uparrow q & & \| \\
F & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}^4_2 & \xrightarrow{p} & S^5 \\
\uparrow & & \uparrow & & \uparrow \\
B & = & B & \longrightarrow & *,
\end{array}$$

where p is the pinch map, $S^3 \to SU(3) \to S^5$ is the canonical fibration and q is the composite $q\colon \Sigma\mathbb{R}\mathrm{P}^4_2 \longrightarrow \Sigma\mathbb{C}\mathrm{P}^2 \hookrightarrow SU(3)$.

Lemma 6.18. *There is a homotopy equivalence*

$$\Omega F \simeq \Omega S^3 \times \Omega B$$

Proof. Since $\pi_3(S^3) \cong \pi_3(SU(3))$, $\pi_3(F) \cong \pi_3(\Sigma\mathbb{R}\mathrm{P}^4_2)$. Thus $\pi_3(\Sigma\mathbb{R}\mathrm{P}^4_2) \cong \pi_3(SU(3))$ and hence the result. □

Recall that $H_*(\Omega\mathbb{R}\mathrm{P}^4_2) \cong T(\bar{H}_*(\mathbb{R}\mathrm{P}^4_2)) \cong T(x_2, u_3, v_4)$ as algebras with $Sq^1_* v_4 = u_3$ and $Sq^2_* v_4 = x_2$. Although $H_*(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2)$ is not primitively generated, we will show that both $H_*(\Omega F)$ and $H_*(\Omega B)$ are primitively generated.

Lemma 6.19. *Let* $\alpha \in PH_*(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2)$. *Then* $[\alpha, v_4] + x_2[\alpha, x_2]$ *is primitive.*

Proof. Observe that $\psi(v_4) = v_4 \otimes 1 + 1 \otimes v_4 + x_2 \otimes x_2$ and $\psi(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$. We have

$$\psi([\alpha, v_4]) = [\alpha, v_4] \otimes 1 + 1 \otimes [\alpha, v_4] + x_2 \otimes [x_2, \alpha] + [x_2, \alpha] \otimes x_2,$$
$$\psi(x_2[\alpha, x_2]) = x_2[\alpha, x_2] \otimes 1 + 1 \otimes x_2[\alpha, x_2] + x_2 \otimes [x_2, \alpha] + [x_2, \alpha] \otimes x_2$$

and hence the result. □

Notation 6.20. For $\alpha \in H_*(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2) \cong T(x_2, u_3, v_4)$, we define $\overline{\mathrm{ad}}^0(v_4)(\alpha) = \alpha$, $\overline{\mathrm{ad}}^1(v_4)(\alpha) = [\alpha, v_4] + x_2[\alpha, x_2]$ and $\overline{\mathrm{ad}}^k(v_4)(\alpha) = \overline{\mathrm{ad}}^1(\overline{\mathrm{ad}}^{k-1}(v_4)(\alpha))$ for $k \geq 2$.

By Lemma 6.19, there are operations $\overline{\mathrm{ad}}^k(v_4)\colon PH_*(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2) \to PH_*(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2)$.

Proposition 6.21. $H_*(\Omega F)$ *is primitively generated and* $H_*(\Omega F) \cong T(V_F)$ *as sub-Hopf algebra of* $H_*(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2) \cong T(x_2, u_3, v_4)$, *where* $V_F \subseteq PH_*(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2)$ *has a basis* $\overline{\mathrm{ad}}^k(v_4)(x_2)$ *and* $\overline{\mathrm{ad}}^k(v_4)(u_3)$ *for* $k \geq 0$.

Proof. Observe that $\Omega p_* : H_*(\Omega\Sigma\mathbb{RP}^4_2) \to H_*(\Omega S^5)$ is an epimorphism. The Serre spectral sequence for the fibre sequence $\Omega F \to \Omega\Sigma\mathbb{RP}^4_2 \to \Omega S^5$ collapses and so there is a short exact sequence of Hopf algebras

$$H_*(\Omega F) \longrightarrow H_*(\Omega\Sigma\mathbb{RP}^4_2) \to H_*(\Omega S^5).$$

Thus there is an exact sequence

$$0 \longrightarrow PH_*(\Omega F) \longrightarrow PH_*(\Omega\Sigma\mathbb{RP}^4_2) \longrightarrow PH_*(\Omega S^5).$$

Since both x_2 and u_3 are primitive, the elements $\overline{\mathrm{ad}}^k(v_4)(x_2), \overline{\mathrm{ad}}^k(v_4)(u_3) \in PH_*(\Omega F)$ for $k \geq 0$ by Lemma 6.19. Let A be the subalgebra of $T(x_2, u_3, v_4)$ generated by $\overline{\mathrm{ad}}^k(v_4)(x_2)$ and $\overline{\mathrm{ad}}^k(v_4)(u_3)$ for $k \geq 0$. We claim that $\mathbb{Z}/2 \otimes_A T(x_2, u_3, v_4) = T(v_4)$. Observe that $\mathbb{Z}/2 \otimes_A T(x_2, u_3, v_4) = T(x_2u_3v_4)/IA \cdot T(x_2, u_3, v_4)$. Since $x_2, u_3 \in IA$, it suffices to show that $v_4^k \cdot y \in IA \cdot T(x_2, u_3v_4)$ for each $k \geq 0$, where $y = x_2$ or u_3.

This is shown by induction on k. If $k = 0$, $v_4^0 y = y \in IA \cdot T(x_2, u_3, v_4)$. Suppose that $v_4^k y \in IA \cdot T(x_2, u_3, v_4)$. Then

$$v_4^k y = \sum_i \overline{\mathrm{ad}}^{t_i}(v_4)(y_i) \cdot \alpha_i$$

for some $t_i \geq 0$, $y_i = x_2$ or u_3 and $\alpha_i \in T(x_2, u_3, v_4)$ and so

$$v_4^{k+1} y = \sum_i v_4\overline{\mathrm{ad}}^{t_i}(v_4)(y_i) \cdot \alpha_i.$$

It suffice to show that

$$v_4\overline{\mathrm{ad}}^{t_i}(v_4)(y_i) \in IA \cdot T(x_2, u_3, v_4)$$

for each i. Now

$$\begin{aligned} \overline{\mathrm{ad}}^{t_i+1}(v_4)(y_i) &= [\overline{\mathrm{ad}}^{t_i}(v_4)(y_i), v_4] + x_2 \cdot [\overline{\mathrm{ad}}^{t_i}(v_4)(y_i), x_2] \\ &= v_4\overline{\mathrm{ad}}^{t_i}(v_4)(y_i) + \overline{\mathrm{ad}}^{t_i}(v_4)(y_i) \cdot v_4 + x_2 \cdot [\overline{\mathrm{ad}}^{t_i}(v_4)(y_i), x_2] \end{aligned}$$

Thus

$$v_4\overline{\mathrm{ad}}^{t_i}(v_4)(y_i) = \overline{\mathrm{ad}}^{t_i+1}(v_4)(y_i) + \overline{\mathrm{ad}}^{t_i}(v_4)(y_i)v_4 + x_2[\overline{\mathrm{ad}}^{t_i}(v_4)(y_i), x_2] \in IA \cdot T(x_2, u_3, v_4).$$

The induction is completed.

Similarly, $T(x_2, u_3, v_4) \otimes_A \mathbb{Z}/2 = T(v_4)$. Thus A is a normal subalgebra of $T(x_2, u_3, v_4)$. Observe that

$$\chi(A) = (1-t^2-t^3-t^4)^{-1} \cdot (1-t^4) = (1-\sum_{k=0}^{\infty}(t^{4k+2}+t^{4k+3}))^{-1} = \chi(H_*\Omega F) = \chi(T(V_F)).$$

Since $A \subseteq H_*(\Omega F)$ and $T(V_F) \to A$ is onto, $A = H_*(\Omega F)$ and so

$$T(V_F) \cong A = H_*(\Omega F).$$

Note that $V_F \subseteq PH_*(\Omega\Sigma\mathbb{RP}^4_2)$. Thus $H_*(\Omega F)$ is primitively generated and hence the result. □

Corollary 6.22. *The Hopf algebra $H_*(\Omega B)$ is primitively generated and*

$$H_*(\Omega B) \cong T(V_B)$$

as a subalgebra of $H_*(\Omega\Sigma\mathbb{RP}^4_2) \cong T(x_2, u_3, v_4)$, where $V_B \subseteq PH_*(\Omega\Sigma\mathbb{RP}^4_2)$ has a basis $\mathrm{ad}^{j_2}(x_2)\overline{\mathrm{ad}}^{j_1}(v_4)(x_2)$ *for $j_2 \geq 0, j_1 \geq 1$ and* $\mathrm{ad}^{j_2}(x_2)\overline{\mathrm{ad}}^{j_1}(v_4)(u_3)$ *for $j_2, j_1 \geq 0$.*

Proof. There is a short exact sequence of Hopf algebras

$$H_*(\Omega B) \hookrightarrow H_*(\Omega F) \twoheadrightarrow H_*(\Omega S^3).$$

Note that $H_*(\Omega F) \cong T(V_F)$ is primitively generated. The assertion follows from [9, Lemma 3.13]. □

Now we are going to decompose ΩB up to dimension 11.

Lemma 6.23. *There exists a map $\theta\colon B \to \Omega S^5$ so that the composite*

$$S^4 \xrightarrow{j} B \xrightarrow{\theta} \Omega S^5$$

is of degree 1.

Proof. Consider the homotopy commutative diagram

$$\begin{array}{ccccc}
SU(3) & \xrightarrow{2} & SU(3) & \longrightarrow & S^5 \\
\uparrow & & \uparrow & & \| \\
\Sigma\mathbb{R}\mathrm{P}^4_2 & \xrightarrow{[2]} & \Sigma\mathbb{R}\mathrm{P}^4_2 & \xrightarrow{p} & S^5 \\
\downarrow p & & \downarrow p & & \| \\
P^5(2) & \xrightarrow{[2]} & P^5(2) & \xrightarrow{\text{pinch}} & S^5.
\end{array}$$

The composite

$$\Sigma\mathbb{R}\mathrm{P}^4_2 \longrightarrow SU(3) \xrightarrow{2} SU(3) \longrightarrow S^5$$

is null homotopic and so there is a homotopy commutative diagram

$$\begin{array}{ccccccccc}
\Omega SU(3) & \longrightarrow & B & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}^4_2 & \longrightarrow & SU(3) & & \\
\downarrow 2 & & \downarrow \theta & & \downarrow & & \downarrow 2 & & \\
\Omega SU(3) & \longrightarrow & \Omega S^5 & \longrightarrow & S^3 & \longrightarrow & SU(3) & \longrightarrow & S^5.
\end{array}$$

Recall that $\pi_4(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/4$. From the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_4(\Omega SU(3)) & \longrightarrow & \pi_4(B) & \longrightarrow & \pi_4(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/4 & \longrightarrow & \pi_4(SU(3)) = 0 \\
& & \downarrow 2 & & \downarrow \theta_* & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_4(\Omega SU(3)) & \longrightarrow & \pi_4(\Omega S^5) & \longrightarrow & \pi_4(S^3) = \mathbb{Z}/2 & \longrightarrow & \pi_4(SU(3)) = 0,
\end{array}$$

the map $\theta_* : \pi_4(B) \to \pi_4(\Omega S^5)$ is of degree 1 and hence the result. □

Corollary 6.24. *The bottom cell S^3 is a stable retract of ΩB*

Proof. There is a retraction: $\Sigma^\infty \Omega^2 S^5 \to \Sigma^\infty S^3$ and hence the result. □

Lemma 6.25. *The 7-skeleton* $\mathrm{sk}_7(\Omega B)$ *of* ΩB *is homotopy equivalent to* $S^3\vee S^7\vee Y^7$*, where* $\bar{H}_*(Y^7)$ *has a basis* $[x_2,u_3],u_3^2,[x_2,v_4],\overline{\mathrm{ad}}(v_4)(u_3)$ *with* $Sq^1_*[x_2,v_4]=[x_2,u_3]$*,* $Sq^2_*\overline{\mathrm{ad}}(v_4)(u_3)=[x_2,u_3]$ *and the second Bockstein* $\beta_2(\overline{\mathrm{ad}}(v_4)(u_3))=u_3^2$*.*

Proof. By the Cartan formula, we have

$$Sq^1_*[x_2,v_4]=[x_2,u_3]\quad\text{and}\quad Sq^2_*(\overline{\mathrm{ad}}(v_4)(u_3))=[x_2,u_3].$$

We first show that $\beta_2(\overline{\mathrm{ad}}(v_4)(u_3))=u_3^2$. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}\Omega B & \longrightarrow & \Omega\Sigma\mathbb{R}\mathrm{P}^4_2 & \longrightarrow & \Omega SU(3)\\ \downarrow{\scriptstyle\Omega\phi} & & \downarrow & & \downarrow\\ \Omega F^5\{2\} & \longrightarrow & \Omega P^5(2) & \longrightarrow & \Omega S^5\end{array}$$

We have $\Omega\phi_*(\overline{\mathrm{ad}}(v_4)(u_3))=[u_3,v_4]$ and $\Omega\phi_*(u_3^2)=u_3^2$ in $H_*\Omega F^5\{2\}$. Since

$$\beta_1(\overline{\mathrm{ad}}(v_4)(u_3))=Sq^1_*(\overline{\mathrm{ad}}(v_4)(u_3))=0,$$

we have $\Omega\phi_*\beta_2(\overline{\mathrm{ad}}(v_4)(u_3))=u_3^2$ and so $\beta_2(\overline{\mathrm{ad}}(v_4)(u_3))=u_3^2+\epsilon[x_2,v_4]\equiv u_3^2$ in Bockstein spectral sequence for $H_*\Omega B$. Now $\bar{H}_*(\mathrm{sk}_7(\Omega B))$ has a basis

$$\{u_3,[x_2,u_3],u_3^2,[x_2,v_4],\overline{\mathrm{ad}}(v_4)(u_3),[[x_2,u_3],x_2].$$

Since x_2 and u_3 are spherical in $H_*(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2)$, the elements $u_3,[x_2,u_3],u_3^2,[[x_2,u_3],x_2]$ are spherical in $\Omega(\Sigma\mathbb{R}\mathrm{P}^4_2)$ by Samelson products and Hopf Invariants 1. Observe that $\Omega SU(3)$ is a commutative H-space. All of Samelson products lift to the fibre ΩB. It follows that $u_3,[x_2,u_3],u_3^2$ and $[[x_2,u_3],u_3]$ are spherical in $H_*(\Omega B)$. Thus the 5-skeleton $\mathrm{sk}_5(\Omega B)\simeq S^3\vee S^5$ and the 6-skeleton $\mathrm{sk}_6(\Omega B)\simeq S^6\vee X$, where $\bar{H}_*(X)$ has a basis $\{u_3,[x_2,u_3],[x_2,v_4]\}$. The space X is the homotopy cofibre of the attaching map $S^5\xrightarrow{f}S^3\vee S^5$. Since $Sq^1_*[x_2,v_4]=[x_2,u_3]$, the composite

$$S^5\xrightarrow{f}S^3\vee S^5\xrightarrow{\text{proj.}}S^5$$

is of degree 2. Let ϕ be the composite $S^5\xrightarrow{f}S^3\vee S^5\xrightarrow{\text{proj.}}S^3$. Then $\phi=k\eta^2$ for some $k=0,1$. If $k=1$, then $\pi_5(X)=\mathbb{Z}/4$ and $\pi_5(S^3)$ is not a summand of $\pi_5(X)$, which contradicts to the facts that

i) $\pi_5(S^3)\cong\pi_5^s(S^3)=\mathbb{Z}/2$;
ii) S^3 is a stable retract of ΩB by Corollary 6.24.
iii) $\pi_5(X)\cong\pi_5(\Omega B)$.

Thus $k=0$ and so $X\simeq S^3\vee P^6(2)$ or $\mathrm{sk}_6(\Omega B)\simeq S^3\vee S^6\vee P^6(2)$. Now we have $\mathrm{sk}_7(\Omega B)\simeq S^7\vee Y$, where $\bar{H}_*(Y)$ has a basis $u_3,[x_2,u_3],u_3^2,[x_2,v_4]$ and $\overline{\mathrm{ad}}(v_4)(u_3)$. There is a cofibre sequence

$$S^6\xrightarrow{\phi}S^3\vee S^6\vee P^6(2)\longrightarrow Y.$$

Since $Sq^2_*\overline{\mathrm{ad}}(v_4)(u_3)=[x_2,u_3]$, the composite $S^6\xrightarrow{\phi}S^3\vee S^6\vee P^6(2)\xrightarrow{\text{proj.}}P^6(2)$ is $\bar\eta$ which is a generator for $\pi_6(P^6(2))=\mathbb{Z}/2$. Note that $\beta_2(\overline{\mathrm{ad}}(v_4)(u_3))=u_3^2$. The composite $S^6\xrightarrow{\phi}S^3\vee S^6\vee P^6(2)\xrightarrow{\text{proj.}}S^6$ is of degree 4. Observe that the composite

$Y \longrightarrow \Omega B \longrightarrow \Omega^2 S^5$ maps into the 7-skeleton $\mathrm{sk}_7(\Omega^2 S^5)$ up to homotopy. There is a commutative diagram of cofibre sequences

$$\begin{array}{ccccccc} S^6 & \xrightarrow{\phi} & S^3 \vee S^6 \vee P^6(2) & \longrightarrow & Y & \longrightarrow & S^7 \\ \downarrow t & & \downarrow \text{proj.} & & \downarrow & & \downarrow t \\ S^6 & \xrightarrow{(\nu', [2])} & S^3 \vee S^6 & \longrightarrow & \mathrm{sk}_7(\Omega^2 S^5) & \longrightarrow & S^7, \end{array}$$

Thus the map $t : S^6 \to S^6$ is of degree 2 and there is a projection

$$p_1 : S^3 \vee S^6 \vee P^6(2) \to S^3$$

such that the composite $S^6 \xrightarrow{\phi} S^3 \vee S^6 \vee P^6(2) \xrightarrow{p_1} S^3$ is $2\nu' = \eta_3^3 = E\eta_2^3$. Hence

$$\phi = (\eta_3^3, 4, \bar{\eta}) = E(\eta_2^3, 4, \bar{\eta}) : S^6 \to S^3 \vee S^6 \vee P^6(2).$$

It follows that $Y \simeq \Sigma Y'$ for some Y'. Now let $f \colon S^3 \vee S^6 \vee P^6(2) \to S^3$ be the composite $S^3 \vee S^6 \vee P^6(2) \xrightarrow{p} P^6(2) \xrightarrow{\bar{\eta}_3^2} S^3$, where p is the projection and $\bar{\eta}_3^2$ is the extension of $\eta_3^2 : S^5 \to S^3$. Then $f \circ \phi = \eta_3^3$. Define $g \colon S^3 \vee S^6 \vee P^6(2) \to S^3$ by $g = p_1 + f$. Then $[g] \circ [\phi] = [p_1 \circ \phi] + [f \circ \phi] = 2\nu' + \eta_2^3 = 4\nu' = 0$ and so g extends to $\tilde{g} \colon Y \to S^3$. Note that the composite $S^3 \hookrightarrow S^3 \vee S^6 \vee P^6(2) \xrightarrow{g} S^3$ is a homotopy equivalence and Y is a suspension. The space S^3 is a retract of Y and hence the result. □

Notation 6.26. Let $\bar{L}_3$ denote $\mathrm{hocolim}_{\beta_3} \Sigma(\mathbb{RP}^4_2)^{(3)}$, where β_3 is the composite

$$\Sigma X^{(3)} \xrightarrow{\mathrm{id} - \Sigma T_{12}} \Sigma X^{(3)} \xrightarrow{\mathrm{id} - \Sigma\sigma} \Sigma X^{(3)}$$

with $T_{12}(x_1 \wedge x_2 \wedge x_3) = (x_2 \wedge x_1 \wedge x_3)$ and $\sigma(x_1 \wedge x_2 \wedge x_3) = x_3 \wedge x_1 \wedge x_2$. Then $\Sigma(\Sigma\mathbb{RP}^4_2)^{(3)} \simeq \bar{L}_3 \vee M_3$, see Chapter 3. Let L_3 denote $\mathrm{sk}_{11}(\Omega(\bar{L}_3))$.

Lemma 6.27. *With the notations above, we have*

1) $\Sigma L_3 \simeq \bar{L}_3$.
2) L_3 *is a retract of* $(\mathbb{RP}^4_2)^{(3)}$.

Proof. The map $L_3 \hookrightarrow \Omega\bar{L}_3$ induces an H-map $\Omega\Sigma L_3 \to \Omega\bar{L}_3$. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} \Omega\Sigma L_3 & \longrightarrow & * & \longrightarrow & \Sigma L_3 \\ \downarrow & & \downarrow & & \downarrow \\ \Omega\bar{L}_3 & \longrightarrow & * & \longrightarrow & \bar{L}_3. \end{array}$$

Since $\bar{L}_3$ is 8-connected and $\dim \bar{L}_3 = 12$, the homology suspension

$$\sigma|_{\bar{H}_*(L_3)} \colon \bar{H}_*(L_3) \longrightarrow \bar{H}_{*+1}(\bar{L}_3)$$

is an isomorphism and hence assertion 1.
(2). Note that $J((\mathbb{RP}^4_2)^{(3)})/(\mathbb{RP}^4_2)^{(3)}$ is 11-connected. The composite

$$L_3 \longrightarrow \Omega\bar{L}_3 \longrightarrow J((\mathbb{RP}^4_2)^{(3)})$$

maps into the subcomplex $(\mathbb{R}\mathrm{P}^4_2)^{(3)}$ up to homotopy. Since $\Omega\bar{L}_3/L_3 \simeq \Omega\Sigma L_3/L_3$ is 13-connected, $(\mathbb{R}\mathrm{P}^4_2)^{(3)} \to \Omega\bar{L}_3$ maps into L_3 up to homotopy. Thus there is a commutative diagram

$$\begin{array}{ccccc}
\Omega\bar{L}_3 & \hookrightarrow & \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(3)} & \xrightarrow{\Omega\,\mathrm{hocolim}\,\beta_3} & \Omega\bar{L}_3 \\
\uparrow & & \uparrow & & \uparrow \\
L_3 & \longrightarrow & (\mathbb{R}\mathrm{P}^4_2)^{(3)} & \longrightarrow & L_3
\end{array}$$

and hence the result. □

Proposition 6.28. *There is a map which is a homotopy equivalence through* 11-*skeleton*

$$\phi\colon S^3 \times \Omega\Sigma Y^7 \times \Omega\Sigma L_3 \times P^{11}(2) \times \Sigma^6\mathbb{R}\mathrm{P}^5_3 \times S^{11} \to \Omega B.$$

Proof. The idea is to construct a map ϕ and show that ϕ induces an isomorphism in homology through dimension 11. The construction of the map ϕ is given by several steps.

Step 1. Let $\phi_1\colon \Omega\Sigma L_3 \to \Omega B$ be the composite

$$\Omega\Sigma L_3 \longrightarrow \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(3)} \xrightarrow{\tilde{S}_3} \Omega B,$$

where $\tilde{S}_3$ is the H-map induced by the Samelson product $S_3\colon (\mathbb{R}\mathrm{P}^4_2)^{(3)} \to \Omega B$. We check the homology information. Observe that $\bar{H}_*(L_3) = PH_*(L_3)$ has a basis $\alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \beta_8, \beta_9, \beta_{10}$ and β_{11} with the Steenrod operations given as follows:

$$Sq^4_*\beta_{11} = \alpha_7, Sq^3_*\beta_{11} = \beta_8, Sq^2_*\beta_{11} = \beta_9, Sq^1_*\beta_{11} = \beta_{10},$$

$$Sq^3_*\alpha_{10} = \alpha_7, Sq^2_*\alpha_{10} = \alpha_8, Sq^1_*\alpha_{10} = \alpha_9, Sq^2_*\alpha_9 = \alpha_8 \quad \text{and} \quad Sq^1_*\alpha_8 = \alpha_7.$$

Note that $H_7(\Omega(\mathbb{R}\mathrm{P}^4_2)^{(3)}) \cong H_7(\mathbb{R}\mathrm{P}^4_2)^{(3)}$ has a basis

$$\{x_2 \otimes x_2 \otimes u_3, x_2 \otimes u_3 \otimes x_2, u_3 \otimes x_2 \otimes x_2\},$$

where both x_2 and u_3 are primitive. Thus $\tilde{S}_3\colon H_7(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2) \to H_7(\Omega B)$ is the Lie bracket and $\phi_{1*}(\alpha_7) = [u_3, x_2^2]$. Now

$$\phi_{1*}(\beta_{11}) = \overline{\mathrm{ad}}^2(v_4)(u_3) + l_1[[x_2, u_3], [x_2, v_4]] + l_2[[[x_2, v_4], x_2], u_3] +$$

$$l_3\,\mathrm{ad}^2(x_2)(\overline{\mathrm{ad}}(v_4)(u_3)) + l_4\,\mathrm{ad}^4(x_2)(u_3) + l_5\,\mathrm{ad}^3(u_3)(x_2),$$

$$\phi_{1*}(\beta_8) = Sq^3_*\phi_{1*}(\beta_{11}) = [x_2, u_3],$$

$$\phi_{1*}(\beta_9) = Sq^2_*\phi_{1*}(\beta_{11} = [[x_2, v_4], u_3] + l_3\,\mathrm{ad}^3(x_2)(u_3),$$

$$\phi_{1*}(\beta_{10}) = Sq^1_*\phi_{1*}(\beta_{11}) = [\overline{\mathrm{ad}}(v_4), u_3] + l_2[[u_3, x_2^2], u_3],$$

$$\phi_{1*}(\alpha_{10}) = \overline{\mathrm{ad}}^2(v_4)(x_2) + k_1[x_2, u_3]^2 + k_2[\overline{\mathrm{ad}}(v_4)(u_3), u_3] + k_3[[u_3, x_2^2], u_3] + k_4\,\mathrm{ad}^2(x_2)([v_4, x_2]),$$

$$\phi_{1*}(\alpha_8) = [v_4, x_2^2] + [x_2, u_3^2] + k_2[x_2, u_3^2],$$

$$\phi_{1*}(\alpha_9) = [\overline{\mathrm{ad}}(v_4)(u_3), x_2] + k_4\,\mathrm{ad}^3(x_2)(u_3).$$

Step 2. Let $\phi_2\colon \mathrm{sk}_8(L_3) \wedge S^3 \to \Omega B$ be the Samelson product of

$$\phi_1|_{\mathrm{sk}_8(L_3)}\colon \mathrm{sk}_8(L_3) \longrightarrow \Omega B$$

and $u_3\colon S^3 \to \Omega B$. Then

$$\begin{aligned} \phi_{2*}(\alpha_7 \otimes \iota_3) &= [[u_3, x_2^2], u_3], \\ \phi_{2*}(\alpha_8 \otimes \iota_3) &= [[v_4, x_2^2], u_3] + (k_2+1)[[x_2, u_3^2], u_3], \\ \phi_{2*}(\beta_8 \otimes \iota_3) &= [[x_2, u_3^2], u_3]. \end{aligned}$$

Step 3. We give a decomposition of $\mathrm{sk}_9(L_3)$ in this step. Observe that $\bar{H}_*(\mathrm{sk}_8(L_3))$ has a basis $\alpha_7, \alpha_8, \beta_8$ with $Sq^1_*\alpha_8 = \alpha_7$. Thus

$$\mathrm{sk}_8(L_3) \simeq S^8 \vee P^8(2).$$

Note that $\bar{H}_*(\mathrm{sk}_9(L_3))$ has a basis $\{\alpha_7, \alpha_8, \alpha_9, \beta_8, \beta_9\}$ with $Sq^2_*(\alpha_9) = \alpha_7, Sq^1_*\alpha_8 = \alpha_7$ and $Sq^1_*\beta_9 = \beta_8$. There is a cofibre sequence

$$S^8 \vee S^8 \xrightarrow{\psi} S^8 \vee P^8(2) \longrightarrow \mathrm{sk}_9(L_3) \longrightarrow S^9 \vee S^9.$$

Let ψ_1 be the composite $S^8 \vee S^8 \overset{\psi}{\hookrightarrow} S^8 \vee P^8(2) \xrightarrow{\text{proj.}} P^8(2)$. Recall that

$$[S^8 \vee S^8, P^8(2)] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

We have $[\psi_1] = \epsilon_1\bar{\eta}^{(1)} + \epsilon_2\bar{\eta}^{(2)}$ for some $\epsilon_i = 0, 1$. Since $Sq^2_* \neq 0$ in $H_*(\mathrm{sk}_9(L_3))$, we have $(\epsilon_1, \epsilon_2) \neq (0, 0)$, that is, $[\psi_1] = \bar{\eta}^{(1)}, \bar{\eta}^{(2)}$ or $\bar{\eta}^{(1)} + \bar{\eta}^{(2)}$. Let $i\colon S^8 \to S^8 \vee S^8$ be $\iota_8^{(2)}, \iota_8^{(1)}$ or $\iota_8^{(1)} + \iota_8^{(2)}$ for the cases where $[\psi_1] = \bar{\eta}^{(1)}, \bar{\eta}^{(2)}$ or $\bar{\eta}^{(1)} + \bar{\eta}^{(2)}$, respectively. Then $\psi_1 \circ i$ is null homotopic and so there is a homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccc} S^8 & \xrightarrow{[k]} & S^8 & \longrightarrow & P^9(k) \\ \downarrow{\scriptstyle i} & & \downarrow & & \downarrow{\scriptstyle j} \\ S^8 \vee S^8 & \xrightarrow{\psi} & S^8 \vee P^8(2) & \longrightarrow & \mathrm{sk}_9(L_3) \\ \downarrow & & \downarrow{\scriptstyle \text{proj.}} & & \downarrow \\ S^8 & \xrightarrow{\bar{\eta}} & P^8(2) & \longrightarrow & \Sigma^4\mathbb{R}\mathrm{P}^5_3. \end{array}$$

Let v be a basis for $H_9(P^9(k))$. Then $Sq^2_* j_*(v) = 0$ and $j_*(v) = \beta_9$. Thus $Sq^1_*(v) \neq 0$ and so $k = 2l$ with $l \equiv 1 \mod 2$. Consider the cofibre sequence

$$P^9(2) \xrightarrow{j} \mathrm{sk}_9(L_3) \longrightarrow \Sigma^4\mathbb{R}\mathrm{P}^5_3 \xrightarrow{\theta} P^{10}(2).$$

There is a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma^4\mathbb{R}\mathrm{P}^5_3 & \xrightarrow{\theta} & P^{10}(2) \\ \downarrow{\scriptstyle \text{pinch}} & & \uparrow{\scriptstyle \theta'} \\ S^9 & = & S^9. \end{array}$$

Note that $\theta_*\colon \bar{H}_*(\Sigma\mathbb{R}\mathrm{P}^5_3) \longrightarrow \bar{H}_*(P^{10}(2))$ is zero. Thus θ' is null homotopic and so is θ. Observe that the 9-skeleton $\mathrm{sk}_9(L_3)$ is the 14-skeleton of the homotopy fibre of θ and $\mathrm{sk}_9(L_3)$ is a suspension. We have $\mathrm{sk}_9(L_3) \simeq P^9(2) \vee \Sigma^4\mathbb{R}\mathrm{P}^5_3$.

Step 4. Let $\bar{\phi}_3\colon \Sigma^4\mathbb{R}\mathrm{P}^5_3 \wedge S^2 \to \Omega\Sigma\mathbb{R}\mathrm{P}^4_2$ be Samelson product of the map $x_2\colon S^2 \to \Omega\Sigma\mathbb{R}\mathrm{P}^4_2$ and the composite

$$\Sigma^4\mathbb{R}\mathrm{P}^5_3 \hookrightarrow \mathrm{sk}_9(L_3) \hookrightarrow L_3 \xrightarrow{\phi_1} \Omega B \longrightarrow \Omega\Sigma\mathbb{R}\mathrm{P}^4_2.$$

Then $\bar{\phi}_3$ lifts to ΩB. Let $\phi_3\colon \Sigma^4\mathbb{R}\mathrm{P}^5_3 \wedge S^2 \longrightarrow \Omega B$ be a lifting of $\bar{\phi}_3$. Let x_7, x_8, x_9 be a basis for $\bar{H}_*(\Sigma^4\mathbb{R}\mathrm{P}^5_3)$. Let $f\colon \Sigma^4\mathbb{R}\mathrm{P}^5_3 \to L_3$ be the inclusion. Then $f_*(x_7) = \alpha_7$, $f_*(x_9) = \alpha_9$ and $f_*(x_8) = \alpha_8 + \epsilon\beta_8$ for some $\epsilon = 0, 1$. Then

$$\phi_{3*}(x_7 \otimes \iota_2) = [\phi_{1*}(\alpha_7), x_2] = \mathrm{ad}^3(x_2)(u_3),$$

$$\phi_{3*}(x_8 \otimes \iota_2) = [\phi_{1*}(\alpha_8), x_2] + \epsilon[\phi_{1*}(\beta_8), x_2] = [[v_4, x_2^2], x_2] + (1 + \epsilon + k_2)[[x_2, u_3^2], x_2],$$

$$\phi_{3*}(x_9 \otimes \iota_2) = [\phi_{1*}(\alpha_9), x_2][[\overline{\mathrm{ad}}(v_4)(u_3), x_2], x_2] + k_4\,\mathrm{ad}^4(x_2)(u_3).$$

Note that $\mathrm{ad}^4(x_2)(u_3)$ is spherical.

Step 5. Let $\phi_4\colon S^{11} \longrightarrow \Omega B$ be a map such that $\phi_{4*}(\iota_{11}) = \mathrm{ad}^4(x_2)(u_3)$ and let $\pi_{-1}\colon S^3 \to \Omega B$ be a map such that $\phi_{-1*}(\iota_3) = u_3$. Let $\phi_0\colon \Omega\Sigma Y^7 \to \Omega B$ be the H-map induced by the inclusion $Y^7 \to \Omega B$. We obtain a map

$$\phi\colon S^3 \times \Omega\Sigma Y^7 \times \Omega\Sigma L_3 \times P^{11}(2) \times \Sigma^6\mathbb{R}\mathrm{P}^5_3 \times S^{11} \longrightarrow \Omega B$$

by taking the product of the maps ϕ_j for $-1 \le j \le 4$. Now we show that ϕ induces an isomorphism in homology. Let Z denote $S^3 \times \Omega\Sigma Y^7 \times \Omega\Sigma L_3 \times P^{11}(2) \times \Sigma^6\mathbb{R}\mathrm{P}^5_3 \times S^{11}$. Let V be the image of $\phi_*\colon PH_*(Z) \to PH_*(\Omega B)$. Then Clearly $V_r = PH_r(\Omega B)$ for $r \le 8$. Since V_9 is generated by

$$[[x_2, v_4], u_3] + l_3\,\mathrm{ad}^3(x_2)(u_3), [[\overline{\mathrm{ad}}(v_4)(u_3), x_2] + k_4\,\mathrm{ad}^3(x_2)(u_3), \mathrm{ad}^3(x_2)(u_3),$$

we have $V_9 = PH_9(\Omega B)$. Observe that V_{10} is generated by

$$[x_2, u_3]^2, [\overline{\mathrm{ad}}(v_4)(u_3), u_3] + l_2[[u_3, x_2^2], u_3],$$

$$\overline{\mathrm{ad}}^2(v_4)(x_2) + k_1[x_2, u_3]^2 + k_2[\overline{\mathrm{ad}}(v_4)(u_3), u_3] + k_3[[u_3, x_2^2], u_3] + k_4\,\mathrm{ad}^2(x_2)([v_4, x_2]),$$

$$[[u_3, x_2^2], u_3], [[v_4, x_2^2], x_2] + (1 + \epsilon + k_2)[[x_2, u_3^2], x_2].$$

Thus $V_{10} = PH_{10}(\Omega B)$. Now V_{11} is generated by

$$[[x_2, u_3], [x_2, v_4]],$$

$$\overline{\mathrm{ad}}^2(v_4)(u_3) + l_1[[x_2, u_3], [x_2, v_4] + l_2[[x_2, v_4], x_2], u_3] + l_3\,\mathrm{ad}^2(x_2)\overline{\mathrm{ad}}(v_4)(u_3) +$$

$$+ l_4\,\mathrm{ad}^4(x_2)(u_3) + l_5\,\mathrm{ad}^3(u_3)(x_2),$$

$$[[x_2, u_3^2], u_3], \mathrm{ad}^2(x_2)\overline{\mathrm{ad}}(v_4)(u_3) + k_4\,\mathrm{ad}^4(x_2)(u_3),$$

$$[[v_4, x_2^2], u_3] + (k_2 + 1)[[x_2, u_3^2], u_3], \mathrm{ad}^4(x_2)(u_3).$$

Thus $V_{11} = PH_{11}(\Omega B)$. Note that $\dim PH_r(Z) = \dim PH_r(\Omega B)$ for $r \le 11$. Thus $\phi_*\colon PH_*(Z) \to PH_*(\Omega B)$ is an isomorphism and so $\phi_*\colon H_r(Z) \to H_r(\Omega B)$ is a monomorphism for $r \le 11$. Since $\dim H_r(Z) = \dim H_r(\Omega B)$ for $r \le 11$, the map $\phi_*\colon H_r(Z) \to H_r(\Omega B)$ is an isomorphism for $r \le 11$. We finish the proof now. $\square$

Corollary 6.29. *There is a decomposition*

$$\pi_r(B)\cong\pi_{r-1}(S^3)\oplus\pi_r(\Sigma Y^7)\oplus\pi_r(\Sigma L_3)\oplus\pi_r(P^{12}(2))\oplus\pi_r(\Sigma^7\mathbb{R}\mathrm{P}^5_3)$$

for $r\leq 11$.

Observe that ΣY^7 is 5-connected.

Proposition 6.30. *The homotopy groups* $\pi_r(\Sigma Y^7)$ *for* $6\leq r\leq 11$ *are as follows.*

1) $\pi_6(\Sigma Y^7)=\mathbb{Z}/2$.
2) $\pi_7(\Sigma Y^7)=\mathbb{Z}/8$.
3) $\pi_8(\Sigma Y^7)=\mathbb{Z}/2\oplus\mathbb{Z}/2$.
4) $\pi_9(\Sigma Y^7)=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$.
5) $\pi_{10}(\Sigma Y^7)=\mathbb{Z}/4\oplus\mathbb{Z}/4$.
6) $\pi_{11}(\Sigma Y^7)=\mathbb{Z}/2\oplus\mathbb{Z}/4$.

Proof. **1.** $\pi_r(\Sigma Y^7)$ for $r\leq 9$: Observe that $\bar{H}_*(Y^7)$ has a basis $[x_2,u_3], u_3^2, [x_2,v_4]$ and $\overline{\mathrm{ad}}(v_4)(u_3)$ with

$$Sq^1_*[x_2,v_4]=[x_2,u_3], Sq^2_*\overline{\mathrm{ad}}(v_4)(u_3)=[x_2,u_3]\quad\text{and}\quad\beta_2\overline{\mathrm{ad}}(v_4)(u_3)=u_3^2.$$

Thus Y^7 is the homotopy cofibre of $(4,\bar{\eta})\colon S^6\longrightarrow S^6\vee P^6(2)$ and so there is a cofibre sequence

$$S^7\xrightarrow{(4,\bar{\eta})}S^7\vee P^7(2)\longrightarrow\Sigma Y^7\longrightarrow S^8.$$

Since $S^7\vee P^7(2)$ is the 12-skeleton of the homotopy fibre of the pinch map $\Sigma Y^7\to S^8$, the sequence

$$\pi_r(S^7)\xrightarrow{(4,\bar{\eta})_*}\pi_r(S^7\vee P^7(2))\longrightarrow\pi_r(\Sigma Y^7)\longrightarrow\pi_{r-1}(S^7)\longrightarrow\cdots$$

is exact for $r\leq 11$. Thus $\pi_6(\Sigma Y^7)\cong\pi_6(P^7(2))=\mathbb{Z}/2$. By expecting the exact sequence $\pi_7(S^7)\xrightarrow{(4,\bar{\eta})_*}\pi_7(S^7)\oplus\pi_7(P^7(2))\pi_7(\Sigma Y^7)\longrightarrow 0$, we have $\pi_7(\Sigma Y^7)\cong\mathbb{Z}/8$ and there is an exact sequence

$$\pi_8(S^7)\xrightarrow{(4,\bar{\eta})_*}\pi_8(S^7)\oplus\pi_8(P^7(2))\longrightarrow\pi_8(\Sigma Y^7)\longrightarrow 0.$$

Observe that $(4,\bar{\eta})_*(\eta_7)=2\alpha$, where α is a generator for $\pi_8(P^7(2))\cong\mathbb{Z}/4$. We obtain $\pi_8(\Sigma Y^7)\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$ and there is an exact sequence

$$\pi_9(S^7)\xrightarrow{(4,\bar{\eta})_*}\pi_9(S^7)\oplus\pi_9(P^7(2))\longrightarrow\pi_9(\Sigma Y^7)\longrightarrow 0.$$

Since $(4,\bar{\eta})_*\colon\pi_9(S^7)\to\pi_9(S^7)\oplus\pi_9(P^7(2))$ is zero, we have

$$\pi_9(\Sigma Y^7)\cong\pi_9(S^7)\oplus\pi_9(P^7(2))\cong\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2.$$

2. $\pi_{10}(\Sigma Y^7)$: Now there is an exact sequence

$$\pi_{10}(S^7)\xrightarrow{(4,\bar{\eta})_*}\pi_{10}(S^7)\oplus\pi_{10}(P^7(2))\longrightarrow\pi_{10}(\Sigma Y^7)\longrightarrow\pi_9(S^7)\longrightarrow 0,$$

where $(4,\bar{\eta})_*(\nu_7)=4\nu_7+\bar{\eta}_*(\nu_7)=4\nu_7+j_*(\eta_6\circ\nu_7)=4\nu_7$ and $j\colon S^6\to S^7$ is the inclusion of bottom cell. Thus there is a short exact sequence

$$0\longrightarrow\pi_{10}(S^7)/4\oplus\pi_{10}(P^7(2))\longrightarrow\pi_{10}(\Sigma Y^7)\longrightarrow\pi_9(S^7)\longrightarrow 0$$

Note that $\pi_{10}(P^7(2))\cong\pi^s_{10}(P^7(2))\cong\mathbb{Z}/2$. There is a short exact sequence

$$0\longrightarrow\mathbb{Z}/4\oplus\mathbb{Z}/2\longrightarrow\pi_{10}(\Sigma Y^7)\longrightarrow\mathbb{Z}/2\longrightarrow 0.$$

We need to solve the group extension problem. Consider the homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccccc}
P^7(2) & \xrightarrow{[2]} & P^7(2) & \xrightarrow{j_1} & P^6(2)\wedge P^2(2) & \xrightarrow{p_1} & P^8(2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p_2} \\
S^7 & \xrightarrow{(4,\bar\eta)} & S^7\vee P^7(2) & \xrightarrow{j_2} & \Sigma Y^7 & \longrightarrow & S^8 \\
\downarrow{\scriptstyle 2} & & \downarrow & & \downarrow & & \| \\
S^7 & \xrightarrow{[2]} & S^7 & \longrightarrow & P^8(2) & \longrightarrow & S^8.
\end{array}$$

Let $\alpha\in\pi_{10}(P^6(2)\wedge P^2(2))$ such that $p_{2*}\circ p_{1*}(\alpha)=\eta_8^2$. Then 2α is the composite

$$S^{10}\xrightarrow{\alpha}P^6(2)\wedge P^2(2)\xrightarrow{p_1}P^8(2)\xrightarrow{\eta\wedge\mathrm{id}}P^7(2)\xrightarrow{j_1}P^6(2)\wedge P^2(2).$$

Thus $2\alpha=j_{1*}(\bar\eta^3)$, where $\bar\eta^3\in\pi_{10}(P^7(2))\cong\mathbb{Z}/2$ is the generator. Hence $j_{2*}(\bar\eta^3)$ is also divisible by 2 in $\pi_{10}(\Sigma Y^7)$ and so $\pi_{10}(\Sigma Y^7)\cong\mathbb{Z}/4\oplus\mathbb{Z}/4$.

3. $\pi_{11}(\Sigma Y^7)$: Consider the exact sequence

$$\pi_{11}(S^7)=0\longrightarrow\pi_{11}(S^7)\oplus\pi_{11}(P^7(2))\longrightarrow\pi_{11}(\Sigma Y^7)\longrightarrow 2\cdot\pi_{10}S^7\longrightarrow 0.$$

There is a short exact sequence $0\longrightarrow\pi_{11}(P^7(2))\longrightarrow\pi_{11}(\Sigma Y^7)\longrightarrow 2{\cdot}\pi_{10}(S^7)\longrightarrow 0$ or $0\longrightarrow\mathbb{Z}/2\longrightarrow\pi_{11}(\Sigma Y^7)\longrightarrow\mathbb{Z}/4\longrightarrow 0$. Thus $\pi_{11}(\Sigma Y^7)\cong\mathbb{Z}/2\oplus\mathbb{Z}/4$ or $\mathbb{Z}/8$. We show that $\pi_{11}(\Sigma Y^7)\neq\mathbb{Z}/8$. From this, we will obtain $\pi_{11}(\Sigma Y^7)\cong\mathbb{Z}/2\oplus\mathbb{Z}/4$. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc}
P^8(2) & \xrightarrow{\eta\wedge\mathrm{id}} & P^7(2) & \longrightarrow & \mathbb{C}\mathrm{P}^2\wedge P^5(2) & \longrightarrow & P^9(2) \\
\uparrow & & \| & & \uparrow & & \uparrow \\
S^7 & \xrightarrow{\bar\eta} & P^7(2) & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}^5_3 & \longrightarrow & S^8 \\
\| & & \uparrow{\scriptstyle \mathrm{proj.}} & & \uparrow & & \| \\
S^7 & \xrightarrow{(4,\bar\eta)} & S^7\vee P^7(2) & \xrightarrow{j_2} & \Sigma Y^7 & \longrightarrow & S^8,
\end{array}$$

where the rows are cofibre sequences. Observe that $P^7(2)$ is the 12-skeleton of the homotopy fibre of the pinch map $\mathbb{C}\mathrm{P}^2\wedge P^5(2)\longrightarrow P^9(2)$. There is an exact sequence

$$\pi_{11}(P^8(2))\xrightarrow{\eta\wedge\mathrm{id}_*}\pi_{11}(P^7(2))\longrightarrow\pi_{11}(\mathbb{C}\mathrm{P}^2\wedge P^5(2))\longrightarrow\pi_{11}(P^9(2)).$$

Let α_n be a generator for $\pi_{n+1}(P^n(2))\cong\mathbb{Z}/4$ for $n\geq 3$. Note that $\pi_{11}(P^8(2))=\mathbb{Z}/2$ and $p_*(\alpha_8\circ\eta^2)=\eta^3=4\nu$ in $\pi_{11}(S^8)$, where $p: P^8(2)\to S^8$ is the pinch map. Thus $\pi_{11}(P^8(2))$ is generated by $\alpha_8\circ\eta^2$. Also note that $\pi_9(P^7(2))\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$ generated by $\alpha_7 \circ \eta$ and $j_*(\nu_6)$, where $j\colon S^6 \to P^7(2)$ the inclusion, and $p_*((\eta \wedge \mathrm{id}) \circ \alpha_8) = \eta_7^2$. We have

$$(\eta \wedge \mathrm{id}) \circ \alpha_8 = \alpha_7 \circ \eta + \epsilon \cdot j_*(\nu_6) \quad \text{and}$$

$$(\eta \wedge \mathrm{id}) \circ \alpha_8 \circ \eta = \alpha_7 \circ \eta^2 + \epsilon j_*(\nu_6 \circ \eta) = \alpha_7 \circ \eta^2$$

for some $\epsilon = 0, 1$. Thus

$$(\eta \wedge \mathrm{id}) \circ \alpha_8 \circ \eta^2 = \alpha_7 \circ \eta^3 = \alpha_7 \circ 4 \circ \nu_8 = 0.$$

Hence $(\eta \wedge \mathrm{id})_* : \pi_{11}(P^8(2)) \to \pi_{11}(P^7(2))$ is zero and so there is an exact sequence

$$0 \longrightarrow \pi_{11}(P^7(2)) = \mathbb{Z}/2 \longrightarrow \pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^5(2)) \longrightarrow \pi_{11}(P^9(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Now suppose that $\pi_{11}(\Sigma Y^7) = \mathbb{Z}/8$. Then $j_{2*}(\beta)$ is divisible by 4, where $\beta \in \pi_{11}(P^7(2))$ the generator. From the commutative diagram

$$\begin{array}{ccc} \pi_{11}(P^7(2)) & \longrightarrow & \pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^5(2)) \\ \| & & \uparrow f_* \\ \pi_{11}(P^7(2)) & \xrightarrow{j_{2*}} & \pi_{11}(\Sigma Y^7), \end{array}$$

the element $f_* \circ (j_2)_*(\beta)$ is divisible by 4 in $\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^5(2))$. By the exact sequence above, $4\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^5(2)) = 0$. This is a contradiction. Thus $\pi_{11}(\Sigma Y^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/4$. This completes the calculation. □

Now we compute $\pi_*(\Sigma L_3)$. Observe that ΣL_3 is 7-connected.

Proposition 6.31. *The homotopy groups $\pi_r(\Sigma L_3)$ for $8 \leq r \leq 11$ are as follows.*

1) $\pi_8(\Sigma L_3) = \mathbb{Z}/2$.
2) $\pi_9(\Sigma L_3) = \mathbb{Z}/2$.
3) $\pi_{10}(\Sigma L_3) = \mathbb{Z}/2$.
4) $\pi_{11}(\Sigma L_3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof. **1.** $\pi_8(\Sigma L_3)$: By Step 3 of Proposition 6.28, $\mathrm{sk}_{10}(\Sigma L_3) \simeq P^{10}(2) \vee \Sigma^5 \mathbb{R}\mathrm{P}^5_3$. Thus $\pi_8(\Sigma L_3) \cong \pi_8(\Sigma^5 \mathbb{R}\mathrm{P}^5_3) \cong \mathbb{Z}/2$.
2. $\pi_9(\Sigma L_3)$: Consider the exact sequence

$$\pi_9(S^9) \xrightarrow{\bar{\eta}_*} \pi_9(P^9(2)) \longrightarrow \pi_9(\Sigma^5 \mathbb{R}\mathrm{P}^5_3) \longrightarrow 0.$$

We have $\pi_9(\Sigma \mathbb{R}\mathrm{P}^5_3) = 0$ and so $\pi_9(\Sigma L_3) = \mathbb{Z}/2$.
3. A decomposition of $\mathrm{sk}_{11}(\Sigma L_3)$: There is a cofibre sequence

$$S^{10} \vee S^{10} \xrightarrow{\phi} P^{10}(2) \vee \Sigma^5 \mathbb{R}\mathrm{P}^5_3 \longrightarrow \mathrm{sk}_{11}(\Sigma L_3).$$

Let ϕ_1 be the composite $S^{10} \vee S^{10} \xrightarrow{\phi} P^{10}(2) \vee \Sigma \mathbb{R}\mathrm{P}^5_3 \xrightarrow{\text{proj.}} P^{10}(2)$. Note that $[S^{10} \vee S^{10}, P^{10}(2)] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $P^{10}(2)$ is not a retract of $\mathrm{sk}_{11}(\Sigma L_3)$. There exists an inclusion $i : S^{10} \to S^{10} \vee S^{10}$ such that $i_*\colon H_{10}(S^{10}) \to H_{10}(S^{10} \vee S^{10})$ is a monomorphism and $\phi_1 \circ i$ is null homotopic. Thus there is a homotopy commutative diagram

$$\begin{array}{ccccccc}
S^{10} & \xrightarrow{\theta} & \Sigma^5\mathbb{R}\mathrm{P}^5_3 & \xrightarrow{q} & A & \xleftarrow{\psi} & S^{10} \\
\downarrow{\scriptstyle i} & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \bar\eta} \\
S^{10}\vee S^{10} & \xrightarrow{\phi} & P^{10}(2)\vee\Sigma^5\mathbb{R}\mathrm{P}^5_3 & \xrightarrow{\theta} & \mathrm{sk}_{11}(\Sigma L_3) & \xleftarrow{\theta|_{P^{10}(2)}} & P^{10}(2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{10} & \xrightarrow{\bar\eta} & P^{10}(2) & \longrightarrow & \Sigma^6\mathbb{R}\mathrm{P}^5_3 & = & \Sigma^6\mathbb{R}\mathrm{P}^5_3,
\end{array}$$

where the columns are cofibre sequences and the rows except the terms in the last column are cofibre sequences. We claim that ψ is null.

First we compute $\pi_{10}(A)$. Consider the exact sequence

$$\pi_{10}(S^9) \xrightarrow{\bar\eta_*} \pi_{10}(P^9(2)) \xrightarrow{j_*} \pi_{10}(\Sigma^5\mathbb{R}\mathrm{P}^5_3) \xrightarrow{p_*} \pi_9(S^9) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Note that $\bar\eta_*(\eta_9) = 2\alpha_9$. We have

$$\pi_{10}(\Sigma^5\mathbb{R}\mathrm{P}^5_3) = \{j_*(\alpha_9)\} \oplus \{\bar\iota_9\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}$$

with $p_*(\bar\iota_9) = 2\iota_9$. Observe that $g_*\colon \bar H_*(A) \to \bar H_*(\mathrm{sk}_{11}(\Sigma L_3))$ is a monomorphism. Thus $\bar H_*(A)$ has a basis x_8, x_9, x_{10}, x_{11} with $Sq^1_* x_{11} = x_{10}$, $Sq^2_* x_{11} = x_9$, $Sq^1_* x_9 = x_8$ and $Sq^2_* x_{10} = x_8$. The composite

$$S^{10} \xrightarrow{\theta} \Sigma^5\mathbb{R}\mathrm{P}^5_3 \xrightarrow{\text{pinch}} S^{10}$$

is of degree ± 2. Thus

$$[\theta] = \pm\bar\iota_9 + \epsilon j_*(\alpha_9)$$

in $\pi_{10}(\Sigma^5\mathbb{R}\mathrm{P}^5_3)$ for some $\epsilon = 0, 1$. Consider the exact sequence

$$\pi_{10}(S^{10}) \xrightarrow{\theta_*} \pi_{10}(\Sigma^5\mathbb{R}\mathrm{P}^5_3) \longrightarrow \pi_{10}(A) \longrightarrow 0$$

with $\theta_*(\iota_9) = \pm\bar\iota_9 + \epsilon j_*(\alpha_9)$. Thus $\pi_{10}(A) = \mathbb{Z}/2$ generated by $q \circ j_*(\alpha_9)$.

Now suppose that $\psi\colon S^{10} \to A$ is essential. Then $[\psi] = q \circ j_*(\alpha_9)$ and ψ maps into the subcomplex $P^9(2)$ of A up to homotopy. Thus there is a homotopy commutative diagram

$$
\begin{array}{ccccccc}
S^{10} & \xrightarrow{\alpha_9} & P^9(2) & \longrightarrow & C' & \longrightarrow & S^{11} \\
\| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} & & \| \\
S^{10} & \xrightarrow{\psi} & A & \longrightarrow & C & \longrightarrow & S^{11} \\
\downarrow{\scriptstyle \bar{\eta}} & & \downarrow & & \| & & \\
P^{10}(2) & \xrightarrow{j'=\theta|_{P^{10}(2)}} & \mathrm{sk}_{11}(\Sigma L_3) & \xrightarrow{p'} & C, & &
\end{array}
$$

where $\phi = q \circ j$ and the rows are cofibre sequences. Let u_9 and v_{10} be the basis for $\bar{H}_*(P^{10}(2))$. Note that j'_* is a monomorphism. Let $j'_*(v_{10}) = l_1\alpha_{10} + l_2\beta_{10}$. Since $Sq^2_* v_{10} = 0$, we have $l_1 = 0$ and so $j'_*(v_{10}) = \beta_{10}$. It follows that $j'_*(u_9) = \beta_9$. Now $\bar{H}_*(C)$ has a basis $p'_*(\alpha_8), p'_*(\alpha_9), p'_*(\alpha_{10}), p'_*(\alpha_{11})$ and $p'_*(\beta_{11})$. Observe that $\bar{H}_*(C')$ has a basis $\{y_8, y_9, y_{11}\}$ $Sq^2_* y_{11} = y_9$ and $Sq^1_* y_9 = y_8$. Since $f_*(y_8) = p'_*(\alpha_8)$ in H_*C, we have $f_*(y_9) = p'_*(\alpha_9)$. Let

$$f_*(y_{11}) = k_1 p'_*(\alpha_{11}) + k_2 p'_*(\beta_{11}).$$

Then

$$0 \neq f_*(y_9) = f_*(Sq^2_* y_{11}) = Sq^2_* f_*(y_{11}) = k_1 Sq^2_* p'_*(\alpha_{11}) + k_2 Sq^2_* p'_*(\beta_{11})$$
$$= k_1 p'_*(\alpha_9) + k_2 p'_*(\beta_9) = k_1 p'_*(\alpha_9)$$

Thus $k_1 = 1$ and so $f_*(y_{11}) = p'_*(\alpha_{11}) + k_2 p'_*(\beta_{11})$. It follows that

$$0 = f_*(Sq^1_* y_{11}) = Sq^1_* f_*(y_{11}) = Sq^1_* p'_*(\alpha_{11}) + Sq^1_* k_2 p'_*(\beta_{11}) = p'_*(\alpha_{10}) \neq 0$$

This is a contradiction. Thus $\psi\colon S^{10} \to A$ is null homotopic and so is $j' \circ \bar{\eta}$. Hence $j'\colon P^{10}(2) \longrightarrow \mathrm{sk}_{11}(\Sigma L_3)$ extends to $\tilde{j}\colon \Sigma^6\mathbb{R}\mathrm{P}^5_3 \longrightarrow \mathrm{sk}_{11}(\Sigma L_3)$. Now the composite

$$\Sigma^6\mathbb{R}\mathrm{P}^5_3 \xrightarrow{\tilde{j}} \mathrm{sk}_{11}(\Sigma L_3) \longrightarrow \Sigma^6\mathbb{R}\mathrm{P}^5_3$$

is a homotopy equivalence and therefore $\mathrm{sk}_{11}(\Sigma L_3) \simeq A \vee \Sigma^6\mathbb{R}\mathrm{P}^5_3$.
4. $\pi_{10}(\Sigma L_3)$:

$$\pi_{10}(\mathrm{sk}_{11}(\Sigma L_3)) \cong \pi_{10}(A) \oplus \pi_{10}(\Sigma^6\mathbb{R}\mathrm{P}^5_3) = \mathbb{Z}/2 \oplus 0 = \mathbb{Z}/2.$$

5. $\pi_{11}(\Sigma L_3)$: By expecting the Steenrod operation on $\bar{H}_*(A)$, we have

$$A \simeq \mathbb{C}\mathrm{P}^2 \wedge P^7(2).$$

From the exact sequence

$$\pi_{11}(P^{10}(2)) \xrightarrow{\eta\wedge\mathrm{id}_*} \pi_{11}(P^9(2)) \longrightarrow \pi_{11}(A) \longrightarrow \pi_{10}(P^{10}(2)) \xrightarrow{\eta\wedge\mathrm{id}_*} \pi_{10}(P^9(2)),$$

we have $\pi_{11}(A) = \mathbb{Z}/2$ and $\pi_{11}(P^9(2)) \to \pi_{11}(A)$ is onto. Consider the cofibre sequence

$$S^{11} \xrightarrow{\theta} A \vee \Sigma^6\mathbb{R}\mathrm{P}^5_3 \longrightarrow \Sigma L_3.$$

Since $Sq^4_* \neq 0$ in $\bar{H}_*(\Sigma L_3)$, the composite

$$S^{11} \xrightarrow{\theta} A \vee \Sigma^6\mathbb{R}\mathrm{P}^5_3 \xrightarrow{\text{proj.}} A$$

is a generator for $\pi_{11}(A) = \mathbb{Z}/2$. Note that $Sq^1_*\beta_{11} = \beta_{10}$. The composite

$$S^{11} \xrightarrow{\theta} A \vee \Sigma^6\mathbb{R}\mathrm{P}^5_3 \xrightarrow{\text{proj.}} \Sigma^6\mathbb{R}\mathrm{P}^5_3 \xrightarrow[p]{\text{proj.}} S^{11}$$

is of degree 2. Observe that

$$\pi_{11}(\Sigma^6\mathbb{R}\mathrm{P}^5_3) = \{j_*(\alpha_{10})\} \oplus \{\bar{\iota}_{11}\} = \mathbb{Z}/2 \oplus \mathbb{Z}$$

with $p_*(\bar{\iota}_{11}) = 2\iota_{11}$. There is an exact sequence

$$\mathbb{Z} \xrightarrow{\theta_*} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \longrightarrow \pi_{11}(\Sigma L_3) \longrightarrow 0$$

with the composite

$$\mathbb{Z} \xrightarrow{\theta_*} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \xrightarrow{\text{proj.}} \mathbb{Z}$$

of degree 1. Thus $\pi_{11}(\Sigma L_3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. This completes the calculation. □

Now we are ready to compute $\pi_*(\Sigma\mathbb{R}\mathrm{P}^4_2)$. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc}
\Omega S^5 & \xrightarrow{\delta} & F & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}^4_2 & \longrightarrow & S^5 \\
\uparrow E & & \uparrow i & & \| & & \| \\
S^4 & \xrightarrow{(\eta,[2])} & S^3 \vee S^4 & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}^4_2 & \longrightarrow & S^5
\end{array}$$

where the top row is fibre sequence and the bottom row is cofibre sequence. By Lemmas 6.18 and 6.23 and Proposition 6.28, there is a homotopy commutative diagram

$$\begin{array}{ccc}
\Omega^2 S^5 & \xrightarrow{\delta} & \Omega F \\
\uparrow & & \uparrow \phi \\
 & & \Omega S^3 \times S^3 \times \Omega\Sigma Y^7 \times \Omega\Sigma L_3 \times P^{11}(2) \times \Sigma^6\mathbb{R}\mathrm{P}^5_3 \times S^{11} \\
 & & \uparrow \\
 & & \Omega S^3 \times S^3 \times \Omega\Sigma Y^7 \\
 & & \uparrow \psi \\
\Omega S^4 & \xrightarrow{\Omega(\eta,[2])} & \Omega S^3 \times S^3 \times \Omega S^7 \simeq \Omega S^3 \times \Omega S^4,
\end{array}$$

where $\psi = (\mathrm{id}_{\Omega S^3}, \mathrm{id}_{S^3}, \Omega j)$, ϕ is a homotopy equivalence through the 11-skeleton and $\Omega j_*(\iota_6) = u_3^2$ in $H_*(\Omega\Sigma Y^7)$.

Lemma 6.32. [23, pp.217] *The homotopy group* $\pi_{11}(SU(3)) = \mathbb{Z}/4$.

Theorem 6.33. *The homotopy groups* $\pi_r(\Sigma\mathbb{R}\mathrm{P}^4_2)$ *for* $r \leq 11$ *are as follows.*

1) $\pi_3(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}$.
2) $\pi_4(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/4$.
3) $\pi_5(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/2$.
4) $\pi_6(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
5) $\pi_7(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/8$.
6) $\pi_8(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4$.
7) $\pi_9(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/2^{\oplus 4}$.
8) $\pi_{10}(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.
9) $\pi_{11}(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/2^{\oplus 4} \oplus \mathbb{Z}/4 \oplus \pi_{11}(SU(3)) = \mathbb{Z}/2^{\oplus 4} \oplus \mathbb{Z}/4^{\oplus 2}$.

Proof. **1.** $\pi_r(\Sigma\mathbb{R}\mathrm{P}^4_2)$ for $r \leq 7$: Since L_3 is 6-connected, the sequence

$$\pi_r(S^5) \xrightarrow{\delta_*} \pi_{r-2}(\Omega S^3 \times S^3 \times \Omega\Sigma Y^7) \longrightarrow \pi_{r-1}(\Sigma\mathbb{R}\mathrm{P}^4_2) \longrightarrow \pi_{r-1}(S^5) \longrightarrow \cdots$$

is exact for $r \leq 8$. Hence $\pi_3(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}$. Note that $\delta_*(\iota_5) = \eta_3 + 2\iota_3$. We have $\pi_4(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/4$. Observe that $\delta_*(\eta_5) = \delta_*(E\eta_4) = \eta_3^2$. Thus

$$\pi_5(\Sigma\mathbb{R}\mathrm{P}^4_2) = \mathbb{Z}/2.$$

Note that $\delta_*(\eta_5^2) = \delta_*(E^2\eta_3^2) = \eta_3^3 + 0 + 0 = \eta_3^3$. Thus

$$\pi_6(\Sigma\mathbb{R}\mathrm{P}^4_2) = \pi_6(S^3)/2 \oplus \pi_5(S^3) \oplus \pi_6(\Sigma Y^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Notice that $\delta_*(\nu_5) = \delta_* E\nu_4 = (1_{\Omega S^3}, 1_{S^3}, j)_* \circ \Omega(\eta, [2])_*(\nu_4)$. Now

$$\Omega(\eta, [2])_*(\nu_4) = \Omega\eta_*(\nu_4) + \Omega[2]_*(\nu_4) = \eta_3 \circ \nu_4 + 4\nu_4 - E\nu'.$$

Thus

$$\delta_*(\nu_5) = \eta_3 \circ \nu_4 - \nu' + 4j_*(\iota_7),$$

where $j_*(\iota_7)$ is a generator for $\pi_7(\Sigma Y^7) = \mathbb{Z}/8$. It follows that

$$\pi_7(\Sigma\mathbb{R}\mathrm{P}^4_2) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2,$$

which are generated by $\phi_* \circ j_*(\iota_7)$ and $\phi_*(\nu')$. Furthermore

$$\mathrm{Ker}(\delta_*\colon \pi_8(S^5) \to \pi_6(\Omega S^3 \times S^3 \times \Omega\Sigma Y^7)) = 4 \cdot \pi_8(S^5)$$

because $2\delta_*(\nu_5) = 2\nu' \neq 0$.
2. $\pi_8(\Sigma\mathbb{R}\mathrm{P}^4_2)$: Consider the exact sequence

$$\pi_9(S^5) \xrightarrow{\delta_*} \pi_8(S^3) \oplus \pi_7(S^3) \oplus \pi_8(\Sigma Y^7) \oplus \pi_8(\Sigma L_3) \longrightarrow \pi_8(\Sigma\mathbb{R}\mathrm{P}^4_2) \longrightarrow 4 \cdot \pi_8(S^5) \longrightarrow 0$$

with $\delta_*(\nu_5 \circ \eta_8) = \delta_*(\nu_5) \circ \eta_8 = \eta_3 \circ \nu_4 \circ \eta_7 + \nu' \circ \eta_6 = \nu' \circ \eta_6^2 + \nu' \circ \eta_6 \neq 0$. Recall that $\pi_8(\Sigma Y^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi_8(\Sigma L_3) = \mathbb{Z}/2$. There is a short exact sequence

$$0 \longrightarrow (\pi_8(S^3) \oplus \pi_7(S^3))/\{\nu' \circ \eta_6^2 + \nu' \circ \eta_6\} \oplus \pi_8(\Sigma Y^7) \oplus \pi_8(\Sigma L_3) \longrightarrow \pi_8(\Sigma\mathbb{R}\mathrm{P}^4_2)$$
$$\longrightarrow 4 \cdot \pi_8(S^5) \longrightarrow 0 \quad \text{or} \quad 0 \longrightarrow \mathbb{Z}/2^{\oplus 4} \longrightarrow \pi_8(\Sigma\mathbb{R}\mathrm{P}^4_2) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega^2 F & \longrightarrow & \Omega^2\Sigma\mathbb{R}\mathrm{P}^4_2 & \xrightarrow{p} & \Omega^2 S^5 & \xrightarrow{\delta} & \Omega F \\
\uparrow & & \uparrow e' & & \uparrow E^2 & & \uparrow e'' \\
\Omega^2 S^3\times\Omega S^3 & \longrightarrow & X & \longrightarrow & S^3 & \xrightarrow{(\eta,2)} & \Omega S^3\times S^3.
\end{array}$$

Then the universal cover $X\langle 1\rangle$ is the homotopy fibre of $S^3 \xrightarrow{(\eta,2)} \Omega(S^3\langle 3\rangle)\times S^3$ and so there is a commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega^2(S^3\langle 3\rangle)\times\Omega S^3 & \xrightarrow{i} & X\langle 1\rangle & \xrightarrow{p'} & S^3 & \xrightarrow{(\eta,2)} & \Omega(S^3\langle 3\rangle)\times S^3 \\
\downarrow & & \downarrow f & & \| & & \downarrow \pi \text{ proj.} \\
\Omega^2(S^3\langle 3\rangle) & \longrightarrow & \Omega^2 S^5 & \xrightarrow{\Delta} & S^3 & \xrightarrow{\eta} & \Omega(S^3\langle 3\rangle),
\end{array}$$

where the bottom fibre sequence is the EHP sequence

$$S^3 \longrightarrow \Omega(S^3\langle 3\rangle) \longrightarrow \Omega S^5.$$

Consider the commutative diagram of exact sequence

$$\begin{array}{ccccccccc}
\pi_7(S^3) & \xrightarrow{0} & \pi_8(S^3)\oplus\pi_7(S^3) & \xrightarrow{i_*} & \pi_6(X) & \xrightarrow{p'_*} & \pi_6(S^3) & \xrightarrow{(\bar{\eta}_*,2)} & \pi_7(S^3)\oplus\pi_6(S^3) \\
\| & & \downarrow \pi_* & & \downarrow f_* & & \| & & \downarrow \pi_* \\
\pi_7(S^3) & \xrightarrow{0} & \pi_8(S^3) & \longrightarrow & \pi_8(S^5) & \xrightarrow{\Delta} & \pi_6(S^3) & \xrightarrow{0} & \pi_7(S^3).
\end{array}$$

There is a short exact sequence

$$0 \longrightarrow \pi_8(S^3)\oplus\pi_7(S^3) \longrightarrow \pi_6(X) \xrightarrow{p'_*} 2\cdot\pi_6(S^3) \longrightarrow 0.$$

Let $\alpha\in\pi_6(X)$ such that $p'_*(\alpha)=2\nu'$. Then $f_*(\alpha)=\pm 2\nu_5$ and so $\pi_*(2\alpha)=\nu'\circ\eta_6^2$ or $2\alpha=i_*(\nu'\circ\eta_6^2)+\epsilon i_*(\nu'\circ\eta_6)$ for some $\epsilon=0,1$. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega^2 S^3\times\Omega S^3 & \longrightarrow & X & \longrightarrow & S^3 & \xrightarrow{(\eta,2)} & \Omega S^3\times S^3 \\
\downarrow & & \downarrow g & & \| & & \downarrow \pi' \text{ proj.} \\
\Omega S^3 & \longrightarrow & S^3\{2\} & \xrightarrow{p''} & S^3 & \xrightarrow{2} & S^3.
\end{array}$$

There is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_8(S^3)\oplus\pi_7(S^3) & \xrightarrow{i_*} & \pi_6(X) & \xrightarrow{p'_*} & 2\pi_6(S^3) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \pi'_*} & & \downarrow{\scriptstyle g_*} & & \| & & \\
0 & \longrightarrow & \pi_7(S^3) & \xrightarrow{j_*} & \pi_6(S^3\{2\}) & \xrightarrow{p''_*} & 2\pi_6 S^3 & \longrightarrow & 0
\end{array}$$

with $p''_*(g_*(\alpha)) = 2\nu'$. Recall that $\Omega(S^3\{2\}) \simeq (\Omega S^3)\{2\}$. We have

$$2g_*(\alpha) = j_*((2\nu')\circ\eta) = 0$$

or $g_*(2\alpha) = 0$. Thus $0 = \pi'_*(\nu'\circ\eta_6^2 + \epsilon\nu'\circ\eta_6) = \epsilon\nu'\circ\eta_6$, that is, $\epsilon = 0$ and so

$$2\alpha = i_*(\nu'\circ\eta_6^2).$$

Now consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\pi_9(S^5) & \xrightarrow{\delta_*} & \pi_8(F) & \xrightarrow{\phi_*} & \pi_8(\Sigma\mathbb{RP}^4_2) & \xrightarrow{p_*} & 4\pi_8(S^5) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle e''_*} & & \uparrow{\scriptstyle e'_*} & & \cong\uparrow{\scriptstyle E^2} & & \\
0 & \longrightarrow & \pi_8(S^3)\oplus\pi_7(S^3) & \xrightarrow{i_*} & \pi_6(X) & \xrightarrow{p'_*} & 2\pi_6(S^3) & \longrightarrow & 0
\end{array}$$

Since $p'_*(\alpha) = 2\nu'$, we have $p_*(e'_*(\alpha)) = 4\nu_5$. Since $2\alpha = i_*(\nu'\circ\eta_6^2))$, we have $2e'_*(\alpha) = \phi_*(\nu'\circ\eta_6^2)$. Since $\delta_*(\nu_5\circ\eta_8) = \nu'\circ\eta_6^2 + \nu'\circ\eta_6$, the element $\nu'\circ\eta_6^2$ does not lie in the image of δ_* and so

$$2e'_*(\alpha) = \phi_*(\nu'\circ\eta_6^2) \neq 0.$$

Thus $e'_*(\alpha)$ is of order 4 and so $\pi_8(\Sigma\mathbb{RP}^4_2) \cong \mathbb{Z}/4\oplus\mathbb{Z}/2^{\oplus 3}$.

3. $\pi_9(\Sigma\mathbb{RP}^4_2)$: Consider the exact sequence

$$\pi_{10}(S^5) \xrightarrow{\delta_*} \pi_9(S^3)\oplus\pi_8(S^3)\oplus\pi_9(\Sigma Y^7)\oplus\pi_9(\Sigma L_3) \longrightarrow \pi_9(\Sigma\mathbb{RP}^4_2) \longrightarrow 0$$

$$\text{with}\quad \delta_*(\nu_5\circ\eta_8^2) = \delta(\nu_5\circ\eta_8)\circ\eta_9 = \nu'\circ\eta_6^2\circ\eta_8 + \nu'\circ\eta_6\circ\eta_7 = \nu'\circ\eta_6^2.$$

Thus $\pi_9(\Sigma\mathbb{RP}^4_2) \cong \pi_9(\Sigma Y^7)\oplus\pi_9(\Sigma L_3) \cong \mathbb{Z}/2^{\oplus 4}$.

4. $\pi_{10}(\Sigma\mathbb{RP}^4_2)$: Consider the exact sequence

$$\pi_{11}(S^5) \xrightarrow{\delta_*} \pi_{10}(S^3)\oplus\pi_9(S^3)\oplus\pi_{10}(\Sigma Y^7)\oplus\pi_{10}(\Sigma L_3)\oplus\pi_{10}(\Sigma^7\mathbb{RP}^5_3) \longrightarrow \pi_{10}(\Sigma\mathbb{RP}^4_2) \longrightarrow 0.$$

Recall that $\pi_9(S^3) = \pi_{10}(S^3) = 0$ and $4\cdot\pi_{10}(\Sigma Y^7) = 0$. We have

$$\delta_*(\nu_5^2) = \delta_*(E\nu_4^2) = j_*\circ\Omega(\eta,2)_*(\nu_4^2) = j_*(4\nu_4^2) = 0$$

and so $\pi_{10}(\Sigma\mathbb{RP}^4_2) \cong \pi_{10}(\Sigma Y^7)\oplus\pi_{10}(\Sigma L_3)\oplus\pi_{10}(\Sigma^7\mathbb{RP}^5_3) \cong \mathbb{Z}/4\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$.

5. $\pi_{11}(\Sigma\mathbb{RP}^4_2)$: There is an exact sequence

$$\pi_{12}(S^5) \xrightarrow{\delta_*} \pi_{11}(S^3)\oplus\pi_{10}(S^3)\oplus\pi_{11}(\Sigma Y^7)\oplus\pi_{11}(\Sigma L_3)\oplus\pi_{11}(P^{12}(2))\oplus\pi_{11}(\Sigma^7\mathbb{RP}^5_3)$$

$$\longrightarrow \pi_{11}(\Sigma\mathbb{RP}^4_2) \longrightarrow \pi_{11}(S^5) \longrightarrow 0.$$

Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc} \Omega S^5 & \xrightarrow{\delta} & F & \longrightarrow & \Sigma\mathbb{R}\mathrm{P}^4_2 & \longrightarrow & S^5 \\ \| & & \downarrow & & \downarrow & & \| \\ \Omega S^5 & \xrightarrow{\tilde{\eta}} & S^3 & \longrightarrow & SU(3) & \longrightarrow & S^5 \end{array}$$

By [42, pp.64], $\eta_4 \circ \sigma''' = 0$ and so $\tilde{\eta}_*(\sigma''') = p \circ \eta_4 \circ \sigma''' = 0$, where $p\colon \Omega S^4 \longrightarrow S^3$ is a retraction. Thus

$$\mathrm{Im}(\delta_*\colon \pi_{11}(\Omega S^5) \to \pi_{11}(F)) \subseteq \pi_{10}(\Omega B).$$

Let $X^4 = \mathrm{sk}_7(\Omega^2 S^5)$. There is a commutative diagram

$$\begin{array}{ccc} \pi_{10}(\Omega^2 S^5) & \xrightarrow{H} & \pi_{10}(\Omega^2 S^9) \\ \uparrow{\scriptstyle i_*} & & \uparrow \\ \pi_{10}(X^4) & \xrightarrow{p_*} & \pi_{10}(S^7), \end{array}$$

where p is the pinch map. By Step 5 of Theorem 5.8, $\pi_{10}(X^4) \xrightarrow{p_*} 4\cdot\pi_{10}(S^7)$ is onto. By [42, Lemma 5.13], $\pi_{10}(\Omega^2 S^5) \cong 4 \cdot \pi_{10}(\Omega^2 S^9)$ under the Hopf invariant H and so $i_*\colon \pi_{10}(X^4) \longrightarrow \pi_{10}(\Omega^2 S^5)$ is onto. Let δ' be the composite

$$\Omega^2 S^5 \xrightarrow{\delta} \Omega F \xrightarrow{\text{proj.}} \Omega B.$$

Then δ' maps X^4 into the 7-skeleton of ΩB up to homotopy and so there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega^2 S^5 & \xrightarrow{\delta'} & \Omega B \\ \uparrow & & \uparrow \\ X^4 & \xrightarrow{\delta''} & S^3 \times \Omega\Sigma Y^7. \end{array}$$

Let δ''' be the composite

$$X^4 \xrightarrow{\delta''} S^3 \times \Omega\Sigma Y^7 \xrightarrow{\text{proj.}} \Omega\Sigma Y^7.$$

We show that $\delta'''_*\colon \pi_{10}(X^4) \longrightarrow \pi_{10}(\Omega\Sigma Y^7)$ *is zero.* Since X^4 is the homotopy cofibre of $(\nu', [2])\colon S^6 \to S^3 \vee S^6$ and Y^7 is 4-connected, there is a homotopy commutative diagram

$$\begin{array}{ccc} X^4 & \xrightarrow{\delta'''} & \Omega\Sigma Y^7 \\ \Big\downarrow{\scriptstyle \text{pinch}} & & \Big\uparrow{\scriptstyle E} \\ P^7(2) & \xrightarrow{\tilde{\delta}'''} & Y^7. \end{array}$$

Recall that $\pi_{10}(P^7(2)) \cong \pi^s_{10}(P^7(2)) = \mathbb{Z}/2$ generated by $\alpha_7 \circ \eta_7^2$, where α_7 is a generator for $\pi_8(P^7(2)) = \mathbb{Z}/4$. Since $\tilde{\delta}'''_*\colon H_7(P^7(2)) \to H_7(Y^7)$ is zero, the composite

$$P^7(2) \xrightarrow{\tilde{\delta}'''} Y^7 \xrightarrow{\text{pinch}} S^7$$

is null homotopic and so $\tilde{\delta}'''$ maps into the subcomplex $P^6(2) \vee S^6$. Let ϕ_1 be the composite

$$P^7(2) \xrightarrow{\tilde{\delta}'''} P^6(2) \vee S^6 \xrightarrow{\text{proj.}} P^6(2)$$

and let ϕ_2 be the composite $P^7(2) \xrightarrow{\tilde{\delta}'''} P^6(2) \vee S^6 \xrightarrow{\text{proj.}} S^6$. Then ϕ_2 is homotopic to either the constant map or the composite $P^7(2) \longrightarrow S^7 \xrightarrow{\eta_6} S^6$. In the second case, $\phi_2 \circ \alpha_7 = \eta_6^2$ and $\phi_2 \circ \alpha_7 \circ \eta_8^2 = 0$. Thus, in both cases, $\phi_2 \circ \alpha_7 \circ \eta_8^2 = 0$. Observe that $[P^7(2), P^6(2)] = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. There are three possibilities of the map ϕ_1:

Case 1. The map ϕ_1 is homotopic to the composite

$$P^7(2) \xrightarrow{\text{pinch}} S^7 \xrightarrow{\alpha_6} P^6(2).$$

Then $\phi_1 \circ \alpha_7 = \alpha_6\eta_7$ and $\phi_1 \circ \alpha_7 \circ \eta_8^2 = \alpha_6 \circ \eta_7^3 = \alpha_6 \circ 4\nu_7 = 0$.

Case 2. The map ϕ_1 is homotopic to the composite $P^7(2) \xrightarrow{\bar{\eta}} S^5 \xhookrightarrow{j} P^6(2)$, where $\bar{\eta}$ is an extension of $\eta\colon S^6 \to S^5$. Then $\phi_1 \circ \alpha_7 = j \circ (k\nu_5)$ for some k and so $\phi_1 \circ \alpha_7 \circ \eta^2 = j \circ (k\nu_5 \circ \eta^2)$. It follows that $E(\phi_1 \circ \alpha_7 \circ \eta^2) = 0$.

Case 3. The map $\phi_1 = \eta \wedge \mathrm{id}\colon P^7(2) \longrightarrow P^6(2)$. Then $\phi_1 \circ \alpha_7 = \alpha_6 \circ \eta + \epsilon j_*(\nu_5)$ for some $\epsilon = 0, 1$ and $\phi_1 \circ \alpha_7 \circ \eta^2 = \alpha_6\eta^3 + \epsilon j_*(\nu_5 \circ \eta^2) = \epsilon j_*(\nu_5 \circ \eta^2)$. It follows that $E(\phi_1 \circ \alpha_7 \circ \eta^2 = 0$.

Thus, in any case, we have

$$E \circ \tilde{\delta}'''_*\colon \pi_{10}(P^7(2)) \longrightarrow \pi_{10}(\Omega\Sigma Y^7)$$

is zero and so is $\delta'''_* : \pi_{10}(X^4) \longrightarrow \pi_{10}(\Omega\Sigma Y^7)$.

Now we show that the boundary $\delta_*\colon \pi_{12}(S^5) \to \pi_{11}(F)$ is zero. Since $\pi_{10}(S^3)$ is zero, $\delta''_*\colon \pi_{10}(X^4) \longrightarrow \pi_{10}(S^3 \times \Omega\Sigma Y^7)$ is zero. By using the fact that $\pi_{10}(X^4) \to \pi_{10}(\Omega^2 S^5)$ is onto, we have $\delta_*\colon \pi_{10}(\Omega^2 S^5) \longrightarrow \pi_{10}(\Omega F)$ is zero and so there is a short exact sequence

$$0 \longrightarrow \pi_{11}(S^3) \oplus \pi_{11}(\Sigma Y^7) \oplus \pi_{11}(\Sigma L_3) \oplus \pi_{11}(P^{12}(2)) \longrightarrow \pi_{11}(\Sigma\mathbb{R}\mathrm{P}^4_2) \xrightarrow{p_*} \pi_{11}(S^5) \longrightarrow 0.$$

We are going to solve the group extension problem now. Let $\alpha \in \pi_{11}(\Sigma\mathbb{R}\mathrm{P}^4_2)$ such that $p_*(\alpha) = \nu_5^2$ and let $2\alpha = \beta_1 + \beta_2 + \beta_3 + \beta_4$, where $\beta_1 \in \pi_{11}(S^3)$, $\beta_2 \in \pi_{11}(\Sigma Y^7)$, $\beta_3 \in \pi_{11}(\Sigma L_3)$ and $\beta_4 \in \pi_{11} P^{12}(2)$.

First we show that $\beta_4 = 0$. Consider the Hurewicz homomorphism

$$h\colon \pi_{10}(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2) \to H_{10}(\Omega\Sigma\mathbb{R}\mathrm{P}^4_2).$$

Then

$$0 = h(2\alpha) = h(\beta_1) + h(\beta_2) + h(\beta_3) + h(\beta_4).$$

If $\beta_4 \neq 0$, then β_4 is a generator for $\pi_{11}(P^{12})(2) = \mathbb{Z}/2$ and $h(\beta_4) = [[x_2, u_3], u_3]$. Note that $h(\beta_1), h(\beta_2)$ and $h(\beta_3)$ lie in $H_*(S^3), H_*(\Omega\Sigma Y^7)$ and $H_*(\Omega\Sigma L_3)$, respectively. There is a contradiction. Thus $\beta_4 = 0$.
Now we show that $\beta_3 = 0$. Let ϕ be the composite

$$\Omega\Sigma L_3 \longrightarrow \Omega B \longrightarrow \Omega\Sigma\mathbb{R}\mathrm{P}^4_2 \xrightarrow{H_3} \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(3)} \longrightarrow \Omega\Sigma L_3.$$

Then $\phi_* \colon H_7(\Omega\Sigma L_3) \longrightarrow H_7(\Omega\Sigma L_3)$ is an isomorphism. Hence

$$\phi_* \colon PH_r(\Omega\Sigma L_3) \longrightarrow PH_r(\Omega\Sigma L_3)$$

is an isomorphism for $r \leq 13$ and so

$$\phi_* \colon \pi_r(\Omega\Sigma L_3) \longrightarrow \pi_r(\Omega\Sigma L_3)$$

is an isomorphism for $r \leq 12$. Let ψ be the composite

$$\Omega\Sigma\mathbb{R}\mathrm{P}^4_2 \xrightarrow{H_3} \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(3)} \longrightarrow \Omega\Sigma L_3.$$

From the commutative diagram

$$\begin{array}{ccc}
\Omega\Sigma\mathbb{R}\mathrm{P}^4_2 & \xrightarrow{H_3} & \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(3)} \\
\uparrow & & \uparrow \\
\Omega S^3 & \xrightarrow{H_3} & \Omega\Sigma S^6,
\end{array}$$

we have $\psi_*(\beta_1) = 0$. Recall that $\pi_{11}(\Sigma Y^7) \cong \pi_{11}(P^7(2)) \oplus 2 \cdot \pi_{10}(S^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/4$. By Theorem 5.13, $j_*(\omega_6)$ is a generator for $\pi_{11}(P^7(2)) = \mathbb{Z}/2$. From the homotopy commutative diagram

$$\begin{array}{ccccc}
S^{10} & \xrightarrow{j_*(\omega_6)} & \Omega\Sigma\mathbb{R}\mathrm{P}^4_2 & \xrightarrow{H_3} & \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(3)} \\
\downarrow{\scriptstyle \omega_6} & & \uparrow & & \uparrow \\
\Omega S^6 & \xrightarrow{J([x_2, u_3])} & \Omega\Sigma(S^2 \vee S^3) & \xrightarrow{H_3} & \Omega\Sigma(S^2 \vee S^3)^{(3)},
\end{array}$$

we have $\psi_*(j_*(\omega_6)) = 0$. By expecting the homotopy commutative diagram

$$\begin{array}{ccccccc}
S^6 & \xrightarrow{([4], \bar{\eta})} & S^6 \vee P^6(2) & \longrightarrow & Y^7 & \xrightarrow{\text{pinch}} & S^7 \\
\uparrow{\scriptstyle 2\nu_6} & & \uparrow & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle 2\nu_7} \\
S^9 & \longrightarrow & * & \longrightarrow & S^{10} & = & S^{10},
\end{array}$$

the element $E\delta$ is a generator for the summand $2\pi_{10}(S^7)$ in $\pi_{11}(\Sigma Y^7)$. Since $\Omega\Sigma L_3$ is 6-connected, there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\Omega\Sigma\mathbb{R}\mathrm{P}^4_2 & \xrightarrow{H_3} & \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(3)} & \longrightarrow & \Omega\Sigma L_3 \\
\big\uparrow & & & & \big\uparrow{\scriptstyle \psi'} \\
Y^7 & & \xrightarrow{\quad\text{pinch}\quad} & & S^7.
\end{array}$$

Thus $\psi_*(E\delta) = \psi'_*(2\nu_7) = 2\psi'_*(\nu_7) = 0$ in $\pi_{10}(\Omega\Sigma L_3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. And therefore $\psi_*(\beta_2) = 0$. It follows that

$$0 = \psi_*(2\alpha) = \psi_*(\beta_3) = \phi_*(\beta_3).$$

Thus $\beta_3 = 0$.

Continuation of the proof. Now consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
\Omega F^5\{2\} & \longrightarrow & \Omega P^5(2) & \longrightarrow & \Omega S^5 \\
\big\uparrow{\scriptstyle \Omega\bar{f}} & & \big\uparrow{\scriptstyle \Omega f} & & \big\| \\
\Omega F & \longrightarrow & \Omega\Sigma\mathbb{R}\mathrm{P}^4_2 & \longrightarrow & \Omega S^5.
\end{array}$$

Since $\Omega f|_{\Omega S^3} \colon \Omega S^3 \to \Omega P^5(2)$ is null homotopic, $\Omega f_*(\beta_1) = 0$. By the relative Samelson product, the composite

$$\Omega S^6 \xrightarrow{J([x_2,u_3])} \Omega\Sigma\mathbb{R}\mathrm{P}^4_2 \longrightarrow \Omega P^5(2)$$

is null homotopic and so $\Omega f_*(j_*(\omega_6)) = 0$. Observe that there is a commutative diagram

$$\begin{array}{ccccccc}
\Omega B & \longrightarrow & \Omega F & \xrightarrow{\bar{\Omega} f} & \Omega F^5\{2\} & = = & \Omega F^5\{2\} \\
\big\uparrow & & \big\uparrow{\scriptstyle g} & & \big\uparrow & & \\
Y^7 & \xrightarrow{f'} & \Omega F^5\{2\} & \xrightarrow{g} & X^5 & & \\
\big\downarrow{\scriptstyle \text{pinch}} & & \big\downarrow & & \big\downarrow & & \\
S^7 & = = & = = & = = & S^7, & &
\end{array}$$

where X^5 is the homotopy fibre of $S^7 \xrightarrow{2\omega_4} S^4$. Note that $\pi_{10}(X^5) \cong 2 \cdot \pi_{10}(S^7)$. The element $g \circ f' \circ \delta \colon S^{10} \to X^5$ is a generator for $\pi_{10}(X^5)$. Let $\beta_2 = k_1 j_*(\omega_6) + k_2 E(\delta)$. Then

$$f_*(2\alpha) = f_*(\beta_1 + \beta_2) = f_*(\beta_1 + k_1 j_*(\omega_6) + k_2 E(\delta)) = k_2 f_*(E(\delta)).$$

Consider the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{11}(X^5) & \longrightarrow & \pi_{11}(P^5(2)) & \longrightarrow & \pi_{11}(S^5) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle f_*} & & \| & & \\
0 & \longrightarrow & \pi_{11}(F) & \longrightarrow & \pi_{11}(\Sigma\mathbb{R}\mathrm{P}^4_2) & \longrightarrow & \pi_{11}(S^5) & \longrightarrow & 0
\end{array}$$

By Step 5 of Theorem 5.10, the short exact sequence

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \pi_{11}(P^5(2)) \longrightarrow \pi_{11}(S^5) \longrightarrow 0$$

splits. Since $f_*(E\delta)$ is a generator for $\mathbb{Z}/4$-summand in $\pi_{11}(P^5(2))$, we have $k_2 \equiv 0(2)$. Let $\tilde{\alpha} = \alpha + k_2/2E(\delta)$. Then $p_*(\tilde{\alpha}) = \nu_5^2$ and

$$2\tilde{\alpha} = 2\alpha + k_2E(\delta) = \beta_1 + k_1 j_*(\omega_6).$$

We Show that $k_1 = 0$. Consider the commutative diagram

$$\begin{array}{ccccc}
\Omega S^6 & \xrightarrow{J([x_2,u_3])} & \Omega\Sigma(S^2\vee S^3) & \xrightarrow{\Omega\Sigma j} & \Omega\Sigma\mathbb{R}\mathrm{P}^4_2 \\
 & & \downarrow{\scriptstyle H_2} & & \downarrow{\scriptstyle H_2} \\
 & & \Omega\Sigma(S^2\vee S^3)^{(2)} & \xrightarrow{\Omega\Sigma j^{(2)}} & \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(2)}.
\end{array}$$

By [47, Corollary 1.2], the composite $H_2 \circ J([x_2,u_3])$ is a loop map and so

$$H_2 \circ \Omega\Sigma j \circ J([x_2,u_3]) = \Omega\lambda,$$

where $\lambda\colon S^6 \to \Sigma(\mathbb{R}\mathrm{P}^4_2)^{(2)}$ is a map with the property that $\lambda_*(\iota_6) = s(x_2\otimes u_3 + u_3\otimes x_2)$ in $H_*(\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(2)})$ and s is the suspension. Observe that $s(x_2\otimes x_2)$, $s(x_2\otimes u_3)$ and $s(u_3\otimes u_3)$ are spherical classes in $H_*(\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(2)})$. There is a map

$$f\colon S^5\vee S^6\vee S^7 \longrightarrow \Sigma(\mathbb{R}\mathrm{P}^4_2)^{(2)}$$

such that the elements $s(x_2\otimes x_2)$, $s(x_2\otimes u_3)$ and $s(u_3\otimes u_3)$ lie in the image of f_*. Let $X_1 = C_f$ be the homotopy cofibre of f. Then $\mathrm{sk}_7(X_1) = P^7(2)\vee S^7$ and $s(x_2\otimes v_4)$ is spherical in $H_*(X_1)$. Let $X_2 = X_1/S^7$ which cancels the element $s(x_2\otimes v_4)$. Then $\bar{H}_*(X_2)$ has a basis

$$\{s[x_2,u_3], s[x_2,v_4] + s(u_3\otimes u_3), s[u_3,v_4], s(v_4\otimes v_4), s(u_3\otimes v_4)\}.$$

There is a cofibre sequence

$$S^7\vee S^7 \longrightarrow P^7(2) \longrightarrow \mathrm{sk}_8(X_2).$$

Since $[S^7\vee S^7, P^7(2)] \cong \mathbb{Z}/2\oplus\mathbb{Z}/2$, $\mathrm{sk}_8(X_2) \simeq \Sigma^3\mathbb{R}\mathrm{P}^5_3\vee S^8$. Thus $s(u_3\otimes v_4)$ is spherical in $H_*(X_2)$. By expecting the Steenrod operations, we have $X_2/S^8 \simeq \mathbb{C}\mathrm{P}^2\wedge P^5(2)$. Thus there is a pinch map $g\colon \Sigma(\mathbb{R}\mathrm{P}^4_2)^{(2)} \to \mathbb{C}\mathrm{P}^2\wedge P^5(2)$ such that $g_*(s[x_2,u_3]) \neq 0$. Consider the exact sequence

$$0 \longrightarrow \pi_{11}(P^7(2)) \xrightarrow{i_*} \pi_{11}(\mathbb{C}\mathrm{P}^2\wedge P^5(2)) \xrightarrow{p_*} \pi_{11}(P^9(2)) \xrightarrow{\partial} \pi_{10}P^7(2).$$

Recall that $\pi_{11}(P^9(2)) \cong \mathbb{Z}/2\oplus\mathbb{Z}/2$ generated by $\alpha_9\circ\eta_{10}$ and $j_*(\nu_8)$. Observe that

i) $\partial\circ j_*(\nu_8) = j_*(\eta_7\circ\nu_8) = 0$.

ii) $\partial(\alpha_9) = \alpha_7 \circ \eta + \epsilon j_*(\nu_6)$.
iii) $\partial(\alpha_9 \circ \eta_{10}) = \delta_*(\alpha_9) \circ \eta_{10} = \alpha_7 \circ \eta^2 + \epsilon j_*(\nu_6 \circ \eta) = \alpha_7 \circ \eta^2$.

The kernel $\mathrm{Ker}(\partial\colon \pi_{11}(P^9(2)) \to \pi_{10}(P^7(2)))$ is generated by $j_*(\nu_8)$. Let

$$\beta \in \pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^5(2))$$

so that $p_*(\beta) = j_*(\nu_8)$. Then 2β is the composite

$$S^{11} \longrightarrow \mathbb{C}\mathrm{P}^2 \wedge P^5(2) \xrightarrow{\text{pinch}} \Sigma^5\mathbb{C}\mathrm{P}^2 \xrightarrow{\eta\wedge \mathrm{id}} \Sigma^4\mathbb{C}\mathrm{P}^2 \longrightarrow \mathbb{C}\mathrm{P}^2 \wedge P^5(2).$$

Note that $\eta \wedge \mathrm{id}\,|_{S^7}$ is null homotopic and $\pi_9(S^6) \longrightarrow \pi_9(\Sigma^4\mathbb{C}\mathrm{P}^2)$ is onto. Thus $\eta \wedge \mathrm{id}\colon \Sigma^5\mathbb{C}\mathrm{P}^2 \to \Sigma^4\mathbb{C}\mathrm{P}^2$ is homotopic to the composite

$$\Sigma^5\mathbb{C}\mathrm{P}^2 \xrightarrow{\text{pinch}} S^9 \xrightarrow{k\nu_6} S^6 \longrightarrow \Sigma^4\mathbb{C}\mathrm{P}^2.$$

Consider the commutative diagram of pinch maps

$$\begin{array}{ccc} \Sigma^5\mathbb{C}\mathrm{P}^2 & \xrightarrow{p'} & S^9 \\ \uparrow{\scriptstyle q} & & \uparrow{\scriptstyle q'} \\ \mathbb{C}\mathrm{P}^2 \wedge P^5(2) & \xrightarrow{p} & P^9(2) \end{array}$$

Since $p'_* \circ q_*(\beta) = q'_* \circ p_*(\beta) = q'_* \circ j_*(\nu_8) = 0$, we have $2\beta = 0$ and so

$$\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^5(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

generated by $j_*(\omega_6)$ and $\overline{j_*(\nu_8)}$. Now let μ be the composite

$$\Omega\Sigma\mathbb{R}\mathrm{P}^4_2 \xrightarrow{H_2} \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(2)} \xrightarrow{\Omega g} \Omega(\mathbb{C}\mathrm{P}^2 \wedge P^5(2)).$$

Then $\mu_*(j_*(\omega_6)) = j_*(\omega_6)$ the generator for $\pi_{11}(P^7(2))$. From the commutative diagram

$$\begin{array}{ccc} \Omega\Sigma\mathbb{R}\mathrm{P}^4_2 & \xrightarrow{H_2} & \Omega\Sigma(\mathbb{R}\mathrm{P}^4_2)^{(2)} \\ \uparrow & & \uparrow \\ \Omega S^3 & \xrightarrow{H_2} & \Omega S^5, \end{array}$$

we have $\mu_*(\beta_1) = 0$ and so

$$0 = \mu_*(2\tilde{\alpha}) = \mu_*(\beta_1) + k_1\mu_*(j_*(\omega_6)) = k_1 j_*(\omega_6).$$

Thus $k_1 = 0$ and so $2\tilde{\alpha} = \beta_1 \in \pi_{11}(S^3) = \mathbb{Z}/2$.

Continuation for solving the group extension problem. Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_{11}(B) & \hookrightarrow & \pi_{11}(F) & \longrightarrow & \pi_{11}(S^3) \\
\| & & \downarrow \phi_1 & & \downarrow \phi_2 \\
\pi_{11}(B) & \hookrightarrow & \pi_{11}(\Sigma\mathbb{R}\mathrm{P}^4_2) & \xrightarrow{q_*} & \pi_{11}(SU(3)) \\
 & & \downarrow & & \downarrow \\
 & & \pi_{11}(S^5) & = & \pi_{11}(S^5),
\end{array}$$

where ϕ_1 and ϕ_2 are monomorphisms. Let G be the subgroup of $\pi_{11}(\Sigma\mathbb{R}\mathrm{P}^4_2)$ generated by $\tilde{\alpha}$ and ϵ_3. Then G has four elements and $q_*|_G\colon G \to \pi_{11}(SU(3))$ is onto. Thus $G \cong \pi_{11}(SU(3))$ and so

$$\pi_{11}(\Sigma\mathbb{R}\mathrm{P}^4_2) \cong \pi_{11}(SU(3)) \oplus \pi_{11}(B) = \pi_{11}(SU(3)) \oplus \mathbb{Z}/2^{\oplus 4} \oplus \mathbb{Z}/4.$$

We finish the proof now. □

6.3.2. *The Homotopy Groups $\pi_*(P^3(2))$.* Now we compute the homotopy groups of other factors in the decomposition of $\Omega^3_0(P^3(2))$. Observe that $\mathbb{C}\mathrm{P}^2 \wedge P^6(2)$ is 6-connected.

Lemma 6.34. *The homotopy groups of $\pi_r(\mathbb{C}\mathrm{P}^2 \wedge P^6(2))$ for $7 \leq r \leq 11$ are as follows.*

1) $\pi_7(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = \mathbb{Z}/2$.
2) $\pi_8(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = 0$.
3) $\pi_9(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = \mathbb{Z}/2$.
4) $\pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = \mathbb{Z}/2$.
5) $\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = \mathbb{Z}/2$.

Proof. Consider the cofibre sequence

$$P^9(2) \xrightarrow{\eta\wedge\mathrm{id}} P^8(2) \longrightarrow \mathbb{C}\mathrm{P}^2 \wedge P^6(2) \longrightarrow P^{10}(2) \xrightarrow{\eta\wedge\mathrm{id}} P^9(2).$$

The sequence

$$\pi_r(P^9(2)) \xrightarrow{\eta\wedge\mathrm{id}_*} \pi_r(P^8(2)) \longrightarrow \pi_r(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \longrightarrow \pi_{r-1}(P^9(2)) \longrightarrow \cdots$$

is exact for $r \leq 11$. Thus

$$\pi_7(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \cong \pi_7(P^8(2)) = \mathbb{Z}/2 \quad \text{and} \quad \pi_8(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = 0.$$

Consider the short exact sequence

$$0 \longrightarrow \pi_9(P^9(2)) \xrightarrow{\eta\wedge\mathrm{id}_*} \pi_9(P^8(2)) \longrightarrow \pi_9(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \longrightarrow 0.$$

We have $\pi_9(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = \mathbb{Z}/2$ and there is an exact sequence

$$\pi_{10}(P^9(2)) \xrightarrow{\eta\wedge\mathrm{id}_*} \pi_{10}(P^8(2)) \longrightarrow \pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \longrightarrow 0.$$

Thus $\pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = \mathbb{Z}/2$ and there is an exact sequence

$$\pi_{11}(P^9(2)) \xrightarrow{\eta\wedge \mathrm{id}_*} \pi_{11}(P^8(2)) \longrightarrow \pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \longrightarrow 2\cdot\pi_{10}(P^9(2)) \longrightarrow 0.$$

Notice that $(\eta\wedge)\circ\alpha_9 = \alpha_8\circ\eta + \epsilon j_*(\nu_7)$. Then

$$(\eta\wedge 1)\circ\alpha_9\circ\eta = \alpha_8\circ\eta^2 + \epsilon j_*(\nu_7\circ\eta) = \alpha_8\circ\eta^2.$$

Thus $\eta\wedge \mathrm{id}_*\colon \pi_{11}(P^9(2)) \longrightarrow \pi_{11}(P^8(2)) = \mathbb{Z}/2$ if onto and so

$$\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \cong 2\pi_{10}(P^9(2)) = \mathbb{Z}/2.$$

This completes the calculation. □

Lemma 6.35. *The homotopy groups $\pi_r((\mathbb{C}\mathrm{P}^2)^{(2)} \wedge P^6(2))$ for $9 \le r \le 11$ are as follows.*

1) $\pi_9(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = \mathbb{Z}/2.$
2) $\pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = 0.$
3) $\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$

Proof. Clearly $\pi_9(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \cong \pi_9(\mathbb{C}\mathrm{P}^2 \wedge P^8(2)) = \mathbb{Z}/2$. There is an exact sequence

$$\pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge P^9(2)) \longrightarrow \pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge P^8(2)) \longrightarrow \pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \longrightarrow 0.$$

Thus $\pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = 0$ and there is an exact sequence

$$\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^9(2)) \longrightarrow \pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^8(2)) \longrightarrow \pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2))$$
$$\longrightarrow \pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge P^9(2)) \longrightarrow 0.$$

Recall that $\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^9(2)) = 0$. There is a short exact sequence

$$0 \longrightarrow \pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge P^8(2)) \longrightarrow \pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2)) \longrightarrow \pi_{10}(\mathbb{C}\mathrm{P}^2 \wedge P^9(2)) \longrightarrow 0.$$

From the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2) & \xrightarrow{[2]} & \mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2) \\
\downarrow & & \uparrow \\
\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge S^6 & \xrightarrow{\mathrm{id}\wedge\eta} & \mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge S^5 \\
\downarrow & & \uparrow \\
\mathbb{C}\mathrm{P}^2 \wedge S^{10} & \xrightarrow{\mathrm{id}\wedge k\nu_7} & \mathbb{C}\mathrm{P}^2 \wedge S^7,
\end{array}$$

we have $2\cdot\pi_{11}(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2)) = 0$ and hence the result. □

Now we list a table for $\pi_r(P^3(2))$ with $r \le 11$. In the following table, an integer n indicates a cyclic group $\mathbb{Z}/n$, the symbol " + " the direct sum of groups and n^k indicates the direct sum of k-copies of cyclic group $\mathbb{Z}/n$.

Theorem 6.36. *$\pi_2(P^3(2)) = \mathbb{Z}/2$, $\pi_3(P^3(2)) = \mathbb{Z}/4$, $\pi_4(P^3(2)) = \mathbb{Z}/4$ and $\pi_r(P^3(2))$ for $5 \le r \le 11$ is as follows*

$\pi_r(P^3(2))$	$G_1\oplus$	$G_2\oplus$	$G_3\oplus$	$G_4\oplus$	$G_5\oplus$	$G_6\oplus$	others
$r=5$	2	2	2	0	0	0	0
$r=6$	2	2^3	2	0	0	0	0
$r=7$	4	$2+8$	4	2	0	0	0
$r=8$	2	2^3+4	2^2	0	2	0	0
$r=9$	2	2^4	2^2	2	2	2	0
$r=10$	0	2^2+4^2	8	2	4	0	2^2
$r=11$	0	2^4+4^2	2^2	2	2^2	2^2	2^3

where $G_1 = \pi_{r-1}(S^3)$, $G_2 = \pi_r(\Sigma\mathbb{R}\mathrm{P}^4_2)$,$G_3 = \pi_r(P^6(2))$, $G_4 = \pi_r(\mathbb{C}\mathrm{P}^2 \wedge P^6(2))$, $G_5 = \pi_r(P^9(2))$, and $G_6 = \pi_r(\mathbb{C}\mathrm{P}^2 \wedge \mathbb{C}\mathrm{P}^2 \wedge P^6(2))$.

Appendix A. The Table of the Homotopy Groups of $\Sigma^n\mathbb{R}\mathrm{P}^2$

The homotopy groups of the mod 2 Moore spaces, up to $8+$ dimension, are listed in the following table. In the following table, an integer k indicates a cyclic group $\mathbb{Z}/k\mathbb{Z}$, the symbol " $+$ " indicates the direct sum of groups, k^l indicates the direct sum of l-copies of the cyclic group $\mathbb{Z}/k\mathbb{Z}$.

$r\backslash n$	n=3	4	5	6	7	8	9	10	11	≥ 12
r=-1	2	2	2	2	2	2	2	2	2	2
0	4	2	2	2	2	2	2	2	2	2
1	4	4	4	4	4	4	4	4	4	4
2	2^3	$2 + 4$	$2 + 4$	2^2	2^2	2^2	2^2	2^2	2^2	2^2
3	2^5	2^2	2^2	2^2	2	2	2	2	2	2
4	$2^2 + 4^2 + 8$	2^3	2^2	8	2	0	0	0	0	0
5	$2^7 + 4$	2^3	$2 + 4$	2^2	2	2	2	2	2	2
6	2^{10}	$2^2 + 4$	$2^2 + 4$	2^2	2^2	$2 + 4$	$2 + 4$	2^2	2^2	2^2
7	$2^5 + 4^3 + 8$	$2^3 + 4$	2^3	2^3	$2^3 + 4$	2^4	2^4	2^4	2^3	2^3
8	$2^{14} + 4^2$	$2^5 + 4$	$2^2 + 4^2$	$2^4 + 4$	$2 + 4^2 + 8$	$2^3 + 4^2$	$2^3 + 4^2$	$2 + 4^2 + 8$	$2^2 + 4^2$	$2 + 4^2$

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