

NOTE ON CHOW RINGS OF NONTRIVIAL G -TORSORS OVER A FIELD

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ABSTRACT. Let G_k be a split reductive group over a field k corresponding to a compact Lie group G . Let E be a nontrivial G_k -torsor over a field k . In this paper we study the Chow ring of nontrivial G_k -torsors. For example when $(G, p) = (F_4, 3)$, we have the isomorphism $CH^*(E)_{(3)} \cong \mathbb{Z}_{(3)}$.

1. INTRODUCTION

Let k be a subfield of \mathbb{C} which contains primitive p -th root of the unity. Let G be a compact connected Lie group. Let us denote by G_k the split reductive group over k which corresponds G . By definition, a G_k -torsor E over k is a variety over k with a free G_k -action such that the quotient variety is $\text{Spec}(k)$. A G_k -torsor over k is called trivial, if it is isomorphic to G_k or equivalently it has a k -rational point.

Let H be a subgroup of G . Given a torsor E over k , we can form the twisted form of G/H by

$$(E \times G_k/H_k)/G_k \cong E/H_k.$$

In this paper, we always assume that E is a (inner) *nontrivial* torsor over k . We mainly study the cases that G are exceptional Lie groups and the (p component) torsion index $t(G)_{(p)} = p$. In particular, when $(G, p) = (G_2, 2), (F_4, 3)$ and $H = T$; the maximal torus, $H = P$ maximal parabolic subgroups, we compute $CH^*(E/H_k)_{(p)}$ explicitly. Moreover we show $CH^*(E)_{(p)} = \mathbb{Z}_{(p)}$ for these cases. We also study the case $(G, p) = (SO_{2n+1-1}, 2), n \geq 3$. This case $CH^*(E)_{(2)} \subset CH^*(G_k)_{(2)}$ but it is not isomorphic to $\mathbb{Z}_{(2)}$ nor $CH^*(G_k)_{(2)}$.

For these groups, Petrov, Semenov and Zainoulline [Pe-Se-Za] showed that the Chow motive of E/P_k and E/T_k are isomorphic to direct sums of the generalized Rost motive ([Vo4],[Ro2],[Su-Jo],[Vi-Za]). The algebraic cobordism $MGL^{2*,*}(-)$ of the Rost motives are given in [Vi-Ya],[Ya3]. From this, we show the multiplicative structures of

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$CH^*(E/P_k)$ and $CH^*(E/T_k)$. The algebraic cobordism $MGL^{2*,*}(G_k)$ is studied in [Ya1]. By using arguments in [Ya1], we can compute $CH^*(E)_{(p)}$.

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2. ROST MOTIVE

Let k be a field of $ch(k) = 0$ and X the smooth variety. We consider the Chow ring $CH^*(X)$ generated by cycles modulo rational equivalence. For a non zero symbol $a = \{a_0, \dots, a_n\}$ in the mod 2 Milnor K-theory $K_{n+1}^M(k)/2$, let $\phi_a = \langle \langle a_0, \dots, a_n \rangle \rangle$ be the $(n+1)$ -fold Pfister form. Let X_{ϕ_a} be the projective quadric of dimension $2^{n+1} - 2$ defined by ϕ_a . The Rost motive $M_a (= M_{\phi_a})$ is a direct summand of the motive $M(X_{\phi_a})$ representing X_{ϕ_a} so that $M(X_{\phi_a}) \cong M_a \otimes M(\mathbb{P}^{2^n-1})$.

Moreover for an odd prime p and nonzero symbol $0 \neq a \in K_{n+1}^M/p$, we can define ([Ro],[Vo],[Su],[Vi-Za]) the generalized Rost motive M_a , which is irreducible and is split over K/k if and only if $a|_K = 0$ as the case $p = 2$.

The Chow ring of the Rost motive is well known. Let \bar{k} be an algebraic closure of k , $X|_{\bar{k}} = X \otimes_k \bar{k}$, and $i_{\bar{k}} : CH^*(X) \rightarrow CH^*(X|_{\bar{k}})$ the restriction map.

Lemma 2.1. (*Rost [R1],[Vo4], [Vi-Ya], [Ya3]*) *The Chow ring $CH^*(M_a)$ is only dependent on n . There are isomorphisms*

$$CH^*(M_a) \cong \mathbb{Z}\{1\} \oplus (\mathbb{Z}\{c_0\} \oplus \mathbb{Z}/p\{c_1, \dots, c_{n-1}\})[y]/(c_i y^{p-1})$$

$$\text{and } CH^*(M_a|_{\bar{k}}) \cong \mathbb{Z}[y]/(y^p)$$

where $|y| = 2(p^{n-1} + \dots + p + 1)$ and $|c_i| = |y| + 2 - 2p^i$. Here the multiplications are given by $c_i \cdot c_j = 0$ for all $0 \leq i, j \leq n-1$. Moreover the restriction map is given by $i_{\bar{k}}(c_0) = py$ and $i_{\bar{k}}(c_i) = 0$ for $i > 0$.

Remark. The element y does not exist in $CH^*(M_a)$ while $c_i y$ exists. Usually $CH^*(M_a)$ is defined only additively, however we give the ring structure as above in this paper.

Remark. In this paper the degree $|x|$ of an element $x \in CH^*(X)$ means the 2-times of the usual degree of the Chow ring so that it is compatible with the degree of the (topological) cohomology $H^*(X(\mathbb{C}))$.

Let us use notation $\Omega^*(X)$ for the motivic cobordism $MGL^{2*,*}(X)_{(p)}$ defined by Voevodsky (but not the algebraic cobordism of Levine and Morel, because Lemma 2.2 bellow is not proved for this theory). It is known that

$$\Omega^* = \Omega^*(pt.) \cong MU^{2*}(pt.)_{(p)} \cong \mathbb{Z}_{(p)}[x_1, x_2, \dots]$$

where $MU^{2*}(pt.)$ is the complex cobordism ring and $|x_i| = -2i$. There is the relation ([Ya2])

$$(2.1) \quad \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(p)} \cong CH^*(X)_{(p)}.$$

We can take for x_{p^i-1} the cobordism class of a $2(p^i - 1)$ -dimensional manifold whose characteristic numbers are divisible by p but the additive characteristic number s_{p^i-1} is not by p^2 . Let us denote x_{p^i-1} as v_i . Let us denote by

$$(2.2) \quad I_n = (p = v_0, v_1, \dots, v_{n-1}) \subset \Omega^*$$

the ideal of Ω^* generated by p, \dots, v_{n-1} . Then it is well known that I_n and I_∞ are the only prime ideals stable under the Landweber-Novikov cohomology operations ([Ra]) in Ω^* .

The category of cobordism motives is defined and studied in [Vi-Ya]. In particular, we can define the algebraic cobordism of motives. The following is the main result in [Vi-Ya] (in [Ya3] for odd primes).

Lemma 2.2. ([Vi-Ya], [Ya3]) *The restriction map*

$$i_{\bar{k}} : \Omega^*(M_a) \rightarrow \Omega^*(M_a|_{\bar{k}}) \cong \Omega^*[y]/(y^p)$$

is injective and there is the Ω^ -algebra isomorphism*

$$\Omega^*(M_a) \cong \Omega^*\{1\} \oplus I_n\{y, \dots, y^{p-1}\} \subset \Omega^*[y]/(y^p)$$

such that $v_i y = c_i$ in $\Omega^(M_a) \otimes_{\Omega^*} \mathbb{Z}_{(p)} \cong CH^*(M_a)_{(p)}$ in (2.2).*

Remark. When $n = 1$, the restriction map $i_{\bar{k}} : CH^*(M_1) \rightarrow CH^*(M_1|_{\bar{k}})$ is injective and

$$\begin{aligned} \text{Im}(i_{\bar{k}}) &\cong \mathbb{Z}_{(p)}\{1\} \oplus \mathbb{Z}_{(p)}[y]/(y^{p-1})\{py\} \\ &\subset \mathbb{Z}_{(p)}[y]/(y^p) \cong CH^*(X|_{\bar{k}})_{(p)}. \end{aligned}$$

Remark. Let $BP\langle n \rangle^* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$. Recall ([Ya2]) that

$$ABP\langle n \rangle^{2*,*}(X) = \Omega^*(X) \otimes_{\Omega^*} BP\langle n \rangle^*$$

for smooth X . Then we also see that

$$i_k : ABP\langle n-1 \rangle^{2*,*}(M_a) \rightarrow ABP\langle n-1 \rangle^{2*,*}(M_a|_{\bar{k}})$$

is injective.

3. COMPACT LIE GROUP G

Let G be a compact connected Lie group. By the Borel theorem, we have the ring isomorphism for p odd

$$(3.1) \quad H^*(G; \mathbb{Z}/p) \cong P(y)/(p) \otimes \Lambda(x_1, \dots, x_l)$$

$$\text{with } P(y) = \mathbb{Z}[y_1, \dots, y_k]/(y_1^{p^{r_1}}, \dots, y_k^{p^{r_k}})$$

where $|y_i| = \text{even}$ and $|x_j| = \text{odd}$. When $p=2$, for each y_i , there is x_j with $x_j^2 = y_i$. Hence we have $grH^*(G; \mathbb{Z}/2) \cong P(y)/(2) \otimes \Lambda(x_1, \dots, x_l)$.

Let T be the maximal torus of G and BT the classifying space of T . We consider the fibering

$$(3.2) \quad G \xrightarrow{\pi} G/T \xrightarrow{i} BT$$

and the induced spectral sequence

$$E_2^{*,*} = H^*(BT; H^*(G; \mathbb{Z}/p)) \implies H^*(G/T; \mathbb{Z}/p).$$

The cohomology of the classifying space of the torus is given by

$$H^*(BT) \cong \mathbb{Z}[t_1, \dots, t_\ell] \quad \text{with } |t_i| = 2.$$

where ℓ is also the number of the odd degree generators x_i in $H^*(G; \mathbb{Z}/p)$. It is known that y_i are permanent cycles and that there is a regular sequence $([\text{Tod}], [\text{Mi-Ni}]) (\bar{b}_1, \dots, \bar{b}_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i) = \bar{b}_i$. Thus we get

$$E_\infty^{*,*'} \cong grH^*(G/T; \mathbb{Z}/p) \cong P(y) \otimes \mathbb{Z}/p[t_1, \dots, t_\ell]/(\bar{b}_1, \dots, \bar{b}_\ell).$$

Moreover we know that G/T is a manifold of torsion free, and we get

$$(3.3) \quad H^*(G/T)_{(p)} \cong \mathbb{Z}_{(p)}[y_1, \dots, y_k, t_1, \dots, t_\ell]/(f_1, \dots, f_k, b_1, \dots, b_\ell)$$

where $b_i = \bar{b}_i \text{ mod}(p)$ and $f_i = y_i^{p^{r_i}} \text{ mod}(t_1, \dots, t_\ell)$. Since G/T is cellular, we also know

$$(3.4) \quad \Omega^*(G/T) \cong \Omega^*[y_1, \dots, y_k, t_1, \dots, t_\ell]/(\tilde{f}_1, \dots, \tilde{f}_k, \tilde{b}_1, \dots, \tilde{b}_\ell)$$

where $\tilde{b}_i = b_i \text{ mod}(\Omega^{<0})$ and $\tilde{f}_i = f_i \text{ mod}(\Omega^{<0})$.

Let G_k be the split reductive algebraic group corresponding G and T_k the split maximal Torus. Hence

$$CH^*(G_k) \cong CH^*(G_{\mathbb{C}}) \cong H^*(G).$$

Similarly $CH^*(G_k/T_k) \cong H^*(G/T)$ and $CH^*(BT_k) \cong H^*(BT)$. Next we consider the relation between $CH^*(G_k)$ and $CH^*(G_k/T_k)$.

Theorem 3.1. (*Grothendieck [Gr], [Ya1]*) *Let E be a G_k -torsor over k . (Here we do not assume the nontiviality of E). Let $h^*(-) = CH^*(-)$ or $\Omega^*(X)$. Then*

$$h^*(E) \cong h^*(E/T_k)/(i^*h^*(BT_k)) \cong h^*(E/T_k)/(t_1, \dots, t_\ell).$$

Proof. Let $L_i \rightarrow E/T_k$ be the line bundle corresponding the element $t_i \in h^2(E/T_k)$. Then we can embed the T_k -bundle $E \rightarrow E/T_k$ into the associated bundle $\oplus_i L_i \rightarrow E/T_k$. such that E is an open subscheme of $\oplus_i L_i$. Consider the localization exact sequence

$$\oplus_i h^*(\oplus_{j \neq i} L_j) \xrightarrow{\oplus s_i^*} h^*(\oplus_i L_i) \rightarrow h^*(E) \rightarrow 0$$

where $s_i : E/T_k \rightarrow L_i$ is the zero section. Since L_i are vector bundles

$$h^*(E) \cong h^*(\oplus_{i \neq j} L_j) \cong h^*(\oplus_i L_i).$$

By the definition the first Chern class, we know $t_i = c_1(L_i) = s_i^* s_{i*}(1)$. Thus we get the desired result $h^*(E) \cong h^*(E/T_k)/(t_1, \dots, t_\ell)$. \square

Since $CH^*(G_k) \cong CH^*(G_{\mathbb{C}})$, from the result for $H^*(G/T)$ (3.3), we know

Corollary 3.2. (*[Kac [Ka]]*) $Ch^*(G_k)_{(p)} \cong P(y)/(py_i)$.

The result $\Omega^*(G_{\mathbb{C}})$ is one of the main result in [Ya1]. Let Q_i be the Milnor primitive operation in $H^*(X; \mathbb{Z}/p)$ inductively defined by $Q_i = [Q_{i-1}, P^{p^{i-1}}]$ and $Q_0 = \beta$; the Bockstein operation where $P^{p^{p-1}}$ is the p^{p-1} -th reduced power operation. It is known that we can take generators such that $Q_i(x_{odd}) \in P(y_{even})/(p)$ for all $i \geq 0$ ([Mi-Ni]).

Theorem 3.3. (*[Ya1]*) *Take generators so that $Q_i(x_{odd}) \in P(y_{even})/(p)$ for all $i \geq 0$. Then there is the Ω^* -module isomorphism*

$$\Omega^*(G_k)/I_\infty^2 \cong \Omega^* \otimes P(y_{even})/(I_\infty^2, \sum_i v_i Q_i(x_{odd})).$$

Let P be a parabolic subgroup. Then the inclusion $T \subset P$ induces the fibering

$$(3.5) \quad P/T \rightarrow G/T \xrightarrow{p} G/P$$

and the spectral sequence (see [Tod])

$$E(E/T)_2^{*,*'} \cong H^*(G/P) \otimes H^*(P/T) \implies H^*(G/T).$$

Since these spaces have no torsion and even dimensionally generated, this spectral sequence is collapses, that is

$$(3.6) \quad gr H^*(G/T) \cong H^*(G/P) \otimes H^*(P/T).$$

Hence $H^*(G/P)$ can be computed from $H^*(G/T)$ (while many cases $H^*(G/P)$ is more easily computed than $H^*(G/T)$).

The cohomology $H^*(P/T)$ can be compute by the fibering

$$P/T \rightarrow BT \xrightarrow{i} BP.$$

Indeed, if i^* is injective, then

$$(3.7) \quad H^*(P/T) \cong H^*(BT)/(i^*\tilde{H}^*(BP)).$$

Note that for the Borel subgroup B , we have the isomorphisms $CH^*(X/T) \cong CH^*(X/B)$ and $CH^*(BB) \cong CH^*(BT)$.

4. EXCEPTIONAL GROUPS OF TYPE (I)

Let G be a compact connected Lie group of $\dim(G) = 2d$. The torsion index is defined by

$$t(G) = |H^{2d}(G/T; \mathbb{Z})/i^*H^{2d}(BT; \mathbb{Z})|.$$

By Grothendieck, it is known that any G_k -torsor E splits over any fields L over k of index dividing $t(G)$. By Totaro all $t(G)$ are recently known [To1,2]. Let us write by $t(G)_{(p)}$ the p -component of $t(G)$. In this section, we restrict the cases $t(G)_{(p)} = p$ (for ease of arguments) and G are exceptional Lie groups. We call such (G, p) is of type (I), that is

$$(G_2, 2), \quad (F_4, 2), \quad (E_6, 2) \\ (F_4, 3), \quad (E_6, 3), \quad (E_7, 3), \quad \text{and} \quad (E_8, 5).$$

Throughout this section, we assume (G, p) are type of (I).

Remark. In [Pe-Se-Za], Petrov, Semenov and Zainoulline defined the J -invariant $J_G(i_1, \dots, i_k)$ of G from the smallest number i_s such that $y_s^{p^{i_s}} \in \text{Im}(i_{\bar{k}})$. (More accurate definition, see 4.5 in [Pe-Se-Za].) Hence type (I) group has J -invariant $J_G(1)$.

For these cases, the ordinary $\text{mod}(p)$ cohomology is well known

$$H^*(G; \mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^p) \otimes \Lambda(x_1, \dots, x_\ell)$$

where $\ell = \text{rank}(G) \geq 2$, $|y| = 2p + 2$, $|x_1| = 3$, $|x_2| = 2p + 1$. Moreover

$$Q_1(x_1) = y, \quad Q_0(x_2) = y.$$

Hence we have the isomorphisms as (3.3) and (3.4)

$$\text{gr}H^*(G; \mathbb{Z}/p) \cong \mathbb{Z}/p[y, t_1, \dots, t_\ell]/(y^p, \bar{b}_1, \dots, \bar{b}_\ell),$$

$$H^*(G) \cong \mathbb{Z}[y, t_1, \dots, t_\ell]/(f_1, b_1, \dots, b_\ell),$$

$$\Omega^*(G) \cong \mathbb{Z}/p[y, t_1, \dots, t_\ell]/(\tilde{f}_1, \tilde{b}_1, \dots, \tilde{b}_\ell),$$

where $b_i = \bar{b}_i \text{ mod}(p)$ and $f_1 = y_i^p \text{ mod}(t_1, \dots, t_\ell)$, and where $\tilde{b}_i = b_i \text{ mod}(\Omega^{<0})$ and $\tilde{f}_1 = f_1 \text{ mod}(\Omega^{<0})$. From Corolary 3.2, we see

Corollary 4.1. $CH^*(G_k)_{(p)} \cong \mathbb{Z}/p[y]/(y^p)$.

From Theorem 3.3 and the Q_i -actions, we see

$$\Omega^*(G_k)/I_\infty^2 \cong \Omega^*[y]/(py, v_1y, y^p, I_\infty^2),$$

while we have more strong result (Theorem 5.1 in [Ya1]) than Theorem 3.3.

Corollary 4.2. $\Omega^*(G_k) \cong \Omega^*[y]/(py, v_1y, y^p)$.

Remark. In the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*,*''} \cong H^{*,*'}(G_k; MU^{*''}) \implies MGL^{*,*'}(G_k)$$

we know that

$$d_{2p-1}(x_1) = v_1 \otimes Q_1(x_1) = v_1y.$$

Thus we get also $E_\infty^{2*,*,*''} \cong MU^*[y]/(py, v_1, y^p)$.

In general, torsors E are parametrized by elements $\xi \in H^1(k; \text{Aut}(G))$. A torsor is said to be inner if it is in the image of the canonical map $H^1(k; G)$ to $H^1(k; \text{Aut}(G))$. In this paper, we *assume* all torsors are *inner* type.

Petrov, Semenov and Zainoulline [Pe-Se-Za] developed the theory of generally splitting varieties. We say that L is splitting field of a variety of X if $M(X|_L)$ is isomorphic to a direct sum of twisted Tate motives $\mathbb{T}^{\otimes i}$ and the restriction map $i_L : M(X) \rightarrow M(X|_L)$ is isomorphic after tensoring \mathbb{Q} . A smooth scheme X is said to be generically split over k if its function field $L = k(X)$ is a splitting field. Petrov Semenov and Zainoulline showed that torsors of type G of (I), (restricting E_6 to the adjoint type E_6^{ad}), are all generally split.

Theorem 4.3. (Theorem 4.9 in [Pe-Se-Za]) *There is a mod(p) motive $R_p(G)$ such that*

$$(1) \quad CH^*(R_p(G)|_{\bar{k}})/p \cong \mathbb{Z}/p[y]/(y)$$

$$(2) \quad M(E/T_k; \mathbb{Z}/p) \cong \oplus_s R_p(G) \otimes \mathbb{T}^{\otimes i_s} \cong R_p(G) \otimes H^*(G/T; \mathbb{Z}/p)/(y)$$

where we identify $H^*(G/T; \mathbb{Z}/p)/(y)$ as the sum of Tate motives $\oplus \mathbb{T}^{\otimes i_s}$.

Theorem 4.4. (Theorem 3.8 in [Pe-Se-Za]) *Let $Q_k \subset P_k$ be parabolic subgroups of G_k which are generically split over k . There is a decomposition of motive $M(E/Q_k)_{(p)} \cong M(E/P_k)_{(p)} \otimes H^*(P/Q)$.*

For $p = 2, 3$ (i.e., except for E_8 , $p = 5$), from Proposition 5.5 (for $m = p$) and §7 in [Pe-Se-Za], we have the *integral* motivic decopostion which deduces the *mod*(p) decomposition in Theorem 4.3. Moreover from Corollary 6 in [Vi-Za] (see also [Se],[Bo]), we see the integral motive corresponding $R_p(G)$ is really generalized Rost motive.

Theorem 4.5. *Let G of type (I) except for E_8 and E_6 . Then there is a parabolic subgroup P_k such that E/P_k is generically split and*

$$CH^*(G_k/P_k)_{(p)} \cong \mathbb{Z}[y]/(y^p) \otimes A \quad \text{and} \quad M(E/P_k)_{(p)} \cong M_2 \otimes A$$

where A is a sum of twisted Tate motives and $M_2 = M_a$ is the generalized Rost motive for some $0 \neq a \in K_3^M(k)/p$.

Now we state the main theorem of this paper.

Theorem 4.6. *Let G of type (I) except for E_8 and E_6 . The chow ring $CH^*(E/T_k)_{(p)}$ is multiplicatively generated by t_1, \dots, t_ℓ . Hence $CH^*(E)_{(p)} \cong \mathbb{Z}_{(p)}$.*

Proof. From the above theorems, we have the decomposition of motive

$$M(E/T_k)_{(p)} \cong M_3 \otimes H^*(G/T)/(y).$$

We consider the restriction map

$$i_{\bar{k}} : \Omega^*(E/T_k) \rightarrow \Omega^*(E/T_k|_{\bar{k}}) \cong \Omega^*(G/T).$$

Since $i_{\bar{k}}|_{\Omega(M_2)}$ is injective, so is $i_{\bar{k}}$ above. Let us write

$$Im(i_{\bar{k}}) = i_{\bar{k}}(\Omega^*(E/T_k)) \subset \Omega^*(G_k/T_k) = \Omega^*(G/T).$$

Of course $py^i, v_1y^i \in Im(i_{\bar{k}})$ for $i > 0$ since so in $\Omega^*(M_2|_{\bar{k}})$. Note that $t_1, \dots, t_\ell \in Im(i_{\bar{k}})$ because they exist in $CH^*(E/T_k)$ since so in $CH^*(BT_k)$.

Recall that each element $x \in \Omega^*(E/T_k|_{\bar{k}}) \cong \Omega^*(G/T)$ is represented as

$$(*) \quad x = \sum_{i=0}^{p-1} \sum_s v(s, i) t(s, i) y^i, \quad v(s, i) \in \Omega^*, \quad t(s, i) \in \mathbb{Z}_{(p)}[t_1, \dots, t_\ell]$$

while if $x \in Im(i_{\bar{k}})$, then $v(s, i) \in Ideal(p, v_1)$ for $i > 0$.

From Corollary 4.2, we see $py = v_1y = 0$ in $\Omega^*(G_k)$. From Theorem 3.1, this means

$$py, v_1y \in (t_1, \dots, t_\ell)\Omega^*(G_k/T_k).$$

(But note that this does not mean $\in (t_1, \dots, t_\ell)Im(i_{\bar{k}})$.) Let us write $v_1y = \sum v(s, i)t(s, i)y^i$ as (*). The above fact implies $|t(s, i)| > 0$ and hence $|v(s, i)| < 0$.

Now let us write

$$\Omega\langle 1 \rangle^*(X) = \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(p)}[v_1] = ABP\langle 1 \rangle^{2*,*}(X).$$

In $\Omega\langle 1 \rangle^*(G_k/T_k)$, the fact $|v(s, i)| < 0$ means

$$v(s, i) \in (v_1) = \mathbb{Z}_{(p)}[v_1]^{<0} = \Omega\langle 1 \rangle^{<0}.$$

Hence $v_1y \in (t_1, \dots, t_\ell)Im(i_{\bar{k}})$ in $\Omega\langle 1 \rangle^*(-)$ theory. Similarly we can prove that py^i, v_1y^i have the same property. (We note $y^p \in (p, v)\Omega^*(G_k/T_k)$ so in $Im(i_{\bar{k}})$.)

Thus we can write

$$v_1y = \sum_i \sum_s v(s, i)'t(s, i)v_1y^i \quad \text{in } \Omega\langle 1 \rangle^*(E/T_k).$$

If $v(s, i)' \neq 0$ for $i > 1$, then apply the same equation to the right hand side v_1y in the above equation. Since $t(s, i) = 0$ when $|t(s, i)| > \dim(G/T)$, we can write

$$v_1y = \sum_s v(s, 0)t(s, 0).$$

The same property holds for py^i and v_1y^i when $i > 1$. Hence $i_{\bar{k}}(\Omega\langle 1 \rangle^*(E/T_k))$ is generated as an $\Omega\langle 1 \rangle^*$ -algebra by t_1, \dots, t_ℓ .

Since we know the isomorphisms

$$CH^*(E/T_k)_{(p)} \cong \Omega^*(E/T_k) \otimes_{\Omega^*} \mathbb{Z}_{(p)} \cong \Omega\langle 1 \rangle^*(E/T_k) \otimes_{\Omega\langle 1 \rangle^*} \mathbb{Z}_{(p)},$$

the elements t_1, \dots, t_ℓ are multiplicatively generate $CH^*(E/T_k)_{(p)}$. \square

5. EXCEPTIONAL LIE GROUP G_2

In this section we study the case $(G, p) = (G_2, 2)$. We recall the cohomologies from Toda-Watanabe [To-Wa]

$$H^*(G/T; \mathbb{Z}) \cong \mathbb{Z}[t_1, t_2, y]/(t_1^2 + t_1t_2 + t_2^2, t_2^3 - 2y, y^2)$$

with $|t_i| = 2$ and $|y| = 6$. Let P be one of the maximal parabolic subgroups. Then from (3.6) and $H^*(P/T) \cong \mathbb{Z}\{1, t_1\}$

$$H^*(G/P; \mathbb{Z}) \cong \mathbb{Z}[t_2, y]/(t_2^3 - 2y, y^2) \cong \mathbb{Z}\{1, y\} \otimes \{1, t_2, t_2^2\}.$$

By Bonnet, we have the decomposition

Theorem 5.1. (*[Bo], Corollary 5.6 in [Pe-Se-Za]*)

$$M(E/P_k) \cong M_2 \oplus M_2(1) \oplus M_2(2).$$

Theorem 5.2. *There is the ring isomorphism*

$$\begin{aligned} CH^*(E/P_k)_{(2)} &\cong \mathbb{Z}_{(2)}[t_2, u]/(t_2^6, 2u, t_2^3u, u^2) \\ &\cong \mathbb{Z}_{(2)}[t_2]/(t_2^6) \oplus \mathbb{Z}/2[t_2]/(t_2^3)\{u\} \end{aligned}$$

with $|t_2| = 2, |u| = 4$.

Proof. From Lemma 2.2, we know

$$\Omega^*(M_2) \cong \Omega^*\{1, 2y, vy\} \subset \Omega^*\{1, y\}.$$

From the preceding theorem, we have the Ω^* -algebra isomorphism

$$\Omega^*(E/P_k) \cong \Omega^*\{1, v_1y, 2y\} \otimes \{1, t_2, t_2^2\} \subset \Omega^*(G/P).$$

Since $CH^*(X)_{(p)} \cong \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(p)}$, we have the isomorphism

$$CH^*(E/P_k)_{(2)} \cong \mathbb{Z}_{(2)}\{1, 2y\}\{1, t_2, t_2^2\} \oplus \mathbb{Z}/2\{v_1y\}\{1, t_2, t_2^2\}.$$

Here the multiplications are given as follows. Since $2y = t_2^3 \pmod{\Omega^{<0}}$ in $\Omega^*(G_k/T_k)$, we can take $2y = t_2^3$ so that

$$\mathbb{Z}_{(2)}\{1, 2y\}\{1, t_2, t_2^2\} = \mathbb{Z}_{(2)}[t_2]/(t_2^6).$$

Let us write $u = v_1y$ in $CH^*(E/T_k)_{(2)}$. Then $t_2^3u = 2yv_1y = 0$ and $u^2 = v_1^2y^2 = 0$ in $\Omega^*(E/T_k) \otimes_{\Omega^*} \mathbb{Z}_{(2)}$. Hence we have the isomorphism in the theorem. \square

Next consider $CH^*(E/T_k)_{(2)}$.

Theorem 5.3. *There is the ring isomorphism*

$$CH^*(E/T_k)_{(2)} \cong \mathbb{Z}_{(2)}[t_1, t_2]/(t_2^6, 2u, t_2^3u, u^2)$$

where $u = t_1^2 + t_1t_2 + t_2^2$.

Proof. The Chow ring is isomorphic to

$$\begin{aligned} (*) \quad CH^*(E/T_k)_{(2)} &\cong CH^*(E/P_k)\{1, t_1\} \\ &\cong (\mathbb{Z}_{(2)}\{1, 2y\} \oplus \mathbb{Z}/2\{v_1y\})\{1, t_2, t_2^2\}\{1, t_1\}. \end{aligned}$$

Here $2y = t_2^3$. Since $v_1y \in (t_1, t_2)$ and $v_1y = 0 \in CH^*(G/T)$, we see

$$v_1y = \lambda(t_1^2 + t_1t_2 + t_2^2) \pmod{(t_1, t_2)\Omega^{<0}\Omega^*(G/T)}$$

for $\lambda \in \mathbb{Z}_{(2)}$. We can take $\lambda = 1 \pmod{2}$. Otherwise $v_1y = 0 \in \Omega^*(G/T)/2$, which is a $\Omega^*/2$ -free, and this is a contradiction. Hence we can take $t_1^2 + t_1t_2 + t_2^2$ as v_1y . (This is also proved by Lemma 4.3 in [Ya1], since $Q_1(x_1) = y$ and $d_3(x_1) = t_1^2 + t_1t_2 + t_2^2$.) Hence in $CH^*(E/T_k)$ we have the relation

$$(t_2^3)^2 = 0, (t_2^3)u = 0, u^2 = 0, 2u = 0.$$

We consider the mod 2 Poincare polynomial

$$\begin{aligned} \sum_i \text{rank}_{\mathbb{Z}/2}(CH^i(E/T_k)/2)t^i &= (1 + t^2 + t^4)(1 + t + t^2)(1 + t) \\ &= 1 + 2t + 3t^2 + 4t^3 + 4t^5 + 3t^5 + t^6 = \frac{(1 - t^6)(1 - t^4)}{(1 - t)(1 - t)} - t^5(1 + t)^2 \end{aligned}$$

which is the ($\text{mod}(2)$) Poincare series of the right hand side ring of the theorem. (Note (t_2^6, u^2) is a regular sequence in $\mathbb{Z}/2[t_1, t_2]$ but $(t_2^6, u^2, (t_2^3)u)$ is not.) \square

The author learned the following remarks by Zainouline.

Remark. It is well known that there is the bijection between $H^1(k; G_2)$ and the class of Cayley algebras C from the fact $G_2 = \text{Aut}(C|_{\bar{k}})$. Hence each torsor E over k corresponds a Cayley algebra. Moreover E/T_k and E/P_k correspond the following varieties [Ca-Pe-Se-Za]. By an i -space ($i = 1, 2$), we mean i -dimensional subspace V_i of C such that $u \cdot v = 0$ for every $u, v \in V_i$. The flag variety corresponding E/T_k is the full flag variety

$$X(1, 2) = \{V_1 \subset V_2 | V_i; i - \text{subspaces} \subset C\}$$

and that corresponding E/P_k is $X(2) = \{V_2 | V_2; 2 - \text{subspace} \subset C\}$. Let

$$g : H^1(k; G_2) \rightarrow H^3(k; \mathbb{Z}/2) \cong K_3^M(k)/2$$

be the Arason invariant (which is know to be isomorphic). The symbol of the Rost motive in Theorem 5.1 is $g(E)$ i.e., $M_2 = M_{g(E)}$.

Remark. Similar facts hold for $(G, p) = (F_4, 3)$. This case, the corresponding algebras are exceptional Jordan algebras of dimension 27 over k , and the symbol of the generalizedf motive is the Rost-Serre invariant.

6. EXCEPTIONAL GROUP F_4 FOR $p = 3$

Let $(G, p) = (F_4, 3)$ throughout this section. Let E be a nontrivial G_k -torsor as previous sections. Let P_k be a maximal parabolic subgroup of G_k given by the the last vertex of the Dynkin diagram.

Theorem 6.1. ([Pa-Se-Za]) *Let M_2 be the generalized Rost motive. Then there is an isomorphism $M(E/P_k) \cong \bigoplus_{i=0}^7 M_2(i)$.*

We first recall the ordinary cohomology of G/P ([Is-To], [Du-Za]).

$$H^*(G/P)_{(3)} \cong \mathbb{Z}[t, y]/(r_8, r_{12}), \quad |t| = 2, \quad |y| = 8$$

where $r_8 = 3y^2 - t^8$ and $r_{12} = 26y^3 - 5t^{12}$. Hence we can rewrite

$$H^*(G/P)_{(3)} \cong \mathbb{Z}_{(3)}\{1, t, \dots, t^7\} \otimes \{1, y, y^2\}.$$

Recall the Rost motive

$$CH^*(M_2|_{\bar{k}}) \cong \mathbb{Z}[y]/(y^3),$$

$$CH^*(M_2) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}\{3y, 3y^2\} \oplus \mathbb{Z}/3\{v_1y, v_1y^2\}.$$

Of course, the above y is that in $H^*(G/P)_{(3)}$ by the dimensional reason. From the above theorem, we have the decomposition

$$(*) \quad CH^*(E/P_k) \cong \mathbb{Z}_{(3)}\{1, t, \dots, t^7\} \otimes (\mathbb{Z}_{(3)}\{1, 3y, 3y^2\} \oplus \mathbb{Z}/3\{v_1y, v_1y^2\}).$$

The ring structure is given as follows.

Theorem 6.2.

$$\begin{aligned} CH^*(E/P_k)_{(3)} &\cong \mathbb{Z}_{(3)}[t, b, a_1, a_2]/(t^{16}, t^8b, b^2 = 3t^8, ba_i, 3a_i, t^8a_i, a_1a_2) \\ &\cong \mathbb{Z}_{(3)}\{1, t, \dots, t^7\} \otimes (\mathbb{Z}_{(3)}\{1, \sqrt{3}t^4, t^8\} \oplus \mathbb{Z}/3\{a_1, a_2\}) \end{aligned}$$

where $|b| = 8$ and $|a_1| = 4, |a_2| = 12$.

Proof. From the relation r_8 in $CH^*(G/P)$, we have

$$3y^2 = t^8 + vx \in \Omega^*(G/P) \quad \text{for } v \in \Omega^{<0}.$$

Hence we can take t^8 instead of $3y^2$ in (*). Of course

$$(3y)^2 = 3t^8 + 3vx \in \Omega^*(G_k/P_k).$$

Hence we write by $b = \sqrt{3}t^4$ the element $3y$. Write by a_1, a_2 the elements v_1y, v_1y^2 respectively. Elements in $I_\infty\Omega^{<0} \subset \Omega(G_k/P_k)$ reduces to zero in $CH^*(E/T_k)$. Therefore we have the desired multiplicative results. \square

The cohomology $H^*(G/T)$ is given by Toda-Watanabe [To-Wa]

$$H^*(G/T)_{(3)} \cong \mathbb{Z}_3[t_1, t_2, t_3, t_4, y]/(\rho_2, \rho_4, \rho_6, \rho_8, \rho_{12}).$$

Here relations ρ_i is written by the elementary symmetric function $c_i = \sigma_i(t_1, t_2, t_3, t_4)$, that is,

$$\begin{aligned} \rho_2 &= c_2 - (1/2)c_1^2, \quad \rho_4 = c_4 - c_3c_1 + (1/2)^3c_1^4 - 3y, \quad \rho_6 = -c_4c_1^2 + c_3^2, \\ \rho_8 &= 3c_4c_1^4 - (1/2)^4c_1^8 + 3y(2^4y + 2^3c_3c_1), \quad \rho_{12} = y^3 \end{aligned}$$

By the arguments similar to the proof of Theorem 5.3 (or Lemma 4.3 in [Ya1]).

Theorem 6.3. *Let $\pi : E/T_k \rightarrow E/P_k$. Then*

$$\pi^*(t) = c_1, \quad \pi^*(a_1) = \rho_2, \quad \pi^*(a_2) = \rho_6, \quad \pi^*(b) = c_4 - c_3c_1 - (2)^{-3}c_1^4.$$

Hence there is the epimorphism

$$\begin{aligned} \mathbb{Z}_{(3)}[t_1, t_2, t_3, t_4]/(c_1^{16}, c_1^8\pi^*(b), \pi^*(b)^2 - 3c_1^8, \pi^*(b)\rho_j, 3\rho_j, c_1^8\rho_j, \rho_2\rho_6) \\ \rightarrow CH^*(E/T_k)_{(3)} \end{aligned}$$

where $j = 2, 6$.

7. THE ORTHOGONAL GROUP $SO(m)$ AND $p = 2$

We consider the orthogonal groups $G = SO(m)$ and $p = 2$. The mod 2-cohomology is written as (see for example [Ni])

$$grH^*(SO(m); \mathbb{Z}/2) \cong \Lambda(x_1, x_2, \dots, x_{m-1})$$

where the multiplications are given by $x_s^2 = x_{2s}$. We write $y_{2(\text{odd})} = x_{\text{odd}}^2$. Hence we can write

$$H^*(SO(m); \mathbb{Z}/2) \cong \mathbb{Z}/2[y_{4i+2} | 2 \leq 4i+2 \leq m-1] / (y_{4i+2}^{s(i)}) \otimes \Lambda(x_1, x_3, \dots, x_{\bar{m}})$$

where $s(i)$ is the smallest number such that $2^{s(i)}(4i+2) \geq m$ and $\bar{m} = m-1$ (resp. $\bar{m} = m-2$) if m is even (resp. odd).

The Q_i -operations are given by Nishimoto [Ni]

$$Q_n x_{\text{odd}} = x_{\text{odd}+|Q_n|}, \quad Q_n x_{\text{even}} = Q_n y_{\text{even}} = 0$$

Relations in $\Omega^*(SO(m))$ are given by

$$\sum_n v_n Q_n(x_{\text{odd}}) = \sum_n v_n x_{\text{odd}+|Q_n|} = 0 \quad \text{mod}(I_\infty^2).$$

For example, the relations in $\Omega^*(SO(m))/I_\infty^2$ starting with $2y_6$ are given by

$$2y_6 + v_1 y_2^4 + v_2 y_6^2 + v_3 y_{10}^2 + \dots = 0,$$

Theorem 7.1. ([Ya1]) *There are $\Omega_{(2)}^*$ -algebra isomorphisms*

$$(1) \quad \Omega^*(SO(m))/I_\infty^2 \cong \Omega^*[y_{4i+2} | 2 \leq 4i+2 \leq m-1] / (R, I_\infty^2)$$

where $R = \{\text{relations starting with } y_{4i+2}^{2^{s(i)}}, 2y_{4i+2}, v_1 y_{4i+2} \mid i' \neq 0.\}$

For ease of arguments, we only consider the case $G = SO(\text{odd})$. Let $G = SO(2m'+1)$ and $P = SO(2m'-1) \times SO(2)$. Then it is well known [To-Wa]

Lemma 7.2. $H^*(G/P) \cong \mathbb{Z}[t, y] / (t^{m'} - 2y, y^2) \quad |y| = 2m'$.

By Toda-Watanabe [To-Wa], we also know

Theorem 7.3. ([To-Wa])

$$H^*(G/T) \cong \mathbb{Z}[t_i, y_{2i}, t_{m'}, y] / (c_i - 2y_{2i}, J_{2i}, t_{m'}^{m'} - 2y, y^2)$$

where $1 \leq i \leq m'-1$, $c_i = \sigma(t_1, \dots, t_{m'})$ and

$$J_{2i} = 1/4 \left(\sum_{j=0}^{2i} (-1)^j c_j c_{2i-j} \right) = y_{4i} - \sum_{0 < j < 2i} (-1)^j y_{2j} y_{4i-2j}.$$

Hence we can write

$$grH^*(G/T) \cong H^*(G/P) \otimes A \quad A = \mathbb{Z}[t_i, y_i]/(c'_i - 2y_i, J_{2i})$$

where $c'_i = \sigma(t_1, \dots, t_{m'-1})$. More precisely, we can write

$$grA = P(y)' \otimes P(t)'$$

where $P(y)' = \otimes_{i < 2^{n-1}-1} \mathbb{Z}[y_{4i+2}]/(y^{2^{s_i}})$ so that $P(y) = P(y)' \otimes \mathbb{Z}[y]/(y^2)$. Moreover we see

$$P(t)' = H^*(BT_{m'-1})/(H^*(BU(m'-1))) \cong \mathbb{Z}[t_1, \dots, t_{m'-1}]/(c'_1, \dots, c'_{m'-1})$$

Indeed, it is also known that

$$grH^*(G/(U(m'-1) \times SO(2))) \cong P(y)' \otimes H^*(G/P).$$

Now we recall an argument of quadrics. Let $m = 2m' + 1$. and let us write the quadratic form $q(x)$ defined by

$$q(x_1, \dots, x_m) = x_1x_2 + \dots + x_{m-2}x_{m-1} + x_m^2$$

and the projective quadric X_q defined by the quadratic form q . Then it is well known that (in fact $SO(m)$ acts on the affine quadric in $\mathbb{A}^m - 0$)

$$X_q \cong SO(m)/(SO(m-2) \times SO(2)).$$

Let $G = SO(m)$ and $P = SO(m-2) \times SO(2)$. Then the quadric q is always split over k and we know $CH^*(G_k/P_k) \cong CH^*(X_q)$.

In particular we consider the case $m = 2^{n+1} - 1$. Let $\rho = \{-1\} \in K_1^M(k) = k^*$. We consider the field k such that

$$(**) \quad 0 \neq \rho^{n+1} \in K_{n+1}^M(k)/2.$$

Throughout this section, we assume the condition (**). (If k is a real number field, then (**) is satisfied.) Define the quadratic form q' by

$$q'(x_1, \dots, x_m) = x_1^2 + \dots x_m^2.$$

Then this q is a subform of

$$\langle\langle -1, \dots, -1 \rangle\rangle = \phi_{\rho^{n+1}}$$

the $(n+1)$ -th Pfister form associated to ρ^{n+1} . (That is, q' is the maximal neighbor of the $(n+1)$ -th Pfister form.) Of course $q|_{\bar{k}} = q'|_{\bar{k}}$ and we can identify $E/P_k \cong X_{q'}$. From Lemma 7.4 (or Rost's result), we know

$$CH^*(X_{q'}|_{\bar{k}}) \cong \mathbb{Z}[t, y]/(t^{2^n-1} - 2y, y^2).$$

The multiplicative structure of Chow ring of the anisotropic quadric is easily compute from Lemma 2.2 [Ya4]

Lemma 7.4. ([Ya4])

$$CH^*(E/P_k) \cong \mathbb{Z}[t]/(t^{2^{n+1}-2}) \oplus \mathbb{Z}/2[t]/(t^{2^n-1})\{u_1, \dots, u_{n-1}\}$$

where $u_i = v_i y \in \Omega^*(E/p) \otimes_{\Omega^*} \mathbb{Z}_{(2)}$ so $u_i u_j = 0$.

By the projection $E/T_k \rightarrow E/B_k$, Petrov, Semenov and Zainoulline also show the following (the example of the last page in [Pe-Se-Za], indeed, they show the J -invariant $J_2(G) = (0, \dots, 0, 1)$).

Theorem 7.5. When $m = 2^{n+1} - 1$. The restriction map $i_{\bar{k}} : \Omega^*(E/T_k) \rightarrow \Omega^*(E/T_k|_{\bar{k}}) = \Omega^*(G_k/T_k)$ is injective and

$$grCH^*(E/T_k) = grCH^*(E/P_k) \otimes A,$$

$$gr\Omega^*(E/T_k) = gr\Omega^*(E/P_k) \otimes A$$

where $A = \mathbb{Z}[t_i, y_{2i}]/(c'_i - 2y_i, J_{2i})$.

As a corollary, we see that t_i, y_{2i} are all in $CH^*(E/T_k)$. Hence $CH^*(E/T_k)$ is multiplicatively generated by t_i, y_i, t and u_1, \dots, u_{n-1} .

Theorem 7.6. Assume $(**)$ and $m = 2^{n+1} - 1$. Then

$$CH^*(E)_{(2)} \cong P(y)'/(2) \subset P(y)' \otimes \mathbb{Z}/2[y]/(y^2) \cong CH^*(G_k)_{(2)}.$$

Proof. The proof is quite similar to that of Theorem 3.4. Let us write

$$\Omega\langle n-1 \rangle^*(X) = \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(2)}[v_1, \dots, v_{n-1}] \cong ABP\langle n-1 \rangle^{2^*,*}(X).$$

By Theorem 3.1, We want to prove

$$(1) \quad u_1, \dots, u_{n-1} \in (t_1, \dots, t_{m'})CH^*(E/T_k).$$

This means

$$u_1, \dots, u_{n-1} \in ((t_1, \dots, t_{m'}) + \Omega^{<0})\Omega\langle n-1 \rangle^*(E/T_k).$$

Let us write

$$Im(i_{\bar{k}}) = i_{\bar{k}}^*(\Omega\langle n-1 \rangle^*(E/T_k)) \subset \Omega\langle n-1 \rangle^*(G_k/T_k),$$

$$I(t, \Omega^{<0}) = ((t_1, \dots, t_{m'}) + \Omega^{<0})Im(i_{\bar{k}}).$$

(Note $I_{\infty}^2 \subset \Omega^{<0}Im(i_{\bar{k}})$.) Thus it is sufficient for (1) to prove

$$(2) \quad 2y, \dots, v_{n-1}y \in I(t, \Omega^{<0}).$$

At first we will show $v_{n-1}y \in I(t, \Omega^{<0})$. Recall $y = y_{2^{n+1}-2} = x_{2^{n+1}-2}$. From Theorem 3.3 and Nishimoto's result, we see

$$(3) \quad \begin{aligned} x &= 2x_{2^n} + v_1x_{2^{n+2}} + \dots + v_{n-2}x_{2^{2^n+2^{n-1}-2}} + v_{n-1}x_{2^{n+1}-2} \\ &= 0 \quad \text{in} \quad \Omega\langle n-1 \rangle^*(G_k)/(I_{\infty}^2). \end{aligned}$$

So $x \in ((t_1, \dots, t_{m'}) + I_{\infty}^2)\Omega\langle n-1 \rangle^*(G_k/T_k)$.

Each element $z \in \Omega\langle n-1 \rangle^*(G_k/T_k)$ is written (not uniquely) by

$$(4) \quad z = \sum v_I t_J y_K + \sum v_{I'} t_{J'} y_{K'} y$$

with $v_I, v_{I'} \in \Omega\langle n-1 \rangle^*$, $t_J, t_{J'} \in \mathbb{Z}_{(2)}[t_1, \dots, t_{m'}]$ and $y_K, y_{K'} \in P(y)'$. Note that if $z \in (t_1, \dots, t_{m'})\Omega\langle n-1 \rangle^*(G_k/T_k)$, then we can take $|t_J| > 0$ and $|t_{J'}| > 0$.

Consider the case $z = x$ in (3). Since $y_K \in \text{Im}(i_{\bar{k}})$, we know

$$v_I t_J y_K \in (t_1, \dots, t_{m'})\text{Im}(i_{\bar{k}}).$$

Since $|y| < |t_{J'} y_{K'} y|$, we know $|v_{I'}| < 0$, i.e., $v_{I'} y \in \text{Im}(i_{\bar{k}})$ because $v_{I'} \in \Omega\langle n-1 \rangle^* = \mathbb{Z}_{(2)}[v_1, \dots, v_{n-1}]$. Thus we know $v_{I'} t_{J'} y_{K'} y \in (t_1, \dots, t_{m'})\text{Im}(i_{\bar{k}})$. Therefore we see

$$(5) \quad x \in I(t, \Omega^{<0}).$$

In (3), $x_{2^{n+2}}, \dots, x_{2^{n+2}n-2}$ are in $\text{Im}(i_{\bar{k}})$. So we see

$$v_1 x_{2^{n+2}} + \dots + v_{n-2} x_{2^{n+2}n-2} \in \Omega^{<0} \text{Im}k(i_k).$$

Hence we see

$$(6) \quad 2x_{2^n} + v_{n-1}y \in I(t, \Omega^{<0}).$$

Next we will see

$$(7) \quad 2y_2, \dots, 2y_{2^{n-2}} \in I(t, \Omega^{<0}).$$

Then in particular, $(y_2)^{2^{n-1}} = x_{2^n}$ implies $v_{n-1}y \in I(t, \Omega^{<0})$ from (6). Similarly we can prove $v_{n-2}y, \dots, 2y \in I(t, \Omega^{<0})$ by using the arguments (3) – (7). Thus we see (2) and so (1).

We prove (7) for $2y_2$ and the other cases are similar. By also using Nishimoto's result and Theorem 3.3, we have the relation

$$x' = 2x_2 + v_1x_4 + \dots v_{n-1}x_{2^n} = 0 \in \Omega\langle n-1 \rangle^*(G_k)/I_\infty^2.$$

By using arguments similar to (3)-(5), we have $x' \in I(t, \Omega^{<0})$. Of course $v_1x_4 + \dots v_{n-1}x_{2^n} \in \Omega^{<0} \text{Im}(i_{\bar{k}})$. Thus we see $2y_2 \in I(t, \Omega^{<0})$. \square

Remark. Note that $i_{\bar{k}}(CH^*(E))_{(2)} = CH^*(G_k)_{(2)} - \{y\}$. The similar fact happens for $\Omega\langle n-1 \rangle$ -theory

$$\text{Im}(i_{\bar{k}}) = \Omega\langle n-1 \rangle^*(G_k) - \{y\}.$$

Indeed $2y, \dots, v_{n-1}y \in \Omega\langle n-1 \rangle^*(G_k)$ are in the $\Omega\langle n-1 \rangle^*$ -subalgebra generated by y_i .

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