EXTENSIONS OF FILTERED λ-RING STRUCTURES OVER THE DUAL NUMBER RING

DONALD YAU

Abstract. We study problems related to the existence and uniqueness of filtered λ-ring structures over the truncated polynomial ring $\mathbf{Z}[x]/x^3$ that extend a given filtered λ-ring structure over $\mathbf{Z}[x]/x^2$.

1. Introduction

A λ-ring is, roughly speaking, a commutative ring R with unit together with operations λ^i on it that act like the exterior power operations. It is widely used in Algebraic Topology, Algebra, and Representation Theory. For example, the complex representation ring $R(G)$ of a group G is a λ-ring, where λ^i is induced by the map that sends a representation to its ith exterior power. Another example of a λ-ring is the complex K-theory of a topological space X. Here, λ^i arises from the map that sends a complex vector bundle η over X to the ith exterior power of η. In the algebra side, the universal Witt ring $\mathbf{W}(R)$ of a commutative ring R is a λ-ring. In fact, in some sense, it is the universal λ-ring that can be associated to R.

The λ-operations impose severe restrictions on the structure of the underlying ring, and not every commutative ring can be given the structure of a λ-ring. Given an arbitrary commutative ring R with unit, it is, therefore, natural to ask whether it admits a λ-ring structure. If so, how many λ-ring structures does it admit and is there a natural structure on the moduli set of λ-ring structures? If R is a λ-ring and the underlying ring of R is a quotient of another ring S, can the λ-ring structure on R be extended to one on S? If so, in how many ways can it be done and what is the structure of the moduli set of such extensions?

Some of these problems have been studied before. In [2] Clauwens showed that over the integral polynomial ring $\mathbf{Z}[x]$, there are essentially only two λ-ring structures. As far as the author is aware, this is the first paper in the literature in which the question of how many λ-ring structures a given ring admits is studied explicitly. In contrast to the polynomial case, the author [6] showed the existence of uncountably many pairwise non-isomorphic λ-ring structures on the power series ring $\mathbf{Z}[[x_1, \ldots, x_k]]$, $k \geq 1$. In fact, the

Date: April 27, 2004.
2000 *Mathematics Subject Classification.* 55S25,16E20,16W99.

uncountably many λ-ring structures in [6] are the K-theory of some spaces that are closely related to the classifying spaces of compact connected Lie groups. It is probably not true that every λ-ring structure over the power series ring arises this way.

Sometimes it is more natural to think about *filtered* λ-ring, which is just a λ-ring whose underlying ring is filtered by a decreasing sequence of ideals that are closed under the λ-operations. For example, the K-theory of a CW complex is naturally a filtered λ-ring. The uncountably many λ-rings in [6] mentioned in the previous paragraph are filtered λ-ring structures over the power series ring with a certain filtration, and they are non-isomorphic as filtered λ-rings. Since a filtered ring R has a topology on it, one is tempted to ask if the moduli set of filtered λ-ring structures on it has a natural topology that is, in some sense, compatible with the topology on R. This is indeed the case, as was shown by the author in [5]. Another result in [5] classifies the set of isomorphism classes of filtered λ-ring structures on the dual number ring $\mathbf{Z}_{(J)}[x]/x^2$ with the x-adic filtration over the J-local integers, where J is an arbitrary set of primes. They are in bijection with the set of sequences $\{b_p\}$, p primes, where b_p is divisible by p in $\mathbf{Z}_{(J)}$.

The main purpose of this note is to continue the work in [5] and study extensions of a filtered λ-ring structure R over the dual number ring $\mathbf{Z}_{(J)}[x]/x^2$ to the truncated polynomial ring $\mathbf{Z}_{(J)}[x]/x^3$. A necessary and sufficient condition for such an extension to exist is that, for all odd primes q, the linear coefficient b_q of $\psi^q(x)$ in R is congruent to its square b_q^2 modulo a certain power of 2, depending on b_2 but not q (Theorem 3.1). In particular, such an extension always exists if either 2 is invertible (Corollary 3.2) or there is exactly one factor of 2 in b_2 (Corollary 3.5). It also implies that there are uncountably many non-isomorphic extensions when b_2 is equal to 0 (Example 3.3) and that there are uncountably many filtered λ-ring structures on $\mathbf{Z}[x]/x^2$ that do not admit any extension (Example 3.4).

Next we study when two extensions are isomorphic. Theorem 3.6 gives a necessary and sufficient condition for two extensions of R to be isomorphic. One formulation of it only depends on the coefficients in the Adams operation ψ^2 when applied to x. There are several interesting consequences of this result. First, over the 2-local integers, if b_2 has exactly one factor of 2, then R admits a unique extension to a filtered λ-ring structure on $\mathbf{Z}_{(2)}[x]/x^3$ (Corollary 3.7). This last statement is not true over the integers (Example 3.8), but the additional assumption, $b_2 = 2$, would make it true (Corollary 3.9). Moreover, over the integers, as long as b_2 is non-zero, there are always at most finitely many extensions (Corollary 3.11). A version of this result, with the integers replaced by the J-local integers, is given in Corollary 3.10. Finally, we note that, although integrally the moduli set of isomorphism classes of extensions is finite, it could be made arbitrarily large by taking different filtered λ-ring structures (Example 3.12).

In the following section, we will review some basic results about λ-rings and observe that many truncated power series rings have uncountably many non-isomorphic filtered λ-ring structures (Theorem 2.2).

Here is a list of some of the notations that are used in this note.

- $\mathbf{Z}_{(J)}$: The J-local integers, where J is a set of primes.
- $r\mathbf{Z}_{(J)}$, $r \in \mathbf{Z}_{(J)}$: The subgroup of $\mathbf{Z}_{(J)}$ consisting of elements of the form rs for some $s \in \mathbf{Z}_{(J)}$.
- $\mathbf{Z}_{(J)}^{\times}$: The invertible elements in $\mathbf{Z}_{(J)}$.
- $\theta_p(r)$, p prime, r rational: $\theta_p(n)$ is the largest integer for which $p^{\theta_p(n)}$ divides n, when n is an integer; $\theta_p(r) = \theta_p(m) - \theta_p(n)$ if $r = m/n$. By convention, $\theta_p(0) = -\infty$ for all primes p.

2. Filtered λ-rings

The purpose of this section is to give a brief account of the basic definitions about λ-rings and their filtered analogues. The result in [5] about the classification of filtered λ-ring structures over the dual number ring is then recalled. This section ends with a result that says that many truncated power series rings have uncountably many non-isomorphic filtered λ-ring structures.

The reader may refer to [1, 3] for more discussions about the basic algebraic properties of λ-rings. We should point out that what we call a λ-ring here is referred to as a "special" λ-ring in [1].

By a λ-ring, we mean a commutative ring R with unit that is equipped with functions

$$\lambda^i\colon R \;\to\; R \quad (i \geq 0),$$

called λ-operations. These functions are required to satisfy the following conditions. For any integers $i, j \geq 0$ and elements r and s in R:

- $\lambda^0(r) = 1$.
- $\lambda^1(r) = r$.
- $\lambda^i(1) = 0$ for $i > 1$.
- $\lambda^i(r + s) = \sum_{k=0}^{i} \lambda^k(r)\lambda^{i-k}(s)$.
- $\lambda^i(rs) = P_i(\lambda^1(r), \ldots, \lambda^i(r); \lambda^1(s), \ldots, \lambda^i(s))$.
- $\lambda^i(\lambda^j(r)) = P_{i,j}(\lambda^1(r), \ldots, \lambda^{ij}(r))$.

The P_i and $P_{i,j}$ are certain universal polynomials with integer coefficients, and they are defined using the elementary symmetric polynomials. See the references mentioned above for their exact definitions. A map f of λ-rings is a ring map of the underlying rings that is compatible with the λ-operations, in the sense that $f\lambda^i = \lambda^i f$ for all i.

There are some very useful operations inside a λ-ring, the so-called Adams operations

$$\psi^n \colon R \to R \quad (n \geq 1).$$

They are defined by the Newton formula:

$$\psi^n(r) - \lambda^1(r)\psi^{n-1}(r) + \cdots + (-1)^{n-1}\lambda^{n-1}(r)\psi^1(r) + (-1)^n n\lambda^n(r) = 0.$$

The Adams operations have the following properties:

- All the ψ^n are ring maps.
- $\psi^1 = \mathrm{Id}$.
- $\psi^m\psi^n = \psi^{mn} = \psi^n\psi^m$.
- $\psi^p(r) \equiv r^p \pmod{pR}$ for each prime p and element r in R.

A λ-ring map f is compatible with the Adams operations, in the sense that $f\psi^n = \psi^n f$ for all n.

As one observes in the Newton formula, one can almost retrieve the λ-operations from the Adams operations. This is, in fact, possible provided that R is $\mathbf{Z}$-torsionfree. More explicitly, a theorem of Wilkerson [4] says that if R is a $\mathbf{Z}$-torsionfree ring equipped with ring endomorphisms ψ^n $(n \geq 1)$ satisfying the last three conditions above, then there exists a unique λ-ring structure on R whose Adams operations are exactly the given ψ^n. We will refer to this as Wilkerson's Theorem. In particular, over the truncated polynomial ring $\mathbf{Z}_{(J)}[x]/x^n$, a λ-ring structure is specified by polynomials $\psi^p(x)$, p primes, such that

$$\psi^p(\psi^q(x)) \;=\; \psi^q(\psi^p(x))$$

and

$$\psi^p(x) \;\equiv\; x^p \pmod{p\mathbf{Z}_{(J)}[x]/x^n}.$$

By a *filtered* ring, we mean a commutative ring R with unit together with a decreasing sequence of ideals

$$R \;=\; I^0 \supseteq I^1 \supseteq I^2 \supseteq \cdots$$

such that $I^nI^m \subseteq I^{m+n}$. A *filtered* λ-ring is a λ-ring R which is also a filtered ring in which each ideal I^n is closed under λ^i for $i \geq 1$. Suppose that R and S are two filtered λ-rings. Then a filtered λ-ring map

$$f \colon R \to S$$

is a λ-ring map that also preserves the filtration ideals, in the sense that $f(I^n) \subseteq I^n$ for all n.

We can now recall the classification of filtered λ-ring structures over the dual number ring $\mathbf{Z}_{(J)}[x]/x^2$. The variable x is given degree 1 here.

Proposition 2.1 (Corollary 4.1.2 in [5])**.** *Let J be a set of primes. Then there is a one-to-one correspondence between the set of isomorphism classes of filtered λ-ring structures over $\mathbf{Z}_{(J)}[x]/x^2$ and the set $\prod_{p \notin J} \mathbf{Z}_{(J)} \times \prod_{p \in J} p\mathbf{Z}_{(J)}$. This correspondence associates (b_p) to the filtered λ-ring structure with Adams operations, $\psi^p(x) = b_p x$.*

In particular, $\mathbf{Z}_{(J)}[x]/x^2$ admits uncountably many isomorphism classes of filtered λ-ring structures.

Therefore, we can and will specify a filtered λ-ring structure over $\mathbf{Z}_{(J)}[x]/x^2$ by a sequence of J-local integers $\{b_p\}$, one for each prime p. The J-local integer b_p is the linear coefficient in $\psi^p(x)$.

Before we go any further, we would like to mention a simple but interesting fact about the abundance of filtered λ-ring structures over truncated power series rings. One can think of it as a partial generalization of the above proposition.

Theorem 2.2. *Let $x_1, \ldots, x_k$ be algebraically independent variables of degree 1 and let J be a set of primes. Let $n_1, \ldots, n_k$ be elements in $\{2, 3, \ldots, \infty\}$ with at least one $n_j < \infty$. Then the truncated power series ring $R = \mathbf{Z}_{(J)}[[x_1, \ldots, x_k]]/(x_1^{n_1}, \ldots, x_k^{n_k})$ admits uncountably many pairwise non-isomorphic filtered λ-ring structures.*

In the statement above, we take x_i^∞ to mean 0. So if $n_i = \infty$, then no positive power of x_i is equal to 0 in R.

Proof. Let N be the maximal of those n_j that are finite. For each prime $p \geq N$ and each index j for which $n_j < \infty$, choose an arbitrary positive integer $b_{p,j} \in p\mathbf{Z}$. Consider the following power series in R:

$$\psi^p(x_i) \;=\; \begin{cases} (1 + x_i)^{b_{p,i}} - 1 & \text{if } p \geq N \text{ and } n_i < \infty, \\ (1 + x_i)^p - 1 & \text{otherwise.} \end{cases} \tag{2.1}$$

Here p runs through the primes and $i = 1, 2, \ldots, k$. Each one of these power series extends to a filtered ring endomorphism of R.

We first claim that these power series are the Adams operations (applied to the x_i) of a filtered λ-ring structure S on R. Since R is $\mathbf{Z}$-torsionfree, by Wilkerson's Theorem [4], it suffices to show that

$$\psi^p \psi^q \;=\; \psi^q \psi^p \tag{2.2}$$

and that

$$\psi^p(r) \;\equiv\; r^p \pmod{pR} \tag{2.3}$$

for all primes p and q and elements $r \in R$. Both of these are verified easily using (2.1). Equation (2.2) is true because it is true when applied to each

x_i and that the x_i are algebra generators of R. Equation (2.3) is true, since it is true for $r = x_i$.

Now suppose that $\bar{S}$ is another filtered λ-ring structure on R constructed in the same way with the integers $\{\bar{b}_{p,j} \in p\mathbf{Z}\}$. (Here again p runs through the primes $\geq N$ and j runs through the indices for which $n_j < \infty$.) So in $\bar{S}$, $\psi^p(x_i)$ looks just like it is in (2.1), except that $b_{p,i}$ is replaced by $\bar{b}_{p,i}$. Suppose, in addition, that there is a prime $q \geq N$ such that

$$\{b_{q,j}\} \cup \{q\} \;\neq\; \{\bar{b}_{q,j}\} \cup \{q\}$$

as sets. We claim that S and $\bar{S}$ are not isomorphic as filtered λ-rings.

To see this, suppose to the contrary that there exists a filtered λ-ring isomorphism

$$\sigma \colon S \;\to\; \bar{S}.$$

Let j be one of those indices for which n_j is finite. Then

$$\sigma(x_j) \;\equiv\; a_1x_1 + \cdots + a_kx_k \pmod{\text{degree } 2}$$

for some $a_1, \ldots, a_k \in \mathbf{Z}_{(J)}$, not all of which are equal to 0. Equating the linear coefficients on both sides of the equation

$$\sigma\psi^q(x_j) \;=\; \psi^q\sigma(x_j),$$

one infers that

$$b_{q,j} \;=\; q \quad \text{or} \quad \bar{b}_{q,i}$$

for some i. In particular, it follows that

$$\{b_{q,j}\} \cup \{q\} \;\subset \{\bar{b}_{q,j}\} \cup \{q\},$$

and therefore the two sets are equal by symmetry. This is a contradiction.

This finishes the proof of the theorem. □

3. The dual number ring

Let R be a filtered λ-ring structure over the dual number ring $\mathbf{Z}_{(J)}[x]/x^2$. A filtered λ-ring *extension* of R to $\mathbf{Z}_{(J)}[x]/x^3$ is a filtered λ-ring structure S over $\mathbf{Z}_{(J)}[x]/x^3$ such that $S/(x^3)$ is identical to R as a λ-ring. In other words, when applied to x, the Adams operations in R are exactly those of S truncated by x^3. Sometimes we will just say that S is an extension of R.

First we would like to have a usable criterion that guarantees that an extension of R exists.

Theorem 3.1. *Let J be a set of primes and let R be a filtered λ-ring structure over $\mathbf{Z}_{(J)}[x]/x^2$ corresponding to the sequence (b_p). Then R admits a filtered λ-ring extension to $\mathbf{Z}_{(J)}[x]/x^3$ if, and only if, for all odd primes q,*

$$b_q \;\equiv\; b_q^2 \pmod{2^{\theta_2(b_2)}\mathbf{Z}_{(J)}}.$$

Proof. We will do the proof for **Z**, which corresponds to the case when J is the set of all primes. The case when J is not the set of all primes can be proved with essentially the same argument.

The existence of an extension of R is equivalent to the existence of integers c_p, p primes, such that the polynomials $\psi^p(x) = b_p x + c_p x^2$ in $\mathbf{Z}[x]/x^3$ satisfy

$$\psi^p(\psi^q(x)) \;=\; \psi^q(\psi^p(x)) \tag{3.1}$$

and

$$\psi^p(x) \;\equiv\; x^p \pmod{pR} \tag{3.2}$$

for all primes p and q. The equation (3.2) is equivalent to $c_2 \equiv 1 \pmod 2$ when $p = 2$ and $c_q \equiv 0 \pmod q$ when q is an odd prime. In $\mathbf{Z}[x]/x^3$ one computes

$$\begin{aligned}\psi^p(\psi^q(x)) &= b_p(b_q x + c_q x^2) + c_p(b_q x + c_q x^2)^2 \\ &= (b_p b_q)x + (b_p c_q + b_q^2 c_p)x^2.\end{aligned}$$

Using symmetry and equating the coefficients of x^2, it follows that (3.1) is equivalent to

$$(b_q^2 - b_q)c_p \;=\; (b_p^2 - b_p)c_q. \tag{3.3}$$

Let's consider first the case $b_2 = 0$. If an extension exists, then since $c_2 \neq 0$, (3.3) implies that $b_q = 0$ for all odd primes q because b_q, being divisible by q, is not equal to 1. This is exactly the condition stated in the theorem when $b_2 = 0$. Conversely, suppose that the stated condition is satisfied, meaning that $b_q = 0$ for all odd primes q. Then (3.3) is a vacuous statement, and therefore one obtains an extension of R by simply choosing integers c_p with $c_2 \equiv 1 \pmod 2$ and $c_q \equiv 0 \pmod q$ when q is an odd prime.

Now let's consider the case $b_2 \neq 0$. Consider (3.3) with $p = 2$. If an extension of R exists, then since c_2 is not divisible by 2, it follows that $2^{\theta_2(b_2)}$ divides $(b_q^2 - b_q)$. Conversely, suppose that $(b_q^2 - b_q)$ is divisible by $2^{\theta_2(b_2)}$. We take

$$c_p \;=\; \begin{cases} \dfrac{b_2(b_2-1)}{2^{\theta_2(b_2)}} & \text{if } p = 2 \\ \dfrac{b_p(b_p-1)}{2^{\theta_2(b_2)}} & \text{if } p > 2. \end{cases}$$

Then (3.3) is clearly satisfied. Moreover, c_2 is an odd integer, since both $b_2/2^{\theta_2(b_2)}$ and $(b_2 - 1)$ are. Likewise, for an odd prime p, $(b_p^2 - b_p)$ is divisible by p (since p divides b_p) and therefore so is $(b_p^2 - b_p)/2^{\theta_2(b_2)}$. This shows that R admits an extension. □

One consequence of this theorem is that there must be an extension when 2 is invertible, since in this case the congruence relation in the theorem is automatically satisfied.

Corollary 3.2. *If J is a set of primes that does not contain 2, then every filtered λ-ring structure over $\mathbf{Z}_{(J)}[x]/x^2$ can be extended to one over $\mathbf{Z}_{(J)}[x]/x^3$.*

Another consequence of the theorem is that integrally, when $b_2 = 0$, the moduli set of isomorphism classes of filtered λ-ring extensions is uncountable.

Example 3.3 ($\theta_2(b_2) = -\infty$)**.** In the proof of Theorem 3.1, we saw that when $b_2 = 0$ (and therefore $b_q = 0$ for all odd primes q as well), an extension of R to a filtered λ-ring structure over $\mathbf{Z}[x]/x^3$ is given by a sequence of integers (c_p) satisfying $c_2 \equiv 1 \pmod 2$ and $c_q \equiv 0 \pmod q$ if $q > 2$. If (c_p) and $(\bar{c}_p)$ correspond to two such extensions, then a filtered λ-ring isomorphism between them is given by a polynomial

$$\sigma(x) \;=\; \epsilon x + a x^2$$

for some $\epsilon \in \{\pm 1\}$ and some integer a. It is compatible with the Adams operations in the sense that $\sigma\psi^p = \psi^p\sigma$, which is equivalent to

$$c_p \;=\; \epsilon \bar{c}_p$$

for all primes p. Therefore, the set of isomorphism classes of filtered λ-ring structures over $\mathbf{Z}[x]/x^3$ extending R is parametrized by

$$\left((1 + 2\mathbf{Z}) \times \prod_q q\mathbf{Z}\right)/\{\pm 1\}$$

in which q runs through the odd primes.

In the integral case, since b_2 is an even integer, we have either $\theta_2(b_2) \geq 1$ or $= -\infty$. In contrast to Example 3.3, there are also many filtered λ-ring structures over the dual number ring for which the moduli set is empty.

Example 3.4 ($\theta_2(b_2) \geq 2$)**.** Over the dual number ring $\mathbf{Z}[x]/x^2$, there exist uncountably many isomorphism classes of filtered λ-ring structures that do not admit any filtered λ-ring extension to $\mathbf{Z}[x]/x^3$. Indeed, let b_2 be any even integer with $\theta_2(b_2) \geq 2$. For each odd prime q, let l_q be any positive integer and set

$$b_q \;=\; 2q^{l_q}.$$

Denote by R the filtered λ-ring structure over $\mathbf{Z}[x]/x^2$ corresponding to (b_p). Then $2^{\theta_2(b_2)}$ does not divide b_q for any odd prime q. Since $(b_q - 1)$ is an odd integer, it is not divisible by $2^{\theta_2(b_2)}$ either. Therefore, Theorem 3.1 implies that R does not admit any filtered λ-ring extension to $\mathbf{Z}[x]/x^3$.

Observe that the product $b_2(b_2-1)$ is always divisible by 2 in $\mathbf{Z}_{(J)}$. In particular, the congruence condition in Theorem 3.1 is automatically satisfied if $\theta_2(b_2) = 1$, and so the moduli set is non-empty.

Corollary 3.5. *Let R be a filtered λ-ring structure over $\mathbf{Z}_{(J)}[x]/x^2$ corresponding to a sequence (b_p) with $\theta_2(b_2) = 1$. Then R admits a filtered λ-ring extension to $\mathbf{Z}_{(J)}[x]/x^3$.*

Of course, this result is only interesting if $2 \in J$, since otherwise we already know from Corollary 3.2 that an extension exists.

We will see below that over $\mathbf{Z}$, R admits a *unique* extension if $b_2 = 2$. Integrally the condition $\theta_2(b_2) = 1$ is not enough to guarantee the existence of a unique extension. However, over the integers localized at 2, $\theta_2(b_2) = 1$ does guarantee that the extension is unique.

In order to further understand uniqueness of extensions and the moduli set of extensions, we would like to know when two extensions are isomorphic.

Let R be a filtered λ-ring structure over $\mathbf{Z}_{(J)}[x]/x^2$ corresponding to a sequence (b_p), where J is a set of primes containing 2. Assume that $b_2 \neq 0$. Suppose that S and $\bar{S}$ are two filtered λ-ring extensions of R to $\mathbf{Z}_{(J)}[x]/x^3$ corresponding to, respectively, (c_p) and $(\bar{c}_p)$. In other words, the Adams operations of S is given by

$$\psi^p(x) \;=\; b_p x + c_p x^2,$$

and similarly for $\bar{S}$. Denote the set of invertible elements in $\mathbf{Z}_{(J)}$ by $\mathbf{Z}_{(J)}^{\times}$.

Theorem 3.6. *With the assumptions and notations as above, the following statements are equivalent:*

1. *S and $\bar{S}$ are isomorphic filtered λ-rings.*
2. *There exist $u \in \mathbf{Z}_{(J)}^{\times}$ and $a \in \mathbf{Z}_{(J)}$ such that*
$$a b_p(1 - b_p) \;=\; u\bar{c}_p - u^2 c_p$$
for all primes p.
3. *There exist $u \in \mathbf{Z}_{(J)}^{\times}$ and $a \in \mathbf{Z}_{(J)}$ such that*
$$a b_2(1 - b_2) \;=\; u\bar{c}_2 - u^2 c_2.$$

If one of these equivalent conditions is satisfied, then a filtered λ-ring isomorphism

$$\sigma \colon S \to \bar{S}$$

is given by

$$\sigma(x) \;=\; ux + ax^2,$$

extended linearly and multiplicatively to all of $\mathbf{Z}[x]/x^2$.

Proof. (1) $\Rightarrow$ (2). Let $\sigma\colon S \to \bar{S}$ be a filtered λ-ring isomorphism. Then

$$\sigma(x) \;=\; ux + ax^2 \tag{3.4}$$

for some J-local unit u and J-local integer a. Applying the map $\sigma\psi^p$ to the generator x, one obtains

$$\begin{aligned}\sigma\psi^p(x) &= b_p(ux + ax^2) + c_p(ux + ax^2)^2 \\ &= ub_px + (ab_p + u^2c_p)x^2.\end{aligned}$$

Similarly, one has

$$\psi^p\sigma(x) \;=\; ub_px + (ab_p^2 + u\bar{c}_p)x^2.$$

Condition (2) now follows by equating the coefficients of x^2.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Since 2 is not invertible in $\mathbf{Z}_{(J)}$, it follows that c_2 cannot be 0. Therefore, (3.3) implies that $c_p = 0$ if and only if $b_p \in \{0, 1\}$. For those $b_p \notin \{0, 1\}$, we infer from (3.3) again that

$$a \;=\; \frac{u\bar{c}_2 - u^2c_2}{b_2 - b_2^2} \;=\; \frac{u\bar{c}_p - u^2c_p}{b_p - b_p^2}.$$

The argument for the proof of (1) $\Rightarrow$ (2) now implies that the filtered ring isomorphism $\sigma\colon S \to \bar{S}$ defined by (3.4) satisfies $\sigma\psi^p = \psi^p\sigma$ for all primes p, since they agree on x. Therefore, σ is a filtered λ-ring isomorphism. □

We will discuss several consequences of this theorem.

Suppose that we are working over the 2-local integers and assume that $\theta_2(b_2) = 1$. Then

$$\bar{c}_2 \;\equiv\; c_2 \pmod{2\mathbf{Z}_{(2)}},$$

since both c_2 and $\bar{c}_2$ are congruent to 1 modulo $2\mathbf{Z}_{(2)}$. Moreover, $(1 - b_2)$ is a 2-local unit, and therefore so is $\frac{b_2}{2}(1 - b_2)$. It follows that

$$\frac{\bar{c}_2 - c_2}{b_2 - b_2^2} \;=\; \frac{\bar{c}_2 - c_2}{2} \cdot \frac{2}{b_2(1 - b_2)}$$

lies in $\mathbf{Z}_{(2)}$. This shows that the third condition in Theorem 3.6 is satisfied.

Corollary 3.7. *Any filtered λ-ring structure over $\mathbf{Z}_{(2)}[x]/x^2$ with $\theta_2(b_2) = 1$ admits, up to isomorphism, a unique filtered λ-ring extension to $\mathbf{Z}_{(2)}[x]/x^3$.*

This corollary is not true over the integers, as the following example shows.

Example 3.8. Consider the filtered λ-ring structure R over $\mathbf{Z}[x]/x^2$ with

$$\psi^p(x) \;=\; \begin{cases} -2x & \text{if } p = 2 \\ 0 & \text{if } p > 2. \end{cases}$$

We claim that R admits, up to isomorphism, exactly two filtered λ-ring extensions to $\mathbf{Z}[x]/x^3$.

To see this, first note that, since $b_q = 0$ for odd primes q, in any extension corresponding to, say, $\{c_p\}$, c_q must be equal to 0 for all odd primes q. In other words, in any extension of R, the Adams operation ψ^q for q odd sends x to 0. Therefore, an extension of R determines and is determined by a choice of an odd integer c_2. Consider the two extensions S and $\bar{S}$ with Adams operations

$$\psi^2(x) \;=\; -2x + x^2$$

and

$$\psi^2(x) \;=\; -2x + 3x^2,$$

respectively. We will show that S and $\bar{S}$ are non-isomorphic and any extension of R is isomorphic to either S or $\bar{S}$.

In order to see that they are not isomorphic, observe that $b_2(1-b_2) = -6$, which divides neither 2 nor 4. Therefore, the third condition in Theorem 3.6 implies that S and $\bar{S}$ cannot be isomorphic as filtered λ-rings.

Now let T be any extension of R with

$$\psi^2(x) \;=\; -2x + cx^2.$$

Since c is an odd integer, it follows that

$$c \;\equiv\; 1,\ 3,\ \text{ or } 5 \pmod{6}.$$

In particular, for some choice of $\epsilon \in \{\pm 1\}$, either $(\epsilon c - 1)$ or $(c - 3)$ is divisible by 6. Therefore, by the third condition in Theorem 3.6, T is isomorphic as a filtered λ-ring to either S or $\bar{S}$.

We can achieve uniqueness of an extension over the integers if we insist that $b_2 = 2$. Indeed, in this case, c_2 is an odd integer, and so $\bar{c}_2 - c_2$ is divisible by 2. In particular, the third condition in Theorem 3.6 is satisfied with $u = 1$ and $a = (c_2 - \bar{c}_2)/2$. Since Corollary 3.5 tells us that in this case there is at least one extension, we infer that the moduli set contains exactly one point.

Corollary 3.9. *Any filtered λ-ring structure over $\mathbf{Z}[x]/x^2$ with $\psi^2(x) = 2x$ admits, up to isomorphism, a unique filtered λ-ring extension to $\mathbf{Z}[x]/x^3$.*

We saw in Example 3.3 that the filtered λ-ring structure over $\mathbf{Z}[x]/x^2$ with $b_2 = 0$ admits uncountably many isomorphism classes of filtered λ-ring extensions to $\mathbf{Z}[x]/x^3$. It is natural to ask if there are other filtered λ-ring structures over the dual number ring that admit uncountably many, or at least countably infinitely many, extensions to $\mathbf{Z}[x]/x^3$ up to isomorphism. The answer, as we will see shortly, is negative. Employing Theorem 3.6 once again, we will prove a slightly more general result here and obtain the integral result as an immediate consequence.

Corollary 3.10. *Let J be a set of primes containing 2 and let R be a filtered λ-ring structure over $\mathbf{Z}_{(J)}[x]/x^2$ with $\psi^2(x) = b_2 x$. Assume that $b_2 \notin \{0,1\}$ and that all the prime factors of the numerator of $b_2(b_2-1)$, written in lowest terms, lie in J. Then R admits at most finitely many filtered λ-ring extensions to $\mathbf{Z}_{(J)}[x]/x^3$.*

Proof. By the third condition in Theorem 3.6, it suffices to show that in $\mathbf{Z}_{(J)}$, there are only finitely many cosets modulo $b_2(b_2-1)\mathbf{Z}_{(J)}$. This will follow from two elementary facts about the integers.

First, if r is an element in $\mathbf{Z}_{(J)}$, write it in lowest terms as $r = k/l$. Then there is a canonical isomorphism of groups

$$\mathbf{Z}_{(J)}/r\mathbf{Z}_{(J)} \xrightarrow{\cong} \mathbf{Z}_{(J)}/k\mathbf{Z}_{(J)}.$$

Indeed, since l is a J-local unit, it follows that the two subsets, $r\mathbf{Z}_{(J)}$ and $k\mathbf{Z}_{(J)}$, of $\mathbf{Z}_{(J)}$ are equal. The isomorphism is given by multiplication by l.

Second, if $k \neq 0$ is an integer, all of whose prime factors lie in J, then the natural inclusion

$$i \colon \mathbf{Z} \hookrightarrow \mathbf{Z}_{(J)}$$

induces an isomorphism of groups

$$i_* \colon \mathbf{Z}/k\mathbf{Z} \xrightarrow{\cong} \mathbf{Z}_{(J)}/k\mathbf{Z}_{(J)}.$$

To show that i_* is surjective, it is enough to show that for every prime p that does not lie in J, there exists an integer m such that $\frac{1}{p} - m$ lies in $k\mathbf{Z}_{(J)}$. Since p and k are relatively prime, there exist integers m and h such that

$$1 = mp + hk.$$

This implies that

$$\frac{1}{p} - m = k \cdot \frac{h}{p},$$

which of course lies in $k\mathbf{Z}_{(J)}$. The argument for the injectivity of i_* is equally easy.

Now write $b_2(b_2-1)$ in lowest terms as k/l. Then there are isomorphisms

$$\mathbf{Z}_{(J)}/b_2(b_2-1)\mathbf{Z}_{(J)} \cong \mathbf{Z}_{(J)}/k\mathbf{Z}_{(J)} \cong \mathbf{Z}/k\mathbf{Z}.$$

Therefore, there are only finitely many cosets in $\mathbf{Z}_{(J)}$ modulo $b_2(b_2-1)\mathbf{Z}_{(J)}$. As mentioned above, this implies that there are at most finitely many filtered λ-ring extensions of R to $\mathbf{Z}_{(J)}[x]/x^3$. □

Over the integers, b_2 is an even integer, so it cannot be equal to 1. In particular, the hypothesis in the previous corollary is satisfied, provided that $b_2 \neq 0$.

Corollary 3.11. *Let R be a filtered λ-ring structure over $\mathbf{Z}[x]/x^2$ with $\psi^2(x) = b_2 x$, $b_2 \neq 0$. Then R admits at most finitely many filtered λ-ring extensions to $\mathbf{Z}[x]/x^3$.*

One might wonder if there is some uniform upper bound for the number of elements in the moduli set of isomorphism classes of filtered λ-ring extensions of $\mathbf{Z}[x]/x^2$. The example below illustrates that there is no such upper bound. This means that by allowing different filtered λ-ring structures on $\mathbf{Z}[x]/x^2$, one can obtain (finitely) arbitrarily large moduli sets of isomorphism classes of filtered λ-ring extensions to $\mathbf{Z}[x]/x^3$.

Example 3.12. Let l be a positive odd integer and let n_l be $l(2l-1)$. Consider the filtered λ-ring structure R on $\mathbf{Z}[x]/x^2$ with Adams operations

$$\psi^p(x) \;=\; \begin{cases} 2lx & \text{if } p = 2 \\ 0 & \text{if } p > 2. \end{cases}$$

We claim that R admits exactly $(n_l + 1)/2$ isomorphism classes of filtered λ-ring extensions to $\mathbf{Z}[x]/x^3$.

The argument here is very similar to that of Example 3.8. Consider the following filtered λ-ring extensions of R: For each odd integer $i \in \{1, 3, \ldots, n_l\}$, let S_i be the filtered λ-ring structure on $\mathbf{Z}[x]/x^3$ with Adams operations

$$\psi^p(x) \;=\; \begin{cases} 2lx + ix^2 & \text{if } p = 2 \\ 0 & \text{if } p > 2. \end{cases}$$

Then each S_i is an extension of R. We will show that (i) they are pairwise non-isomorphic filtered λ-rings and that (ii) each extension of R is isomorphic to some S_i. To see (i), notice that for two distinct odd integers $i, j \in \{1, 3, \ldots, n_l\}$, we have

$$i \;\not\equiv\; \pm j \pmod{2l(2l-1)}.$$

The third condition in Theorem 3.6 now tells us that S_i and S_j cannot be isomorphic as filtered λ-rings.

To prove (ii), one observes that if S is a filtered λ-ring extension of R to $\mathbf{Z}[x]/x^3$, then in S,

$$\psi^2(x) \;=\; 2lx + cx^2$$

for some odd integer c. Since $2l(2l-1)$ is an even integer, it follows that there exists an element $i \in \{1, 3, \ldots, n_l\}$ such that either i or $-i$ is congruent to c modulo $2l(2l-1)$. Using Theorem 3.6 once again, we conclude that S is isomorphic to S_i.

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E-mail address: `dyau@math.uiuc.edu`

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801 USA