On adic genus, Postnikov conjugates, and lambda-rings

Donald Yau

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Department of Mathematics, MIT Room 2-588, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Abstract

Sufficient conditions on a space are given which guarantee that the K-theory ring and the ordinary cohomology ring with coefficients over a principal ideal domain are invariants of, respectively, the adic genus and the SNT set. An independent proof of Notbohm's theorem on the classification of the adic genus of BS^3 by KO-theory λ-rings is given. An immediate consequence of these results about adic genus is that for any positive integer n, the power series ring $\mathbf{Z}[[x_1, \ldots, x_n]]$ admits uncountably many pairwise non-isomorphic λ-ring structures.

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1 Introduction and statement of results

In this paper we study some problems about the adic genus and the SNT set of a space. For a nilpotent finite type space X, we denote by $\mathrm{Genus}(X)$, call the *adic genus* of X, the set of homotopy types of nilpotent finite type spaces Y such that the p-completions of X and Y are homotopy equivalent for each prime p and also their rationalizations are homotopy equivalent.

The adic genus of an infinite dimensional space X is often a very big set. For instance, Møller [17] proved that whenever G is a compact connected non-abelian Lie group, the adic genus of its classifying space BG is uncountably large. It is an important problem to find computable homotopy invariants which can distinguish between spaces with the same adic genus. Such a result was achieved by Notbohm and Smith [19] (see also [11]). Recall that a space X is said to be an *adic fake Lie group of type* G if (1) $X = \Omega BX$ is a finite loop space and (2) BX lies in the adic genus of BG. Notbohm and Smith showed that if X is an adic fake Lie group of type G, where G is a simply-connected compact Lie group, then BX is homotopy equivalent to BG if and only if $K(BX) \cong K(BG)$ as λ-rings. (Here $K(Y)$ denotes the complex K-theory of Y.) In fact, Notbohm [18] showed that K-theory λ-rings classify the genus of BG, provided G is a simply-connected compact Lie group. Note that what we call a λ-ring is what used to be called a "special" λ-ring.

With this result in mind, a natural question is then the following:

> *Is it really necessary to take into account the λ-operations in order to distinguish spaces in the adic genus of BG?*

To describe our answer to this question, we need the following definition. A *filtered ring* is a commutative ring R with unit together with a decreasing filtration

$$R \;=\; I^0 \supset I^1 \supset I^2 \supset \cdots$$

of ideals such that $I^m I^n \subset I^{m+n}$. A map of filtered rings is required to respect filtrations. The filtration in the K-theory of a space X is given by a skeletal filtration of X; that is, $K(X)$ is filtered by the kernels

$$I^n(X) \;=\; \ker(K(X) \to K(X_{n-1}))$$

of the restriction maps, where X_{n-1} denotes the $(n-1)$-skeleton of X. By using the Cellular Approximation Theorem, it is easy to see that the filtered ring isomorphism type of a given space X is well-defined, though there are many different filtrations on X. Clearly, a map between two spaces induces a filtered ring map between the respective K-theories.

Our first main result is then the following, which shows that for a torsion free classifying space, K-theory filtered ring cannot tell the difference between spaces in its adic genus. Consequently, λ-operations are necessary in order to distinguish these spaces.

Theorem 1.1. *Let X be a simply-connected space of finite type whose integral homology is torsion free and is concentrated in even dimensions and whose K-theory filtered ring is a finitely generated power series ring over $\mathbf{Z}$. If a space Y belongs to the adic genus of X, then there exists a filtered ring isomorphism $K(X) \cong K(Y)$.*

For example, in Theorem 1.1 the space X can be $BSp(n)$ $(n \geq 1)$, $BSU(n)$ $(n \geq 2)$, or any finite product of copies of these spaces and of infinite complex projective space.

Remark 1.2. The proof of Theorem 1.1 is independent of Notbohm's result mentioned above. It consists of first showing the weaker statement that the filtered rings $K(Y)/I^n(Y)$ and $K(X)/I^n(X)$ are isomorphic for all n sufficiently large. Since $K(Y)$ can be recovered from the quotients $K(Y)/I^n(Y)$ by taking inverse limits, a filtered ring analog of a classification result of Wilkerson [25] implies that to prove Theorem 1.1 it is sufficient to prove the triviality of a $\varprojlim^1$ set.

Remark 1.3. This theorem has a variant in which complex K-theory (resp. $\mathbf{Z}$) is replaced with KO^*-theory (resp. $KO^* = KO^*(\mathrm{pt})$), provided the integral homology of X (which is simply-connected of finite type) is torsion free and is concentrated in dimensions divisible by 4 (e.g. $X = B\mathrm{Sp}(n)$). This variant admits a proof which is essentially identical with that of Theorem 1.1.

Using Theorem 1.1 we will show the following KO-theory analog of Notbohm's result (applied to BS^3) mentioned above.

Theorem 1.4. *Let X and Y be spaces in the adic genus of BS^3. Then X and Y are homotopy equivalent if and only if there exists a filtered λ-ring isomorphism $KO^*(X) \cong KO^*(Y)$.*

In Theorem 1.4, by a filtered λ-ring we mean a filtered ring R which is also a λ-ring for which the filtration ideals are all closed under the operations

λ^i $(i > 0)$. The proof of of this theorem uses directly Rector's classification of the genus of BS^3 [20] and the KO-theory analog of Theorem 1.1 when $X = BS^3$.

Our next result is a purely algebraic application of Theorem 1.1 to the study of λ-rings.

Now since the adic genus of $BSp(n)$, n any positive integer, is uncountable [17], Theorems 1.1 and Notbohm's result alluded to above imply the following purely algebraic statement about λ-rings.

Corollary 1.5. *Let n be any positive integer. There exist uncountably many distinct isomorphism classes of λ-ring structures over the power series ring $\mathbf{Z}[[x_1, \ldots, x_n]]$ on n indeterminates.*

Remark 1.6. As far as the author is aware, this result is new for every n. It exhibits a huge diversity of λ-ring structures over some rings. The author has recently obtained results about classifying *filtered* λ-ring structures over a filtered ring, exhibiting this set as a moduli space, and also results about how some filtered λ-rings give rise to unstable algebras over the mod p Steenrod algebra.

Remark 1.7. Notice that the way we obtain Corollary 1.5 is by combining three *topological* results about spaces in the genus of classifying spaces and their K-theories, even though the statement of the result is purely algebraic. It would be nice if this statement can be demonstrated in an algebraic way.

Remark 1.8. The analogous question of how many λ-ring structures the *polynomial ring* $\mathbf{Z}[x]$ on one indeterminate supports has been studied by Clauwens [7]. Employing the theory of commuting polynomials, he showed that there are essentially only two non-isomorphic λ-ring structures on the polynomial ring $\mathbf{Z}[x]$. We, however, have not been able to establish any connections between Clauwens' result and our Corollary 1.5. As pointed out by Clauwens, the same question for the polynomial ring on n indeterminates, $n > 1$, is still open.

The last main result of this paper concerns the SNT set of a space. For a given nilpotent finite type space X, $\mathrm{SNT}(X)$ is by definition the set of homotopy types of nilpotent finite type spaces Y such that for every n the Postnikov approximations through dimension n of X and Y are homotopy equivalent. Spaces in the same SNT set are also known as spaces of the same n-type for all n (hence the notation SNT). The SNT set of a space is closely related to its adic genus. Indeed, according to a result of Wilkerson [25], if X is a connected nilpotent finite type space, then there is an inclusion of pointed sets

$$\mathrm{SNT}(X) \;\subset\; \mathrm{Genus}(X).$$

The adic genus of a space, when it is nontrivial, is often strictly bigger than its SNT set.

There are many interesting spaces whose SNT sets are nontrivial (in fact, uncountable); see, for example, [9, 15, 22, 23]. Several homotopy invariants which can distinguish between spaces of the same n-type for all n have been found. These include [16] $\mathrm{Aut}(-)$ and $\mathrm{WI}(-)$, the group of homotopy classes of homotopy self-equivalences and the group of weak identities, and [24] $\mathrm{End}(-)$, the monoid of homotopy classes of self-maps. One might wonder if there are more familiar and more computable homotopy invariants, such as integral cohomology and K-theory, which can distinguish between spaces in the same SNT set. Of course, our Theorem 1.1 together with the above inclusion imply that if a space X is as in Theorem 1.1 and if Y lies in $\mathrm{SNT}(X)$, then the integral cohomology (resp. K-theory) rings of X and Y are isomorphic.

Does this hold for more general spaces?

The following result gives an answer to this question for the ordinary cohomology case.

Theorem 1.9. *Let Λ be a principal ideal domain. Let X be a connected space of finite type whose ordinary cohomology ring with coefficients over Λ is a finitely generated graded algebra over Λ. If Y has the same n-type* *as X for all n, then there exists an isomorphism $H^*(X;\Lambda) \cong H^*(Y;\Lambda)$ of cohomology rings.*

Remark 1.10. It should be emphasized that in Theorem 1.9 the hypothesis $Y \in \mathrm{SNT}(X)$ *cannot* be weakened to the condition $Y \in \mathrm{Genus}(X)$. Indeed, in [5] Bokor constructed two spaces, both two cell complexes, with the same genus but whose integral cohomology rings are non-isomorphic.

This finishes the presentation of the results in this paper. The rest of the paper is organized as follows. In §2 we recall the definitions of a (filtered) λ-ring and of Adams operations. The proofs of Theorems 1.1, 1.4, and 1.9 are given in this order, one in each section, in §3 - §5.

2 λ-rings and Adams operations

In this section, we recall the definitions of a (filtered) λ-ring and of Adams operations. The reader is referred to Atiyah and Tall [3] or Knutson [13] for more information about λ-rings.

2.1 λ-rings

A *λ-ring* is a commutative ring R with unit together with functions

$$\lambda^i \colon R \;\to\; R$$

for $i = 0, 1, \ldots$ satisfying the following properties: For any elements r and s in R one has

- $\lambda^0(r) = 1$
- $\lambda^1(r) = r$, $\lambda^n(1) = 0$ for all $n > 1$
- $\lambda^n(r+s) = \sum_{i=0}^n \lambda^i(r)\lambda^{n-i}(s)$
- $\lambda^n(rs) = P_n(\lambda^1(r), \ldots, \lambda^n(r); \lambda^1(s), \ldots, \lambda^n(s))$
- $\lambda^n(\lambda^m(r)) = P_{n,m}(\lambda^1(r), \ldots, \lambda^{nm}(r))$

Here the P_n and $P_{n,m}$ are certain universal polynomials with integer coefficients. Note that in the literature (for example, Atiyah and Tall [3]) the terminology "special" λ-ring is used.

A λ-ring map $f\colon R \to S$ between two λ-rings is a ring map between the underlying rings which respects the operations λ^i: $f\lambda^i = \lambda^i f$ $(i \geq 0)$.

2.2 Filtered λ-rings

A *filtered λ-ring* is a filtered ring $R = (R, \{I^n\})$ which is also a λ-ring such that the ideals I^n are all closed under the operations λ^i $(i > 0)$.

A filtered λ-ring map is a λ-ring map which is also a filtered ring map.

2.3 Adams operations

Given a λ-ring R, the *Adams operations*

$$\psi^k\colon R \;\to\; R$$

for $k = 1, 2, \ldots$ are defined inductively by the Newton formula:

$$\psi^k(a) - \lambda^1(a)\psi^{k-1}(a) + \cdots + (-1)^{k-1}\lambda^{k-1}(a)\psi^1(a) = (-1)^{k-1}k\lambda^k(a).$$

The Adams operations satisfy the following properties.

1. All the $\psi^k\colon R \to R$ are λ-ring maps.

2. $\psi^1 = \mathrm{Id}$ and $\psi^k\psi^l = \psi^{kl}$ for any $k, l \geq 1$.

3. $\psi^p(a) \equiv a^p \pmod{pR}$ for each prime p and element a in R.

If R is a filtered λ-ring, then the Adams operations are all filtered λ-ring maps.

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1, which consists of a few lemmas. Throughout this section we work in the category of filtered rings.

We will make use of the following observation, whose proof is a straightforward adaptation of Wilkerson's proof of the classification theorem [25, Theorem I] of spaces of the same n-type for all n. Now let X be as in Theorem 1.1.

Lemma 3.1 (Wilkerson). *There is a bijection between the following two pointed sets:*

1. *The pointed set of isomorphism classes of filtered rings* $(R, \{I_n\})$ *with the properties:*
   - *(a) The natural map* $R \to \varprojlim_n R/I_n$ *is an isomorphism, and*
   - *(b)* R/I_n *and* $K(X)/K_n(X)$ *are isomorphic as filtered rings for all* $n > 0$.
2. *The pointed set* $\varprojlim_n^1 \mathrm{Aut}(K(X)/K_n(X))$.

Here $\mathrm{Aut}(-)$ denotes the group of filtered ring automorphisms, and the $\varprojlim^1$ of a tower of not-necessarily abelian groups is as defined in Bousfield-Kan [6].

The two lemmas below will show that, for every space Y in the genus of X, the object $K(Y)$ lies in the first pointed set in Lemma 3.1.

Lemma 3.2. *For every space* Y *in the genus of* X *(as in Theorem 1.1), the natural map*

$$K(Y) \to \varprojlim_j K(Y)/K_j(Y)$$

is an isomorphism.

Proof. According to [4, 2.5 and 7.1] the natural map from $K(Y)$ to $\varprojlim_j K(Y)/K_j(Y)$ is an isomorphism if the following condition holds:

$$\varprojlim_r{}^1 E_r^{p,q} \;=\; 0 \quad \text{for all pairs} \quad (p, q). \tag{3.1}$$

Here $E_r^{*,*}$ is the E_r-term in the K^*-Atiyah-Hirzebruch spectral sequence (AHSS) for Y. This condition is satisfied, in particular, when the AHSS degenerates at the E_2-term. Thus, to prove (3.1) it suffices to show that

$H^*(Y;\mathbf{Z})$ is concentrated in even dimensions, since in that case there is no room for differentials in the AHSS. So pick an odd integer N. We must show that

$$H^N(Y;\mathbf{Z}) \;\cong\; 0. \tag{3.2}$$

By the Universal Coefficient Theorem it suffices to show that the integral homology of Y is torsionfree and is concentrated in even dimensions, which hold because Y lies in the genus of X.

This finishes the proof of Lemma 3.2. $\square$

Lemma 3.3. *For every space Y in the genus of X (as in Theorem 1.1), the filtered rings $K(Y)/K_n(Y)$ and $K(X)/K_n(X)$ are isomorphic for each $n > 0$.*

Proof. We have to show that for each $j > 0$ there is an isomorphism of filtered rings

$$K(Y)/K_j(Y) \;\cong\; K(X)/K_j(X). \tag{3.3}$$

It follows from the hypothesis that for each $j > 0$, the filtered ring $K(Y)/K_j(Y)$ belongs to $\mathrm{Genus}(K(X)/K_j(X))$, where $\mathrm{Genus}(R)$ for a filtered ring R is defined in terms of $R \otimes \mathbf{Q}$ and $R \otimes \widehat{\mathbf{Z}}$ in exactly the same way the genus of a space is defined. To finish the proof we will adapt two results of Wilkerson [26, 3.7 and 3.8], which we now recall.

For a nilpotent finite type space X, Wilkerson showed that there is a surjection

$$\sigma\colon\ \mathrm{Caut}(\hat{X}_0) \to \mathrm{Genus}(X),$$

where $\hat{X}_0$ is the rationalization of the formal completion $\hat{X} = \prod_p X_p^\wedge$ of X. Notice that each homotopy group $\pi_*(\hat{X}_0)$ is a $\mathbf{Q}\otimes\widehat{\mathbf{Z}}$-module, and $\mathrm{Caut}(\hat{X}_0)$ is by definition the group of homotopy classes of self-homotopy equivalences of $\hat{X}_0$ whose induced maps on homotopy groups are $\mathbf{Q}\otimes\widehat{\mathbf{Z}}$-module maps.

Note that the definitions $\mathrm{Genus}(-)$ and $\mathrm{Caut}(-)$ also make sense in both the categories of nilpotent groups and of filtered rings. For instance, if $R = (R, \{I^n\})$ is a filtered ring, then $\mathrm{Caut}(R\otimes\mathbf{Q}\otimes\widehat{\mathbf{Z}})$ is the group of filtered ring automorphisms of $R\otimes\mathbf{Q}\otimes\widehat{\mathbf{Z}}$ which are also $\mathbf{Q}\otimes\widehat{\mathbf{Z}}$-module maps. Now if A is a finitely generated abelian group, then Wilkerson showed that for any class $[\varphi] \in \mathrm{Caut}(A \otimes \mathbf{Q} \otimes \widehat{\mathbf{Z}})$, the image $\sigma([\varphi])$ is isomorphic to A as groups; that is, the image of σ is constant at A.

It is straightforward to adapt Wilkerson's proofs of these results to show that for each j the map

$$\sigma\colon\ \mathrm{Caut}((K(X)/K_j(X)) \otimes \mathbf{Q} \otimes \widehat{\mathbf{Z}}) \to \mathrm{Genus}(K(X)/K_j(X))$$

is surjective and that the image of σ is constant at $K(X)/K_j(X)$. In other words, the genus of $K(X)/K_j(X)$ is the one-point set. Therefore, $K(Y)/K_j(Y)$ is isomorphic to $K(X)/K_j(X)$.

This finishes the proof of Lemma 3.3. □

In view of Lemma 3.1, to finish the proof of Theorem 1.1 we are only left to show that the classifying object

$$\varprojlim{}_n^1\ \mathrm{Aut}(K(X)/K_n(X))$$

is the one point set. To do this, it suffices to show that almost all the structure maps in the tower are surjective. This is shown in the following lemma.

Lemma 3.4. *The maps*

$$\mathrm{Aut}(K(X)/K_{j+1}(X)) \to \mathrm{Aut}(K(X)/K_j(X))$$

are surjective for all j sufficiently large.

Proof. First note that by hypothesis the K-theory filtered ring of X has the form

$$K(X) = \mathbf{Z}[[c_1, \ldots, c_n]] \tag{3.4}$$

in which the generators c_i are algebraically independent over $\mathbf{Z}$ and $K_j(X)$ is the ideal generated by the monomials of filtrations at least j. Suppose that d_i is the largest integer k for which c_i lies in filtration k. Let N be the integer

$$N \ = \ \max\{d_i \colon 1 \leq i \leq n\} + 1.$$

We will show that the structure maps

$$\mathrm{Aut}(K(X)/K_{j+1}(X)) \to \mathrm{Aut}(K(X)/K_j(X))$$

are surjective for all $j > N$.

So fix an integer $j > N$ and pick a filtered ring automorphism σ of $K(X)/K_j(X)$. We must show that σ can be lifted to a filtered ring automorphism of $K(X)/K_{j+1}(X)$. For $1 \leq i \leq n$ pick any lift of the element $\sigma(c_i)$ to $K(X)/K_{j+1}(X)$ and call it $\hat{\sigma}(c_i)$. Since there are no relations among the c_i in $K(X)$, it is easy to see that $\hat{\sigma}$ extends to a well-defined filtered ring endomorphism of $K(X)/K_{j+1}(X)$, and it will be a desired lift of σ once it is shown to be bijective.

To show that

$$\hat{\sigma} \colon K(X)/K_{j+1}(X) \to K(X)/K_{j+1}(X)$$

is surjective, it suffices to show that the image of each c_i in $K(X)/K_{j+1}(X)$ lies in the image of $\hat{\sigma}$, since $K(X)/K_{j+1}(X)$ is generated as a filtered ring by the images of the c_i. So fix an integer i with $1 \leq i \leq n$. We know that there exists an element $g_i \in K(X)/K_j(X)$ such that

$$\sigma(g_i) \;=\; c_i. \tag{3.5}$$

Pick any lift of g_i to $K(X)/K_{j+1}(X)$, call it g_i again, and observe that (3.5) implies that

$$\hat{\sigma}(g_i) \;=\; c_i + \alpha_i \tag{3.6}$$

in $K(X)/K_{j+1}(X)$ for some element $\alpha_i \in K_j(X)/K_{j+1}(X)$. We will alter g_i to obtain a $\hat{\sigma}$-pre-image of c_i as follows.

Observe that the ideal $K_j(X)/K_{j+1}(X)$ is generated by certain monomials in $c_1, \ldots, c_n$. Namely, the monomials

$$\mathbf{c}^{\mathbf{i}} \;=\; c_1^{i_1} \cdots c_n^{i_n}, \quad \mathbf{i} \;=\; (i_1, \ldots, i_n) \in J_j \tag{3.7}$$

where J_j is the set of ordered n-tuples $\mathbf{i} = (i_1, \ldots, i_n)$ of nonnegative integers satisfying

$$\mathbf{d} \cdot \mathbf{i} \;=\; \sum_{l=1}^{n} d_l i_l \;=\; j.$$

Thus, for every element $\mathbf{i}$ in the set J_j, there exists a corresponding integer $a_{\mathbf{i}}$ such that we can write the element α_i as the sum

$$\alpha_i \;=\; \sum_{\mathbf{i}\,\in\, J_j} a_{\mathbf{i}}\, \mathbf{c}^{\mathbf{i}}. \tag{3.8}$$

Now define the element $\bar{g}_i$ in $K(X)/K_{j+1}(X)$ by the formula

$$\bar{g}_i \;\stackrel{\text{def}}{=}\; g_i \;-\; \sum_{\mathbf{i}\,\in\, J_j} a_{\mathbf{i}}\, \mathbf{g}^{\mathbf{i}}, \quad \text{where} \quad \mathbf{g}^{\mathbf{i}} = g_1^{i_1} \cdots g_n^{i_n}. \tag{3.9}$$

We claim that $\bar{g}_i$ is a $\hat{\sigma}$-pre-image of c_i. That is, we claim that

$$\hat{\sigma}(\bar{g}_i) \;=\; c_i \quad \text{in} \quad K(X)/K_{j+1}(X). \tag{3.10}$$

In view of (3.6), (3.8), and (3.9), it clearly suffices to prove the following equality for each element $\mathbf{i}$ in J_j:

$$\hat{\sigma}\left(\mathbf{g}^{\mathbf{i}}\right) \;=\; \mathbf{c}^{\mathbf{i}} \quad \text{in} \quad K(X)/K_{j+1}(X). \tag{3.11}$$

Now in the quotient $K(X)/K_{j+1}(X)$, one computes

$$\begin{aligned}
\hat{\sigma}\left(\mathbf{g}^{\mathbf{i}}\right) \;&=\; \prod_{j=1}^{n} \hat{\sigma}(g_j)^{i_j} \\
&=\; \prod_{j=1}^{n} (c_j + \alpha_j)^{i_j} \quad \text{by (3.6)} \\
&=\; \mathbf{c}^{\mathbf{i}} \;+\; (\text{terms of filtrations } > j) \\
&=\; \mathbf{c}^{\mathbf{i}}.
\end{aligned}$$

This proves (3.11), and hence (3.10), and therefore $\hat{\sigma}$ is surjective.

It remains to show that $\hat{\sigma}$ is injective. Since any surjective endomorphism of a finitely generated abelian group is also injective and since $K(X)/K_{j+1}(X)$ is a finitely generated abelian group, it follows that $\hat{\sigma}$ is injective as well. Thus, $\hat{\sigma}$ is a filtered-ring automorphism of $K(X)/K_{j+1}(X)$ and is a lift of σ.

This finishes the proof of Lemma 3.4. □

This proof of Theorem 1.1 is complete.

4 Proof of Theorem 1.4

In this section we prove Theorem 1.4. The arguments in this section, especially Lemma 4.2 below, are inspired by Rector's [21, §4].

We begin by noting that an argument entirely similar to the proof of Theorem 1.1 implies that whenever X belongs to Genus(BS^3), one has

$$KO^*(X) \cong KO^*[[x]]$$

as filtered rings, where $x \in KO^4_4(X)$ is a representative of an integral generator $x_4 \in H^4(X;\mathbf{Z}) = E_2^{4,0}$ in the KO^*-Atiyah-Hirzebruch spectral sequence for X. Here $KO^a_b(X)$ denotes the subgroup of $KO^a(X)$ consisting of elements u which restrict to 0 under the natural map

$$KO^a(X) \ \to \ KO^a(X_{b-1}).$$

Such an element u is said to be in *degree* a and *filtration* b.

Now we recall the relevant notations, definitions, and results regarding Rector's classification of the genus of BS^3 [20]. Let $\xi \in \pi_{-4}KO$ and $b_R \in \pi_{-8}KO$ be the generators so that $\xi^2 = 4b_R$. As usual, denote by ψ^k ($k = 1, 2, \ldots$) the Adams operations. Since ΩX is homotopy equivalent to S^3, it follows as in [21, §4] that there exists an integer a, depending on the choice of the representative x, such that the following statements hold.

1. $\psi^2(\xi x) = 4\xi x + 2ab_R x^2 \pmod{KO^0_9(X)}$.

2. The integer a is well-defined (mod 24). This means that if x' is another representative of x_4 with corresponding integer a', then $a \equiv a'$ (mod 24), and if x_4 is replaced with $-x_4$, then a will be replaced with $-a$. We can therefore write $a(X)$ for a.

3. $a(X) \equiv \pm 1, \pm 5, \pm 7$, or ± 11 (mod 24).

The last condition above follows from the examples constructed by Rector in [21, §5] and James' result [10] which says that there are precisely eight homotopy classes of homotopy-associative multiplications on S^3. These eight classes can be divided into four pairs with each pair consisting of a homotopy class of multiplication and its inverse.

Rector's invariant (X/p) for p an odd prime is defined as follows [20]. The Adem relation $P^1P^1 = 2P^2$ implies that

$$P^1\,\overline{x}_4 \;=\; \pm\, 2\overline{x}_4^{(p+1)/2}$$

in $H^*(X;\mathbf{Z}/p)$, where $\overline{x}_4$ is the mod p reduction of the integral generator x_4. Then $(X/p) \in \{\pm 1\}$ is defined as the sign on the right-hand side of this equation.

The invariant $(X/2)$ and a canonical choice of orientation of the integral generator are given as follows. Using the mod 24 integer $a(X)$, define

$$((X/2),(X/3)) = \begin{cases} (1,1) & \text{if } a(X) \equiv \pm 1 \text{ mod } 24; \\ (1,-1) & \text{if } a(X) \equiv \pm 5 \text{ mod } 24; \\ (-1,1) & \text{if } a(X) \equiv \pm 7 \text{ mod } 24; \\ (-1,-1) & \text{if } a(X) \equiv \pm 11 \text{ mod } 24. \end{cases} \tag{4.1}$$

The orientation is then chosen so that $(X/3)$ is as given in (4.1). This definition of Rector's invariants coincides with the original one (cf. [14, §9]). Now we can recall the classification theorem of the genus of BS^3 [20].

Theorem 4.1 (Rector). *The (X/p) for p primes provide a complete list of classification invariants for the (adic) genus of BS^3. Any combination of values of the (X/p) can occur. If X is BS^3 then $(X/p) = 1$ for any prime p.*

From now on in this section, X and Y are as in Theorem 1.4. Now we can prove Theorem 1.4.

Proof of Theorem 1.4. In view of Theorem 4.1, to prove Theorem 1.4 it suffices to show that if there exists a filtered λ-ring isomorphism

$$\sigma\colon KO^*(X) \;\cong\; KO^*(Y),$$

then

$$a(X) \;\equiv\; \pm\, a(Y) \pmod{24}$$

and

$$(X/p) \;=\; (Y/p)$$

for all odd primes p. We will prove these in Lemmas 4.2 and 4.3 below, thereby proving Theorem 1.4. $\square$

As explained above, $KO^*(X) = KO^*[[x]]$ and $KO^*(Y) = KO^*[[y]]$ with $x \in KO^4_4(X)$ and $y \in KO^4_4(Y)$ representing, respectively, the integral generators $x_4 \in H^4(X;\mathbf{Z})$ and $y_4 \in H^4(Y;\mathbf{Z})$.

Lemma 4.2. *If there exists a filtered λ-ring isomorphism*

$$\sigma\colon KO^*(X) \;\overset{\cong}{\to}\; KO^*(Y),$$

then

$$a(X) \;\equiv\; \pm\, a(Y) \quad (\mathit{mod}\ 24).$$

Proof. Since σ is a ring isomorphism, we have

$$\sigma(\xi x) \;=\; \varepsilon\xi y + \sigma_2 b_R y^2 \quad (\mathrm{mod}\ KO^0_9(Y))$$

for some integer σ_2 and $\varepsilon \in \{\pm 1\}$. Computing modulo $KO^0_9(X)$ we have

$$\begin{aligned} 4\sigma(b_R x^2) \;&=\; \sigma(\xi x)^2 \\ &=\; \xi^2 y^2 \\ &=\; 4b_R y^2. \end{aligned}$$

Therefore, one has

$$\sigma(b_R x^2) \;=\; b_R y^2 \quad (\mathrm{mod}\ KO^0_9(Y)).$$

First we claim that there is an equality

$$a(X) \;=\; 6\sigma_2 + \varepsilon a(Y). \tag{4.2}$$

To prove (4.2) we will compute both sides of the equality

$$\sigma\psi^2(\xi x) \;=\; \psi^2\sigma(\xi x) \quad (\mathrm{mod}\ KO^0_9(Y)).$$

Working modulo $KO_9^0(Y)$ we have, on the one hand,

$$\begin{aligned}\sigma\psi^2(\xi x) &= \sigma(4\xi x + 2a(X)b_R x^2)\\ &= 4(\varepsilon\xi y + \sigma_2 b_R y^2) + 2a(X)b_R y^2\\ &= 4\varepsilon\xi y + (4\sigma_2 + 2a(X))b_R y^2.\end{aligned}$$

On the other hand, still working modulo $KO_9^0(Y)$, we have

$$\begin{aligned}\psi^2\sigma(\xi x) &= \varepsilon\psi^2(\xi y) + \sigma_2\psi^2(b_R y^2)\\ &= \varepsilon(4\xi y + 2a(Y)b_R y^2) + \sigma_2(2^4 b_R y^2)\\ &= 4\varepsilon\xi y + (16\sigma_2 + 2\varepsilon a(Y))b_R y^2.\end{aligned}$$

Equation (4.2) now follows by equating the coefficients of $b_R y^2$.

In view of (4.2), to finish the proof of Lemma 4.2 it is enough to establish

$$\sigma_2 \;\equiv\; 0 \pmod 4. \tag{4.3}$$

To prove (4.3), note that since σ is a KO^*-module map, we have

$$\xi\sigma(x) \;=\; \sigma(\xi x).$$

Since σ is a ring isomorphism, we also have

$$\sigma(x) \;=\; \varepsilon' y + \sigma_2'\xi y^2 \pmod{KO_9^4(Y)}$$

for some integer σ_2' and $\varepsilon' \in \{\pm 1\}$. Therefore, working modulo $KO_9^0(Y)$ we have

$$\begin{aligned}\xi\sigma(x) &= \varepsilon'\xi y + \sigma_2'\xi^2 y^2\\ &= \varepsilon'\xi y + 4\sigma_2' b_R y^2\\ &= \varepsilon\xi y + \sigma_2 b_R y^2.\end{aligned}$$

In particular, by equating the coefficients of $b_R y^2$ we obtain

$$\sigma_2 \;=\; 4\sigma_2',$$

thereby proving (4.3).

This completes the proof of Lemma 4.2. $\square$

Lemma 4.3. *If there exists a filtered λ-ring isomorphism*

$$\sigma\colon KO^*(X) \xrightarrow{\cong} KO^*(Y),$$

then

$$(X/p) = (Y/p)$$

for each odd prime p.

Proof. It follows from Theorem 1.1 that

$$K^*(X) \cong K^*[[u_x]]$$

with $u_x \in K_4^4(X)$ a representative of the integral generator $x_4 \in H^4(X;\mathbf{Z}) = E_2^{4,0}$ in the K^*-Atiyah-Hirzebruch spectral sequence. Moreover, we may choose u_x so that

$$c(x) = u_x,$$

where

$$c\colon KO^*(X) \to K^*(X)$$

is the complexification map. Similar remarks apply to Y so that

$$K^*(Y) \cong K^*[[u_y]].$$

Now denote by $b \in \pi_{-2}K$ the Bott element and let p be a fixed odd prime. We first claim that

$$\psi^p(b^2u_x) = (b^2u_x)^p + 2(X/p)\,p\,(b^2u_x)^{(p+1)/2} + p\,w_x + p^2x_0 \tag{4.4}$$

for some $w_x \in K^0_{2p+3}(X)$ and some $x_0 \in K^0_4(X)$. To see this, note that since $b^2u_x \in K_4^0(X)$, it follows from Atiyah's theorem [2, 5.6] that

$$\psi^p(b^2u_x) = (b^2u_x)^p + p\,x_1 + p^2x_0$$

for some $x_i \in K^0_{4+2i(p-1)}(X)$ $(i = 0, 1)$. Moreover, one has

$$\overline{x}_1 = P^1\,\overline{b^2u_x},$$

where $\overline{z}$ is the mod p reduction of z and P^1 is the Steenrod operation of degree $2(p-1)$ in mod p cohomology. Thus, to prove (4.4) it is enough to show that

$$x_1 \;=\; 2(X/p)(b^2u_x)^{(p+1)/2} + w_x + p\,z_x \tag{4.5}$$

for some $w_x \in K^0_{2p+3}(X)$ and some $z_x \in K^0_{2p+2}(X)$. Now in $H^*(X;\mathbf{Z})\otimes \mathbf{Z}/p$ we have

$$\begin{aligned} \overline{x}_1 &= P^1\,\overline{b^2u_x} \\ &= P^1\,\overline{x}_4 \\ &= 2(X/p)\,\overline{x}_4^{(p+1)/2} \\ &= 2(X/p)\,\overline{b^2u_x}^{(p+1)/2}. \end{aligned}$$

Now (4.5) follows immediately. As remarked above, this also establishes (4.4).

Now the λ-ring isomorphism σ induces via c a λ-ring isomorphism

$$\sigma_c\colon K^*(X) \;\overset{\cong}{\longrightarrow}\; K^*(Y).$$

By composing σ_c with a suitable λ-ring automorphism of $K^*(Y)$ if necessary, we obtain a λ-ring isomorphism

$$\alpha\colon K^*(X) \;\overset{\cong}{\longrightarrow}\; K^*(Y)$$

with the property that

$$\alpha(b^2u_x) \;=\; b^2u_y + \text{ higher terms in } b^2u_y. \tag{4.6}$$

Using (4.4) and (4.6) it is then easy to check that

$$\alpha\psi^p(b^2u_x) \;=\; 2(X/p)\,p\,(b^2u_y)^{(p+1)/2} \quad (\text{mod } K^0_{2p+3}(Y) \text{ and } p^2) \tag{4.7}$$

and

$$\psi^p\alpha(b^2u_x) \;=\; 2(Y/p)\,p\,(b^2u_y)^{(p+1)/2} \quad (\text{mod } K^0_{2p+3}(Y) \text{ and } p^2). \tag{4.8}$$

Since $\alpha\psi^p = \psi^p\alpha$ it follows from (4.7) and (4.8) that

$$2(X/p)\,p \;\equiv\; 2(Y/p)\,p \quad (\text{mod } p^2),$$

or, equivalently,

$$2(X/p) \equiv 2(Y/p) \pmod{p}.$$

But p is assumed odd, and so

$$(X/p) \equiv (Y/p) \pmod{p}.$$

Hence $(X/p) = (Y/p)$, as desired.

This finishes the proof of Lemma 4.3. □

The proof of Theorem 1.4 is complete.

5 Proof of Theorem 1.9

In this final section we give the proof of Theorem 1.9. The argument is somewhat similar to the proof of Theorem 1.1. All cohomology groups and rings have coefficients over a fixed principal ideal domain Λ. Throughout this section we are working in the category of (non-negatively) graded algebras over Λ.

We need the following two lemmas.

Lemma 5.1. *For each $n > 0$, there is an isomorphism*

$$H^{\leq n}(X) \cong H^{\leq n}(Y)$$

of graded algebras over Λ.

Proof. This follows immediately from the assumption that X and Y have the same n-type for all n. □

By hypothesis the cohomology ring of X has the form

$$H^*(X) = \Lambda[x_1, \ldots, x_s]/J$$

for some homogeneous generators x_i $(1 \leq i \leq s)$ and some homogeneous ideal J. Suppose that $|x_i| = d_i$. Note that since the polynomial ring

$\Lambda[x_1, \ldots, x_s]$ is Noetherian, the ideal J is generated by finitely many polynomials, say, f_m $(1 \leq m \leq k)$. We can thus choose an integer N which is strictly greater than the d_i $(1 \leq i \leq s)$ and the degree of any nontrivial monomial in any f_m $(1 \leq m \leq k)$.

Lemma 5.2. *For each $j > N$, the natural map*

$$\mathrm{Aut}\left(H^{\leq j+1}(X)\right) \to \mathrm{Aut}\left(H^{\leq j}(X)\right)$$

is surjective, where $\mathrm{Aut}(-)$ *denotes the group of graded* Λ*-algebra automorphisms.*

Proof. Let j be any integer strictly greater than N and let σ be an automorphism of the graded algebra $H^{\leq j}(X)$. We want to show that σ can be lifted to an automorphism of $H^{\leq j+1}(X)$. Now $\sigma(f_m)$ is 0 in $H^{\leq j}(X)$ for $m = 1, \ldots, k$. But since each $\sigma(x_i)$ is homogeneous of degree d_i, each monomial of

$$\sigma(f_m) = f_m(\sigma(x_1), \ldots, \sigma(x_s))$$

still has degree less than N, and in this degree the natural map

$$H^{\leq j+1}(X) \to H^{\leq j}(X)$$

is an isomorphism. Therefore, there is a well-defined graded algebra map

$$\hat{\sigma} \colon H^{\leq j+1}(X) \to H^{\leq j+1}(X)$$

satisfying

$$\hat{\sigma}|H^{\leq j}(X) = \sigma \colon H^{\leq j}(X) \to H^{\leq j}(X).$$

We will be done once we show that $\hat{\sigma}$ is bijective.

It is clear that $\hat{\sigma}$ is surjective because $H^{\leq j+1}(X)$ is generated as an algebra by the x_i $(1 \leq i \leq s)$ and each x_i is in the image of $\hat{\sigma}$, since this is true for σ.

Finally, since $H^{\leq j+1}(X)$ is a finitely generated Λ-module, the injectivity of $\hat{\sigma}$ now follows from its surjectivity. Therefore, $\hat{\sigma}$ is an automorphism of $H^{\leq j+1}(X)$ and is a lift of σ.

This finishes the proof of Lemma 5.2. □

Proof of Theorem 1.9. This is now an immediate consequence of Lemma 5.1, Lemma 5.2, and the analog of Wilkerson's Theorem 3.1 in the context of (non-negatively) graded Λ-algebras. □

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References

[1] J. F. Adams, Localisation and completion, University of Chicago Press, Chicago, Ill., 1975.

[2] M. F. Atiyah, Power operations in K-theory, Quart. J. Math. Oxford 17 (1966) 163-193.

[3] M. F. Atiyah and D. O. Tall, Group representations, λ-rings and the J-homomorphism, Topology 8 (1969) 253-297.

[4] J. Michael Boardman, Conditionally convergent spectral sequences, Contemp. Math. 239, Amer. Math. Soc., Providence, RI, 1999, p. 49-84.

[5] I. Bokor, On the connection between the topological genus of certain polyhedra and the algebraic genus of their Hilton-Hopf quadratic forms, Publ. Mat. 34 (1990) 323-333.

[6] A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972.

[7] F. J. B. J. Clauwens, Commuting polynomials and λ-ring structures on $\mathbf{Z}[x]$, J. Pure Appl. Algebra 95 (1994) 261-269.

[8] E. Dror, W. G. Dwyer, and D. M. Kan, Self-homotopy equivalences of Postnikov conjugates, Proc. Amer. Math. Soc. 74 (1979) 183-186.

[9] J. R. Harper and J. Roitberg, Phantom maps and spaces of the same n-type for all n, J. Pure Appl. Algebra 80 (1992) 123-137.

[10] I. M. James, Multiplication on spheres. II, Trans. Amer. Math. Soc. 84 (1957) 545-558.

[11] A. Jeanneret and A. Osse, The K-theory of p-compact groups, Comment. Math. Helv. 72 (1997) 556-581.

[12] D. C. Johnson and W. S. Wilson, BP operations and Morava's extraordinary K-theories, Math. Z. 144 (1975) 55-75.

[13] D. Knutson, λ-rings and the representation theory of the symmetric group, Lecture Notes in Mathematics, Vol. 308. Springer-Verlag, Berlin-New York, 1973.

[14] C. A. McGibbon, The Mislin genus of a space, The Hilton Symposium 1993 (Montreal, PQ), 75-102, CRM Proc. Lecture Notes, 6, Amer. Math. Soc., Providence, RI, 1994.

[15] C. A. McGibbon and J. M. Møller, On spaces with the same n-type for all n, Topology 31 (1992) 177-201.

[16] ———, How can you tell two spaces apart when they have the same n-type for all n?, Adams Memorial Symposium on Algebraic Topology, 1 (Manchester, 1990), 131-143, London Math. Soc. Lecture Note Ser., 175, Cambridge Univ. Press, Cambridge, 1992.

[17] J. M. Møller, The normalizer of the Weyl group, Math. Ann. 294 (1992) 59-80.

[18] D. Notbohm, Maps between classifying spaces and applications, J. Pure Appl. Algebra 89 (1993) 273-294.

[19] D. Notbohm and L. Smith, Fake Lie groups and maximal tori. I – III, Math. Ann. 288 (1990) 637-661, 663-673 and 290 (1991) 629-642.

[20] D. L. Rector, Loop structures on the homotopy type of S^3, Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle, Wash., 1971), pp. 99-105. Lecture Notes in Math., Vol. 249, Springer, Berlin, 1971.

[21] ———, Subgroups of finite dimensional topological groups, J. Pure Appl. Algebra 1 (1971) 253-273.

[22] Y. Shitanda, Uncountably many loop spaces of the same n-type for all n. I and I, Yokohama Math. J. 41 (1993) 17-24 and 41 (1994) 85-93.

[23] ———, Uncountably many infinite loop spaces of the same n-type for all n, Math. J. Okayama Univ. 34 (1992) 217-223.

[24] ———, Spaces of the same clone type, J. Math. Soc. Japan 48 (1996) 705-714.

[25] C. Wilkerson, Classification of spaces of the same n-type for all n, Proc. Amer. Math. Soc. 60 (1976) 279-285.

[26] ———, Applications of minimal simplicial groups, Topology 15 (1976) 111-130.

[27] ———, Lambda-rings, binomial domains, and vector bundles over $CP(\infty)$, Comm. Algebra 10 (1982) 311-328.