A FAITHFUL UNITARY REPRESENTATION OF THE 2-COMPACT GROUP $DI(4)$

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Abstract. We construct a monomorphism from the 2-compact group $DI(4)$ into a 2-compact unitary group.

Let p be a fixed prime. A p-Compact group X is a triple (X, BX, ε), where

- X is a p-complete space with finite $\mathbb{F}_p$-homology,
- BX is a pointed p-complete space,
- $\varepsilon : \Omega BX \to X$ is a weak homotopy equivalence.

p-Compact groups has been defined by Dwyer and Wilkerson [6] and since they possess many remarkable properties of compact Lie groups, there are considered as homotopy-theoretic analogues of compact Lie groups. In particular, if G is a compact Lie group such that $\pi_0(G)$ is a finite p-group, then a triple $(G^\wedge_p, BG^\wedge_p, \varepsilon)$ is a p-compact group. It is an interesting question what properties of compact Lie groups generalize onto p-compact groups. Let us recall the theorem of Peter and Weyl:

Theorem (Peter-Weyl)**.** *Every compact Lie group admits a monomorphism into a unitary group $U(n)$ for some integer n.*

To formulate the analogue of this theorem we need some further definitions. If $X := (X, BX, \varepsilon_X)$ and $Y := (Y, BY, \varepsilon_Y)$ are p-compact groups, then a homomorphism $f : X \to Y$ is a pointed map $Bf : BX \to BY$. The homomorphism f is a monomorphism iff the homotopy fibre of Bf has finite $\mathbb{F}_p$-homology.

Conjecture. *Every p-compact group X admits a monomorphism into a p-compact unitary group $U(n)^\wedge_p$.*

This conjecture is known to be true in many cases. If X is a completion of a compact Lie group it is obvious and Castellana [4] has proven it for simple p-compact groups if p is odd. The main result of this paper is the following

Main Theorem. *There is a monomorphism of 2-compact groups $DI(4) \to U(2^{46})^\wedge_2$.*

The 2-compact group $DI(4)$. $DI(4)$ is the only known simple exotic (i.e. not being a completion of a Lie group) 2-compact group. Its characteristic property is that the mod 2 cohomology algebra of its classifying space is an algebra of rank 4 mod 2 Dickson invariants. The space $BDI(4)$ has been constructed by Dwyer and Wilkerson [5] as a 2-completion of the homotopy colimit of the following diagram (which is in fact a centralizer decomposition [7]):

$$(0.1) \qquad \overset{GL_4(\mathbb{F}_2)}{\curvearrowright} B\{\pm 1\}^4 \rightrightarrows B(T^3 \overset{GL_3(\mathbb{F}_2)}{\curvearrowright}{\ltimes} \{\pm 1\})^\wedge_2 \rightrightarrows B(SU(2)^3 \overset{GL_2(\mathbb{F}_2)}{\curvearrowright} /\{\pm 1\})^\wedge_2 \rightrightarrows BSpin(7)^\wedge_2.$$

The underlying category $\mathcal{A}$ is isomorphic to the opposite category of $\mathbb{F}_2$-vector spaces of dimensions 1,...,4 and monomorphisms. The groups $Spin(7)$, $SU(2)^3/\{\pm 1\}$, $T^3 \ltimes \{\pm 1\}$ and $\{\pm 1\}^4$ are centralizers of elementary abelian subgroups of $Spin(7)$ of orders 1,2,3,4 respectively and the maps between different spaces are induced by inclusions. Automorphisms are more complicated and can be defined only after completing classifying spaces.

The action of the Weyl group on the maximal torus. A Weyl group $W_{DI(4)}$ of $DI(4)$ is isomorphic to $\{\pm 1\} \times \mathrm{GL}_3(\mathbb{F}_2)$ and since $Spin(7)$ and $DI(4)$ have a common maximal torus it contains $W_{Spin(7)}$ as a subgroup. Let $a : W_{Spin(7)} \to \mathrm{GL}_3(\mathbb{Z})$ be a natural homomorphism and let V be a subgroup of $W_{Spin(7)}$ of index 2 such that the composition

$$i : V \subset W_{Spin(7)} \xrightarrow{a} \mathrm{GL}_3(\mathbb{Z}) \xrightarrow{\mathrm{mod}\ 2} \mathrm{GL}_3(\mathbb{F}_2)$$

is a monomorphism (one can easily see that V is isomorphic to $C_2 \wr \Sigma_3$). By [6, 4.1] there is a section j of the mod 2 reduction $\mathrm{GL}_3(\mathbb{Z}_2^\wedge) \to \mathrm{GL}_3(\mathbb{F}_2)$ such that compositions

$$V \xrightarrow{\subset} W_{Spin(7)} \xrightarrow{a} \mathrm{GL}_3(\mathbb{Z}) \xrightarrow{\subset} \mathrm{GL}_3(\mathbb{Z}_2^\wedge) \tag{0.2}$$

and

$$V \xrightarrow{i} \mathrm{GL}_3(\mathbb{F}_2) \xrightarrow{j} \mathrm{GL}_3(\mathbb{Z}_2^\wedge) \tag{0.3}$$

are equal. The original construction contains no explicit description of j; however it is sufficient to give a value of j on any element of $\mathrm{GL}(\mathbb{F}_2)$ of order 7. Elementary calculations show that j determined by

$$j\left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -3h+1 & -h \\ -1 & h-1 & -h \\ 0 & h & 3h-1 \end{pmatrix}, \tag{0.4}$$

where h is the only 2-adic integer satifying equation $4x^2 - 3x + 1 = 0$, satisfies the required conditions. Now the homomorphism

$$W_{DI(4)} \simeq \{\pm 1\} \times \mathrm{GL}_3(\mathbb{F}_2) \ni (e, M) \to e \cdot j(M) \in \mathrm{GL}_3(\mathbb{Z}_2^\wedge) \tag{0.5}$$

provides an action of the Weyl group on that maximal torus of $DI(4)$.

The decomposition diagram up to homotopy. The action of $\mathrm{GL}_4(\mathbb{F}_2)$ on $\{\pm 1\}^4$ is the obvious one. The action of $\mathrm{GL}_3(\mathbb{F}_2)$ on $B(T^3 \ltimes \{\pm 1\})_2^\wedge$ is an extension of an action j on $B(T^3)_2^\wedge \simeq K((\mathbb{Z}_2^\wedge)^3, 2)$. Finally, the following proposition provides an action of $\mathrm{GL}_2(\mathbb{F}_2) \simeq N_{W_{DI(4)}}(W_{SU(2)^3/\{\pm 1\}}) \subseteq N_{\mathrm{GL}_3(\mathbb{Z}_2^\wedge)}(W_{SU(2)^3/\{\pm 1\}})$ on $B(SU(2)^3/\{\pm 1\})_2^\wedge$:

Proposition 0.6. *If $H \simeq SU(2)^n$ or $H \simeq SU(2)^n/\{\pm 1\}$, then the group of homotopy classes of self-equivalences of $BH_2^\wedge$ is isomorphic to $N_{\mathrm{GL}_n(\mathbb{Z}_2^\wedge)}(W_H)/W_H$.*

Proof. This is a direct corollary from [5, 5.5]. □

Sketch of the proof of the main theorem. The main idea of the proof is to start with a representation of a subgroup of $DI(4)$ and then extend it successively to larger and larger subgroups using homotopy decompositions. The first step is to find a suitable of the representation of the common 2-normalizer of the maximal torus of both $DI(4)$ and $Spin(7)$ (4.6). Next we check that it extends to a homotopy compatible family of maps from the subgroup decomposition diagram of $Spin(7)$ (4.14) and then conclude (using the result of [20]) that it extends to a map $f_G : BSpin(7) \to BU(m)^{\wedge}_2$. A similar procedure is performed to produce an extension of this map to $BDI(4)$. The most difficult part is to show that the restriction of f_G to $B(SU(2)^3/\{\pm 1\})^{\wedge}_2$ is $\mathrm{GL}_2(\mathbb{F}_2)$-invariant; the problem is that there are many non-homotopic maps from $B(SU(2)^3/\{\pm 1\})^{\wedge}_2$ to $BU(m)^{\wedge}_2$ which are homotopic after restriction to the normalizer of a maximal torus. To overcome this problem we classify maps $B(SU(2)^3/\{\pm 1\})^{\wedge}_2 \to BU(m)^{\wedge}_2$ (Section 3) and introduce some additional structure on the set of extensions of a given representation of a maximal torus normalizer (3.24). Technical results on obstruction theory applied here are proven in Section 2.

Notation and terminology. Throughtout the whole paper $D := DI(4)$, $G := Spin(7)$, $L := SU(2)$, $H := L^3/\{\pm 1\}$, $p = 2$ (unless stated otherwise) and A is a ring of p-adic integers. The group T is a maximal torus of both G and H and its 2-completion is a maximal torus of D. Let W_D, W_G, W_H be the Weyl groups of D, G and H respectively. If R is a commutative ring and $\mathcal{C}$ is a small category, then *an* $R[\mathcal{C}]$*-module* M is a contravariant functor from $\mathcal{C}$ into the category of R-modules and $H^n(\mathcal{C}; M)$ is an n-th higher limit of M. A trivial 1-dimensional complex representation of any group is denoted by θ and a non-trivial irreducible representation of an order 2 group is denoted by ι.

1. Homotopy representations of compact Lie groups

Let p be a fixed prime and let S be a compact Lie group. By *a homotopy representation* of S we mean a map from BS into the p-completion of the classifying space of the unitary group $BU(m)^{\wedge}_p$. In this section we recall the method of constructing homotopy representations using a subgroup decomposition due to Jackowski, McClure and Oliver [10]. This approach is presented with more details for example in [12].

Subgroup decomposition. We say that a group is *p-toral* iff it is an extension of a torus by a finite p-group. A p-toral subgroup $P \subseteq S$ is *stubborn* iff $N_S(P)/P$ is finite and has no non-trivial normal p-subgroups. Let $\mathcal{R}_p(S)$ be a category of S-orbits having the form S/P for p-stubborn P and S-maps. By [10] the map

$$\operatorname{hocolim}_{S/P \in \mathcal{R}_p(S)} ES \times_S S/P \longrightarrow BS \tag{1.1}$$

induced by projections induces isomorphism on mod p homology.

Dwyer-Zabrodsky theorem. A group is *p-discrete toral* if it is an extension of a *p-discrete torus* $(\mathbb{Z}/p^\infty)^r$ be a finite p-group. Every p-toral group P has a unique (up to conjugacy) dense p-discrete toral subgroup P^∞ called *a p-discrete approximation* of P. The following theorem provides an algebraic description of homotopy representations of p-toral groups:

Theorem 1.2 ([11, Thm. 1.1])**.** *Let P be a p-toral group and let H be a compact connected Lie group. Then:*

(1) *The maps*

$$\operatorname{Hom}(P^\infty,H)/\operatorname{Inn}(H)=:\operatorname{Rep}(P^\infty,H)\xrightarrow{B}[BP^\infty,BH^\wedge_p]\longleftarrow[BP,BH^\wedge_p]$$

are bijections.

(2) *For any $\varphi : P^\infty \to H$ the pairing $BC_H(\varphi(P^\infty))\times BP^\infty\to BH$ induces a homotopy equivalence*

$$BC_H(\varphi(P^\infty))^\wedge_p\longrightarrow \operatorname{map}(BP^\infty,BH^\wedge_p)_{B\varphi}.$$

Complex representations of p-discrete toral groups. By 1.2 for any p-toral group P we have $[BP,BU(n)^\wedge_p]\cong\operatorname{Rep}(P^\infty,U(n))$. A group P^∞ is not finite and therefore the classical representation theory cannot be applied to calculate $\operatorname{Rep}(P^\infty,U(n))$. However, p-discrete groups are countable locally finite groups and therefore their representations behave similarly to the finite case. Namely every complex representation of P^∞ admits a unique unitary structure ([20, 1.9]) so $\operatorname{Rep}(P^\infty,U(n))\cong\operatorname{Rep}(P^\infty,\mathrm{GL}_n(\mathbb{C}))$, and every complex representation is semisimple and its decomposition into irreducible factors is unique up to permutation of summands ([17], [20, 1.4]). Moreover, for every $\varphi : P^\infty\to U(n)$ the centralizer of $\varphi(P^\infty)$ is a product of unitary groups with one factor for every isomorphism class of irreducible subrepresentations of the rank equal to a multiplicity of a given representation ([20, 1.11]).

$\mathcal{R}_p(S)$-invariant representations. Let N_S be a p-normalizer of a maximal torus T_S of S. A representation $\varphi : N_S^\infty\to U(n)$ is $\mathcal{R}_p(S)$*-invariant* iff $B\varphi^\wedge_p$ determines a homotopy compatible family of maps $\{f_P : BP\to BU(n)^\wedge_p\}_{S/P\in\mathcal{R}_p(S)}$ (or, equivalently the compatible family of representations $\{\varphi_P\in\operatorname{Rep}(P^\infty,U(n))\}$). Obviously every homotopy representation of S determines an $\mathcal{R}_p(S)$-invariant representation of N_S^∞ but a map

$$[BS,BU(n)^\wedge_p]\longrightarrow\lim\nolimits_{S/P\in\mathcal{R}_p(S)}\operatorname{Rep}(P^\infty,U(n))\cong\operatorname{Rep}_{inv}(N_S^\infty,U(n)) \tag{1.3}$$

(where Rep_{inv} stands for the set of $\mathcal{R}_p(S)$-invariant representations) is in general neither surjection nor injection.

Obstruction theory. Fix an $\mathcal{R}_p(S)$-invariant representation φ of N_S^∞. An existence of a homotopy representation of S being an extension of φ (resp. and its uniqueness) is controlled by obstructions lying in groups $H^{i+1}(\mathcal{R}_p(S);\Pi_i^\varphi)$ for $i>0$ (resp. $H^i(\mathcal{R}_p(S);\Pi_i^\varphi)$), where

$$\Pi_i^\varphi(S/P):=\pi_i(\operatorname{map}(BP,BU(n)^\wedge_p)_{B\varphi^\wedge_p|_{BP^\wedge_p}}). \tag{1.4}$$

By Theorem 1.2

$$\Pi_i^\varphi(S/P)=\pi_i(BC_{U(n)}(\varphi(P^\infty)))=\pi_{i-1}(C_{U(n)}(\varphi(P^\infty))) \tag{1.5}$$

Since $C_{U(n)}(\varphi(P^\infty))$ is a product of unitary groups, then $\Pi_1^\varphi(S/P)=\Pi_3^\varphi(S/P)=0$. Let $\mathrm{IR}(\Gamma)$ be a set of isomorphism classes of irreducible representations of a group Γ, and let $\mathrm{IR}(\Gamma,\omega)\subseteq\mathrm{IR}(\Gamma)$ be a subset of isomorphism classes of irreducible subrepresentations of ω. There is a functorial isomorphism [20, 2.3]

$$\Pi_2^\varphi(S/P)\simeq A[\mathrm{IR}(P^\infty,\operatorname{res}^{N_S^\infty}_{P^\infty}\varphi)]. \tag{1.6}$$

As a consequence we obtain the following criterion of extensibility:

Corollary 1.7. Let $\varphi : N_S^\infty \to U(n)$ be an $\mathcal{R}_p(S)$-invariant representation. If $H^3(\mathcal{R}_p(S); A[\mathrm{IR}((-)^\infty, \varphi)]) = 0$ and $H^i(\mathcal{R}_p(S); \mathbf{M}) = 0$ for $i > 4$ and any $A[\mathcal{R}_p(S)]$-module $\mathbf{M}$, then $B\varphi_2^\wedge$ extends to a map $BS \to BU(n)_p^\wedge$. □

2. Obstruction theory

Let $\mathcal{C}$ be a small category and let $F : \mathcal{C} \to \mathbf{Sp}$ be a functor into the category of spaces. Fix a space Z and a homotopy compatible family of maps $f := \{f_C : X(C) \to Z\}$. Assume that for every $C \in \mathcal{C}$ a space $\mathrm{map}(X(C), Z)_{f_C}$ is 1-connected and let $\Pi_i^f(C) := \pi_i(\mathrm{map}(X(C), Z)_{f_C})$. A theorem provided by [18] states that if $H^{i+1}(\mathcal{C}; \Pi_i) = 0$ for all $i > 1$, then there is a map $\tilde{f} : \mathrm{hocolim}_\mathcal{C} F \to Z$ which extends f. If additionally $H^i(\mathcal{C}; \Pi_i) = 0$, then $\tilde{f}$ is determined uniquely up to homotopy. In this section we consider the case when

- $H^{i+1}(\mathcal{C}; \Pi_i^f) = 0$ for all i
- $H^i(\mathcal{C}; \Pi_i^f) = 0$ for all $i \neq 2$.

We will prove that the set of homotopy classes of extensions E_f of f to $\mathrm{hocolim}_\mathcal{C} F$ has a structure of a free and transitive $H^2(\mathcal{C}; \Pi_2^f)$-set and this structure is functorial in some sense.

Definitions and notation. Let $N(\mathcal{C})$ denotes the nerve of $\mathcal{C}$ and let $N(\mathcal{C})_i$ be its i-th skeleton. Define a cochain complex

$$C_i^j = \prod_{\sigma \in N(\mathcal{C})_j} \Pi_i^f(\sigma(0))$$

and let for $u \in C_i^j$

$$\delta_i^j(u)(\sigma) = F(\sigma(0 \to 1))^* u(d_0(\sigma)) + \sum_{k=1}^{j+1} (-1)^k u(d_k(\sigma)) \in C_i^{j+1}.$$

By [14, Lemma 2], $H^*(C_i^*, \delta_i^*) = H^*(\mathcal{C}; \Pi_i^f)$. For each i let Z_i^j, B_i^j, H_i^j denote the cocycles, the coboundaries and the cohomology of the cochain complex C_i^* (note that $H_i^j = H^j(\mathcal{C}; \Pi_i^f)$). Denote for short $X := \mathrm{hocolim}_\mathcal{C}\ F$ and $X_i := \mathrm{hocolim}_\mathcal{C}^{(i)} F$. Let $g : X_i \to Z$ be any map extending f (i.e. such that $g|_{F(C)} = f_C$) and let $\mathrm{sk}_i \Delta^j$ denote i-skeleton of Δ^j. For any $\sigma \in N(\mathcal{C})_j$ let

$$Ad_\sigma(g) : \mathrm{sk}_i \Delta^j \to \mathrm{map}(F(\sigma(0)), Z)_{f_{\sigma(0)}} \tag{2.1}$$

be an adjoint map to a composition

$$\mathrm{sk}_i \Delta^j \times F(\sigma(0)) \cong \{\sigma\} \times \mathrm{sk}_i \Delta^j \times F(\sigma(0)) \longrightarrow X_n \xrightarrow{g} Z.$$

Obviously any map $g : X_n \to Z$ (resp. $g : X \to Z$) is determined uniquely by a collection of maps $Ad_\sigma(g)$, where $\sigma \in N(\mathcal{C})_i$, $0 \le i \le n$ (resp. $i \ge 0$) which satisfies suitable compatibility conditions. Now define an obstruction cochain $o_n(g) \in C_n^{n+1}$ by

$$o_n(g)(\sigma) = Ad_\sigma(g)_*[\partial \Delta^{n+1}] \in \pi_{n+1} \mathrm{map}(F(\sigma(0)), Z)_{f_{\sigma(0)}}. \tag{2.2}$$

(Note that we do not need to care about basepoints since $\mathrm{map}(F(\sigma(0)), Z)_{f_{\sigma(0)}}$ is supposed to be 1-connected).

For a 1-connected space Y, a map $a : D^n \to Y$ and $\omega \in \pi_{n+1}(Y)$ let $a + \omega : D^n \to Y$ be a map such that $a|_{S^n} = (a+\omega)|_{S^n}$ and

$$S^{n+1} \cong D^n \cup_{S^n} D^n \xrightarrow{a \cup (a+\omega)} Y \tag{2.3}$$

represents ω (obviously $g + u$ is defined only up to mod S^n homotopy). For any $u \in C^n_n$ and any $g : X_n \to Z$ which extends f let $g + u : X_n \to Z$ be a map such that $g + u|_{X_n} = g|_{X_n}$ and $Ad_\sigma(g+u) = Ad_\sigma(g) + u(\sigma)$ for each $\sigma \in N(\mathcal{C})_n$. Notice that the set of homotopy classes mod X_{n-1} of extensions of $g|_{X_n-1}$ to X_n is a free and transitive C^n_n-set.

Properties. Here follow well-known properties of concepts introduced above:

Proposition 2.4. *Fix $g : X_n \to Z$ such that $g|_{F(C)} = f_C$ for each $C \in \mathcal{C}$. Then*
(a) g extends to X_{n+1} if and only if $o_n(g) = 0$.
(b) $o_n(g) \in Z^{n+1}_n$.
(c) $o_n(g+u) = o_n(g) + \delta^n_n(u)$ for each $u \in C^n_n$
(d) $g|_{X_{n-1}}$ extends to X_{n+1} if and only if $o_n(g) \in B^{n+1}_n$
(e) Fix $u \in C^n_n$. Then g is homotopic to $g + u$ mod X_{n-2} if and only if $u \in B^n_n$.
□

Proposition 2.5. *Let $g, g' : X \to Z$ be any extensions of f. Then*
(a) $g|_{X_1} \sim g'|_{X_1}$.
(b) If $g|_{X_2} \sim g'|_{X_2}$, then $g \sim g'$.
(c) If $u \in C^2_2$, then $g|_{X_2} + u$ extends to X if and only if $u \in Z^2_2$.

Proof. First statement follows from 1-connectivity of the mapping spaces. Let $i > 2$ and assume $g|_{X_{i-1}} \sim g'|_{X_{i-1}}$. We can replace g' by a homotopic map and assume that $g'|_{X_{i-1}} = g|_{X_{i-1}}$. Let $u \in C^i_i$ such that $g' = g + u$. Moreover, $0 = o_i(g'|_{X_i}) = o_i(g|_{X_i}) + \delta^i_i(u) = \delta^i_i(u)$. Thus $u \in Z^i_i$ and since $H^i_i = 0$ also $u \in B^i_i$. By 2.4.(e) we have $g|_{X_i} \simeq g'|_{X_i}$ and an induction on i starting from 2 implies (b). To prove (c) note that the condition $u \in Z^n_n$ is necessary to extensibility of $g|_{X_2} + u$ to X_3 (by 2.4.(a) and 2.4.(c)) and each map from X_3 extends to X (by 2.4.(b) and 2.4.(d)). □

Corollary 2.6. The action of H^2_2 on E_f is transitive and free.

Proof. Transitivity follows form 2.5. Since for each $C \in \mathcal{C}$ the mapping spaces $\mathrm{map}(F(C), Z)_{f_C}$ are 1-connected, then any two homotopic maps $g \sim g' : X_2 \to Z$ are homotopic modulo X_0. Now the second part follows from 2.4.(e). □

Functoriality of H^2_2-action.

Definition 2.7. *The category of diagrams on a category $\mathcal{A}$*, denoted by $\mathbf{Diag}_{\mathcal{A}}$, is the category whose objects are pairs $(\mathcal{C}_A, A)$, where $\mathcal{C}_A$ is a small category and $A : \mathcal{C}_A \to \mathcal{A}$ is a functor. A morphism φ from $A : \mathcal{C}_A \to \mathcal{A}$ to $B : \mathcal{C}_B \to \mathcal{A}$ is a pair (T_φ, t_φ), where $T_\varphi : \mathcal{C}_A \to \mathcal{C}_B$ is a functor and $t_\varphi : A \to B \circ T_\varphi$ is a natural transformation.

Remark. Every morphism $\varphi : A \to B$ in the category $\mathbf{Diag}_{\mathbf{Sp}}$ induces a map $\varphi_* : \mathrm{hocolim}_{\mathcal{C}_A} A \to \mathrm{hocolim}_{\mathcal{C}_B} B$.

Obviously, F is an object in the category $\mathbf{Diag}_{\mathbf{Sp}}$. Let $\varphi : F \to F$ be an automorphism in the category $\mathbf{Diag}_{\mathbf{Sp}}$ such that for each $C \in \mathcal{C}$ maps f_C and $(\varphi^* f)_C := f_{T_\varphi(C)} \circ t_\varphi(C)$ are homotopic. As before E_f (resp. $E_{\varphi^* f}$) denotes a set of homotopy classes of extensions of f (resp. $\varphi^* f$). Obviously there is a natural bijection $E_f \cong E_{\varphi^* f}$. The morphism φ induces a bijection $\varphi^* : E_f \to E_f$ by

$$E_f \ni [g] \mapsto [X \xrightarrow{\varphi_*} X \xrightarrow{g} Z] \in E_{\varphi^* f} \cong E_f \tag{2.8}$$

and automorphisms of chain complexes C_k^* by

$$(\varphi^*(u))(\sigma) = t_\varphi^*(u(T_\varphi^*(\sigma))), \qquad \text{for } u \in C_k^l,\ \sigma \in N(\mathcal{C})_l \tag{2.9}$$

which obviously induces automorphisms $\varphi^* : H_2^2 \to H_2^2$.

Now we are ready to prove the main result of this section:

Theorem 2.10. *The diagram*

$$\begin{array}{ccc} H_2^2 \times E_f & \xrightarrow{\ +\ } & E_f \\ \Big\downarrow{\scriptstyle \varphi^* \times \varphi^*} & & \Big\downarrow{\scriptstyle \varphi^*} \\ H_2^2 \times E_f & \xrightarrow{\ +\ } & E_f \end{array}$$

commutes.

Proof. Let $g : X \to Z$ be a map extending f, and let $g' : X \to Z$ be a map homotopic to $\varphi^* g$ such that $g'|_{X_1} = g|_{X_1}$.

Note that for any simplex σ we have

$$Ad_\sigma(\varphi^* g) = t_\varphi^*(\sigma(0)) \circ Ad_{T_\varphi \sigma}(g)$$

Then for any $u \in Z_2^2$ and any $\sigma \in N(\mathcal{C})_2$

$$\begin{aligned} Ad_\sigma(\varphi^*(g+u)) &= t_\varphi^* u(\sigma(0)) \circ Ad_{T_\varphi \sigma}(g+u) \\ &= t_\varphi^* u(\sigma(0)) \circ [Ad_{T_\varphi \sigma}(g) + u(T_\varphi(g))] \\ &= t_\varphi^* u(\sigma(0)) \circ Ad_{T_\varphi \sigma}(g) + t_\varphi^*(u(T_\varphi(g))) = Ad_\sigma(\varphi^* g) + \varphi^*(u)(\sigma) \end{aligned}$$

Then $\varphi^*(g+u)|_{X_2} = \varphi^*(g)|_{X_2} + \varphi^* u$. Now the conclusion follows by 2.5.(b). □

3. Homotopy representations of $SU(2)^n$ and $SU(2)^n/\{\pm 1\}$

2-Stubborn subgroups of L^n. By [13] there are two (up to conjugacy) 2-stubborn subgroups of L, namely

$$N_L := \left\langle \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{-2\pi i t} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \tag{3.1}$$

which is in fact the normalizer of the maximal torus T_L of L, and

$$Q := \left\langle m_i := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, m_j := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \tag{3.2}$$

which is a group of order 8 isomorphic to the quaternion group. Morphisms in $\mathcal{J} := \mathcal{R}_2(L)$ are given by $\mathrm{Aut}(L/Q) \simeq \mathrm{Out}(Q) \simeq \Sigma_3$, $\mathrm{Aut}(L/N) = 1$, $\mathrm{Mor}(L/N, L/Q) = \emptyset$ and $\mathrm{Mor}(L/Q, L/N) \simeq \Sigma_3/\Sigma_2$. By [10, 1.6] there are equivalences of categories

$$\mathcal{J}^n = \mathcal{R}_2(L)^n \ni (L/P_1, \ldots, L/P_n) \mapsto L^n/(P_1 \times \ldots P_n) \in \mathcal{R}_2(L^n) \tag{3.3}$$

(3.4)
$$\mathcal{R}_2(L^n) \ni L^n/(P_1 \times \dots P_n) \mapsto (L^n/\{\pm 1\})/(P_1 \times \dots P_n/\{\pm 1\}) \in \mathcal{R}_2(L^n/\{\pm 1\}).$$

Representations of discrete approximations. There are 5 isomorphism classes of irreducible representations of Q, namely the trivial one θ, three non-trivial 1-dimensional representations

$$\tau_i := \mathrm{res}^Q_{Q/\langle m_i\rangle}\ \iota, \qquad \tau_j := \mathrm{res}^Q_{Q/\langle m_j\rangle}\ \iota, \qquad \tau_k := \tau_i \otimes \tau_j$$

and one 2-dimensional representation λ. The group $\mathrm{Aut}_{\mathcal{J}}(N/Q) \simeq \Sigma_3$ permutes τ_i, τ_j, τ_k and fixes θ and λ.

Denote $K := N^\infty_L$. All irreducible representations of K can be found as subrepresentations of representations induced from T^∞_L. For any $k \in A$ define

$$\varrho_k : T^\infty_L \ni \begin{pmatrix} \exp(2\pi i \frac{l}{2^s}) & 0 \\ 0 & \exp(2\pi i \frac{-l}{2^s}) \end{pmatrix} \mapsto \exp\left(2\pi i \frac{kl}{2^s}\right) \in U(1). \tag{3.5}$$

Obviously $\mathrm{IR}(T^\infty_L) = \{\varrho_k\}_{k\in A}$. A representation $\mathrm{ind}^K_{T^\infty_L}\ \varrho_k$ which splits to a sum $\theta \oplus \tau$ if $k = 0$; otherwise it is irreducible. Furthermore, $\mathrm{ind}^K_{T^\infty_L}\ \varrho_k \simeq \mathrm{ind}^K_{T^\infty_L}\ \varrho_l$ iff $k = \pm l$. Therefore

$$\mathrm{IR}(K) = \{\theta, \tau\} \cup \{\mathrm{ind}^K_{T^\infty_L}\ \varrho_k\}_{k\in(\mathbb{Z}_2^\wedge)^*/\{\pm 1\}}$$

Here follows a table of restriction of representations of K to Q:

$$\mathrm{res}^K_Q\ \omega = \begin{cases} \theta & \text{for } \omega \simeq \theta \\ \tau_i & \text{for } \omega \simeq \tau \\ \theta \oplus \tau_i & \text{for } \omega \simeq \mathrm{ind}^K_{T^\infty_L}\ \varrho_{4k},\ k \neq 0 \\ \tau_j \oplus \tau_k & \text{for } \omega \simeq \mathrm{ind}^K_{T^\infty_L}\ \varrho_{4k+2} \\ \lambda & \text{for } \omega \simeq \mathrm{ind}^K_{T^\infty_L}\ \varrho_{2k+1}, \end{cases} \tag{3.6}$$

Next, we provide a useful criterion of $\mathcal{J}$-invariance of representations of K. Define

$$\mathrm{IR}^+(K) := \{\tau\} \cup \{\mathrm{ind}^K_{T^\infty_L}\ \varrho_k\}_{k\in 4\mathbb{Z}_2^\wedge\setminus\{0\}}$$
$$\mathrm{IR}^-(K) := \{\mathrm{ind}^K_{T^\infty_L}\ \varrho_k\}_{k\in 2+4\mathbb{Z}_2^\wedge}$$
$$\mathrm{IR}^0(K) := \{\theta\} \cup \{\mathrm{ind}^K_{T^\infty_L}\ \varrho_k\}_{k\in 1+2\mathbb{Z}_2^\wedge}.$$

Proposition 3.7. *A representation φ of K is $\mathcal{J}$-invariant iff a total multiplicity of irreducible factors of φ in $\mathrm{IR}^+(K)$ is equal to the total multiplicity of factors in $\mathrm{IR}^-(K)$.* □

Let $\mathbf{n} := \{1, 2, \dots, n\}$ and let φ be a representation of $K^{\mathbf{n}}$. For each $k \in \mathbf{n}$ there is a unique presentation

$$\varphi \simeq \bigoplus_{\omega\in\mathrm{IR}(N^\infty)^{\mathbf{n}\setminus\{k\}}} \omega \bar{\otimes} \psi_\omega. \tag{3.8}$$

Proposition 3.9. *The representation φ is $\mathcal{J}^n$-invariant if and only if for each k and each $\omega \in \mathrm{IR}(N^\infty)^{\mathbf{n}\setminus\{k\}}$ the representation ψ_ω is $\mathcal{J}$-invariant.*

Proof. If φ is $\mathcal{J}^n$-invariant, then it obviously satisfies the conditions above. Since each morphism of $\mathcal{J}^n$ is a composition of morphisms with all coordinates but one being an identity, the inverse follows. □

In particular irreducible $\mathcal{J}^n$-invariant representations of K^n are not necessarily tensor products of $\mathcal{J}$-irreducible representations of N^∞.

Obstruction modules. Fix a $\mathcal{J}^n$-invariant representation $\varphi : K^n \to U(m)$. As mentioned before (Section 1), obstructions to existence (uniqueness) of an extension of $B\varphi_2^\wedge$ to a map from BL^n lie in groups $H^{i+1}(\mathcal{J}^n; \Pi_i^\varphi)$ (resp. $H^i(\mathcal{J}^n; \Pi_i^\varphi)$). Furthermore, $\Pi_2(L^n/P) \simeq \mathbb{Z}_2^\wedge[\mathrm{IR}(P^\infty, \varphi)]$ and $\Pi_1^\varphi = \Pi_3^\varphi = 0$. The following lemma states that in fact obstruction modules for L^n and $L^n/\{\pm 1\}$ are isomorphic:

Proposition 3.10. *If ψ is an $\mathcal{J}^n$-invariant representation of $K^n/\{\pm 1\}$, then there is a natural equivalence of $\mathcal{J}^n$-modules $\Pi_i^\psi \cong \Pi_i^{\mathrm{res}_{K^n}^{K^n/\{\pm 1\}} \psi}$.*

Proof. By 1.2 we have

$$\begin{aligned}\Pi_i^\psi((L^n/\{\pm 1\})/(P/\{\pm 1\})) &= \pi_i(BC_{U(m)}(\psi(P^\infty/\{\pm 1\}))) \\ &= \pi_i(BC_{U(m)}(\mathrm{res}_{K^n}^{K^n/\{\pm 1\}} \psi(P^\infty))) = \Pi_i^{\mathrm{res}_{K^n}^{K^n/\{\pm 1\}} \psi}(L^n/P). \quad \square\end{aligned}$$

Spectral sequence. The groups $H^*(\mathcal{J}^n; \Pi_*^\varphi)$ are calculated in two steps: first we calculate cohomology of $\mathcal{J}^n$ with coefficients being atomic functors (i.e. concentrated on a single object). Next, we use a spectral sequence provided by [9]. Define a function

$$\mathrm{ht} : \mathrm{Ob}(\mathcal{J}^n) \ni L^n/(Q^B \times K^{\mathbf{n}\setminus B}) \mapsto |B| \in \mathbb{Z} \tag{3.11}$$

By [9, 1.3] there is a spectral sequence of a cohomological type with the first term

$$E_1^{s,t} := \bigoplus_{\mathrm{ht}(L/P)=s} \Lambda^{s+t}(\mathrm{Aut}_{\mathcal{J}^n}(L^n/P); \Pi_2^\varphi(L/P)) \tag{3.12}$$

which strongly converges to $H^*(\mathcal{R}_p(L); \Pi_2^\varphi)$. By 1.6 we have

$$E_1^{s,t} := \bigoplus_{\mathrm{ht}(L/P)=s} \Lambda^{s+t}(A[\mathrm{IR}(Q^B \times K^{\mathbf{n}\setminus B}); \mathrm{res}_{Q^B \times K^{\mathbf{n}\setminus B}}^{K^{\mathbf{n}}} \varphi]). \tag{3.13}$$

If $\mathbf{M}$ is an atomic $A[\mathcal{J}^n]$-module concentrated on L^n/P, then $H^*(\mathcal{J}^n; \mathbf{M}) = \Lambda^*(\mathrm{Aut}_{\mathcal{J}^n}(L^n/P); \mathbf{M}(L^n/P))$, where group Λ^* were defined by [10]. Let V be a free rank 3 A-module with Σ_3 acting by permutations on its basis.

Proposition 3.14. *Let $B \subseteq \mathbf{n}$ and for every $b \in B$ let M_b be a $A[\Sigma_3]$-module isomorphic either to V or to A. Then*

$$\Lambda^l(\Sigma_3^B; \bar{\bigotimes_{b\in B}} M_b) = 0$$

for each $l \neq |B|$. If $M_b \simeq V$ for every $b \in B$, then and the homomorphism

$$A \simeq A^{\otimes B} \cong \bigotimes_{b\in B} \Lambda^1(\Sigma_3; V) \longrightarrow \Lambda^{|B|}(\Sigma_3^B; \bar{\bigotimes_{b\in B}} M_b)$$

is an isomorphism; otherwise $\Lambda^{|B|}(\Sigma_3^B; \bar{\bigotimes}_{a\in B} M_b) = 0$.

Proof. By [10, 6.2.ii] $\Lambda^i(\Sigma_3; V) \simeq A$ if $i = 1$ and it vanishes otherwise and $\Lambda^*(\Sigma_3; A) = 0$. The isomorphism follows from [10, 6.1]. $\square$

Description of groups $E_1^{*,*}$. Let $Z_\varphi(n)$ be a subset of $(\mathrm{IR}(K)\cup\{*\})^{\mathbf{n}}$ containing all elements $(\omega_1,\dots,\omega_n)$ such that there is a sequence $(\eta_1,\dots,\eta_n)\in\mathrm{IR}(K)^{\mathbf{n}}$ which satisfies the following conditions:

(a) $\eta_1\bar{\otimes}\dots\bar{\otimes}\eta_n\subseteq\varphi$,
(b) If $\omega_l\in\mathrm{IR}(K)$, then $\eta_l=\omega_l$,
(c) If $\omega_l=*$, then $\eta_l\in\mathrm{IR}^+(K)$.

Note that (c) is equivalent to the condition that $\mathrm{res}_Q^K\,\eta_l$ contains a subrepresentation isomorphic to τ_i (or τ_j or τ_k). For each $B\subseteq\mathbf{n}$ let

$$Z_\varphi^B(n):=\{(\omega_1,\dots,\omega_n)\in Z_\varrho(n):(\omega_l=*)\Leftrightarrow(l\in B)\}\tag{3.15}$$

$$Z_\varphi^r(n):=\bigcup_{|B|=r}Z_\varrho^B(n)\tag{3.16}$$

Proposition 3.17.

$$\Lambda^r(\Sigma_3^B;A[\mathrm{IR}(Q^B\times K^{\mathbf{n}\setminus B});\mathrm{res}^{K^{\mathbf{n}}}_{Q^B\times K^{\mathbf{n}\setminus B}}\,\varphi])\cong\begin{cases}A[Z_\varphi^B(n)] & \text{for } r=|B|\\ 0 & \text{for } r\neq|B|\end{cases}$$

Proof. Since $\mathrm{res}^{K^{\mathbf{n}}}_{Q^B\times K^{\mathbf{n}\setminus B}}\,\varphi$ is Σ_3^B-invariant, then we have a presentation

$$\mathrm{res}^{K^{\mathbf{n}}}_{Q^B\times K^{\mathbf{n}\setminus B}}\,\varphi\cong\bigoplus_{\substack{(\mu_a)\in(\mathrm{IR}(Q)^{\Sigma_3})^B\\(\nu_b)\in\mathrm{IR}(K)^{\mathbf{n}\setminus B}}}\left(\Big(\bar{\bigotimes_{a\in B}}\mu_a\Big)\bar{\otimes}\Big(\bar{\bigotimes_{b\in\mathbf{n}\setminus B}}\nu_b\Big)\right)^{\oplus l((\mu_a),(\nu_b))}$$

where $\mathrm{IR}(Q)^{\Sigma_3}=\{\theta,\lambda,\tau_i\oplus\tau_j\oplus\tau_k\}$. Then we obtain

$$A[\mathrm{IR}(Q^B\times K^{\mathbf{n}\setminus B};\mathrm{res}^{K^{\mathbf{n}}}_{Q^B\times K^{\mathbf{n}\setminus B}}\,\varphi)]=\bigoplus_{\substack{(\mu_a)\in(\mathrm{IR}(Q)^{\Sigma_3})^B\\(\nu_b)\in\mathrm{IR}(K)^{\mathbf{n}\setminus B}\\l((\mu_a),(\nu_b))>0}}\Big(\bar{\bigotimes_{\substack{a\in B\\ \mu_a=(\tau_i\oplus\tau_j\oplus\tau_k)}}}V\Big)$$

Finally, by 3.14

$$\Lambda^{|B|}(\Sigma_3^B;A[\mathrm{IR}(Q^B\times K^{\mathbf{n}\setminus B});\mathrm{res}^{K^{\mathbf{n}}}_{Q^B\times K^{\mathbf{n}\setminus B}}\,\varphi])=\bigoplus_{\substack{(\mu_a)=\tau_i\oplus\tau_j\oplus\tau_k\\(\nu_b)\in\mathrm{IR}(K)^{\mathbf{n}\setminus B}\\l((\mu_a),(\nu_b))>0}}A=A[Z_\varrho^B(n)]$$

and $\Lambda^r(\Sigma_3^B;A[\mathrm{IR}(Q^B\times K^{\mathbf{n}\setminus B});\varphi)=0$ if $r\neq|B|$. □

Proposition 3.18. *If $t\neq0$ or $s\notin\{0,1,\dots,n\}$, then $E_1^{s,t}=0$. In particular, the spectral sequence $E_*^{*,*}$ degenerates to an exact sequence:*

$$0\longrightarrow E_1^{0,0}\xrightarrow{d_1^{0,0}}E_1^{1,0}\xrightarrow{d_1^{1,0}}\dots\xrightarrow{d_1^{n-2,0}}E_1^{n-1,0}\xrightarrow{d_1^{n-1,0}}E_1^{n,0}\longrightarrow0.\tag{3.19}$$

Proof. Immediate from 3.17. □

For each $\omega\in\mathrm{IR}(K)\cup\{*\}$ let

$$|\omega|=\begin{cases}0 & \text{for } \omega\in\mathrm{IR}(K)\\1 & \text{for } \omega=*\end{cases}.$$

Then $d_1^{i,0} = \bigoplus_{j=1}^n \delta_j^i$, where

$$\delta_j^i(\omega_1,...,\omega_n) = \begin{cases} (-1)^{|\omega_1|+\ldots+|\omega_{j-1}|}(\omega_1,...,\omega_{j-1},*,\omega_{j+1},...,\omega_n) & \omega_j \in \mathrm{IR}^+(K) \\ (-1)^{|\omega_1|+\ldots+|\omega_{j-1}|+1}(\omega_1,...,\omega_{j-1},*,\omega_{j+1},...,\omega_n) & \omega_j \in \mathrm{IR}^-(K) \\ 0 & \omega_j \in \mathrm{IR}^0(K)\cup\{*\} \end{cases}$$

Proposition 3.20. *Then $H^k(\mathcal{J}^n;\Pi_2^\varphi) = 0$ for $k \geq n$.*

Proof. Since $E_1^{k,0} = 0$ for $k > n$ it suffices to prove that $H^k(\mathcal{J}^n;\Pi_2^\varphi) = 0$. The set $Z_\varphi^n(n)$ is either empty or it contains only $(*,\ldots,*)$. But if $Z_\varphi^n(n) \neq \emptyset$, then there is $\omega \in \mathrm{IR}^-(N^\infty)$ such that $(\omega,*,\ldots,*) \in Z_\varphi^{n-1}(n)$. Hence $(*,\ldots,*) = \pm d_1^{n-1,0}(\omega,*,\ldots,*)$. □

Corollary 3.21. Let φ be an $\mathcal{J}^n$-invariant representation of K^n (resp. of $K^n/\{\pm1\}$). If $n \leq 3$, then φ extends to a homotopy representation of $SU(2)^n$ (resp. $SU(2)^n/\{\pm1\}$). If $n \leq 2$, then the extension is unique.

Proof. It is straightforward from 3.20, 1.7 and 3.10. □

The case $n = 3$. As proven above, each $\mathcal{J}^3$-invariant representation φ of K^3 extends to a map $(BSU(2)^3)_2^\wedge \to BU(m)_2^\wedge$, although it possibly exists more than one extension. Let E_φ denote the set of extensions and let

$$Z_\varphi^2(3)^\pm := \{(*,\ldots,*,\omega,*,\ldots,*) \in Z_\varphi^2(3) : \omega \in \mathrm{IR}(K)^+ \cup \mathrm{IR}(K)^-\} \tag{3.22}$$
$$Z_\varphi^2(3)^0 := Z_\varphi^2(3) \setminus Z_\varphi^2(3)^\pm$$

Let $\approx$ be a relation on $Z_\varphi^2(3)^\pm$ given by

$$(\sigma_1,*,*) \approx (*,\sigma_2,*) \Leftrightarrow (\sigma_1,\sigma_2,*) \in Z_\varphi^1(3)$$
$$(\sigma_1,*,*) \approx (*,*,\sigma_3) \Leftrightarrow (\sigma_1,*,\sigma_3) \in Z_\varphi^1(3)$$
$$(*,\sigma_2,*) \approx (*,*,\sigma_3) \Leftrightarrow (*,\sigma_2,\sigma_3) \in Z_\varphi^1(3)$$

Let $\sim$ be an equivalence relation spanned by $\approx$ and let $\bar{A}[Z_\varphi^2(3)^\pm/\sim]$ be a kernel of the augmentation $A[Z_\varphi^2(3)^\pm/\sim] \to A$.

Proposition 3.23. $H^2(\mathcal{J}^3;A[Z_\varphi^*(3)]) \simeq \bar{A}[Z_\varrho^2(3)^\pm/\sim]$.

Proof. Let $j : A[Z_\varphi^2(3)^\pm] \to A[Z_\varphi^2(3)^\pm]$ be an automorphism defined as follows:

$$j(*,\ldots,*,\omega_i,*,\ldots,*) = (-1)^{i-1}\, sgn(\omega_i)(*,\ldots,*,\omega_i,*,\ldots,*)$$

where $sgn(\omega_i) = 1$ for $\omega_i \in \mathrm{IR}^+(K)$ and $sgn(\omega_i) = -1$ for $\omega_i \in \mathrm{IR}^-(K)$. If $b \approx b'$, then we have

$$\begin{aligned} j(b-b') &= (-1)^{i-1} sgn(\omega_i)(*,\ldots,*,\omega_i,*,\ldots,*)+ \\ &\quad + (-1)^{i'} sgn(\omega_{i'})(*,\ldots,*,\omega_{i'},*,\ldots,*) = \\ &\quad = d_1^{1,0}(*,\ldots,*,\omega_i,*,\ldots,*,\omega_{i'},*,\ldots,*) \in \mathrm{im}\, d_1^{1,0} \end{aligned}$$

and hence $\mathrm{im}\, d_1^{1,0} = j(\langle b-b'\rangle_{b\approx b'}) = j(\langle b-b'\rangle_{b\sim b'})$. The argument similar to the one used in proof of 3.20 shows that $\bar{A}[Z_\varphi^2(3)^0] \subseteq \mathrm{im}\, d_1^{1,0}$. Moreover,

$$d_1^{2,0}(j((*,\ldots,*,\omega_i,*,\ldots,*)) = (*,\ldots,*) \in Z_\varphi^3(3).$$

It implies that $j(\sum b_i) \in \ker d_1^{2,0}$ if and only if $\sum b_i \in \bar{A}[Z^2_\varphi(3)^\pm]$. □

Here follows the main theorem of this section:

Theorem 3.24. *(a) E_φ is a free and transitive $\bar{A}[Z^2_\varphi(3)^\pm/\sim]$-set.*
(b) Fix a permutation $\sigma \in \Sigma_3$ and assume that $\varphi \simeq \sigma^*\varphi$. If σ acts trivially on $Z^2_\varphi(3)^\pm/\sim$, then it acts trivially on E_φ.

Proof. Part (a) follows immediately from 3.23, 2.10 and 3.10. Let $F : \mathcal{J} \to \mathbf{Sp}$ be a decomposition diagram of L and let (T_σ, t_σ) be an element of $\mathrm{Aut}_{\mathbf{Diag_{Sp}}}(F^3)$ which corresponds to σ. By 2.10 and the assumption the map $\sigma^* : E_\varphi \to E_\varphi$ preserves a structure of $\bar{A}[Z^2_\varphi(3)^\pm/\sim]$-set. But the only such an automorphism of E_φ having a finite order is an identity. □

4. A homotopy representation of $Spin(7)$

In the present section we construct a representation φ of N_G^∞ and then extend it to a map $BG_2^\wedge \to BU(m)_2^\wedge$. We use a representation having a huge dimension. The main reason for this particular choice is that it is easy to prove $\mathcal{R}_2(G)$-invariance of φ.

Bases of 2-discrete tori. Let n be a positive integer and let $l = |\frac{n}{2}|$. Denote

$$e(t) := \begin{pmatrix} \cos 2\pi i t & \sin 2\pi i t \\ -\sin 2\pi i t & \cos 2\pi i t \end{pmatrix} \in SO(2) \tag{4.1}$$

and let $i : \mathbb{R}^l \to T_{Spin(n)}$ be a lift of a map

$$\mathbb{R}^l \ni (t_1, \ldots, t_l) \mapsto diag(e(t_1), \ldots, e(t_l)) \in T_{SO(n)} \tag{4.2}$$

Let $S \simeq (\mathbb{Z}/2^\infty)^r$ be a p-discrete torus. *A basis of S* is an epimorphism $\beta : \mathbb{Z}[\frac{1}{2}]^r \to S$. Any representation $\omega : S \to U(1)$ has a lift of ω along the expotential map $\mathbb{Z}/2^\infty \ni x \mapsto e^{2\pi i x} \in U(1)$ (which is of course unique). Now the composition $\tilde{\omega} \circ \beta \in \mathrm{Hom}(\mathbb{Z}[\frac{1}{2}]^r, \mathbb{Z}/2^\infty) \simeq (\mathbb{Q}_2^\wedge)^r$ determines a sequence of p-adic rationals $(k_1, \ldots, k_r)$ which will be called *a root of ω in the basis β*. Let $\varrho_{k_1,\ldots,k_r}$ be a representation corresponding to root $(k_1 \ldots, k_r)$.

We use the following three bases of T:

$$\beta : \mathbb{Z}[\tfrac{1}{2}]^3 \ni (x_1, x_2, x_3) \mapsto i(x_1, x_2, x_3), \tag{4.3}$$

$$\beta' : \mathbb{Z}[\tfrac{1}{2}]^3 \ni (x_1, x_2, x_3) \mapsto i(x_2 + x_3, x_1 + x_3, x_1 + x_2), \tag{4.4}$$

$$\beta'' : \mathbb{Z}[\tfrac{1}{2}]^3 \ni (x_1, x_2, x_3) \mapsto i(\tfrac{x_1+x_2}{2}, \tfrac{x_1-x_2}{2}, q^{-1}x_3), \tag{4.5}$$

where q is an odd 2-adic integer such that $q^2 - q + 2 = 0$ (note that $-\frac{q-1}{2} = q^{-1}$ and $q \equiv 3 \mod 8$). The bases β, β' and β'' are natural bases of maximal tori of respectively $SO(7)$, G and H.

A representation of N_G^∞. Throughout the present section we use the basis β. Define

$$\varphi := \mathrm{ind}_{T^\infty}^{N_G^\infty} \left(\bigotimes_{w \in W_D/(W_D)_{\varrho_{1,0,0}}} (\theta \oplus w^*\varrho_{1,0,0}) \right). \tag{4.6}$$

Obviously φ is not a restriction of a continuous representation of N_G since coordinates of its roots are not integers (in any basis).

Proposition 4.7. *The representation φ is faithful. Its character is W_D-invariant on T^∞ and vanishes on $N_G^\infty \setminus T^\infty$.*

Proof. It is an immediate consequence of the definition. □

Proposition 4.8. *The isotropy group $(W_D)_{\varrho_{1,0,0}} \subseteq W_D$ has order 8. An W_D-orbit of $\varrho_{1,0,0}$ contains exactly the representations ϱ_{k_1,k_2,k_3}, where*

$$\begin{aligned}(k_1,k_2,k_3) \in \{&(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1),\\ &(\pm\tfrac{q}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}),(\pm\tfrac{1}{2},\pm\tfrac{q}{2},\pm\tfrac{1}{2}),(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{q}{2}),\\ &(\pm\tfrac{q-1}{2},\pm\tfrac{q-1}{2},0),(\pm\tfrac{q-1}{2},0,\pm\tfrac{q-1}{2}),(0,\pm\tfrac{q-1}{2},\pm\tfrac{q-1}{2})\}.\end{aligned}$$

Proof. By acting with a matrix (0.4) and W_G we can easily produce the representations listed above. On the other hand, there are eight obvious elements in W_G which fix $\varrho_{1,0,0}$. Therefore the orbit contains exactly 42 elements. □

Corollary 4.9. $\dim \varphi = 2^{46}$.

Throughout the rest of this section we prove that φ is $\mathcal{R}_2(G)$-invariant.

Stubborn subgroups of orthogonal groups. Let

$$A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.10}$$

For $i < n$ define the matrices $A_i^n, B_i^n \in O(2^n)$ by

$$A_i^n := I_{2^i} \otimes A \otimes I_{2^{n-i-1}}, \qquad B_i^n := I_{2^i} \otimes B \otimes I_{2^{n-i-1}}.$$

Let

$$\Gamma_{2^n} := \langle -I_{2^n}, A_0^n, \ldots, A_{n-1}^n, B_0^n, \ldots, B_{n-1}^n \rangle \subseteq O(2^n)$$

$$\bar{\Gamma}_{2^n} := \langle \{X \otimes I_{2^{n-1}}\}_{X \in SO(2)}, A_0^n, \ldots, A_{n-1}^n, B_0^n, \ldots, B_{n-1}^n \rangle \subseteq O(2^n)$$

Let $\mathcal{T}_{prod}(k)$ be a set of all product of wreath products $\Gamma_{2^n} \wr C_2^{r_1} \wr \cdots \wr C_2^{r_m}$ (where $n \neq 1$ and $(n, r_1) \neq (0,1)$) and $\bar{\Gamma}_{2^n} \wr C_2^{r_1} \wr \cdots \wr C_2^{r_m}$ (where $n > 0$). By [13, Thm. 8] every 2-stubborn subgroup of $O(k)$ is conjugate to an element of $\mathcal{T}_{prod(k)}$. Moreover, every 2-stubborn subgroup of $SO(k)$ is conjugate to $P \cap SO(k)$, where $P \in \mathcal{T}_{prod}(k)$ [13, Prop. 11] and every 2-stubborn subgroup of $Spin(k)$ has the form $\pi^{-1}(Q)$, where Q is stubborn in $SO(k)$.

Proposition 4.11. *Fix $P \in \mathcal{T}_{prod(k)}$. Let $Q := P \cap SO(k)$ and let $Q_0 \subseteq Q$ be a unity component. Assume that $Q' := g^{-1}Qg \subseteq N_{SO(k)}$. Then for each $x = i(t_1, \ldots, t_l) \in (Q \setminus Q_0) \cap T_{SO(k)}$ one of the following conditions holds:*

(1) $t_i \equiv \frac{1}{2} \mod 1$ *for some* i
(2) $t_i \equiv t_j \equiv \frac{1}{4} \mod \frac{1}{2}$ *for some* $i \neq j$

Proof. By [13, Prop. 9] and [13, Thm. 12] we can assume that g permutes irreducible factors of P. Consider the following cases:

- $P = \Gamma_1 = O(1)$ or $P = \bar{\Gamma}_2 = O(2)$ — obvious.
- $P = \Gamma_{2^n}$ and $n > 1$. Then $Q' \cap T = \langle A_0^n B_0^n \rangle$ and
$$(Q' \setminus Q'_0) \cap T = \{i(\tfrac{1}{4}, \ldots, \tfrac{1}{4}),\ i(\tfrac{1}{2}, \ldots, \tfrac{1}{2}),\ i(\tfrac{3}{4}, \ldots, \tfrac{3}{4})\}.$$
- $P = \bar{\Gamma}_{2^n}$. Then $Q' \cap T = Q'_0$.

- $P = P' \wr C_2^t$ where $P' \subseteq O(2^n)$ and $n > 1$. Then $Q' \cap T_{SO(k)} = (P' \cap T_{SO(2^n)})^{2^t}$. Then $x = (y_1, \dots, y_{2^t})$ where $y_j \in P' \cap T_{SO(2^n)}$ for all j and since $x \notin P_0$ we have $y_l \in (P' \backslash P_0') \cap T_{SO(2^n)}$ for some l. Then the conclusion follows by induction.
- $P = \Gamma_1 \wr C_2^t$, $t > 1$. Then $Q' \cap T_{SO(k)} = (\{\pm 1\}^{2^{t-1}} \ltimes C_2) \cap SO(2))$ and either $t_i \equiv 0 \mod \frac{1}{2}$ for all i, or $t_i \equiv \frac{1}{4} \mod \frac{1}{2}$ for all i.
- $P = P_1' \times \dots \times P_l' \times \{\pm 1\}^s$ and $P_i' \neq \{\pm 1\}$, where $P_i' \in SO(k)_i$. Then

$$x = (y_1, \dots, y_s, z) \in (P_1' \cap T_{SO(k_1)}) \times \dots \times (P_1' \cap T_{SO(k_s)}) \times \{\pm 1\}^s \cap T_{SO(s)}$$

If there is i such that $y_i \notin (P_i')_0$, then the conclusion follows from induction (since $k_i > 1$). Otherwise z is not unit and has the form $diag(e(z_1), e(z_2), \dots)$ where $z_j \in \{0, \frac{1}{2}\}$. Hence some z_j equals $\frac{1}{2}$. □

Proposition 4.12. *Let $S = SO(k)$ or $S = Spin(k)$, $P \in \mathcal{T}_{prod}(k)$, $Q := P \cap SO(k)$ and $g \in SO(k)$. Assume that $g^{-1}Pg \subseteq N_{SO(k)}$. Then there exists $w \in W_{SO(k)}$ such that for each $t \in P_0$ holds $g^{-1}tg = w(t)$.*

Proof. If $S = SO(k)$ it is immediate since g permutes irreducible factors of P. The case $S = Spin(7)$ follows from the former since the projection $\pi : Spin(k) \to SO(k)$ induces an equivalence $\mathcal{R}_2(Spin(7)) \cong \mathcal{R}_2(SO(7))$. □

$\mathcal{R}_2(G)$-invariance of φ.

Proposition 4.13. *Let $x = i(t_1, t_2, t_3) \in G$. If any of numbers t_i or $t_i \pm t_j$ (where $i \neq j$) equals $\frac{1}{2} \mod 1$, then $\chi_\varphi(x) = 0$.*

Proof. Notice that for each $w \in W_D$ φ contains $\theta \oplus w^*\varrho_{1,0,0}$ as a tensor factor. If $t_1 \equiv \frac{1}{2} \mod 1$, then $\chi_{\varrho_{1,0,0}}(x) = -1$. Thus

$$\chi_\varphi(x) = \chi_{\theta\oplus\varrho_{1,0,0}}(x)\chi_\psi(x) = (1-1)\chi_\psi(x) = 0,$$

By replacing succesively $\varrho_{1,0,0}$ by $\varrho_{0,1,0}$, $\varrho_{0,0,1}$, $\varrho_{\frac{q-1}{2},\pm\frac{q-1}{2},0}$, $\varrho_{\frac{q-1}{2},0,\pm\frac{q-1}{2}}$, $\varrho_{0,\frac{q-1}{2},\pm\frac{q-1}{2}}$ and using an analogous argument we obtain the conclusion in the other cases (notice that $q \equiv 3 \mod 8$). □

Proposition 4.14. *Let $P \in \mathcal{T}_{prod}(O(7))$, $Q := P \cap SO(7)$ and $\tilde{Q} := \pi^{-1}(Q)$. Assume that $\pi(g)^{-1}Q\pi(g) \subseteq N_G$ for some $g \in G$. Then for each $x \in \tilde{Q}^\infty$ we have $\chi_\varphi(x) = \chi_\varphi(g^{-1}xg)$. In particular, φ is $\mathcal{R}_2(G)$-invariant.*

Proof. If $\pi(x) \notin Q_0$, then by 4.11 $\chi_\varphi(x) = \chi_\varphi(g^{-1}xg) = 0$. If $\pi(x) \in Q_0$, then the conclusion follows by 4.12. □

Proposition 4.15. *The representation φ extends to a map $f_G : BG \to BU(2^{46})^\wedge_2$.*

Proof. As a consequence of [19] every $\mathcal{R}_2(G)$-invariant representation of N_G^∞ has an extension. □

5. A homotopy representation of H

Let $f_H : BH^\wedge_2 \to BU(m)^\wedge_2$ be the composition of f_G with an inclusion $BH^\wedge_2 \to BG^\wedge_2$. The main result of this section is that f_H is homotopy $\mathrm{GL}_2(\mathbb{F}_2)$-invariant, where the action of $\mathrm{GL}_2(\mathbb{F}_2)$ comes from the centralizer decomposition diagram of $DI(4)$.

$\mathrm{GL}_2(\mathbb{F}_2)$**-action on** $BH_2^\wedge$**.** Since H and G have a common maximal torus, then $W_H \subset W_G$. It appears that the group $N_{W_D}(W_H)/W_H$ acts on $BH_2^\wedge$. Since $N_{W_D}(W_H)/W_H \simeq \mathrm{GL}_2(\mathbb{F}_2)$, this determines the action of $\mathrm{GL}_2(\mathbb{F}_2)$ on $BH_2^\wedge$.

The group $\mathrm{GL}_2(\mathbb{F}_2)$ is generated by an order 2 element which stabilizes the inclusion $BH_2^\wedge \to BG_2^\wedge$ and by some element of order 3, say b. The invariance of f_H under the action of s is clear. The main effort will be to show that $f_H \circ b \simeq f_H$. Elementary calculations show that b is represented (in the basis β'') by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.1}$$

Denote $\omega := \mathrm{res}^{N_G^\infty}_{N_H^\infty}\ \varphi$

Proposition 5.2. *The morphism b stabilizes the representation ω. In particular, both maps f_H and $b \circ f_H$ restrict to the same $\mathcal{R}_2(H)$-invariant representation ω.*

Proof. Strightforward from 4.7. □

Extensions of ω**.** By 3.24 the set E_ω of extensions of ω to a map from $B(L^3/\{\pm 1\})_2^\wedge$ is a free and transitive $H^2(\mathcal{J}^3; \Pi_2^\omega)$-set. By 3.24 $E_\omega \simeq \bar{\mathbb{Z}}_2^\wedge[Z^2_\omega(3)^\pm/\sim]$, where $Z^2_\omega(3)^\pm$ is the set defined in 3.22. For simplicity denote $Z := Z^2_\omega(3)$, $Z^\pm := Z^2_\omega(3)^\pm$. In order to calculate the set Z and prove that b acts trivially on it we need to describe the set of irreducible subrepresentations of ω. Let $Rt(\omega) \subset (\mathbb{Q}_2^\wedge)^3$ be the set of roots of ω (in dual basis β''). Obviously W_D acts of $Rt(\omega)$.

Proposition 5.3. *The set $Rt(\omega)$ contains all combinations of roots having the form $w^*(\frac{1}{2}, \frac{1}{2}, 0)$ for $w \in W_D$.*

Proof. Since the sequence $(1, 0, 0)$ in dual basis β has the form $(\frac{1}{2}, \frac{1}{2}, 0)$ in basis β'', then the conclusion follows from the definition of φ (4.6) and properties of the tensor product. □

Roots having the form $w^*(\frac{1}{2}, \frac{1}{2}, 0)$ will be called *basic roots*. The set of all basic roots will be denoted by $BRt(\omega)$. Put $c = \frac{1}{2}$, $d = \frac{q-1}{4}$. Notice that $c \equiv d \equiv \frac{1}{2}$ mod 2.

Proposition 5.4. *We have*

$$\begin{aligned} BRt(\omega) = \{&(\pm c, \pm c, 0), (\pm c, 0, \pm c), (0, \pm c, \pm c), \\ &(\pm 2d, 0, 0), (0, \pm 2d, 0), (0, 0, \pm 2d), \\ &(\pm(c+d), \pm d, \pm d), (\pm d, \pm(c+d), \pm d), (\pm d, \pm d, \pm(c+d))\} \end{aligned}$$

Proof. It is the list (4.8) converted into the basis β''. □

Directly from the definition (4.6) we obtain

Corollary 5.5.

$$Rt(\omega) \cong \left\{ \sum_{(r_1, r_2, r_3) \in BRt(\omega)} l_{(r_1, r_2, r_3)}(r_1, r_2, r_3) : 0 \le l_{(r_1, r_2, r_3)} \le 16 \right\}.$$

Proposition 5.6. *If even integers l_c and l_d satisfy inequalities $|l_c| \le 16 \cdot 8$, $|l_d| \le 16 \cdot 14$ and $|l_c - l_d| \le 16 \cdot 14$, then $(l_c c + l_d d, 0, 0) \in Rt(\omega)$.*

Proof. Introduce a relation $\overset{1}{\sim}$ on $BRt(\omega)$ by

$$(r_1, r_2, r_3) \overset{1}{\sim} (r'_1, r'_2, r'_3) \Leftrightarrow r_1 = r'_1 \wedge r_2 = -r'_2 \wedge r_3 = -r'_3.$$

Let $SRt(\omega) \subseteq Rt(\omega)$ be the set all combinations

$$\sum_{(r_1,r_2,r_3)\in BRt(\omega)} l_{(r_1,r_2,r_3)}(r_1, r_2, r_3)$$

such that the coefficients $l_{(r_1,r_2,r_3)}$ are constant on equivalence classes of the relation $\overset{1}{\sim}$. Then $SRt(\omega)$ contains exactly sequences

$$(m_c(2c) + m_d(2d) + m_{c+d}(2c + 2d), 0, 0),$$

where the integers m_c, m_d and m_{c+d} satisfy inequalities $-2 \cdot 16 \leq m_c \leq 2 \cdot 16$, $-5 \cdot 16 \leq m_d \leq 5 \cdot 16$, $m_{c+d} = -2 \cdot 16 \leq m_{c+d} \leq 2 \cdot 16$. □

Proposition 5.7. *If odd integers l_c, l_d satisfy inequalities $|l_c| \leq 8 \cdot 16$, $|l_d| \leq 14 \cdot 16$ and $|l_c - l_d| \leq 14 \cdot 16$, then $(l_c c + l_d d, c + d, c + d) \in Rt(\omega)$.*

Proof. By 5.6 we have $((l_c - 1)c + (l_d - 1)d, 0, 0) \in Rt(\omega)$ and it may be written without using basic roots $(c + d, d, d)$ and $(0, c, c)$ (since $l_c \leq 8 \cdot 16 - 2$ and $l_d \leq 14 \cdot 16 - 2$). Then

$$((l_c-1)c+(l_d-1)d, 0, 0)+(c+d, d, d)+(0, c, c) = (l_c c+l_d d, c+d, c+d) \in Rt(\omega). \ \square$$

Elementary arguments lead to the following corollaries

Corollary 5.8. If $(l_c^1 c + l_d^1 d, l_c^2 c + l_d^2 d, l_c^3 c + l_d^3 d) \in Rt(\omega)$ is a root with (2-adic) integer coordinates, then for $i = 1, 2, 3$ we have $l_c^i \equiv l_d^i \mod 2$. Moreover, numbers l_c^i and l_d^i satisfy inequalities analogous to the ones in the formulation of 5.6.

Corollary 5.9.

$$\begin{aligned} Rt(\omega) \cup (\mathbb{Z}_2^\wedge)^3 = \{&(l_c^1 c + l_d^1 d, l_c^2 c + l_d^2 d, l_c^3 c + l_d^3 d) : l_c^i, l_d^i \in \mathbb{Z}, \\ &l_c^i \equiv l_d^i \mod 2 \wedge |l_c^i| \leq 8 \cdot 16 \wedge |l_c^i| \leq 14 \cdot 16 \wedge |l_c^i - l_d^i| \leq 14 \cdot 16\} \end{aligned}$$

Proposition 5.10.

$$\begin{aligned} Z^\pm \cong \{&(l_c c + l_d d, *, *), (*, l_c c + l_d d, *), (*, *, l_c c + l_d d) : l_c, l_d \in \mathbb{Z} \\ &\wedge l_c \equiv l_d \mod 2 \wedge |l_c| \leq 8 \cdot 16 \wedge |l_d| \leq 14 \cdot 16 \wedge |l_c - l_d| \leq 14 \cdot 16\} \end{aligned}$$

Proof. Recall that $Z^\pm$ is a set of symbols $(\eta_1, *, *)$, $(*, \eta_2, *)$, $(*, *, \eta_3)$ such that $\eta_1 \bar{\otimes} \eta_2 \bar{\otimes} \eta_3 \subseteq \omega$, and $\eta_i \in \mathrm{IR}^+(K) \cup \mathrm{IR}^-(K)$ for each i. Now the conclusion follows from 5.9 (we identify a representation $\mathrm{ind}_{T_L^\infty}^K \varrho_k$ with a 2-adic integer k and τ with 0). □

Proposition 5.11. *There are two equivalence classes of relation $\sim$ on the set $Z^\pm$, namely*

$$\begin{aligned} Z_0^\pm = \{&(l_c c + l_d d, *, *), (*, l_c c + l_d d, *), (*, *, l_c c + l_d d) : \\ &l_c, l_d \in 2\mathbb{Z} \wedge |l_c| \leq 8 \cdot 16 \wedge |l_d| \leq 14 \cdot 16 \wedge |l_c - l_d| \leq 14 \cdot 16\} \end{aligned}$$

and

$$\begin{aligned} Z_1^\pm = \{&(l_c c + l_d d, *, *), (*, l_c c + l_d d, *), (*, *, l_c c + l_d d) : \\ &l_c, l_d \in 1 + 2\mathbb{Z} \wedge |l_c| \leq 8 \cdot 16 \wedge |l_d| \leq 14 \cdot 16 \wedge |l_c - l_d| \leq 14 \cdot 16\}. \end{aligned}$$

Proof. For even l_c, l_d, by 5.6 $(l_c c + l_d d, 0, 0) \in Rt(\omega)$ (and therefore $(0, l_c c + l_d d, 0), (0, 0, l_c c + l_d d) \in Rt(\omega)$). Moreover, obviously $(0,0,0) \in Rt(\omega)$. Then

$$(l_c c + l_d d, *, *) \sim (*, 0, *) \sim (0, *, *) \sim (*, l_c c + l_d d, *) \simeq (*, *, l_c c + l_d d)$$

Then all elements of $Z_0^{\pm}$ are in the same equivalence class as $(0, *, *)$. Similarly, if l_c and l_d are odd, then

$$(l_c c + l_d d, c+d, c+d), (c+d, l_c c + l_d d, c+d), \\ (c+d, c+d, l_c c + l_d d), (c+d, c+d, c+d) \in Rt(\omega).$$

Therefore all elements of $Z_1^{\pm}$ are in the same equivalence class as $(c+d, *, *)$. □

Proposition 5.12. *The action of b on E_ω is trivial.*

Proof. The action of b on the set $Z^{\pm}$ is trivial since b has order 3 and $Z^{\pm}$ has only 2 elements. The conclusion follows by 3.24.(b) □

Proposition 5.13. *The map f_G is $\mathcal{A}$-invariant, i.e. it extends to a homotopy compatible family of maps from the centralizer decomposition functor of BD into $BU(m)_2^\wedge$.*

Proof. By 5.12 the map $f_H = f_G|_{BH_2^\wedge}$ is $\mathrm{GL}_2(\mathbb{F}_2)$-invariant. The map $f_G|_{B(T^3 \rtimes \{\pm 1\})_2^\wedge}$ is, by Dwyer-Zabrodsky theorem, the completion of the classifying map of the representation $\mathrm{res}^{N_1^\infty}_{(T^3 \rtimes \{\pm 1\})^\infty} \varphi$, which is $\mathrm{GL}_3(\mathbb{F}_2)$-invariant (since the action of $\mathrm{GL}_3(\mathbb{F}_2)$ is the restriction of the action of the Weyl group). Since $\mathrm{res}^{N_1^\infty}_{\{\pm 1\}^4} \varphi$ is a sum of regular representations, then the map $f_G|_{B\{\pm 1\}^4}$ is $\mathrm{GL}_4(\mathbb{F}_2)$-invariant. □

6. Proof of the main theorem

Let $F : \mathcal{A} \to \mathbf{Sp}$ be the centralizer decomposition diagram of BD and let A_i, $i = 1, \ldots, 4$ be objects of $\mathcal{A}$. We have proven that the map

$$f_G : F(A_1) \simeq BG_2^\wedge \to BU(m)_2^\wedge$$

is $\mathcal{A}$-invariant. Now we have to check that the obstructions to the existence of an extension $BDI(4) \to BU(m)_2^\wedge$ in groups $H^{i+1}(\mathcal{A}; \Pi_i)$ vanish, where

$$\Pi_i^f(A_r) := \pi_i(\mathrm{map}(F(A_r), BU(m)_2^\wedge)_{f_G|_{F(A_r)}}).$$

Oliver [14] provided a powerful tool for calculating this kind of cohomology groups:

Theorem 6.1. ([14, Thm. 1, Prp. 6]) *Let $\mathcal{A} := \mathcal{A}_p(X)$ be the centralizer decomposition category of a p-compact group X. Let $\mathcal{A}_i$ be the set of objects of $\mathcal{A}$ which have the form $B(\mathbb{Z}/p)^i \to BX$. Then for any $\mathbb{Z}_p^\wedge[\mathcal{A}]$-module F there is a isomorphism $H^*(\mathcal{A}; F) \simeq C^*_{St}(F)$, where*

$$C^i_{St}(F) \cong \prod_{A \in \mathcal{A}_{i+1}} \mathrm{Hom}_{\mathrm{Aut}_{\mathcal{A}}(A)}(St_A, F(A))$$

However, this theorem applies only to the case when the coefficient functor has abelian values (obviously $\Pi_i(A_r)$ is abelian for $i > 1$). For $r = 3, 4$ the spaces $F(A_r)$ are 2-completed classifying spaces of 2-toral groups. Therefore, by 1.2 the spaces $\mathrm{map}(F(A_r), BU(m)_2^\wedge)_{f_G|_{F(A_r)}}$ are classifying spaces of products of unitary groups. Hence $\Pi_1^f(\mathbb{F}_2^r) = 0$. For $r = 1, 2$ this argument fails, since the spaces $F(A_1)$ and $F(A_2)$ are not classifying spaces of 2-toral groups.

Proposition 6.2. *The fundamental group of* $\mathrm{map}(BH_2^\wedge, BU(m)_2^\wedge)_{f_H}$ *is abelian.*

Proof. By the Dwyer-Zabrodsky Theorem (1.2) we have

$$\begin{aligned}\mathrm{map}(BH_2^\wedge, BU(m)_2^\wedge)_{f_H} &\cong \mathrm{map}(\mathrm{hocolim}_{H/P\in\mathcal{J}^3}(EH\times_H/P)_2^\wedge, BU(m)_2^\wedge)_{f_H}\\ &= \mathrm{holim}_{H/P\in\mathcal{J}^3}\,\mathrm{map}((EH\times_H H/P)_2^\wedge, BU(m)_p^\wedge)_{f_H|_{BP_2^\wedge}}\\ &\simeq (\mathrm{holim}_{H/P\in\mathcal{J}^3} BC_{U(m)}(\omega(P^\infty))_2^\wedge)_2^\wedge\end{aligned}$$

The second term of the Bousfield spectral sequence [3, XI,7.1], which converges to the homotopy groups of the homotopy inverse limit above has the second term

$$\begin{aligned}E_2^{p,q} &= H^p(\mathcal{J}^3;\pi_{p+q}(\mathrm{map}(EH\times_H H/(-), BU(n)_2^\wedge)_{f_H|_{B(-)_2^\wedge}})\\ &= H^p(\mathcal{J}^3;\pi_{p+q}(BC_{U(m)}(\varphi((-)^\infty)_2^\wedge))\Rightarrow \Pi_q(A_2)\end{aligned}$$

The spaces $BC_{U(m)}(\varphi((-)^\infty)_2^\wedge$ are products of 2-completed classifying spaces of unitary groups (cf. Section 1), where the number of multiplies is the number of isomorphism classes of irreducible subrepresentations of $\mathrm{res}^{N_G^\infty}_{P^\infty}\varphi$ and the rank of every one is the multiplicity of the corresponding irreducible representation. Then the groups $\pi_{p+q}(BC_{U(m)}(\varphi((-)^\infty)_2^\wedge)$ are abelian. Moreover, for $p+q=1,3$ they vanish and for $p+q=2,4$ they are isomorphic to Π_2^ω, where as before $\omega=\mathrm{res}^{N_G^\infty}_{N_H^\infty}\varphi$, since all irreducible subrepresentations of φ appear with multiplicity at least 16. By 3.20 we have $H^3(\mathcal{J}^3;\Pi_2^\omega)=0$. Thus $E_2^{p,1}=0$ for $p\neq 1$. Therefore $\Pi_1^f(A_2)$ is a subquotient of the abelian group $E_2^{1,1}$ and hence is abelian. □

The following proposition is a straightforward consequence of 6.1:

Proposition 6.3. *For any* $\mathbb{Z}_2^\wedge[\mathcal{A}]$*-module* M *there is an isomorphism*

$$H^{*-1}(\mathcal{A};M)\cong H^*(\mathrm{Hom}_{\mathrm{GL}_*(\mathbb{F}_2)}(St_{\mathrm{GL}_*(\mathbb{F}_2)}, M(A_*))),$$

where St_Γ *is the Steinberg module of the group* Γ*.* □

In particular, $H^i(\mathcal{A};M)=0$ for $i>3$.

Proposition 6.4. *For each* $i\geq 1$ *holds* $H^{i+1}(\mathcal{A};\Pi_i^f)=0$.

Proof. If $i\geq 3$, then the conclusion is obvious, so it is sufficient to consider cases $i=1,2$ only. Note that the full subcategory of $\mathcal{A}$ with objects A_1 and A_2 is isomorphic to $\mathcal{J}$. Since $\Pi_i^f(A_r)=0$ for $r=3,4$, then we have $H^2(\mathcal{A};\Pi_1^f)\cong H^2(\mathcal{J};\mathrm{res}^{\mathcal{A}}_{\mathcal{J}}\Pi_1^f)=0$. If $i=2$, then by 6.3 $H^3(\mathcal{A};\Pi_2^f)$ is a quotient of the group

$$\mathrm{Hom}_{\mathrm{GL}_4(\mathbb{F}_2)}(St_{\mathrm{GL}_4(\mathbb{F}_2)},\Pi_2^f(A_4)).$$

A $\mathbb{Z}_2^\wedge$-module $\Pi_2^f(A_4)=\pi_2(\mathrm{map}(B\{\pm1\}^4, BU(m)_2^\wedge)_{\mathrm{res}^{N_1^\infty}_{\{\pm1\}^4}\varphi})$ is free and has dimension not larger than 2^4 (there are 2^4 isomorphism classes of irreducible representations of $\{\pm1\}^4$). The Steinberg module $St_{\mathrm{GL}_4(\mathbb{F}_2)}$ is a second homology group of a geometrical realization of a poset of all non-trivial proper subspaces of $\mathbb{F}_2^4$. By an Euler characteristic argument it has dimension 64 and is an irreducible $\mathbb{Z}_2^\wedge[\mathrm{GL}_4(\mathbb{F}_2)]$-module (cf. [16]). Hence there is no non-zero homomorphism $St_{\mathrm{GL}_4(\mathbb{F}_2)}\to\Pi_2^f(A_4)$. Hence $\mathrm{Hom}_{\mathrm{GL}_4(\mathbb{F}_2)}(St_{\mathrm{GL}_4(\mathbb{F}_2)},\Pi_2^f(A_4))=0$ and thus $H^3(\mathcal{A};\Pi_2^f)=0$. □

As a consequence we obtain the main theorem of this paper:

Theorem 6.5. *The map $f_G : BG^\wedge_2 \to BU(m)^\wedge_2$ extends to a faithful complex representation of the* 2*-compact group* $DI(4)$.

Proof. By 6.4 the map f_G extends to $BDI(4)$. By 4.7 and [8, 3.2] the extension is a classifying map of a monomorphism of 2-compact groups. □

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