CHOW RINGS OF NONABELIN p-GROUPS OF ORDER p^3

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Abstract. Let G be a nonabelian p group of order p^3 (i.e., extraspecial p-group), and BG its classifying space. Then $CH^*(BG) \cong H^{2*}(BG)$ where $CH^*(-)$ is the Chow ring over the field $k = \mathbb{C}$. We also compute $mod(2)$ motivic cohomology and motivic cobordism of BQ_8 and BD_8.

1. Introduction

For a smooth algebraic variety over $k = \mathbb{C}$, let $CH^*(X)$ be the Chow ring (over $\mathbb{C}$) and $BP^*(X)$ the Brown-Peterson theory. Then Totaro [To1] defined the modified cycle map

$$\tilde{cl} : CH^*(X)_{(p)} \to BP^{2*}(X) \otimes_{BP^*} \mathbb{Z}_{(p)}$$

such that the composition with the Thom map $\rho : BP^*(X) \to H^*(X)$, is the usual cycle map.

Let G be an algebraic group over $\mathbb{C}$ and BG the classifying space. Totaro conjectured that the map $\tilde{cl}$ is an isomorphism for $X = BG$. This conjecture is correct for connected groups $O(n), SO(n), G_2, Spin_7, Sinn_8, PGL_p$ ([To2],[Mo-Vi],[In-Ya], [Gu], [Ma], [Ka-Ya],[Vi]), and finite abelian groups and (finite) symmetric groups (localized at 2) [To2].

We will show it holds for each non abelian p-group of order p^3.

Theorem 1.1. *If G is an extraspecial p-group of order p^3 (i.e., p_+^{1+2} or p_-^{1+2} for an odd prime, and Q_8 or D_8 for $p = 2$), Then*

$$CH^*(BG)_{(p)} \cong BP^{2*}(BG) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong H^{2*}(BG)_{(p)}.$$

This is the first example for nonabelian p-group ($p > 2$) which satisfies Totaro's conjecture. Note that the cycle map $cl : CH^*(BG) \to H^{2*}(BG)$ is not surjective for $G = (\mathbb{Z}/p)^3$, and not injective for the central product $D_8 \cdot D_8 \times \mathbb{Z}/2$ (see [To1]).

1991 *Mathematics Subject Classification.* Primary 55P35, 57T25; Secondary 55R35, 57T05.

Key words and phrases. Chow ring, motivic cohomology, extraspecial p-groups.

It is known [Te-Ya], that for each of the above groups, the Brown-Peterson cohomology is given

$$BP^*(BG) \cong BP^*[[y_1, y_2, c_1, ..., c_p]]/(relations)$$

where y_1, y_2 are the first Chern classes of liner representations of G, and c_i is the i-th Chern class of some p-dimensional representation of G. Moreover we know

$$BP^{2*}(BG) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong H^{2*}(BG)_{(p)}.$$

It is shown in [Ya1] that if $CH^*(BG)$ is multiplicatively generated by $y_1, y_2, c_1, ..., c_p$, then Totaro's conjecture holds. In this paper, we will prove this fact and hence Totaro's conjecuture for the above extraspecial p-groups.

Let $MU^*(X)$ be the complex cobordism theory so that $MU^*(X)_{(p)} \cong MU^*_{(p)} \otimes_{BP^*} BP^*(X)$. Let $MGL^{*,*'}(X)$ and $MGL^{*,*'}(X; \mathbb{Z}/p)$ be the motivic cobordism defined by Voevodsky and its $mod(p)$ theory.

From the above theorem and Proposition 9.4 in [Ya3], we have

Corollary 1.2. *For an extraspecial p-group G, we have the isomorphism $MGL^{2*,*}(BG)_{(p)} \cong MU^{2*}(BG)_{(p)}$.*

When $p = 2$, we get the rather strong results. Let $H^{*,*'}(X; \mathbb{Z}/2)$ be the $mod(2)$ motivic cohomology and $0 \neq \tau \in H^{0,1}(Spec(\mathbb{C}); \mathbb{Z}/2)$. Then we prove in §6 ;

Theorem 1.3. *Let $G = Q_8$ or D_8. Then there is the filtration of $H^*(BG; \mathbb{Z}/2)$ such that*

$$H^{*,*'}(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes gr^{*'} H^*(BG; \mathbb{Z}/2).$$

Using this theorem, we prove in the last section ;

Theorem 1.4. *Let $G = Q_8$ or D_8. Then there is the isomorphism*

$$MGL^{*,*'}(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes MU^{2*}(BG).$$

2. EXTRASPECIAL p-GROUPS

Throughout this paper, let G be a non abelian p-group of order p^3. Then the group is called an extraspecial p-group so that there is the central extension

$$0 \to C \to G \to V \to 0$$

where $C \cong \mathbb{Z}/p$ is the center and $V \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$. We can take $a, b, c \in G$ such that $[a, b] = c$ here c generates C and a, b generate V. (See [Lw],[Ly],[Gr-Ly],[Te-Ya] for details.)

These groups have two types for each prime p. For an odd prime p, they are written as p_-^{1+2}, p_+^{1+2} where $a^p = c$ for the first type but $a^p = b^p = 1$ for the other type. When $p = 2$, the groups are the quaternion group Q_8 and the dihedral group D_8, where $a^2 = b^2 = c$ for Q_8 but $a^2 = c$, $b^2 = 1$ for D_8.

Let $a^*, b^* : G \to V \to \mathbb{C}^*$ be the linear representation which is the dual of a, b respectively. Let $\tilde{c} = Ind_{\langle a,c\rangle}^G(c^*)$ for $G = p_+^{1+2}$, for other groups cases, let $\tilde{c} = Ind_{\langle a\rangle}^G(a')$ where $a' : \langle a\rangle \to \mathbb{C}^*$ is the dual of a. For example when $G = p_+^{1+2}$, we can take

$$\tilde{c}(c) = diag(\zeta, ..., \zeta), \quad \tilde{c}(a) = diag(1, \zeta, ..., \zeta^{p-1}),$$

and $\tilde{c}(b)$ as the permutation matrix in $GL_p(\mathbb{C})$ where ζ is a primitive p-th root of unity.

For an integer $N \geq 1$, representations $N\tilde{c}$, Na^*, Nb^* give the G-action on

$$U_N = \mathbb{C}^{pN*} \times \mathbb{C}^{N*} \times \mathbb{C}^{N*}$$

where $\mathbb{C}^{pN*} = \mathbb{C}^{pN} - \{0\}$ and $\mathbb{C}^{N*} = \mathbb{C}^N - \{0\}$. Here G acts freely on $U_N = \mathbb{C}^{N(p+2)} - H_N$ with $codim(H_N) \geq N$. Hence given G-variety X, the Borel cohomology can be defined by

$$CH_G^*(X) = CH^*(U_N \times_G X) \quad when * < N.$$

Of course $CH_G^*(pt.) = CH_G^* \cong CH^*(BG)$ the Chow ring of the classifying space BG.

Let us write by $y_1, y_2 \in CH^*(BG)$ the first Chern classes of a^* and b^* respectively. Let c_i be the i-th Chern class of $\tilde{c}$. We consider $CH_G^*(U_N)$ when $N = 1$. We use the stratified methods by Molina-Vistoli [Mo-Vi] which was used to compute the Chow rings of BG for classical groups G.

Lemma 2.1.

$$CH_G^*(\mathbb{C}^{p*} \times \mathbb{C}^* \times \mathbb{C}^*) \cong CH^*(BG)/(y_1, y_2, c_p).$$

Proof. we first consider the localized exact sequence

$$CH_G^*(\{0\} \times \mathbb{C} \times \mathbb{C}) \xrightarrow{i_*} CH_G^{*+p}(\mathbb{C}^p \times \mathbb{C} \times \mathbb{C}) \to CH_G^{*+p}(\mathbb{C}^{p*} \times \mathbb{C} \times \mathbb{C}) \to 0.$$

Here i_* is the multiplying c_p. So we have

$$CH_G^*(\mathbb{C}^{p*} \times \mathbb{C} \times \mathbb{C}) \cong CH_G^*/(c_p).$$

Similarly applying $c_1(a^*) = y_1, c_1(b^*) = y_2$, we have the lemma. □

Corollary 2.2. *The Chow ring $CH^*(BG)$ is multiplicatively generated by elements $dim \leq p + 2$.*

Proof. First note that the G-action on $\mathbb{C}^{p*}\times\mathbb{C}^*\times\mathbb{C}^*$ is free. Hence

$$CH^*_G(\mathbb{C}^{p*}\times\mathbb{C}^*\times\mathbb{C}^*)\cong CH^*((\mathbb{C}^{p*}\times\mathbb{C}^*\times\mathbb{C}^*)/G).$$

Of course $dim(\mathbb{C}^{p*}\times\mathbb{C}^*\times\mathbb{C}^*)=p+2$, and we see $CH^*_G/(y_1,y_2,c_p)$ is generated by elements $dim\le p+2$. □

Recall that the Brown-Peterson theory also have Chern classes. It is known [Te-Ya], that for each of the above groups, the Brown-Peterson cohomology is given

$$BP^*(BG)\cong BP^*[[y_1,y_2,c_1,...,c_p]]/(relations).$$

Moreover we know $BP^{2*}(BG)\otimes_{BP^*}\mathbb{Z}_{(p)}\cong H^{2*}(BG)$. Hence $H^{2*}(BG)$ is multiplicatively generated by Chern classes of $dim\le 2p$.

Lemma 2.3. *([To2]) If $H^{2*}(X)_{(p)}$ is multiplicatively generated by Chern classes for $*\le p$, then for all $*\le p$,*

$$CH^*(X)_{(p)}\cong BP^{2*}(X)\otimes_{BP^*}\mathbb{Z}_{(p)}\cong H^{2*}(X)_{(p)}.$$

Proof. Recall that the usual K-theory $K^*(X)_{(p)}$ localized at p can be decomposed to the connected Morava K-theory $\tilde{K}(1)^*(X)$ with the coefficient ring $\tilde{K}(1)=\mathbb{Z}_{(p)}[v_1]$, $|v_1|=-2p+2$. We consider the Atiyah-Hirzebruch spectral sequence ([Te-Ya])

$$E(K)_2^{*,*'}\cong H^*(X)\otimes\tilde{K}(1)^{*'}\Longrightarrow\tilde{K}(1)^*(X).$$

The first nonzero differential is known

$$d_{2p-1}(x)=v_1\otimes\beta P^1(x)\quad(=v_1\otimes Q_1(x)\ mod(p)).$$

Since $H^{2*}(X)_{(p)}$ is generated by Chern classes, each element is a permanent cycle. In fact

$$E(K)_\infty^{2*,*'}\cong H^{2*}(X)\otimes\tilde{K}(1)^{*'}\quad for\ *\le p.$$

This implies from the definition of $gr^i_{geo}K^0(X)$ ([Th], [To2])

$$gr^i_{geo}K^0(X)_{(p)}\cong H^{2i}(X)_{(p)}\quad for\ i\le p.$$

Next consider the Atiyah-Hirzebruch spectral sequence for $BP^*(X)$

$$E(BP)_2^{*,*'}\cong H^*(X)\otimes BP^{*'}\Longrightarrow BP^*(X).$$

Similarly we have $E(BP)_\infty^{2*,*'}\cong BP^{*'}\otimes H^{2*}(X)$ for $*\le p$. (The differential d_{2p-1} is the same as the case $\tilde{K}(1)^*(-)$.) Hence we have

$$(BP^*(X)\otimes_{BP^*}\mathbb{Z}_{(p)})^{2i}\cong H^{2i}(X)_{(p)}.$$

On the other hand, there is the natural map

$$CH^i(X)\to gr^i_{geo}K^0(X)\xrightarrow{c_i}CH^i(X),$$

which is the multiplication by $(-1)^{i-1}(i-1)!$ by Riemann Roch with denominators. Moreover the first map is epic. (See the proof of Corollary 3.2 in [To2].) Hence $CH^i(X)_{(p)} \cong gr^i_{geo}K^0(X)_{(p)}$. □

Corollary 2.4. *If the cycle map $cl : CH^*(BG) \to H^{2*}(BG)$ is injective for $* \leq 2p-2$, then $CH^*(BG) \cong H^{2*}(BG)$ for all $* \geq 0$.*

Proof. Since $H^{2*}(BG)$ is generated by y_1, y_2, c_i, we see from Corollary 2.2 that $CH^*(BG)$ is generated by the same elements y_1, y_2, c_i. All relations between the above multiplicative generators are in $dim \leq 2p-2$ (for the explicit relations, see the following results of the ordinary cohomology). Hence we get the corollary. □

Of course the usual cohomology of BG is explicitly known as follows.

Theorem 2.5. *(Lewis [Le],see also [Ly],[Te-Ya])*

$$H^{even}(Bp_+^{1+2}) \cong (\mathbb{Z}[y_1,y_2]/(y_1y_2^p - y_1^py_2, py_i) \oplus \mathbb{Z}/p\{c_2,...,c_{p-1}\}) \otimes \mathbb{Z}[c_p]/(p^2c_p),$$

$$H^{odd}(BG) \cong H^{even}(BG)/(p)\{e\} \quad |e| = 3.$$

Here $c_iy_j = c_ic_k = 0$ for $i < p-1$, but $y_jc_{p-1} = y_j^p$, $c_{p-1}^2 = y_1^{p-1}y_2^{p-1}$.

Theorem 2.6. *(Lewis [Le], [Ly])*

$$H^{even}(Bp_-^{1+2}) \cong (\mathbb{Z}[y_2]/(py_2) \oplus \mathbb{Z}/p\{y_1 = c_1, c_2,...,c_{p-1}\}) \otimes \mathbb{Z}[c_p]/(p^2c_p),$$

$$H^{odd}(Bp_-^{1+2}) \cong \mathbb{Z}/p[y_2,c_p]\{e\} \quad with\ |e| = 2p+1$$

Here $c_iy_j = c_ic_k = 0$ for $i \leq p-1$.

Theorem 2.7. *(Evens [Ev])*

$$H^{even}(BD_8) \cong (\mathbb{Z}[y_1,y_2,c_2]/(y_1y_2, 2y_i, 4c_2),$$

$$H^{odd}(BD_8) \cong H^{even}(BD_8)/(2)\{e\} \quad with\ |e| = 3.$$

Theorem 2.8. *(Atiyah [At])*

$$H^{even}(BQ_8) \cong (\mathbb{Z}[y_1,y_2,c_2]/(y_i^2, 2y_i, 4c_2 = y_1y_2),$$

$$H^{odd}(BQ_8) \cong 0.$$

3. THE GROUP $E = p_+^{1+2}$

Throughout this section, we assume $p \geq 3$ and $G = E = p_+^{1+2}$. Recall that E is generated by a, b, c such that $[a,b] = c$, $a^p = b^p = c^p = 1$. Recall also the p-dimensional representation $\tilde{c} = Ind^G_{\langle a,c\rangle}(c^*)$ so that

$$\tilde{c}(c) = diag(\zeta,...,\zeta), \quad \tilde{c}(a) = diag(1,\zeta,...,\zeta^{p-1}),$$

and $\tilde{c}(b)$ as the permutation matrix in $GL_p(\mathbb{C})$.

The group E does not act freely on $\mathbb{C}^{p*}$. We consider fixed points for small subgroups. Let $W = \mathbb{C}^{p*}$. Since $\tilde{c}(a) = diag(1, \zeta, ..., \zeta^{p-1})$, the fixed points of the subgroup $\langle a\rangle$ is given by

$$W^{\langle a\rangle} = \{(x, 0, ..., 0)|x \in \mathbb{C}^*\} = \mathbb{C}^*\{e\} \quad e = (1, 0, ..., 0).$$

Since $b^{-i}ab^i = ac^i$ in E, we see

$$ac^ib^{-i}e = b^{-i}ab^ib^{-i}e = b^{-i}ae = b^{-i}e.$$

This means $W^{\langle ac^i\rangle} = \mathbb{C}^*\{b^{-i}e\}$. Let us write

$$H_0 = \mathbb{C}^*\{e, be, ..., b^{p-1}e\}.$$

Then the group E acts on H_0, namely, H_0 is a smooth E-variety.

In $GL_p(\mathbb{C})$, the elements $\tilde{c}(ab^i)$, $\tilde{c}(b)$ have the trace zero and p-th roots of the identity. Hence there is a $g_j \in GL_p(\mathbb{C}^*)$ for $0 \le j \le p$ such that $g_j^{-1}ag_j = ab^j$ for $j < p$ and $g_p^{-1}ag_p = b$. Then we see $ab^jg_j^{-1}e = g_j^{-1}e$ as above arguments, and so $\mathbb{C}^*\{g_j^{-1}e\} = W^{\langle ab^j\rangle}$. Hence we can define E-equivariant set $H_j = g_j^{-1}H_0$. Let

$$H = H_0 \bigvee H_1 \bigvee .... \bigvee H_p.$$

Here note $H_i \cap H_j = \emptyset$ for $i \ne j$ (in fact $H_j \subset \mathbb{C}^{p*} = \mathbb{C}^p - \{0\}$).

Lemma 3.1. *The group E acts freely on $(\mathbb{C}^{p*} - H)$.*

Proof. If $x \in \mathbb{C}^{p*}$ is a point of $Ex \not\cong E$, the stabilizer group is not trivial. Since all subgroups of E isomorphic to $\mathbb{Z}/p$ are $\langle ab^jc^i\rangle$, $\langle bc^i\rangle$ or $\langle c\rangle$. But c is not a stabilizer of any element in $\mathbb{C}^*$. All points which have non trivial stabilizer groups are contained in H. Thus we have the lemma. □

Let $i : H \subset \mathbb{C}^{p*}$. Let us write $i^*(y_i) \in H_E^*(H)$ by the same letter y_i.

Lemma 3.2. *There is the isomorphism $H_E^*(H_i) \cong H_E^*(H_0)$ and*

$$H_E^*(H_0; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1] \otimes \Lambda(x_1, z), \quad \textit{with } |x| = |z| = 1,$$

$$H_E^*(H_0) \cong \mathbb{Z}[y_1]/(py_1)\{1, z\}.$$

Proof. We consider the Hochschild-Serre spectral sequence

$$E_2^{*,*} \cong H^*(B\langle a\rangle; H^*_{\langle b,c\rangle}(H_0; \mathbb{Z}/p)) \Longrightarrow H_E^*(H_0; \mathbb{Z}/p).$$

Here we have

$$H^*_{\langle b,c\rangle}(H_0; \mathbb{Z}/p) \cong H^*_{\langle b,c\rangle}(\langle b\rangle \times \mathbb{C}^*; \mathbb{Z}/p) \cong H^*_{\langle c\rangle}(\mathbb{C}^*; \mathbb{Z}/p) \cong \Lambda(z).$$

Hence the $E_2^{*,*}$ is isomorphic to

$$H^*(B\langle a\rangle; \Lambda(z)) \cong \mathbb{Z}/p[y_1] \otimes \Lambda(x_1) \otimes \Lambda(z).$$

We will see $d_2(z) = 0$ and get the result. Consider the localized exact sequence for the cohomology

$$H_E^{*+2p-1}(\mathbb{C}^{p*} - H) \to H_E^*(H) \to H_E^{*+2p}(\mathbb{C}^{p*}) \to H_E^{*+2p}(\mathbb{C}^{p*} - H) \to .$$

Since E acts on $\mathbb{C}^{p*} - H$ freely, we see

$$H_E^{*+2p}(\mathbb{C}^{p*} - H) \cong H^{*+2p}((\mathbb{C}^{p*} - H)/E),$$

which is zero if $* > 0$. Thus for $* > 0$, we have the isomorphism

$$H_E^*(H) \cong H_E^{*+2p}(\mathbb{C}^{p*}) \cong H^{*+2p}(BE;\mathbb{Z}/p)/(c_p).$$

Since $y_1^p y_2 - y_1 y_2^p = 0 \in H^*(BE)$ from Theorem 2.5, we see

$$H_E^{2*+2p}(\mathbb{C}^{p*}) \cong \mathbb{Z}/p\{y_1^{*+p}, y_1^{*+p-1}y_2, ..., y_1^{*+1}y_2^{p-1}, y_2^{*+p}\}$$

and $H_E^{2*+2p+3}(\mathbb{C}^{p*}) \cong H_E^{2*+2p}(\mathbb{C}^{p*})\{e\}$. Hence

$$rank_p H_E^{2*'+2p}(H) = rank_p H_E^{2*'+1+2p}(H) = p+1.$$

Since all elements in $H^{*+2p}(BE)/(c_p)$ are p-torsion for $* > 0$, we see

$$rank_p H_E^{2*'+2p}(H;\mathbb{Z}/p) = 2 rank_p H_E^{2*'+2p}(H) = 2(p+1).$$

For each $0 \le j \le p$, we still know $H_E^*(H_j\mathbb{Z}/p) \cong H_E^*(H_0;\mathbb{Z}/p)$. Hence $rank_p H_E^*(H_0;\mathbb{Z}/p) = 2$. Thus the spectral sequence collapses. □

Lemma 3.3. *The cycle map $cl : CH^*(BE) \to H^{2*}(BE)$ is isomorphic for $* \le 2p$.*

Proof. Let $* \ge 1$. Consider the diagram

$$\begin{array}{ccccc}
CH_E^*(H) & \xrightarrow{i_{CH*}} & CH_E^{*+p}(\mathbb{C}^{p*}) & \longrightarrow & CH_E^{*+p}(\mathbb{C}^{p*} - H) = 0 \\
\downarrow{\scriptstyle cl_1} & & \downarrow{\scriptstyle cl_2} & & \downarrow{\scriptstyle cl} \\
\to H_E^{2*}(H) & \xrightarrow{i_{H*}} & H_E^{2*+2p}(\mathbb{C}^{p*}) & \longrightarrow & H_E^{2*+2p}(\mathbb{C}^{p*} - H) = 0.
\end{array}$$

Since $H_E^{2*+2p-1}(\mathbb{C}^{p*} - H) = 0$, we see i_{H*} is an isomorphism. From the preceding lemma, $H_E^{2*}(H_j)$ generated by Chern classes (e.g., y_1 for H_0). Hence the cycle map cl_1 is isomorphic for $* \le p$ from Lemma 2.3. Therefore

$$cl_2 \cdot i_{CH*} = i_{H*} \cdot cl_1$$

is isomorphic. □

From Corollary 2.4, we have the isomorphism $CH^*(BE) \cong H^{2*}(BE)$ for all $* \ge 0$. Thus we prove Theorem 1.1 in the introduction when $G = p_+^{1+2}$.

4. Other groups $M = p_-^{1+2}$, D_8 and Q_8

We consider the other groups in this section. Let $M = p_-^{1+2}$ for an odd prime. This case $a^p = c$ and the representation $\tilde{c}$ is given as

$$\tilde{c}(a) = diag(\xi, \xi^{1+p}, \xi^{1+2p}, ..., \xi^{1+(p-1)p})$$

and $\tilde{c}(b)$ is the permutation matrix as the case E, where ξ is a p^2-th primitive root of the unity, i.e., $\xi^p = \zeta$.

The fixed points set of the subgroup $\langle b\rangle$ is given by

$$W^{\langle b\rangle} = \{(x, ..., x)|x \in \mathbb{C}^*\} = \mathbb{C}^*\{e'\} \quad e' = (1, ..., 1).$$

Since $a^{-i}ba^i = bc^i$, we see $W^{\langle bc^i\rangle} = \mathbb{C}\{a^{-i}e'\}$. So M acts on

$$H = \mathbb{C}^*\{e', ae', ..., .a^{p-1}e'\}.$$

Note $(a^ibc^j)^p = c^i$ for $1 \le i \le p-1$ (but $(ab)^2 = 1$ for $G = D_8$). Hence for all $x \in \mathbb{C}^{p*}$, $a^ibc^j(x) \ne x$. Thus we can see that M acts freely on $U - H$, i.e., Lemma 3.1 holds for $G = M$.

Next we will see Lemma 3.2 by $H = H_0$ for $G = M$. We consider the spectral sequence

$$E_2^{*,*'} = H^*(\langle b\rangle; H^{*'}_{\langle a\rangle}(H; \mathbb{Z}/p)) \Longrightarrow H^*_M(H; \mathbb{Z}/p).$$

Since $\langle a\rangle$ acts freely on H, we see

$$H/\langle a\rangle \cong \mathbb{C}^*\{e', ..., a^{p-1}e'\}/\langle a\rangle \cong \mathbb{C}^*/\langle a^p\rangle.$$

Therefore we have $H_{\langle a\rangle}(H; \mathbb{Z}/p) \cong H^*(\mathbb{C}^*/\langle a^p\rangle; \mathbb{Z}/p) \cong \Lambda(z)$ as the case $G = E$. From Theorem 2.6, we know

$$H_M^{2*+2p}(\mathbb{C}^{p*}) \cong \mathbb{Z}/p\{y_2^{*+p}\}.$$

This implies $rank_p H_M^{2*+2p}(H) = 1$. Therefore the spectral sequence collapses. Lemma 3.3 holds for $G = M$ and we see $CH^*(BM) \cong H^{2*}(BM)$.

Next, we consider the case $G = D_8$ and $p = 2$. Then the represntation can be took as the case $G = M$. Take

$$H_0 = \mathbb{C}^*\{e', ae'\}, \quad H_1 = \mathbb{C}^*\{g^{-1}e', g^{-1}ae'\}$$

where $g \in GL_2(\mathbb{C})$ with $g^{-1}bg = ab$ (note $(ab)^2 = 1$). Let $H = H_0 \bigvee H_1$. Then D_8 acts freely on $\mathbb{C}^{2*} - H$. In fact from Theorem 2.7, we know

$$H_{D_8}^{2*+4}(\mathbb{C}^{2*}) \cong \mathbb{Z}/2\{y_1^{*+2}, y_2^{*+2}\}.$$

Hence all arguments work as the case E or M.

At last we consider the case $G = Q_8$. The representation $\tilde{c}$ is given

$$\tilde{c}(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tilde{c}(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can easily see that Q_8 acts freely on $\mathbb{C}^{2*}$. Therefore

$$CH_{Q_8}(\mathbb{C}^{2*}) \cong CH^*(\mathbb{C}^{2*}/Q_8)$$

which is generated by $dim \leq 2$. In fact

$$H^*(BD_8)/(c_2) \cong \mathbb{Z}/2[y_1, y_2]/(y_i^2, 2y_i, y_1y_2).$$

5. MOTIVIC COHOMOLOGY

We recall the motivic cohomology. Let X be a smooth (quasi projective) variety over a field $k \subset \mathbb{C}$. Let $H^{*,*'}(X;\mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology defined by Voevodsky and Suslin ([Vo1-4]). Recall that the Belinson-Lichtenbaum conjecture holds if

$$H^{m,n}(X;\mathbb{Z}/p) \cong H^m_{et}(X;\mu_p^{\otimes n}) \qquad for\ all\ m \leq n.$$

Recently M.Rost and V.Voevodsky ([Vo5],[Su-Jo]) proved the Bloch-Kato conjecture. The Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture.

We assume that k contains a p-th root ζ of unity. Then there is the isomorphism $H^m_{et}(X;\mu_p^{\otimes n}) \cong H^m_{et}(X;\mathbb{Z}/p)$. Let τ be a generator of $H^{0,1}(Spec(k);\mathbb{Z}/p) \cong \mathbb{Z}/p$, so that

$$colim_i \tau^i H^{*,*'}(X;\mathbb{Z}/p) \cong H^*_{et}(X;\mathbb{Z}/p).$$

We define the weight degree $w(x) = 2m - n$ if $0 \neq x \in H^{m,n}(X;\mathbb{Z}/p)$. Then it is known $w(x) \geq 0$ for smooth X.

Let $H^*(X;H^{*'}_{\mathbb{Z}/p})$ be the cohomology of the Zarisky sheaf induced from the presheaf $H^*_{et}(V;\mathbb{Z}/p)$ for open subsets V of X. This sheaf cohomology is isomorphic to the E_2-term

$$E_2^{*,*'} \cong H^*(X;H^{*'}_{\mathbb{Z}/p}) \Longrightarrow H^*_{et}(X;\mathbb{Z}/p)$$

of the coniveau spectral sequence by Bloch-Ogus [Bl-Og]. We also note

$$H^0(X;H^{*'}_{\mathbb{Z}/p}) \subset H^{*'}(k(X);\mathbb{Z}/p).$$

The relation between this cohomology and the motivic cohomoloy is given as follows.

Theorem 5.1. *([Or-Vi-Vo], [Vo5]) There is the long exact sequence*

$$\to H^{m,n-1}(X;\mathbb{Z}/p) \xrightarrow{\times\tau} H^{m,n}(X;\mathbb{Z}/p)$$

$$\to H^{m-n}(X;H^n_{\mathbb{Z}/p}) \to H^{m+1,n-1}(X;\mathbb{Z}/p) \xrightarrow{\times\tau}.$$

In particular, we have

Corollary 5.2. *The cohomology $H^{m-n}(X;H^n_{\mathbb{Z}/p})$ is (additively) isomorphic to*

$$H^{m,n}(X;\mathbb{Z}/p)/(\tau)\oplus Ker(\tau)|H^{m+1,n-1}(X;\mathbb{Z}/p)$$

where $H^{m,n}(X;\mathbb{Z}/p)/(\tau)=H^{m,n}(X;\mathbb{Z}/p)/(\tau H^{m,n-1}(X;\mathbb{Z}/p))$.

Corollary 5.3. *The map $\times\tau: H^{m,m-1}(X;\mathbb{Z}/p)\to H^{m,m}(X;\mathbb{Z}/p)$ is injective.*

By using above theorems, we can do computations for concrete cases. Suppose $k=\mathbb{C}$. Then the realization (cycle map)

$$t_{\mathbb{C}}=cl: H^{*,*'}(X;\mathbb{Z}/p)\to H^*_{et}(X;\mathbb{Z}/p)\cong H^*(X;\mathbb{Z}/p)$$

can be identified with

$$\times\tau^{*-*'}: H^{*,*'}(X;\mathbb{Z}/p)\to H^{*,*}(X;\mathbb{Z}/p)\cong H^*_{et}(X;\mathbb{Z}/p),$$

from the Beilinson-Lichtembaum conjecture.

We define the motivic filtration of $H^*(X;\mathbb{Z}/p)$ by

$$F_i=Im(t_{\mathbb{C}}^{2*-i,*})=\oplus_n t_{\mathbb{C}}(H^{2n-i,n}(X;\mathbb{Z}/p)),$$

namely, $x\in F_i$ if $x=t_{\mathbb{C}}(x')$ for some $x'\in H^{*,*'}(X;\mathbb{Z}/p)$ with $w(x')=i$. Let us write the associated grade ring

$$\oplus_i F_i/F_{i-1}=gr^{*'}H^*(X;\mathbb{Z}/p),\quad F_i/F_{i-1}=gr^iH^*(X;\mathbb{Z}/p).$$

In [Ya2], we define

$$h^{*,*'}(X;\mathbb{Z}/p)=\oplus_{m,n}H^{m,n}(X;\mathbb{Z}/p)/(Ker(t_{\mathbb{C}}^{m,n})),$$

and compute them for some cases of $X=BG$. (Note if $Ker(t_{\mathbb{C}}^{*,*'})=0$, then $H^{*,*'}(X;\mathbb{Z}/p)\cong h^{*,*'}(X;\mathbb{Z}/p)$.) It is immediate that

$$h^{m,n}(X;\mathbb{Z}/p)\cong\oplus_{i=0}gr^{2(n+i)-m}H^m(X;\mathbb{Z}/p)\{\tau^i\}.$$

We will simply write (for ease of notations) the above isomorphism

$$h^{*,*'}(X;\mathbb{Z}/p)\cong gr^{*'}H^*(X;\mathbb{Z}/p)\otimes\mathbb{Z}/p[\tau].$$

Lemma 5.4. *Suppose $dim(X)\le 2$ and*

$$H^*(\mathbb{C}(X);\mathbb{Z}/p)=H^*(Spec(\mathbb{C}(X);\mathbb{Z}/p)\cong 0\quad for\ *\ge 3.$$

Then $H^{*,*'}(X;\mathbb{Z}/p)\cong 0$ for all $*>4$. Moreover we have the bidegree isomorphism $H^{*,*'}(X;\mathbb{Z}/p)\cong h^{*,*'}(X;\mathbb{Z}/p)$.

Proof. It is know that

$$H^{*,*'}(X;\mathbb{Z}/p)\cong 0\quad if\ \ *-*'>dim(X).$$

So we only need to show $H^{*,*'}(X;\mathbb{Z}/p)\cong 0$ for $*>4$ when

$$*=*',\quad *=*'-1,\quad or\quad *=*'-2.$$

Let $* > 4$. The first case follows from $H^*(X;\mathbb{Z}/p) \cong 0$ and the Beilinson-Lichtenbaum conjecture. The second case follows from that the map $H^{*,*-1}(X;\mathbb{Z}/p) \to H^{*,*}(X;\mathbb{Z}/p)$ is injective.

We will show the last case. Since $\tau^2 H^{*,*-2}(X;\mathbb{Z}/p) = 0$ and the injectivity above, we see $\tau H^{*,*-2}(X;\mathbb{Z}/p) = 0$. Suppose $0 \neq x \in H^{*,*-2}(X;\mathbb{Z}/p)$. Then there is $y \in H^0(X;H^{*-1}_{\mathbb{Z}/p})$ so that $d_2 y = x$ in the coniveau spectral sequence. However we still know

$$H^0(X;H^{*-1}_{\mathbb{Z}/p}) \subset H^{*-1}(\mathbb{C}(X);\mathbb{Z}/p) = 0, \quad for * \geq 4.$$

This is a contradiction and $H^{*,*-2}(X;\mathbb{Z}/p) = 0$.

The above argument also show that $Ker(\tau)|H^{4,2}(X;\mathbb{Z}/2) = 0$,i.e, $* = 4$ case. This means $H^{*,*'}(X;\mathbb{Z}/p) \to H^*(X;\mathbb{Z}/p)$ is injective for all $* \geq 0$. Hence we get the lemma. □

Corollary 5.5. *Let $X = \mathbb{C}^{2*}/Q_8$ or $X = (\mathbb{C}^{2*} - H)/D_8$ for the action given in §4. Then $H^{*,*'}(X;\mathbb{Z}/2) \cong h^{*,*'}(X;\mathbb{Z}/2)$.*

Proof. We only need to prove $H^*(\mathbb{C}(X);\mathbb{Z}/2) = 0$ for $* \geq 3$. We prove it for $G = D_8$, and the case Q_8 is similar.

Let $\mathbb{C}^2//G = Spec(\mathbb{C}[t,s]^G)$ be the geometric quotient by G. Then $X = (\mathbb{C}^2 - H)/G$ is an open set in $\mathbb{C}^2//G$. So $\mathbb{C}(X) \cong \mathbb{C}(t,s)^G$; the quotient field of the invariant ring $\mathbb{C}[t,s]^G$. The group $G = D_8$ satisfies the Nother's probem so that $\mathbb{C}(X)$ is purely transcendental over $\mathbb{C}$, i.e. $\mathbb{C}(X) \cong \mathbb{C}(t', s')$.

(This fact is easily seen since

$$\mathbb{C}[t,s]^{D_8} = \mathbb{C}[ts, t^4 + s^4] \subset \mathbb{C}[t,s],$$

where the action is given by $a : \begin{cases} t \mapsto it \\ s \mapsto -is \end{cases}, \quad b : \begin{cases} t \mapsto s \\ s \mapsto t \end{cases}$.)

Since it is well known $H^*(\mathbb{C}(t', s');\mathbb{Z}/2) \cong \Lambda(x_1, x_2)$ with $|x_i| = 1$, we get the result. □

6. Motivic cohomology of BD_8 and BQ_8

In this section, we compute the $mod(2)$ motivic cohomology of BD_8 and BQ_8.

At first, we consider the case Q_8. The mod 2 (usual) cohomology is well known (see Theorem 2.7)

$$H^*(BQ_8;\mathbb{Z}/2) \cong \mathbb{Z}/2\{1, x_1, y_1, x_2, y_2, w\} \otimes \mathbb{Z}/2[c_2]$$

where $x_i^2 = \beta x_i = y_i$ and $|w| = 3$. The graded algebra $gr^{*'}H^*(BQ_8;\mathbb{Z}/2)$ is given by letting the weight degree by

$$w(y_i) = w(c_2) = 0, \qquad w(x_i) = w(w) = 1.$$

Theorem 6.1. *There is the bidegree isomorphism*

$$H^{*,*'}(BQ_8;\mathbb{Z}/2)\cong \mathbb{Z}/2[\tau]\otimes gr^{*'}H^*(BQ_8;\mathbb{Z}/2).$$

Proof. Let $G=Q_8$. In the usual $mod(2)$ cohomology

$$H_G^*(\mathbb{C}^{2*};\mathbb{Z}/2)\cong H^*(BG;\mathbb{Z}/2)/(c_2)\cong \mathbb{Z}/2\{1,x_1,y_1,x_2,y_2,w\},$$

which is isomorphic to $H^*(\mathbb{C}^{2*}/Q_8;\mathbb{Z}/2)$. Hence we can use Corollary 5.5

$$H_G^{*,*'}(\mathbb{C}^{2*};\mathbb{Z}/2)\cong \mathbb{Z}/2[\tau]\otimes \mathbb{Z}/2\{1,x_1,y_1,x_2,y_2,w\}.$$

Here $deg(w)=(3,2)$ by the folowing reason. The Bockstein exact sequence also exists in the motivic cohomology

$$\to H^{*-1,*'}(BG;\mathbb{Z}/2)\to H^{*,*'}(BG;\mathbb{Z})\xrightarrow{\times 2} H^{*,*'}(BG;\mathbb{Z})\to .$$

Since $c_2\in H^{4,2}(BG)$ and $4c_2=0$, we see $w\in H^{3,2}(BG;\mathbb{Z}/2)$.

Using above facts, we can show the lower sequence in the following diagram is exact

$$\begin{array}{ccccc}
\to H^{*-4,*'-2}(BG;\mathbb{Z}/2) & \xrightarrow{c_2} & H^{*,*'}(BG;\mathbb{Z}/2) & \longrightarrow & H_G^{*,*'}(\mathbb{C}^{2*};\mathbb{Z}/2)\to \\
\downarrow & & \downarrow & & \cong\downarrow \\
\to h^{*-4,*'-2}(BG;\mathbb{Z}/2) & \xrightarrow{c_2} & h^{*,*'}(BG;\mathbb{Z}/2) & \longrightarrow & h_G^{*,*'}(\mathbb{C}^{2*};\mathbb{Z}/2)\to
\end{array}$$

where $h_G^{*,*'}(X;\mathbb{Z}/2)=\mathbb{Z}/2[\tau]\otimes gr^{*'}H_G^*(X;\mathbb{Z}/2)$.

By induction on $*\geq 0$ and the five lemma, we easily see that the vertical maps are isomorphic. □

Now we consider the case $G=D_8$. We recall the $mod(2)$ cohomology.

$$H^*(BD_8;\mathbb{Z}/2)\cong (\mathbb{Z}/2[x_1]\oplus\mathbb{Z}/2[x_2])\otimes\mathbb{Z}/2[u]\cong$$

$$(\mathbb{Z}/2[y_1]\{y_1,x_1,y_1u,x_1u\}\otimes\mathbb{Z}/2[y_2]\{y_2,x_2,y_2u,x_2u\}\oplus\mathbb{Z}/2\{1,u\})\otimes\mathbb{Z}/2[c_2]$$

Here we identify, $y_i=x_i^2$ and $c_2=u^2$. The cohomology operations on $H^*(BD_8;\mathbb{Z}/2)$ is well known, e.g.,

$$Q_0(u)=(x_1+x_2)u=e,\quad Q_1Q_0(u)=(y_1+y_2)c_2.$$

Lemma 6.2. *There exist $u_1',u_2'\in H^{3,2}(BD_8;\mathbb{Z}/2)$ with $\tau u_i'=x_iu\in H^{3,3}(BD_8;\mathbb{Z}/2)$ (so $u_i'=\tau^{-1}x_iu$).*

Proof. First note

$$H^{3,2}(BG;\mathbb{Z})\supset\mathbb{Z}/2\{Q_0(u)\},\quad H^{4,2}(BG;\mathbb{Z})\cong\mathbb{Z}/2\{y_1^2,y_2^2\}\oplus\mathbb{Z}/4\{c_2\}.$$

From the Bockstein exact sequence, we see

$$rank_pH^{3,2}(BG;\mathbb{Z}/2)\geq 1+3=4.$$

From the Beilinson-Lichtenbaum conjecture and Corollary 5.3, we see that $H^{*,*'}(X;\mathbb{Z}/p)\to H^*(X;\mathbb{Z}/p)$ is injective for $*\le 3$. On the other hand

$$H^3(BG;\mathbb{Z}/2)\cong \mathbb{Z}/2\{x_1u,x_2u,x_1y_1,x_2y_3\}.$$

Hence each element in $H^3(BG;\mathbb{Z}/2)$ must be in $H^{3,2}(BG;\mathbb{Z}/2)$. □

Therefore we get $gr^{*'}H^*(BD_8;\mathbb{Z}/2)$ which is isomorphic to

$$(\mathbb{Z}/2[y_1]\{y_1,x_1,x_1u_1',u_1'\}\oplus\mathbb{Z}/2[y_2]\{y_2,x_2,x_2u_2',u_2'\}\oplus\mathbb{Z}/2\{1,u\})\otimes\mathbb{Z}/2[c_2]$$

with $w(y_i)=w(c_2)=0$, $w(x_i)=w(u_i')=1$ and $w(u)=2$ (note $u\notin CH^*(BG)/2$), and $x_iu_i'=y_iu$.

Theorem 6.3. *There is the the bidegree module isomorphism*

$$H^{*,*'}(BD_8;\mathbb{Z}/2)\cong\mathbb{Z}/2[\tau]\otimes gr^{*'}H^*(BD_8;\mathbb{Z}/2).$$

Before the proof of this theorem, we give a lemma.

Lemma 6.4.

$$H^{*,*'}_{D_8}(H_0,\mathbb{Z}/2)\cong\mathbb{Z}/2[\tau]\otimes\mathbb{Z}/2[y_1]\otimes\Lambda(x_1,z)\quad \textit{with } deg(z)=(1,1).$$

Proof. Let $G=D_8$. We consider the exact sequence

$$\to H_G^{*-2,*'-1}(\{0\}\times H_0;\mathbb{Z}/2)\xrightarrow{y_1}H_G^{*,*'}(\mathbb{C}\times H_0;\mathbb{Z}/2)\to H_G^{*,*'}(\mathbb{C}^*\times H_0;\mathbb{Z}/2)\to .$$

Here G acts freely on $\mathbb{C}^*\times H_0$ and

$$H_G^{*,*'}(\mathbb{C}^*\times H_0;\mathbb{Z}/2)\cong H^{*,*'}(\mathbb{C}^*\times H_0/G;\mathbb{Z}/2)$$
$$\cong H^{*,*'}(\mathbb{C}^*/\langle b\rangle\times\mathbb{C}^*/\langle a^2\rangle;\mathbb{Z}/2)$$
$$\cong H^{*,*'}(\mathbb{C}^*/\langle b\rangle;\mathbb{Z}/2)\otimes_{\mathbb{Z}/2[\tau]}H^{*,*'}(\mathbb{C}^*/\langle a^2\rangle;\mathbb{Z}/2)\cong\mathbb{Z}/2[\tau]\otimes\Lambda(x_1,z)$$

since $H^{*,*}(\mathbb{C}^{n*}/(\mathbb{Z}/p);\mathbb{Z}/p)$ holds the Kunneth formula.

The natural map $H_G^{*,*'}(H_0;\mathbb{Z}/2)\to\mathbb{Z}/2[\tau]\otimes H_G^*(H_0;\mathbb{Z}/2)$ induces the diagram for two exact sequences. We can prove the lemma by induction on $*\ge 0$ and the five lemma. □

Proof of Theorem 6.3. Let $G=D_8$. First we consider the exact sequence

$$\to H_G^{*-2}(H;\mathbb{Z}/2)\xrightarrow{i_*}H_G^*(\mathbb{C}^{2*};\mathbb{Z}/2)\to H_G^*(\mathbb{C}^{2*}-H;\mathbb{Z}/2)\to .$$

Recall that

$$H_G^*(\mathbb{C}^{2*};\mathbb{Z}/2)\cong A\{1,u\}\quad \textit{with } A=\mathbb{Z}/2[y_1]\otimes\Lambda(x_1)\oplus\mathbb{Z}/2[y_2]\otimes\Lambda(x_2).$$

Using fact that i_* is isomorphic for $*>4$, The map i_* is given explicitly

$$i_*(1_1)=y_1,\ i_*(1_2)=y_2,\quad i_*(z_1)=x_1u,\ i_*(z_2)=x_2u$$

where we use and $1_i, z_i$ are the generators in $H^*_G(H_{i-1};\mathbb{Z}/2)$. (Note $i_*(x_iz_i) = y_iu$ since $y_i = x_i^2$ in $H^*(BG;\mathbb{Z}/2)$.) Therefore

$$H^*((\mathbb{C}^{2*} - H)/G;\mathbb{Z}/2) \cong \mathbb{Z}/2\{1, x_1, x_2, u\}.$$

Hence from Corollary 5.5, we have

$$H^{*,*'}((\mathbb{C}^{2*} - H)/G;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes \mathbb{Z}/2\{1, x_1, x_2, u\}.$$

Here $deg(u) = (2, 2)$ (i.e., $w(u) = 2$) since $CH^1_G((\mathbb{C}^{2*} - H)/G) = 0$.

Next we consider the following diagram

$$\begin{array}{ccccccc} \to H_G^{*-2,*'-1}(H;\mathbb{Z}/2) & \xrightarrow{i_*} & H_G^{*,*'}(\mathbb{C}^{2*};\mathbb{Z}/2) & \longrightarrow & H_G^{*,*'}(\mathbb{C}^{2*} - H;\mathbb{Z}/2) \to \\ \downarrow & & \downarrow & & \cong\downarrow \\ \to h_G^{*-2,*'-1}(H;\mathbb{Z}/2) & \xrightarrow{i_*} & h_G^{*,*'}(\mathbb{C}^{2*};\mathbb{Z}/2) & \longrightarrow & h_G^{*,*'}(\mathbb{C}^{2*} - H;\mathbb{Z}/2) \to . \end{array}$$

Here the lower sequence is also exact from the above arguments for $H^*_G(\mathbb{C}^{2*} - H;\mathbb{Z}/2)$, and from Lemma 6.4 and the isomorphism just before Theorem 6.3.

By using the induction on $* \geq 0$ and five lemma, we get

$$H_G^{*,*'}(\mathbb{C}^{2*};\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes gr^{*'}H^*_G(\mathbb{C}^{2*};\mathbb{Z}/2)$$

where $w(u) = 2$, $w(u') = 1$.

As the case $G = Q_8$, we can see

$$H^{*,*'}(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes gr^{*'}H^*_G(\mathbb{C}^{2*};\mathbb{Z}/2) \otimes \mathbb{Z}/2[c_2].$$

□

7. MOTIVIC COBORDISM OF BQ_8 AND BD_8

Let $MU^*(X)$ and $MU^*(X;\mathbb{Z}/p)$ be the usual complex cobordism theory and its *mod p* theory. Let $MGL^{*,*'}(X)$ be th motivic cobordism theory defined by Voevodsky [Vo1]. Since $t_{\mathbb{C}}|CH^*(BG)$ is injective, from Proposition 9.4 in [Ya3], we have the isomorphism

$$MGL^{2*,*}(BG) \cong MU^{2*}(BG)$$

for each group of order p^3.

In this section, we give rather strong results for only Q_8 and D_8. Let $MGL^{*,*'}(X;\mathbb{Z}/p)$ be the mod p theory defined by the exact sequence

$$\to MGL^{*,*'}(X) \xrightarrow{\times p} MGL^{*,*'}(X) \xrightarrow{\rho} MGL^{*,*'}(X;\mathbb{Z}/p) \xrightarrow{\delta} .$$

Then we have the following theorem, which holds also for $(\mathbb{Z}/p)^n, O_n, SO_n$. (For accurate definition for $MGL^{2*,*}(BG)$ see [De].)

Theorem 7.1. *Let $G = Q_8$ or D_8. Then there are isomorphisms*

$$MGL^{*,*'}(BG;\mathbb{Z}/p) \cong MGL^{2*,*}(BG;\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau],$$

$$MGL^{2*,*}(BG;\mathbb{Z}/p) \cong MU^{2*}(BG;\mathbb{Z}/p) \cong MU^{2*}(BG)/p.$$

Proof. Let $G = Q_8$ or D_8. Let $E(MGL)_r^{*,*',*''}$ (resp. $E(MU)_r^{*,*''}$) be the Atiyah-Hirzebruch spectral sequence converging to $MGL^{*,*'}(BG;\mathbb{Z}/2)$ (resp. $MU^*(BG;\mathbb{Z}/2)$) (see [Ya3]), namely,

$$E(MGL)_2^{*,*',*''} \cong H^{*,*'}(BG;\mathbb{Z}/2) \otimes MU^{*''} \Longrightarrow MGL^{*,*'}(BG;\mathbb{Z}/2),$$

$$E(MU)_2^{*,*',*''} \cong H^*(BG;\mathbb{Z}/2) \otimes MU^{*''} \Longrightarrow MU^*(BG;\mathbb{Z}/2).$$

The realization map $t_{\mathbb{C}}$ induces the map $t_{\mathbb{C}}^{*,*',*''} : E(MGL)_r^{*,*',*''} \to E(MU)_r^{*,*''}$ of spectral sequences.

From Theorem 6.1 and 6.3, we know

$$H^{*,*'}(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes gr^{*'}H^*(BG;\mathbb{Z}/2).$$

Let us write $gr^{*'}E(MU)_2^{*,*''} = gr^{*'}H^*(BG;\mathbb{Z}/2) \otimes MU^{*''}$ so that we have the the bidegree module isomorphism

$$E(MGL)_2^{*,*',*''} \cong \mathbb{Z}/2[\tau] \otimes gr^{*'}E(MU)_2^{*,*''}.$$

Suppose that for all $x \in gr^{*'}E(MU)_2^{*,*''} \subset E(MGL)_2^{*,*',*''}$,

$$(*) \quad d_2(x) \in gr^{*'}E(MU)_2^{*,*''} \quad (i.e., \ \ d_2(x) \neq \tau y \ for\ some\ \tau y \neq 0).$$

Then from the naturality of the map $t_{\mathbb{C}}^{*,*''}$ of spectral sequences, we have

$$E(MGL)_3^{*,*',*''} \cong \mathbb{Z}/2[\tau] \otimes gr^{*'}E(MU)_3^{*,*''}$$

where $gr^{*'}E(MU)_3^{*,*''}$ is the bidegree module made from $grE(MU)_3^{*,*''}$ giving the same second degree. Moreover, if for all $x \in gr^{*'}E(MU)_r^{*,*''}$, $r \geq 2$

$$(**) \quad d_r(x) \in gr^{*'}E(MU)_r^{*,*''},$$

then we have the bidegree isomorphism

$$E(MGL)_\infty^{*,*',*''} \cong \mathbb{Z}/2[\tau] \otimes gr^{*'}E(MU)_\infty^{*,*''},$$

and we can prove this theorem.

To see $(*), (**)$, we note that $gr^{*'}H^*(BG;\mathbb{Z}/2)$ is generated by elements of degree $w(x) \leq 1$ (resp. $w(x) \leq 2$ e.g., $w(u) = 2$) for $G = Q_8$ (resp. $G = D_8$). Since $w(\tau) = 2$, we have

$$H^{2*,*}(BG;\mathbb{Z}/2) \oplus H^{2*+1,*}(BG;\mathbb{Z}/2) \subset gr^{*'}H^*(BG;bZ/2).$$

Then $(*), (**)$ are immediate since $w(d_r(x)) = w(x) - 1 \leq 1$. □

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