

The answer to an email of Mr. Douglas C. Ravenel

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Due to a mistake I made myself, I announce to withdraw the article "A Reply" that appeared on the September 2003 submissions.

The following is the answer to an email of Mr Douglas C. Ravenel on January 12, 2002. I am sorry for replying so late.

(1) In your email, you said that the Serre spectral sequence of the following sequence

$$\rightarrow \Omega(S^{2m+1}) \rightarrow \Omega(B(mp)) \rightarrow \Omega(S^{2m+2p+1}) \rightarrow$$

collapse and so we have the isomorphism $H_*(\Omega(B(mp))) = H_*(\Omega(S^{2m+1}) \otimes H_*(\Omega(S^{2m+2p+1})))$ as an additive group, but we do not know whether this is a ring isomorphism.

(2) In the second part, you said that you do not believe that as maps from $M(l+m+n+2)$ to $M(l) \wedge M(m) \wedge M(n)$, $P((\alpha \odot \beta) \odot \gamma) = \gamma \odot (\alpha \odot \beta)$ and $\text{id}((\alpha \odot \beta) \odot \gamma) = (\alpha \odot \beta) \odot \gamma$ are homotopic. I agree that it needs a proof. Now, we will give a proof in the last part of this letter. The other part of the appendix on page 48 of my paper remain unchanged. So, I still believe that the two proofs of Toda's result $\alpha_1 \beta_1^p = 0$ for $p \geq 3$ are incorrect.

(3) Besides above, there is still a gap in the proof of the statement that $V(3)$ does not exist for $n \geq 5$. In page 290 of your book "Complex Cobordism and Stable Homotopy Groups of Spheres", you said that $x_{761} = \langle \alpha_1 \beta_3, \beta_4, \gamma_2 \rangle$ is a permanent cycle for $p = 5$. But we have $|\alpha_1| = 7, |\beta_3| = 134, |\beta_4| = 82, |\gamma_2| = 437$, and so the Massey product x_{761} exists modular a coset $u\gamma_2 + (\alpha_1 \beta_3)v$ with $|u| = 324$ and $|v| = 620$. From the table in page 291 we know that u and v may not be trivial. Then, how can you get the conclusion that x_{761} is a permanent cycle?

(4) In what follows, we will prove that $(\alpha \odot \beta) \odot \gamma \approx \alpha \odot (\beta \odot \gamma)$. the Moore space $M(l)$, $M(m)$ and $M(n)$ are the same as defined in my paper. Since $\alpha \odot (\beta \odot \gamma) \approx$

$(-1)^{(l+1)(m+n+2)}(\beta \odot \gamma) \odot \alpha$, if $m = n$, then $\alpha \odot (\beta \odot \gamma) \approx (\beta \odot \gamma) \odot \alpha$. So, we have $P((\alpha \odot \beta) \odot \gamma) \approx (\beta \odot \gamma) \odot \alpha \approx \alpha \odot (\beta \odot \gamma) \approx (\alpha \odot \beta) \odot \gamma$ for $m = n = l$.

Firstly, we introduce the notions that will be used. It is assumed that all maps between spaces with base point keep the base point. For two spaces X, Y , $X = Y$ means that there is a given topological map $f: X \rightarrow Y$. Let $n \geq 1$. As usual, we use D^n to denote the unit disk in R^n and S^n to denote the unit sphere in R^{n+1} . So, $\partial(D^{n+1}) = S^n$. The base point $\#$ of D^n and S^{n-1} is taken to be $(-1, 0, \dots, 0)$. It is obvious that there exists a topological map H from S^n to $\wedge_{i=1}^n S_i^1$, where S_i^1 is just a copy of S^1 . We say that H is a decomposition of S^n . In general, we omit the map H and use the notion $S^n = \wedge_{i=1}^n S_i^1$ to denote it. It is easy to check that $D^n \wedge D^1 = D^{n+1}$. By induction, we have $D^n \wedge D^l = D^{n+l}$ for $n, l \geq 1$. It is also obvious that $D^{n+1} = \partial(D^{n+1}) \wedge D^1 = S^n \wedge D^1$. By the decomposition $\partial(D^{n+1}) = S^n = \wedge_{i=1}^n S_i^1$, any point x of D^{n+1} can be expressed as $x_1 \wedge \dots \wedge x_n \wedge t$ with $x_i \in S_i^1$, $i = 1, \dots, n$ and $-1 \leq t \leq 1$. We say that x_1, \dots, x_n, t is the polar coordinates of x with respect to the decomposition $S^n = \wedge_{i=1}^n S_i^1$ and $D^{n+1} = S^n \wedge D^1$. We also have that $D^{n+1} = (\wedge_{i=1}^n S_i^1) \wedge D^1$ is a decomposition of D^{n+1} . In this appendix, p is always assumed to be a fixed odd prime. Let $\theta: S^1 \rightarrow S^1$ be defined by $\theta(e^{2\pi i x}) = e^{2\pi i p x}$. It is obvious that θ has degree p and we call it the standard p -map of S^1 . Let $\theta_n: S^n \rightarrow S^n$ be the map defined by that $\theta_n(x_1 \wedge \dots \wedge x_n) = \theta(x_1) \wedge \dots \wedge \theta(x_n)$ for all $x_i \in S_i^1$, $i = 1, \dots, n$. We call θ_n the standard p -map of S^n with respect to the decomposition $S^n = \wedge_{i=1}^n S_i^1$. The identification space $\bar{M}(n) = D^{n+1} \cup S^n/x \wedge 1$ in D^{n+1} are identified with $\theta_n(x) \in S^n$ for all $x \in S^n$ and $\tilde{M}(n) = D^{n+1} \cup S^n \times (-1, 1]/x \wedge t$ in D^{n+1} are identified with $(\theta_n(x), t)$ for all $x \in S^n$ and $-1 < t \leq 1$ are respectively call S Moore space and SS Moore space. It is obvious that $\bar{M}(n) \subset \tilde{M}(n)$. We call D^{n+1} the standard $(n+1)$ -cell of $\bar{M}(n)$ and $\tilde{M}(n)$ and call S^n the unit sphere of them. We use λ_{n+1} to denote both inclusion maps from D^{n+1} to $\bar{M}(n)$ and $\tilde{M}(n)$.

Notice that in $\bar{M}(n)$ and $\tilde{M}(n)$, ∂D^{n+1} and the unit sphere S^n are not the same space. For $x \in \partial D^{n+1}$, we have $x \notin \theta_n(x) \in S^n$. In $\tilde{M}(n)$, any point can be expressed as $x \wedge t$ with $x \in S^n$ and $-1 \leq t \leq 1$.

For $l > 0, m > 0$, let $D^{l+1} = S^l \wedge D_1^1 = (\wedge_{i=1}^l) \wedge D_1^1$ and $D^{m+1} = S^m \wedge D_2^1 = (\wedge_{i=1}^m) \wedge D_2^1$ be two decompositions of D^{l+1} and D^{m+1} . Let $D^2 = D^1 \wedge D^2 = (\partial D^2) \wedge \bar{D}_1$, then $D^{l+1} \wedge D^{m+1} = (S^l \wedge D_1^1) \wedge (S^m \wedge D_2^1) = S_l \wedge S_m \wedge (D_1^1 \wedge D_2^1) = (\wedge_{i=1}^{l+m+1}) \wedge (\partial D^2) \wedge \bar{D}_1$.

Thus, we get a decomposition of $D^{l+1} \wedge D^{m+1}$. We call this the natural decomposition of $D^{l+1} \wedge D^{m+1}$ with respect to the decomposition of the factors D^{l+1} and D^{m+1} .

Let $\tilde{M}(l)$ and $\tilde{M}(m)$ be two SS Moore spaces with respectively standard cell D^{l+1} and D^{m+1} and standard spheres S^l and S^m . Then, any point of $\tilde{M}(l) \wedge \tilde{M}(m)$ can be expressed as $(x, t_1) \wedge (y, t_2)$ with $x \in S^l$, $y \in S^m$ and $-1 \leq t_1, t_2 \leq 1$. We write it as $(x \wedge y \wedge (t_1, t_2))$. By the definition of $\tilde{M}(l)$ and $\tilde{M}(m)$, we have $x \wedge t = (\theta_l(x), t)$ and $y \wedge t = (\theta_m(y), t)$ for $x \in \partial D^{l+1}$ and $y \in \partial D^{m+1}$ and $-1 < t \leq 1$. Now, $\partial(D^{l+1} \wedge D^{m+1}) = \{x \wedge t_1 \wedge y \wedge t_2 \mid x \in S^l, y \in S^m, t_1 = 1, -1 \leq t_2 \leq 1 \text{ or } t_2 = 1, -1 \leq t_1 \leq 1\}$. So, we have the following

Proposition 1. For $x \in \partial D^{l+1}$, $y \in \partial D^{m+1}$, $t_1 = 1, -1 \leq t_2 \leq 1$ or $t_2 = 1, -1 \leq t_1 \leq 1$, $\lambda_l \wedge \lambda_m(x \wedge t_1 \wedge y \wedge t_2) = (\theta_l(x), t_1) \wedge (\theta_m(y), t_2)$.

Let $\bar{M}(l+m+1)$ and $\tilde{M}(l+m+1)$ be the S and SS Moore spaces defined above. We defined a map $\bar{\Phi}'(l, m): \bar{M}(l+m+1) \rightarrow \tilde{M}(l) \wedge \tilde{M}(m)$ as follows. For $x \in D^{l+1}$, $y \in D^{m+1}$, $\bar{\Phi}'(l, m)(D_{l+m+1}) = \lambda_l(x) \wedge \lambda_m(y)$. Since $S^{l+m+1} = \{x \wedge t_1 \wedge y \wedge t_2 \mid x \in S^l, y \in S^m, t_1 = 1, -1 \leq t_2 \leq 1 \text{ or } t_2 = 1, -1 \leq t_1 \leq 1\}$, we define a map $\bar{\Phi}''(l, m): S^{l+m+1} \rightarrow \tilde{M}(l) \wedge \tilde{M}(m)$ as follows. For $x \in S^l$ and $y \in S^m$, $\bar{\Phi}''(l, m)(x \wedge t_1 \wedge y \wedge t_2) = x \wedge \theta_m(y) \wedge (t_1, t_2)$ for $t_2 = 1, -1 \leq t_1 \leq 1$ and $t_2 = 1, -1 \leq t_1 \leq 1$. Let θ_{l+m+1} be the p -map defined by the decomposition $D^{l+m+2} = D^{l+1} \wedge D^{m+1}$ with respect to the factors D_{l+1} and D^{m+1} . By Proposition 1, we have $\bar{\Phi}'(l, m)(x \wedge y) = \bar{\Phi}''(l, m)((\theta(x \wedge y)))$ for $x \wedge y \in \partial D^{l+m+2}$. So, $\bar{\Phi}'(l, m)$ and $\bar{\Phi}''(l, m)$ define a map $\bar{\Phi}(l, m): \bar{M}(l+m) \rightarrow \tilde{M}(l) \wedge \tilde{M}(m)$. We extend $\bar{\Phi}(l, m)$ to a map $\tilde{\Phi}(l, m): \tilde{M}(l+m+1) \rightarrow \tilde{M}(l) \wedge \tilde{M}(m)$ as follows. For $-1 \leq t \leq 1$, let $\eta_t: [-1, 1] \rightarrow [-1, t]$ be the linear map such that $\eta_t = -1$ and $\eta_t(1) = t$. Let $D_t^{l+1} = S^l \wedge (-1, t]$, $D_t^{m+1} = S^m \wedge (-1, t]$. Define $\bar{M}_t(l) = D_t^{l+1} \wedge S^l \times t / x \wedge t$ are identified with $\theta_l(x \times t)$ for all $x \in \partial D^{l+1}$. $\bar{M}_t(m)$ is similarly defined. We have $\tilde{M}(l) = \cup_{t \in (-1, 1]} \bar{M}_t(l)$ and $\tilde{M}(m) = \cup_{t \in (-1, 1]} \bar{M}_t(m)$. Let $\eta_t^l: \bar{M}(l) \rightarrow \bar{M}_t(l)$ and $\eta_t^m: \bar{M}(m) \rightarrow \bar{M}_t(m)$ be respectively the maps defined by $\eta_t^l(x \wedge \tau) = x \wedge \eta_t(\tau)$ for $x \in \partial D^{l+1}$ and $\eta_t^l(x' \wedge 1) = (x, t)$ for $x' \in S^l$ and $\eta_t^m(y \wedge \tau) = y \wedge \eta_t(\tau)$ for $y \in \partial D^{m+1}$ and $\eta_t^m(y' \wedge 1) = (y, t)$ for $y' \in S^m$. We extend $\bar{\Phi}(l, m)$ to a map $\tilde{\tilde{\Phi}}(l, m): \tilde{M}(l+m+1) \rightarrow \bar{M}(l+m+1) \cup S^{l+m+1} \times (-1, 1]$ as follows. For $x \in S^{l+m+1}$ and $-1 \leq t \leq 1$, $\tilde{\tilde{\Phi}}(l, m)(x, t) = (\eta_t^l \wedge \eta_t^m) \circ \bar{\Phi}(l, m)(x, 1) \in \bar{M}_t^l \wedge \bar{M}_t^m \subset \tilde{M}(l) \wedge \tilde{M}(m)$. It can be easily seen that the map $\tilde{\tilde{\Phi}}(l, m)$ so defined is continuous and $\tilde{\tilde{\Phi}}(l, m)_*(S^{l+m+1}) = S^l \wedge D^{m+1} + (-1)^{l+1} D^{l+1} \wedge S^m$ and $\tilde{\tilde{\Phi}}(l, m)_*(D_{l+m+2}) = D^{l+1} \wedge D^{m+1}$. So, both $\tilde{\tilde{\Phi}}(l, m)$ and $\tilde{\Phi}(l, m)$ are $\alpha \odot \beta$ maps.

Now, we prove the main proposition of this appendix $(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma)$. Let $\alpha = \tilde{M}(l)$, $\beta = \tilde{M}(m)$, $\gamma = \tilde{M}(n)$ be three SS Moore spaces and D^{l+1} , D^{m+1} , D^{n+1} be respectively the standard cells of α , β and γ . $D^{l+1} = (\wedge_{i=1}^l S_i^1) \wedge D_1^1$, $D^{m+1} = (\wedge_{i=m}^l S_i^1) \wedge D_2^1$, $D^{n+1} = (\wedge_{i=n}^l S_i^1) \wedge D_3^1$. Now, there are two different decompositions of $D^{l+m+n+3}$. The first is $D^{l+m+n+3} = D^{l+m+2} \wedge D_{n+1}$ and the second is $D^{l+m+n+3} = D^{l+1} \wedge D_{m+n+2}$. Let $\theta_{l+m+n+2}^{(1)}$ and $\theta_{l+m+n+2}^{(2)}$ be respectively the standard p -maps with the first and second decomposition. Then, we have the following

Proposition 2. For all $x \in S^{l+m+n+2}$, $\theta_{l+m+n+2}^{(1)}(x) = \theta_{l+m+n+2}^{(2)}(x)$, and there exist two disks \bar{D}^1 and \tilde{D}^1 such that

$$D_1^1 \wedge D_2^1 = \partial(D_1^1 \wedge D_2^1) \wedge \bar{D}^1, \quad D_2^1 \wedge D_3^1 = \partial(D_2^1 \wedge D_3^1) \wedge \tilde{D}^2$$

Proof. We have $D^{l+m+2} = D^{l+1} \wedge D^{m+1} = (\wedge_{i=1}^{l+m} S_i^1) \wedge (D_1^1 \wedge D_2^1) = (\wedge_{i=1}^{l+m} S_i^1) \wedge \partial(D_1^1 \wedge D_2^1) \wedge \bar{D}^1$ and $D^{l+m+n+3} = D^{l+m+2} \wedge D^{n+1} = (\wedge_{i=1}^{l+m} S_i^1) \wedge \partial(D_1^1 \wedge D_2^1) \wedge \bar{D}^1 \wedge D_3^1$. It can be easily seen that $\partial(D_1^1 \wedge D_2^1) \wedge \partial(D_2^1 \wedge D_3^1) = \partial(D_1^1 \wedge D_2^1 \wedge D_3^1)$. Since any point x in $S^{l+m+n+2}$ can be expressed by $\wedge_{i=1}^{l+m+n} x_i \wedge T_1 \wedge T_2$ with $T_1 \in \partial(D_1^1 \wedge D_2^1)$ and $T_2 \in \partial(\bar{D}^1 \wedge D_3^1)$, we have by definition $\theta_{l+m+n+2}^{(1)}(x) = \theta(x_1) \wedge (\wedge_{i=2}^{l+m+n} x_i) \wedge T_1 \wedge T_2$. Since any point T in $\partial(D_1^1 \wedge D_2^1 \wedge D_3^1)$ can be expressed by $T_1 \wedge T_2$ and any point x can be expressed by $(\wedge_{i=1}^{l+m+n} S_i^1) \wedge T$, we have $\theta_{l+m+n+2}^{(1)}(x) = \theta(x_1) \wedge (\wedge_{i=2}^{l+m+n} x_i) \wedge T$. By the same method as above, we also have $\theta_{l+m+n+2}^{(2)}(x) = \theta(x_1) \wedge (\wedge_{i=2}^{l+m+n} x_i) \wedge T$. So, we have $\theta_{l+m+n+2}^{(1)}(x) = \theta_{l+m+n+2}^{(2)}(x)$.

Proposition 3. $\Phi(l, m) \wedge \text{id} \circ \Phi(l + m + 1, n) = \text{id} \wedge \Phi(m, n) \circ \Phi(l, m + n + 1)$.

Proof. Notice that in the definition of $\bar{\Phi}(l + m + 1, n)$ and $\tilde{\Phi}(l, m + n + 1)$, the two standard p -maps are the same, so the two $\bar{M}(l + m + n + 1)$ are the same. Now, we have the following commutative diagram

$$\begin{array}{ccccc} \partial(D^{l+m+n+3}) & \xrightarrow{\text{id}} & \partial(D^{l+m+2} \wedge D^{n+1}) & \xrightarrow{\text{id}} & \partial(D^{l+1} \wedge D^{m+1} \wedge D^{n+1}) \\ \downarrow \bar{\theta}_{l+m+n+3} & & \downarrow \lambda_{l+m} \wedge \lambda_n & & \downarrow \lambda_l \wedge \lambda_m \wedge \lambda_n \\ S^{l+m+n+2} & \longrightarrow & \tilde{M}(l+m+1) \wedge \tilde{M}(n) & \longrightarrow & \tilde{M}(l) \wedge \tilde{M}(m) \wedge \tilde{M}(n) \end{array}$$

where the two maps on the bottom line are respectively $\bar{\Phi}(l+m+1, n)$ and $\tilde{\Phi}(l+m+1, n) \wedge \text{id}$. For $x \in S^{l+m+n+2}$, there exists an $x' \in \partial D^{l+m+n}$ such that $x = \theta_{l+m+n+2}^{(1)}(x') = \theta_{l+m+n+2}^{(2)}(x')$. Since

$$\begin{aligned} & (\tilde{\Phi}(l, m) \wedge \text{id})(\bar{\Phi}(l + m + 1, n))(x) \\ &= (\tilde{\Phi}(l, m) \wedge \text{id})(\bar{\Phi}(l + m + 1, n)) \bar{\theta}_{l+m+n+2}(x') \end{aligned}$$

$$\begin{aligned}
&= (\lambda_l \wedge \lambda_m \wedge \lambda_n)(\text{id})(\text{id})(x') \\
&= (\lambda_l \wedge \lambda_m \wedge \lambda_n)(x')
\end{aligned}$$

Similarly, we have $(\text{id} \wedge \tilde{\Phi}(m, n))(\bar{\Phi}(l, m + n + 1))(x) = (\lambda_l \wedge \lambda_m \wedge \lambda_n)(x')$. So, we have $(k = l + m + n + 2)$

$$(\Phi(l, m) \wedge \text{id})(\bar{\Phi}(l + m + 1, n))|S^k = (\text{id} \wedge \tilde{\Phi}(m, n))(\bar{\Phi}(l, m + n + 1))|S^k.$$

Since both maps on $D^{l+m+n+3}$ are $\lambda_l \wedge \lambda_m \wedge \lambda_n$, we have

$$(\Phi(l, m) \wedge \text{id})(\bar{\Phi}(l + m + 1, n)) = \text{id} \wedge (\tilde{\Phi}(m, n))(\bar{\Phi}(l, m + n + 1)).$$