HOMOTOPY REPRESENTATIONS OF $SO(7)$ AND $Spin(7)$ AT PRIME 2

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Abstract. A homotopy (complex) representation of a compact connected Lie group L at prime p is a map from BL into the p-completion of the classifying space of the unitary group $BU(n)^\wedge_p$. In this paper we give a partial classification of homotopy representations of $SO(7)$ and $Spin(7)$ at prime 2. Motivation for considering this problem is twofold: first, one may hope that it would help to understand maps between classifying spaces. Secondly, construction of a homotopy representation of $Spin(7)$ is a crucial step in the construction of a faithful representation of the 2-compact group $DI(4)$ [14].

Let p be a prime and let L be a compact connected Lie group. A group P is p-toral if it is an extension of a finite p-group by a torus; a p-toral subgroup $P \subseteq L$ is p-stubborn if $N_L(P)/P$ is finite and contains no non-trivial normal p-subgroups. Let $\mathcal{O}_p(L)$ (resp. $\mathcal{R}_p(L)$) be a category of L-orbits L/P for p-toral P (resp. p-stubborn P) and L-equivariant maps. By [7], the map

$$(\operatorname{hocolim}_{L/P\in\mathcal{R}_p(L)}(EL\times_L L/P)^\wedge_p)^\wedge_p \longrightarrow BL^\wedge_p$$

is a weak homotopy equivalence. The category $\mathcal{R}_p(L)$ contains a maximal object G/N, where N is a p-normalizer of a maximal torus T. Let N^∞ be a p-discrete approximation of N, i.e. its dense subgroup such that $T\cap N^\infty$ be a subgroup of all elements of T having a p-power order. By a version of Dwyer-Zabrodsky theorem [8] the map

$$\operatorname{Rep}(N^\infty, U(n)) \ni \varphi \mapsto (B\varphi)^\wedge_p \to [(EL\times L/N)^\wedge_p, BU(n)^\wedge_p],$$

where $\operatorname{Rep}(N^\infty, U(n)) := \operatorname{Hom}(N^\infty, U(n))/\operatorname{Inn}(U(n))$, is an isomorphism. A representation φ of N^∞ is $\mathcal{R}_p(L)$-invariant iff $(B\varphi)^\wedge_p$ defines a homotopy compatible family of maps in $\lim_{L/P\in\mathcal{R}_p(L)}[EL\times_L(\cdot), BU(n)^\wedge_p]$. Obviously if $(B\varphi)^\wedge_p$ extends to a homotopy representation of L (i.e. a map $BL^\wedge_p \to BU(n)^\wedge_p$), then φ is $\mathcal{R}_p(L)$-invariant; the question is whether or not an $\mathcal{R}_p(L)$-invariant representations extends (after completion) to a homotopy representation of L. The main result of the present paper is the following

Theorem. *Let $p=2$. If $L = Spin(7)$ or $L=SO(7)$, then every $\mathcal{R}_p(L)$-invariant representation of N^∞ extends to a homotopy representation of L.*

Throughout the whole paper $G = SO(7)$, $\tilde{G} = Spin(7)$, $\pi:\tilde{G}\to G$ is an obvious projection and $u\in\tilde{G}$ is a non-trivial lift of unity in G. For any subgroup $H\subseteq G$ let $\tilde{H} := \pi^{-1}(H)\subseteq\tilde{G}$.

Organization of the paper. Section 1 contains generalization of some elementary facts from representation theory of finite groups onto locally finite groups (in particular onto discrete approximations of p-toral groups). In Section 2 we describe a general method of construction homotopy representations of compact Lie groups.

Section 3 contains a classification of 2-stubborn subgroups of orthogonal and symmetric groups, and a construction of a full inclusion $\mathcal{R}_2(\Sigma_n) \to \mathcal{R}_2(O(n))$. The last result simplifies calculations of higher limits on $\mathcal{R}_2(O(n))$. In Section 4 some facts about representations of discrete approximations of 2-stubborn subgroups of $SO(7)$ and $Spin(7)$ are provided. Section 5 contains specific computations of higher limits of atomic functors. Finally, the main theorem is proven in Section 6.

1. Representations of countable locally finite groups

A group Γ is countable locally finite if it satisfies the following equivalent conditions:

- Γ is countable and every finitely generated subgroup of Γ is finite.
- There is an ascending sequence of finite groups

$$\{1\} = \Gamma^{(0)} \subseteq \Gamma^{(1)} \subseteq \Gamma^{(2)} \subseteq \cdots \subseteq \Gamma \tag{1.1}$$

  such that $\bigcup_s \Gamma^{(s)} = \Gamma$.

By *a complex representation* of a group Γ we mean a homomorphism $\varphi : \Gamma \to \mathrm{GL}(V)$, where V is a finite dimensional complex vector space. A representation is *unitary* if V is equipped with a hermitian scalar product which is preserved by $\varphi(g)$ for each $G \in \Gamma$.

Throughout this section Γ denotes any countable locally finite group and $\Gamma^{(s)}$ is a fixed sequence (1.1). Let $\mathrm{Rep}(\Gamma) := \bigcup_n \mathrm{Rep}(\Gamma, \mathrm{GL}(\mathbb{C}^n))$ and let $\mathrm{IR}(\Gamma) \subset \mathrm{Rep}(\Gamma)$ be the set of isomorphism classes of irreducible representations of Γ. The following proposition provides a main reason for similarity of representation theory of locally finite groups with the finite case.

Proposition 1.2. *If $\varphi : \Gamma \to \mathrm{GL}(V)$ and $\psi : \Gamma \to \mathrm{GL}(W)$ are representations, then there is $s < \infty$ such that*

$$\mathrm{Hom}_\Gamma(V, W) = \mathrm{Hom}_{\Gamma^{(s)}}(V, W).$$

Proof. A sequence

$$\mathrm{Hom}_{\Gamma^{(0)}}(V, W) \supseteq \mathrm{Hom}_{\Gamma^{(1)}}(V, W) \supseteq \mathrm{Hom}_{\Gamma^{(2)}}(V, W) \supseteq \cdots \supseteq \mathrm{Hom}_\Gamma(V, W)$$

eventually stabilizes since the dimensions of these spaces do. Moreover, we have $\mathrm{Hom}_\Gamma(V, W) = \bigcap_{r=0}^{\infty} \mathrm{Hom}_{\Gamma^{(r)}}(V, W)$. Thus $\mathrm{Hom}_\Gamma(V, W) = \mathrm{Hom}_{\Gamma^{(s)}}(V, W)$ for large enough s. □

Here follow immediate corollaries of Proposition 1.2:

Proposition 1.3. *Fix representations $\varphi : \Gamma \to \mathrm{GL}(V)$ and $\psi : \Gamma \to \mathrm{GL}(W)$.*

1. *If φ is irreducible, then for some s $\mathrm{res}^\Gamma_{\Gamma^{(s)}} \varphi$ is irreducible.*
2. *If for all s $\mathrm{res}^\Gamma_{\Gamma^{(s)}} \varphi$ and $\mathrm{res}^\Gamma_{\Gamma^{(s)}} \psi$ are isomorphic, then φ and ψ are isomorphic.*
3. *(Schur lemma) If φ is irreducible, then $\mathrm{End}_\Gamma(V)$ contains only homoteties. If additionally ψ is irreducible and non-isomorphic to φ, then $\mathrm{Hom}_\Gamma(V, W) = 0$.*

Proof. Fix s such that $\mathrm{End}_\Gamma(V) = \mathrm{End}_{\Gamma^{(s)}}(V)$ and $\mathrm{Hom}_\Gamma(V, W) = \mathrm{Hom}_{\Gamma^{(s)}}(V, W)$

1. If $\mathrm{res}^\Gamma_{\Gamma^{(s)}} \varphi$ is reducible, then there is a projection $f \in \mathrm{End}_{\Gamma^{(s)}}(V)$ onto one of its irreducible summands. Since $f \in \mathrm{End}_\Gamma(V)$ it denies irreducibility of φ.

(2) If $\mathrm{Hom}_{\Gamma^{(s)}}(V,W)$ contains an isomorphism, then $\mathrm{Hom}_\Gamma(V,W)$ also does.
(3) By (1) and (2) it follows from the Schur lemma for finite groups.

□

Proposition 1.4. [12] *Every representation of Γ is semisimple, and its decomposition into irreducible summands is unique up to permutation of summands.*

Proof. Let $\varphi : \Gamma \to \mathrm{GL}(V)$ be any representation and fix s such that $\mathrm{End}_{\Gamma^{(s)}} V = \mathrm{End}_\Gamma(V)$. Let $\bigoplus_i W_i$ be a decomposition of $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}} \varphi$ onto irreducible summands. All W_i's are irreducible Γ-subrepresentations (since any $\Gamma^{(s)}$-projection $V \to W_i$ is also Γ-homomorphism). Then $\bigoplus_i W_i$ is a decomposition of φ.

If $\bigoplus_i W_i$ and $\bigoplus_j X_j$ are decompositions of φ, then there is a $\Gamma^{(s)}$-isomorphism

$$f : V = \bigoplus_i W_i \longrightarrow \bigoplus_j X_j = V$$

permuting irreducible $\Gamma^{(s)}$-summands. But f is also Γ-isomorphism. □

A decomposition into irreducible summands can be obtained functorially. Pick a representative $\sigma : \Gamma \to \mathrm{GL}(W_\sigma)$ of every element of $\mathrm{IR}(\Gamma)$.

Proposition 1.5. *For any representation $\varphi : \Gamma \to \mathrm{GL}(V)$ the evaluation*

$$\bigoplus_{\sigma\in\mathrm{IR}(\Gamma)} W_\sigma \otimes \mathrm{Hom}_\Gamma(W_\sigma, V) \xrightarrow{ev} V$$

is an isomorphism. Furthermore, the homomorphism

$$\bigoplus_{\sigma\in\mathrm{IR}(\Gamma)} \mathrm{End}(\mathrm{Hom}_\Gamma(W_\sigma, V)) \ni \bigoplus_\sigma f_\sigma \mapsto$$
$$\mapsto ev \circ \Big(\bigoplus_\sigma Id_{W_\sigma} \otimes f_\sigma\Big) \circ (ev)^{-1} \in \mathrm{End}_\Gamma(V)$$

is an isomorphism.

Proof. By Proposition 1.4 we can assume that $V = \bigoplus_{\sigma\in\mathrm{IR}(\Gamma)} W_\sigma^{\oplus l_\sigma}$. By the Schur lemma $\mathrm{Hom}_\Gamma(W_\sigma, W_\sigma^{\oplus l_\sigma}) \cong \mathrm{Hom}_\Gamma(W_\sigma, V)$. Then the evaluation is a composition

$$\bigoplus_\sigma W_\sigma \otimes \mathrm{Hom}_\Gamma(W_\sigma, V) \cong \bigoplus_\sigma W_\sigma \otimes \mathrm{Hom}_\Gamma(W_\sigma, W_\sigma^{\oplus l_\sigma}) \cong \bigoplus_\sigma W_\sigma \otimes \mathbb{C}^{l_\sigma} = V.$$

Again by the Schur lemma

$$\mathrm{End}_\Gamma(V) \simeq \bigoplus_{\sigma\in\mathrm{IR}(\Gamma)} \mathrm{End}_\Gamma(W_\sigma \otimes \mathrm{Hom}_\Gamma(W_\sigma, V)).$$

Moreover,

$$\begin{aligned}
&\mathrm{End}_\Gamma(W_\sigma \otimes \mathrm{Hom}_\Gamma(W_\sigma, V)) \\
&\quad = \mathrm{Hom}_\Gamma(W_\sigma \otimes \mathrm{Hom}_\Gamma(W_\sigma, V), W_\sigma \otimes \mathrm{Hom}_\Gamma(W_\sigma, V)) \\
&\quad = \mathrm{Hom}(\mathrm{Hom}_\Gamma(W_\sigma, V), \mathrm{Hom}_\Gamma(W_\sigma, W_\sigma \otimes \mathrm{Hom}_\Gamma(W_\sigma, V))) \\
&\quad = \mathrm{Hom}(\mathrm{Hom}_\Gamma(W_\sigma, V), \mathrm{Hom}_\Gamma(W_\sigma, W_\sigma) \otimes \mathrm{Hom}_\Gamma(W_\sigma, V)) \\
&\quad = \mathrm{End}(\mathrm{Hom}_\Gamma(W_\sigma, V)).
\end{aligned}$$

□

Let $Ch(\Gamma) \subseteq \mathbb{C}^\Gamma$ be a vector subspace spanned by all characters of representations (it does not contain all class functions).

Proposition 1.6. *If $\chi, \chi' \in Ch(\Gamma)$, then the sequence $(\chi|\chi')_s := (\chi|_{\Gamma^{(s)}}|\chi'|_{\Gamma^{(s)}})$ stabilizes. In particular,*

$$(\chi|\chi') := \lim_{s\to\infty} (\chi|\chi')_s$$

is a hermitian product on $Ch(\Gamma)$ and characters of irreducible representations form an orthonormal basis of $Ch(\Gamma)$.

Proof. By Proposition 1.4 it is enough to prove that the sequnce $(\chi|\chi')_s$ stabilizes for $\chi = \chi_\varphi$, $\chi' = \chi_\psi$, where $\varphi, \psi \in \mathrm{IR}(\Gamma)$. By 1.3.(1) for large enough s both $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}} \varphi$ and $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}} \psi$ are irreducible. Then if φ and ψ are isomorphic, then $(\chi_\varphi|\chi_\psi)_s = 1$; if not $(\chi_\varphi|\chi_\psi)_s = 0$ (cf. 1.3.(2)). □

From Propositions 1.4 and 1.6 we obtain

Corollary 1.7. Two representations of a given locally finite group are isomorphic iff their characters are equal.

The next proposition states that representations of products behave similarly to the finite case:

Proposition 1.8. *Let Γ and Δ be countable locally finite groups. The map*

$$\bar{\otimes} : \mathrm{IR}(\Gamma) \times \mathrm{IR}(\Delta) \ni (\varphi, \psi) \mapsto \mathrm{res}^{\Gamma}_{\Gamma\times\Delta} \varphi \otimes \mathrm{res}^{\Delta}_{\Gamma\times\Delta} \psi \in \mathrm{IR}(\Gamma \times \Delta)$$

is a bijection.

Proof. If $\varphi \in \mathrm{IR}(\Gamma)$, $\psi \in \mathrm{IR}(\Delta)$ then for some s both representations $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}} \varphi$ and $\mathrm{res}^{\Delta}_{\Delta^{(s)}} \psi$ are irreducible. Hence $\mathrm{res}^{\Gamma\times\Delta}_{\Gamma^{(s)}\times\Delta^{(s)}} \varphi\bar{\otimes}\psi$ is irreducible and so is $\varphi\bar{\otimes}\psi$. Then the map $\bar{\otimes}$ is well-defined.

Now let $\omega \in \mathrm{IR}(\Gamma \times \Delta)$. There is r such that for $s \geq r$ $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}} \omega$ is irreducible and then it is isomorphic to a tensor product of irreducible representations of factors. Then both $\mathrm{res}^{\Gamma\times\Delta}_{\Gamma^{(s)}\times\{1\}} \omega$ and $\mathrm{res}^{\Gamma\times\Delta}_{\{1\}\times\Delta^{(s)}} \omega$ are sums of pairwise isomorphic irreducible representations. As a consequence of Proposition 1.3.(2) we have

$$\mathrm{res}^{\Gamma\times\Delta}_{\Gamma\times\{1\}} \omega \simeq \varphi^{\oplus \dim \psi}, \qquad \mathrm{res}^{\Gamma\times\Delta}_{\{1\}\times\Delta} \omega \simeq \psi^{\oplus \dim \varphi}$$

for some $\varphi \in \mathrm{IR}(\Gamma)$, $\psi \in \mathrm{IR}(\Delta)$. For each $s \geq r$ the characters of ω and $\varphi\bar{\otimes}\psi$ are equal on $\Gamma^{(s)} \times \Delta^{(s)}$ and hence they are equal on $\Gamma \times \Delta$. Now Corollary 1.7 implies that ω is isomorphic to $\varphi\bar{\otimes}\psi$. □

Up to this point we have considered linear representations when Dwyer-Zabrosky theorem requires unitary ones. The following proposition states that any complex representation admits a unique unitary structure.

Proposition 1.9. *Let V be a complex linear space with a hermitian scalar product. Then the map $\mathrm{Rep}(\Gamma, U(V)) \to \mathrm{Rep}(\Gamma, \mathrm{GL}(V))$ is a bijection.*

Proof. Consider the following (obviously commutative) diagram (lim stands for an inverse limit)

$$\begin{array}{ccc} \mathrm{Rep}(\Gamma, U(V)) & \longrightarrow & \mathrm{Rep}(\Gamma, \mathrm{GL}(V)) \\ \downarrow & & \downarrow \\ \lim\limits_{s\to\infty} \mathrm{Rep}(\Gamma^{(s)}, U(V)) & \longrightarrow & \lim\limits_{s\to\infty} \mathrm{Rep}(\Gamma^{(s)}, \mathrm{GL}(V)) \end{array} \tag{1.10}$$

where the vertical maps assign to any representation the collection of its restrictions to subgroups $\Gamma^{(s)}$. The lower horizontal arrow is bijective since groups $\Gamma^{(s)}$ are finite. For $L = U(V)$ or $L = \mathrm{GL}(V)$ let $\{\varphi_s\} \in \lim_{s\to\infty} \mathrm{Hom}(\Gamma^{(s)}, L)$. Define homomorphisms $\psi_s \in \mathrm{Hom}(\Gamma^{(s)}, L)$ by induction. For $s > 0$ let $\psi_s(g) = h^{-1}\varphi_s(g)h$ where h is any element such that $\psi_{s-1} = h^{-1}\varphi_s|_{\Gamma^{(s-1)}}h$. Then $\psi = \bigcup_s \psi_s$ is a well-defined homomorphism $\Gamma \to L$ such that $\psi|_{\Gamma^{(s)}}$ is conjugate to φ_s. This implies the surjectivity of both vertical maps. Let $\varphi, \psi \in \mathrm{Hom}(\Gamma, U(V))$ be non-conjugate homomorphisms and put $G_s = \{g \in U(V) : g^{-1}\varphi g = \psi\}$. G_s is a non-increasing sequence of closed subsets of $U(V)$. Since $\bigcap_r G_r = \emptyset$ and $U(V)$ is compact, then there is s such that $G_s = \emptyset$. Then $\varphi|_{\Gamma^{(s)}}$ is not conjugate to $\psi|_{\Gamma^{(s)}}$ and the map $\mathrm{Rep}(\Gamma, U(V)) \to \lim_s \mathrm{Rep}(\Gamma^{(s)}, U(V))$ is injective. Injectivity of the right vertical arrow follows from Proposition 1.7. Now the conclusion follows. □

We conclude with a corollary from Propositions 1.5 and 1.9:

Corollary 1.11. For any unitary representation $\varphi : \Gamma \to U(V)$ we have

$$C_{U(V)}(\varphi(\Gamma)) = U_\Gamma(V) \cong \prod_{\sigma\in \mathrm{IR}(\Gamma)} U(\mathrm{Hom}_\Gamma(W_\sigma, V)).$$

2. Homotopy representations of compact Lie groups

The tool which appears the most appriopriate for calculating the set of homotopy representations of L is a subgroup homotopy decomposition due to Jackowski, McClure and Oliver [7]. Recall that holds the following

Theorem 2.1 ([7, 1.4])**.** *The map*

$$(\mathrm{hocolim}_{L/P\in\mathcal{R}_p(L)}\, EL \times_L L/P)^\wedge_p \simeq BL^\wedge_p$$

induced by projection is a weak homotopy equivalence.

Notice that $EL \times_L L/P$ is homotopy equivalent to BP. The next theorem describes the mapping space $\mathrm{map}(BP, BU(n)^\wedge_p)$ if P is p-toral.

Theorem 2.2 ([8, Thm. 1.1])**.** *Let P be a p-toral group and let H be a compact connected Lie group. Then:*

(1) The maps

$$\mathrm{Rep}(P^\infty, H) \xrightarrow{B} [BP^\infty, BH^\wedge_p] \longleftarrow [BP, BH^\wedge_p]$$

are bijections.

(2) For any $\varphi : P^\infty \to H$ the pairing $BC_H(\varphi(P^\infty)) \times BP^\infty \to BH$ induces a homotopy equivalence

$$BC_H(\varphi(P^\infty))^\wedge_p \longrightarrow \mathrm{map}(BP, BH^\wedge_p)_{B\varphi}.$$

Hence $\lim_{L/P}[EL \times_L (\cdot), BU(n)^\wedge_p] \simeq \lim_{L/P} \mathrm{Rep}(P^\infty, U(n))$. Let $N \subseteq L$ be a p-normalizer of a maximal torus and let $\varphi : N^\infty \to U(V)$ be an $\mathcal{R}_p(L)$-invariant representation. For $i > 0$ define contravariant functors Π_i from $\mathcal{R}_p(L)$ to the category of groups (abelian if $i > 1$)

$$\Pi^\varphi_i(L/P) := \pi_i\, \mathrm{map}((EL/P)^\wedge_p, BU(n)^\wedge_p)_{B\varphi^\wedge_p|_{(EL/P)^\wedge_p}}.$$

By the theorem of Wojtkowiak [13], the obstructions to an existence of an extension of $(B\varphi)^\wedge_p$ to $(BL)^\wedge_p$ lie in groups $H^{i+1}(\mathcal{R}_p(L); \Pi_i)$ for $i \geq 1$. Luckily functors Π_i

can be interpreted in terms of irreducible representations of restrictions $\mathrm{res}^{N^\infty}_{P^\infty}\ \varphi$ for p-stubborn P.

For any $\omega \in \mathrm{Rep}(\Gamma, U(n))$ let $\mathrm{IR}(\Gamma,\omega) \subseteq \mathrm{IR}(\Gamma)$ be a subset of isomorphism classes of irreducible subrepresentations of ω. Furthermore, let $R(\Gamma) := \mathbb{Z}[\mathrm{IR}(\Gamma)]$ and $R(\Gamma,\omega) := \mathbb{Z}[\mathrm{IR}(\Gamma,\omega)]$.

Theorem 2.3. *For each $L/P \in \mathcal{R}_p(L)$ we have $\Pi^\varphi_1(L/P) = \Pi^\varphi_3(L/P) = 0$. Furthermore, there is a functorial isomorphism*

$$\Pi^\varphi_2(L/P) \simeq \mathbb{Z}^\wedge_2 \otimes R(P^\infty, \mathrm{res}^{N^\infty}_{P^\infty}\ \varphi).$$

Proof. Let W be a unitary space with a P^∞-action which represents some element of $\mathrm{Rep}(P^\infty, \mathrm{res}^{N^\infty}_{P^\infty}\ \varphi)$. Choose a monomorphism $j : W \to V^{\oplus s}$ and put

$$J_{P,\varphi} : R(P^\infty, \mathrm{res}^{N^\infty}_{P^\infty}\ \varphi) \ni [\psi] \mapsto j_*(l_W) \in \pi_1(U_{P^\infty}(V^{\oplus s})), \tag{2.4}$$

where $l_W \in \pi_1(U(W))$ is represented by a loop $t \mapsto e^{2\pi i t}\cdot I$. Maps $J_{P,\varphi}$ does not depend on the choice of j since all inclusions $W \to V^{\oplus s}$ determine conjugate homomorphisms $U_P(W) \to U_P(V^{\oplus s})$. It also does not depend on the choice of s since for $s < s'$ the map $U_P(V^{\oplus s}) \to U_P(V^{\oplus s'})$ induces an isomorphism on π_1.

Let $a : L/Q \to L/P$ be a morphism in $\mathcal{R}_p(L)$. Commutativity of a diagram

$$\begin{array}{ccc} R(P^\infty, \mathrm{res}^{N^\infty}_{P^\infty}\ \varphi) & \xrightarrow{J_{P,\varphi}} & \pi_1(U_{P^\infty}(V)) \\ \Big\downarrow{\scriptstyle a^*} & & \Big\downarrow \\ R(Q, \mathrm{res}^{N^\infty}_{P^\infty}\ \varphi) & \xrightarrow{J_{Q,\varphi}} & \pi_1(U_{Q^\infty}(V)) \end{array}$$

implies that J_φ is a natural transformation. Moreover, both groups $R(P^\infty, \mathrm{res}^{N^\infty}_{P^\infty}\ \varphi)$ and $\pi_1(U_P(V))$ are free and abelian, the generator set of them both is $\mathrm{IR}(P^\infty, \mathrm{res}^{N^\infty}_{P^\infty}\ \varphi)$ and the generators are mapped to the corresponding generators. Therefore $J_\varphi : R(-,\varphi) \to \pi_1 \mathrm{Aut}_{(-)}(V)$ is a natural equivalence. Thus the composition

$$\begin{aligned} \mathbb{Z}^\wedge_p \otimes R(-,\varphi) &\xrightarrow{Id\otimes J_\varphi} \mathbb{Z}^\wedge_p \otimes \pi_1(U_{(-)}(V)) \cong \mathbb{Z}^\wedge_p \otimes \pi_1(C_{U(V)}(\varphi(-))) \\ &\xrightarrow{\cong} \mathbb{Z}^\wedge_p \otimes \pi_2(BC_{U(V)}(\varphi(-))) \longrightarrow \pi_2(BC_{U(V)}(\varphi(-)))^\wedge_p \\ &\underset{(2.2)}{\xrightarrow{\cong}} \pi_2(\mathrm{map}(EL/(-), BU(V)^\wedge_p)_{B\varphi|_P})^\wedge_p = \Pi^\varphi_2(L/P). \end{aligned}$$

is a required natural equivalence. Furthermore, by 1.11 $U_{P^\infty}(V) = C_{U(n)}(\varphi(P^\infty))$ is a product of unitary groups and hence $\pi_0(\mathrm{Aut}_{P^\infty}(V)) = \pi_2(\mathrm{Aut}_{P^\infty}(V)) = 0$. □

For any $\mathcal{R}_p(L)$-invariant representation φ of N^∞ denote $\Xi(\varphi) := \mathbb{Z}^\wedge_p \otimes R(-,\varphi)$. The following is an immediate consequence of 2.3:

Theorem 2.5. *Let $\varphi : N^\infty \to U(V)$ be an $\mathcal{R}_p(L)$-invariant representation. If $H^3(\mathcal{R}_p(L); \Xi(\varphi)) = 0$ and $H^i(\mathcal{R}_p(L); \mathbf{M}) = 0$ for $i > 4$ and any contravariant functor $\mathbf{M} : \mathcal{R}_p(L) \to \mathbb{Z}^\wedge_p-\mathbf{Mod}$, then $B\varphi^\wedge_2$ extends to a map $BL \to BU(m)^\wedge_p$.* □

3. Stubborn subgroups of symmetric and orthogonal groups

In this section we recall results of Oliver [9], who gave the classification of 2-stubborn subgroups of orthogonal groups and the description of the categories

$\mathcal{R}_2(O(n))$. Furthermore, we recall the classification of 2-stubborn groups of symmetric groups (given by Alperin and Fong [1]) and describe the categories $\mathcal{R}_2(\Sigma_n)$. The section is concluded with a constuction of a full inclusion $\mathcal{R}_2(\Sigma_n) \to \mathcal{R}_2(O(n))$ which makes easier calculations of higher limits on $\mathcal{R}_2(O(n))$.

2-Stubborn subgroups of orthogonal groups. Each 2-stubborn subgroup of an orthogonal group is built up out of some number of "pieces" by taking products and wreath products. (Recall that the wreath product $G \wr H$ is the semi-direct product $G^H \ltimes H$, where H acts on G^H by shifting coordinates). Let $I_n \in \mathrm{GL}_n(\mathbb{R})$ denote the identity matrix. If $M \in \mathrm{GL}_m(\mathbb{R})$ and $N \in \mathrm{GL}_n(\mathbb{R})$, then $M \otimes N \in \mathrm{GL}_{nm}(\mathbb{R}$ is the matrix with entries

$$(M \otimes N)_{am+b-m,cm+d-m} = M_{b,d}N_{a,c}.$$

for $a, c = 1, \ldots, n$, $b, d = 1, \ldots, m$. Let

$$A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

and for $i < n$ define the matrices $A_i^n, B_i^n \in O(2^n)$ by

$$A_i^n := I_{2^i} \otimes A \otimes I_{2^{n-i-1}}, \qquad B_i^n := I_{2^i} \otimes B \otimes I_{2^{n-i-1}}. \tag{3.2}$$

Let

$$\Gamma_{2^n} := \langle -I_{2^n}, A_0^n, \ldots, A_{n-1}^n, B_0^n, \ldots, B_{n-1}^n \rangle \subseteq O(2^n) \tag{3.3}$$

$$\bar{\Gamma}_{2^n} := \langle \{X \otimes I_{2^{n-1}}\}_{X \in SO(2)}, A_0^n, \ldots, A_{n-1}^n, B_0^n, \ldots, B_{n-1}^n \rangle \subseteq O(2^n) \tag{3.4}$$

The group Γ_{2^n} is an extra-special group of order 2^{2n+1} and $\bar{\Gamma}_{2^n}$ is a 2-toral group with 1-dimensional torus. Note that $\Gamma_1 \cong \{\pm 1\} = O(1)$ and $\bar{\Gamma}_2 \cong O(2)$.

Definition 3.5. ([9, Def. 2]) Let $\mathcal{T}_\Gamma(k) \subseteq \mathcal{T}_{irr}(k) \subseteq \mathcal{T}_{prod}(k)$ be the sets of 2-toral subgroups of $O(k)$ defined as follows:

- $\mathcal{T}_\Gamma(1) = \{\Gamma_1 = O(1)\}$,
- $\mathcal{T}_\Gamma(2) = \{\bar{\Gamma}_2 = O(2)\}$,
- $\mathcal{T}_\Gamma(2^n) = \{\Gamma_{2^n}, \bar{\Gamma}_{2^n}\}$ for $n > 1$,
- $\mathcal{T}_\Gamma(k) = \emptyset$ for $k \neq 2^n$,
- $\mathcal{T}_{irr}(2^n)$ is the set of those wreath products in $O(2^n)$ of the form
$$\Gamma \wr C_2^{t_1} \wr \ldots \wr C_2^{t_s},$$
where $\Gamma \in \mathcal{T}_\Gamma(2^l)$, $n = l + t_1 + \ldots + t_s$ and $t_1 > 1$ if $\Gamma = \Gamma_1$,
- $\mathcal{T}_{prod}(k)$ is the set of all products in $O(k)$ of the form
$$P = P_1 \times \ldots \times P_s \subseteq O(k_1) \times O(k_2) \times O(k_s) \subseteq O(k),$$
where $P_i \in \mathcal{T}_{irr}(k_i)$ and $k = k_1 + \ldots + k_s$.

Remark. If $\Gamma \subseteq O(2^l)$, then the embedding $\Gamma \wr C_2^t \subseteq O(2^{l+t})$ is chosen as follows:

$$\Gamma \wr C_2^t \cong \langle \Gamma^{2^t}, B_l^{l+t}, B_{l+1}^{l+t}, \ldots, B_{l+t-1}^{l+t} \rangle \subseteq O(2^{l+t}).$$

By [9, Theorem 8] each 2-stubborn subgroup of $O(k)$ is conjugate to a subgroup in $\mathcal{T}_{prod}(O(k))$. Possibly not all elements $\Gamma \in \mathcal{T}_{prod}(O(k))$ are 2-stubborn — it depends on the Weyl group of Γ. The following proposition follows immediately from [9, Theorem 6]:

Proposition 3.6. *Let $W_L(P) := N_L(P)/P$. Then*

- $W_{O(2^n)}(\Gamma_{2^n}) \simeq O^+(X_n)$,
- $W_{O(2^n)}(\bar{\Gamma}_{2^n}) \simeq Sp(V_n)$,
- $W_{O(2^n)}(\Gamma \wr C_2^{t_1} \wr \ldots \wr C_2^{t_s}) \simeq W_{O(2^m)}(\Gamma) \times \mathrm{GL}_{t_1}(\mathbb{F}_2) \times \ldots \times \mathrm{GL}_{t_s}(\mathbb{F}_2)$, $\Gamma \in \mathcal{T}(2, O(2^m))$.

The following theorem determines which elements of $\mathcal{T}_{prod}(k)$ are in fact 2-stubborn subgroups of $O(k)$:

Theorem 3.7. ([9, Theorem 6]) *Let $P \in O(k)$ be a 2-stubborn subgroup. Then P is conjugate to an element of $\mathcal{T}_{prod}(k)$. If $P \in \mathcal{T}_{prod}(k)$ then P is 2-stubborn if and only if when written as a product*

$$P = P_1 \times \ldots \times P_s, \qquad (P_i \in \mathcal{T}_{irr}(k_i))$$

there is no factor P_i with $W_{O(k_i)}(P_i) = 1$ which occurs with multiplicity exactly 2 or 4.

Theorem 3.8. ([9, Proposition 9]) *Let $P, P' \in \mathcal{T}_{prod}(k)$. If P' is conjugate to a subgroup of P, then $x^{-1}P'x \subseteq P$ for some permutation matrix x which permutates irreducible factors of P'. If $P' \subseteq P$ then the inclusion is a composite of products of the following types:*

(a) $O(1) \times O(1) \subseteq O(2)$,
(b) $O(1) \wr C_2^{2t_1} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s} \subseteq O(2) \wr C_2^{t_1} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s}$,
(c) $(\Gamma \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s})^{t_{s+1}} \subseteq \Gamma \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s} \wr C_2^{t_{s+1}}$,
(d) $\Gamma_{2^k} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s} \subseteq \bar{\Gamma}_{2^k} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s}$,
(e) $\Gamma \wr \cdots \wr C_2^{t_i} \wr C_2^{t_{i+1}} \wr \cdots \wr C_2^{t_s} \subseteq \Gamma \wr \cdots \wr C_2^{t_i+t_{i+1}} \wr \cdots \wr C_2^{t_s}$,
(f) $\Gamma_{2^{k+t_1}} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s} \subseteq \Gamma_{2^k} \wr C_2^{t_1} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s}$,
(g) $\bar{\Gamma}_{2^{k+t_1}} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s} \subseteq \bar{\Gamma}_{2^k} \wr C_2^{t_1} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s}$,

where Γ stands for either Γ_{2^k} or $\bar{\Gamma}_{2^k}$.

Corollary 3.9. Each morphism in $\mathcal{R}_2(O(n))$ is a composition of automorphisms and inclusions enlisted in 3.8.

2-Stubborn subgroups of symmetric groups. The following classification of 2-stubborn subgroups of symmetric groups is due to Alperin and Fong [1] (a subgroup of a finite group is 2-stubborn iff it is 2-radical).

Note that if $G \subseteq \Sigma_m$, $H \subseteq \Sigma_n$, then the product $G \times H$ is a subgroup of Σ_{m+n} and the wreath product $G \wr H$ is a subgroup of Σ_{mn}.

Definition 3.10. For any sequence $t_1, \ldots, t_s$ of positive integers let

$$B(t_1, \ldots, t_s) := 1 \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s} \subseteq \Sigma_{2^t},$$

where $t = t_1 + \cdots + t_s$ (we treat 1 as a subgroup of Σ_1). The groups $B(t_1, \ldots, t_s)$ will be called *basic* subgroups of Σ_{2^t}. The set of all basic subgroups of Σ_{2^t} will be denoted by $\mathcal{B}_{irr}(2^t)$.

Definition 3.11. Let $\mathcal{B}_{prod}(n)$ denotes the family of all products of basic subgroups in Σ_n, i.e.

$$\mathcal{B}_{prod}(n) = \{P_1 \times \cdots \times P_r : P_i \in \mathcal{B}_{irr}(2^{t_i}), n = 2^{t_1} + \cdots + 2^{t_r}\}.$$

Here follow two propositions which are consequences of [1, (2B)]:

Proposition 3.12. *If $t = t_1 + \cdots + t_s$, then*

$$W_{\Sigma_{2^t}}(B(t_1, \ldots, t_s)) \simeq \mathrm{GL}_{t_1}(\mathbb{F}_2) \times \cdots \times \mathrm{GL}_{t_s}(\mathbb{F}_2).$$

Proposition 3.13. *Let $P_i \subseteq \Sigma_{2^{t_i}}$, for $i = 1, \dots, r$, be a collection of pairwise non-isomorphic basic subgroups, and let $k = \sum_{i=1}^{r} 2^{t_i} l_i$. Then*

$$W_{\Sigma_k}\left(\prod_{i=1}^{r} P_i^{l_i}\right) = \prod_{i=1}^{r} W_{\Sigma_{2^{t_i}}}(P_i) \wr \Sigma_{l_i}.$$

Theorem 3.14. *Each 2-stubborn subgroup $G \subseteq \Sigma_k$ is, up to conjugacy, a product of basic subgroups (i.e. $G \in \mathcal{B}_{prod}(k)$). A group $P \in \mathcal{B}_{prod}(k)$ is stubborn if and only if written as a product of basic subgroups $P = P_1 \times \cdots \times P_r$ there is no factor isomorphic to $B(1, \dots, 1)$ which occurs with multiplicity exactly 2 or 4.*

Proof. The first statement is a consequence of [1, (2A)]. Since $\mathrm{GL}_n(\mathbb{F}_2)$ has no non-trivial normal 2-subgroups and Σ_n has a non-trivial normal 2-subgroup if and only if $n = 2$ or $n = 4$, then the second statement follows immediately from Propositions 3.12 and 3.13. □

Proposition 3.15. *If groups $P, Q \in \mathcal{B}_{prod}(k)$ are conjugate, then there exists a conjugacy between them which permutes its basic factors.*

Proof. Since each basic subgroup of a symmetric group acts transitively on the set of letters, basic factors of P (and, similarly, Q) are in bijection with the set of P-orbits (Q-orbits). The conjugacy between P and Q permutes the orbits and therefore it also permutes its basic factors. □

Proposition 3.16. *Fix collections of subgroups $P_i \subseteq \Sigma_{k_i}$ for $i = 1, \dots, r$ and $H_j \subseteq \Sigma_{l_j}$ for $j = 1, \dots, s$. Assume that for each i the group G_i acts transitively on the set of letters, and that $n := k_1 + \cdots + k_r = l_1 + \cdots + l_s$. If*

$$Q := Q_1 \times \cdots \times Q_s \subseteq P := P_1 \times \cdots \times P_r \subseteq \Sigma_n,$$

then $Q = (Q \cap P_1) \times \cdots \times (Q \cap P_r)$.

Proof. For each $i = 1, \dots, r$ let O_i^P be an orbit of $P_i \subseteq P$. Note that $\{1, \dots, k\} = \bigcup_{i=1}^{r} O_i^P$ is a decomposition onto G-orbits. Similarly define Q-sets O_j^Q, for $j = 1, \dots, s$. Since $Q \subseteq P$, then for each j there exists i such that $O_j^Q \subseteq O_i^P$. Hence $Q_j \subseteq P \cap \Sigma_{O_i^P} = P_i$ and the conclusion follows. □

Proposition 3.17. *For any subgroup $P \subseteq \Sigma_k$ let $\delta(P) \subseteq P$ denotes the subgroup generated by all elements $g \in P$ which have a fixed point. The following holds:*

- *if $P \in \mathcal{B}_{prod}(k)$ is a non-trivial product of basic subgroups, then $\delta(P) = P$.*
- *$\delta(B(t_1, \dots, t_r)) = B(t_1, \dots, t_{r-1})^{2^{t_r}}$,*

Proof. The product $Q \times Q'$ is generated by $Q \times \{1\} \cup \{1\} \times Q'$. It implies that the first statement holds, and that $B(t_1, \dots, t_{r-1})^{2^{t_r}} \subseteq \delta(B(t_1, \dots, t_r))$. On the other hand, for $C_2^{t_r} \subseteq \Sigma_{2^{t_r}}$ we have $\delta(C_2^{t_r}) = \{1\}$. Then each element $g \in B(t_1, \dots, t_r) \setminus B(t_1, \dots, t_{r-1})^{2^{t_r}}$ acts freely on the set $\{1, \dots, 2^t\}$ (where $t = t_1 + \cdots + t_r$). As a consequence we obtain the second statement. □

Theorem 3.18. *Let $P, Q \in \mathcal{B}_{prod}(k)$. Assume that $Q \subseteq P$. Then the inclusion $Q \subseteq P$ is a composite of products of inclusions of the following types:*

(a) $B(t_1, \dots, t_{r-1})^{2^{t_r}} \subseteq B(t_1, \dots, t_{r-1}, t_r)$,
(b) $B(t_1, \dots, t_j + t_{j+1}, \dots, t_r) \subseteq B(t_1, \dots, t_j, t_{j+1}, \dots, t_r)$.

Proof. If Q is reducible, then by Proposition 3.16 the inclusion is the product of inclusions $(Q \cap P_i) \subseteq P_i$, where P_i's are irreducible, so assume that $P = B(t_1, \dots, t_r)$. If Q is reducible, then by Proposition 3.17 we obtain

$$Q = \delta(Q) \subseteq \delta(P) = B(t_1, \dots, t_{r-1})^{2^{t_r}} \subseteq B(t_1, \dots, t_{r-1}, t_r) = P.$$

In this case the inclusion is the composition of an inclusion of type (a) with $Q \subseteq \delta(P)$. Finally, if Q is irreducible then $Q = B(t'_1, \dots, t'_s)$. We have

$$\delta(Q) = B(t'_1, \dots, t'_{s-1})^{2^{t'_s}} \subseteq B(t_1, \dots, t_{r-1})^{2^{t_r}} = \delta(P).$$

Since Q-orbits are contained in P-orbits, then $t'_s \geq t_r$. If $t'_s = t_r$, then we are reduced to the case of smaller inclusion of irreducible subgroups. If $t'_s > t_r$, then the inclusion

$$B(t'_1, \dots, t'_{s-1})^{2^{t'_s - t_r}} \subseteq B(t_1, \dots, t_{r-1})$$

factors through $B(t_1, \dots, t_{r-2})^{2^{t_{r-1}}}$. Hence $t'_s - t_r \geq t_{r-1}$ and then $t'_s \geq t_{r-1} + t_r$. Finally, we obtain the factorization

$$Q \subseteq B(t_1, \dots, t_{r-2}, t_{r-1} + t_r) \subseteq B(t_1, \dots, t_{r-2}, t_{r-1}, t_r) = G. \qquad \square$$

As a consequence we obtain

Corollary 3.19. Each morphism in $\mathcal{R}_2(O(n))$ is a composition of automorphisms and inclusions enlisted in 3.18.

A full inclusion $\mathcal{R}_2(\Sigma_n) \to \mathcal{R}_2(O(n))$.

Definition 3.20. For any $P \in \mathcal{B}_{prod}(n)$ let $\bar{P} \in \mathcal{T}_{prod}(n)$ be given by

$$\bar{P} = \{\pm 1\} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_r} \quad \text{for } P = B(t_1, \dots, t_r),\, t_1 > 1$$
$$\bar{P} = O(2) \wr C_2^{t_2} \wr \cdots \wr C_2^{t_r} \quad \text{for } P = B(1, t_2, \dots, t_r)$$
$$\bar{P} = \bar{P}_1 \times \cdots \times \bar{P}_r \quad \text{for } P_i \in \mathcal{B}_{irr}(k_i)$$

Remark. For each $P \in \mathcal{B}_{prod}(n)$ holds

$$\bar{P} \cap \{\pm 1\} \wr \Sigma_k = \{\pm 1\} \wr P. \tag{3.21}$$

Theorem 3.22. *The formulae*

$$\mathcal{R}_2(\Sigma_n) \ni \Sigma_n / P \mapsto O(n)/\bar{P} \in \mathcal{R}_2(O(n))$$
$$\mathrm{Mor}_{\mathcal{R}_2(\Sigma_n)}(Q, P) \ni gP \mapsto g\bar{P} \in \mathrm{Mor}_{\mathcal{R}_2(O(k))}(\bar{Q}, \bar{P})$$

define the functor $I : \mathcal{R}_2(\Sigma_n) \to \mathcal{R}_2(O(n))$ which is an inclusion onto the full subcategory.

Proof. The functor I is well-defined.
It is sufficient to check that for each generating morphism $gP : Q \to P$ holds $g^{-1}\bar{Q}g \subseteq \bar{P}$. It is clear for automorphisms (cf. Propositions 3.12 and 3.13), so assume that $g = 1$ and $Q \to P$ is a product of the inclusions enlisted in Theorem 3.18. If the inclusion $Q \subseteq P$ is a non-trivial product of inclusions $Q_1 \subseteq P_1$ and $Q_2 \subseteq P_2$, then $\bar{Q} \subseteq \bar{P}$ if and only if $\bar{Q}_1 \subseteq \bar{P}_1$ and $\bar{Q}_2 \subseteq \bar{P}_2$. Hence we are reduced to the case when the inclusion is of type (a) or type (b) (cf. 3.18). If

$$Q = B(t_1, \dots, t_{r-1})^{2^{t_r}} \subseteq B(t_1, \dots, t_{r-1}, t_r) = P,$$

then for $t_1 > 1$ we obtain

$$\bar{Q} = (\{\pm 1\} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_{r-1}})^{2^{t_r}} \subseteq \{\pm 1\} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_r} = \bar{P}$$

and for $t_1 = 1$

$$\bar{Q} = (O(2) \wr C_2^{t_2} \wr \cdots \wr C_2^{t_{r-1}})^{2^{t_r}} \subseteq O(2) \wr C_2^{t_2} \wr \cdots \wr C_2^{t_r} = \bar{P}.$$

If

$$Q = B(t_1, \ldots, t_j + t_{j+1}, \ldots, t_r) \subseteq B(t_1, \ldots, t_j, t_{j+1}, \ldots, t_r) = P,$$

then for $j > 1$ we obtain the inclusion

$$\bar{Q} = K \wr C_2^{t_2} \wr \cdots \wr C_2^{t_j + t_{j+1}} \wr \cdots \wr C_2^{t_r} \subseteq K \wr C_2^{t_2} \wr \cdots \wr C_2^{t_j} \wr C_2^{t_{j+1}} \wr \cdots \wr C_2^{t_r} = \bar{P},$$

where $K = O(2)$ if $t_1 = 1$ and $K = \{\pm 1\} \wr C_2^{t_1}$ otherwise. Similarly if $j = 1$ and $t_1 > 1$, then the inclusion $\bar{Q} \subseteq \bar{P}$ is straightforward. The only non-trivial case appears when $j = t_1 = 1$. Then

$$\bar{Q} = \{\pm 1\} \wr C_2^{1+t_2} \wr \cdots \wr C_2^{t_r} \subseteq O(2) \wr C_2^{t_2} \wr \cdots \wr C_2^{t_r} = \bar{P},$$

since $\{\pm 1\} \wr C_2^{1+t_2} \subseteq \{\pm 1\} \wr C_2 \wr C_2^{t_2} \subseteq O(2) \wr C_2^{t_2}$. Hence I is well-defined.

The functor I is faithful.
By combining Propositions 3.6, 3.12 and 3.13 we see that for each subgroup $P \in \mathcal{B}_{prod}(k)$ the homomorphism $I : \mathrm{Aut}_{\mathcal{R}_2(\Sigma_k)}(P) \to \mathrm{Aut}_{\mathcal{R}_2(O(k))}(\bar{P})$ is actually an isomorphism. Now fix $Q \neq P \in \mathcal{B}_{prod}(k)$ and choose morphisms $g_1 P, g_2 P : Q \to P$ in the category of Σ_k-orbits. Let us consider the compositions

$$Q \xrightarrow{g_i P} g_i^{-1} Q g_i \xrightarrow{1 \cdot P} P$$

for $i = 1, 2$. By Proposition 3.15 $g_1^{-1} Q g_1$ and $g_2^{-1} Q g_2$ differ by conjugation by an element h which permutes irreducible factors. Hence the conjugation by $i(h)$, where $i : \Sigma_k \to O(k)$ is an obvious inclusion, sends the group $g_1^{-1} \bar{Q} g_1$ onto $g_2^{-1} \bar{Q} g_2$ and also permutes irreducible factors. By (3.21) $h \in P$ if and only if $i(h) \in \bar{P}$. It shows that $g_1 P$ and $g_2 P$ represent the same morphism $Q \to P$ in $\mathcal{R}_2(\Sigma_k)$ if and only if they represent the same morphism in $\mathcal{R}_2(\Sigma_k)$. As a consequence we get that I is an isomorphism on sets of morphisms. □

4. 2-Stubborn subgroups of G, $\tilde{G}$ and their representations

2-Stubborn subgroups of $O(7)$. By [9, Prop. 11], [9, Th. 12] and [7, Prop. 1.6.(i)] the functors

$$\begin{aligned} &\mathcal{R}_2(O(7)) \ni O(7)/P \mapsto G/(P \cap G) \in \mathcal{R}_2(G) \\ &\mathcal{R}_2(G) \ni G/P \mapsto \tilde{G}/\tilde{P} \in \mathcal{R}_2(\tilde{G}) \end{aligned} \tag{4.1}$$

are natural equivalences. We have

$$\mathcal{T}_{irr}(1) = \{\{\pm 1\}\}, \qquad \mathcal{T}_{irr}(2) = \{O(2)\},$$

$$\mathcal{T}_{irr}(4) = \{O(2) \wr C_2,\ \{\pm 1\} \wr C_2^2,\ \bar{\Gamma}_4,\ \Gamma_4\}.$$

Therefore

$$\begin{aligned} \mathcal{T}_{prod}(7) = \{H \times O(2) \times \{\pm 1\},\ H \times \{\pm 1\}^3\}_{H \in \mathcal{T}_{irr}(4)} \\ \cup \{O(2)^i \times \{\pm 1\}^{7-2i}\}_{i=0,\ldots,3}. \end{aligned} \tag{4.2}$$

All groups in $\mathcal{T}_{prod}(7)$ but $O(2)^2 \times \{\pm 1\}^3$ are 2-stubborn in $O(7)$ (by 3.7). Introduce the following notation for 2-stubborn subgroups of G and $\tilde{G}$. Let $N :=$

$O(2) \wr C_2$, $K := \{\pm 1\} \wr C_2^2$, $J := \bar{\Gamma}_4$ and $M := \Gamma_4$, and for each $H \in \{J, K, M, N\}$ let

$$H_1 := (H \times O(2) \times \{\pm 1\}) \cap G \subseteq (O(4) \times O(2) \times O(1)) \cap G \subset G \tag{4.3}$$

$$H_0 := (H \times \{\pm 1\}^3) \cap G \subseteq (O(4) \times O(3)) \cap G \subset G, \tag{4.4}$$

and for $i = 0, 1, 3$ let

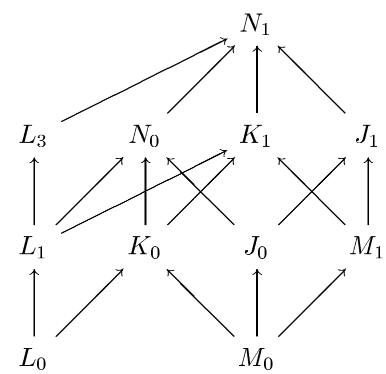
$$L_i := (O(2)^i \times \{\pm 1\}^{7-2i}) \cap G \subseteq G. \tag{4.5}$$

Then $\mathrm{Ob}(\mathcal{R}_2(G)) = \{G/P\}$ and $\mathrm{Ob}(\mathcal{R}_2(\tilde{G})) = \{\tilde{G}/\tilde{P}\}$, where P is conjugate to one of J_i K_i, M_i, N_i $(i = 0, 1)$ or L_i $(i = 0, 1, 3)$.

Here follows the list of Weyl groups of 2-stubborn subgroups of G:

$$\begin{gathered} W_G(N_1) = 1, \qquad W_G(N_0) \cong W_G(L_3) \cong W_G(K_1) \cong W_G(J_1) \cong \Sigma_3 \\ W_G(L_1) \cong \Sigma_5, \qquad W_G(L_0) \cong \Sigma_7, \qquad W_G(K_0) \cong W_G(J_0) \cong \Sigma_3 \times \Sigma_3 \\ W_G(M_1) \cong \Sigma_3 \wr \Sigma_2, \qquad W_G(M_0) \cong \Sigma_3 \wr \Sigma_2 \times \Sigma_3 \end{gathered} \tag{4.6}$$

The set of morphisms of $\mathcal{R}_2(O(7)) \cong \mathcal{R}_2(G) \cong \mathcal{R}_2(\tilde{G})$ is generated by automorphisms and by inclusions presented on the following diagram:



Representations. The remaining part of this section contains a partial classification of complex representations of discrete approximations of 2-stubborn subgroups of G and $\tilde{G}$. At the beginning let us introduce some notation. Let θ denote a trivial irreducible representation of any group and let ι be a non-trivial irreducible representation of an order 2 group. Every irreducible representation of a 2-discrete torus $\mathbb{Z}/2^\infty \cong SO(2)^\infty$ is 1-dimensional and has the form

$$\varrho_k : \mathbb{Z}/2^\infty \ni \frac{n}{2^t} \mapsto \exp\left(2\pi i \frac{kn}{2^t}\right) \in U(1), \qquad k \in \mathbb{Z}_2^\wedge \tag{4.7}$$

Finally, let

$$\alpha_k := \mathrm{ind}_{SO(2)^\infty}^{O(2)^\infty} \varrho_k \qquad \text{for } k \in 1 + \mathbb{Z}_2^\wedge \tag{4.8}$$

$$\beta_k := \mathrm{ind}_{SO(2)^\infty}^{O(2)^\infty} \varrho_k \qquad \text{for } k \in 2\mathbb{Z}_2^\wedge \setminus \{0\}. \tag{4.9}$$

We have $\mathrm{IR}(O(2)^\infty) = \{\alpha_k\}_{k \in 1+\mathbb{Z}_2^\wedge} \cup \{\beta_k\}_{k \in 2\mathbb{Z}_2^\wedge \setminus \{0\}} \cup \{\theta, \tau := \det\}$.

Odd representations of subgroups of $\tilde{G}$. Let Γ be any locally finite subgroup of G and let $\tilde{\Gamma} := \pi^{-1}(\Gamma)$. An irreducible representation φ of $\tilde{\Gamma}$ is

- *even* if it is a restriction of some representation of Γ (and $\chi_\varphi(u) = \dim\varphi$)
- *odd* if it is not (in this case $\chi_\varphi(u) = -\dim\varphi$)

The following lemma will be applied later to groups $\tilde{M}_0$ and $\tilde{L}_0$:

Proposition 4.10. *Let $\Delta \in O(4)$ be a locally finite group and let*

$$\tilde{\Gamma} := \pi^{-1}((\Delta \times \{\pm 1\}^3) \cap G) \subseteq \tilde{G}.$$

Then any odd representation of $\tilde{\Gamma}$ is invariant under conjugation by an element $l \in \tilde{G}$ such that $\pi(l) = -(I_5 \oplus B)$ (cf. 3.1).

Proof. Let φ be an odd representation of $\tilde{\Gamma}$ and let $g \in \tilde{\Gamma}$. We have $\pi(g) = (h,t)$, where $h \in O(4)$, $t \in \{\pm 1\}^3$. If $t = (1,1,1)$ or $t = (-1,-1,-1)$, then g and l commute and hence $\chi_{l^*\varphi}(g) = \chi_\varphi(l^{-1}gl) = \chi_\varphi(g)$. Otherwise $l^{-1}gl$ is conjugate to $l^{-1}glu$ and g is conjugate to gu. Since φ is odd, we have $\chi_\varphi(g) = \chi_\varphi(gu) = -\chi_\varphi(g)$. Therefore $\chi_\varphi(g) = 0$ (and similarly $\chi_\varphi(l^{-1}gl) = 0$). Then representations φ and $l^*\varphi$ have equal characters and they are isomorphic. □

Representations of L_0 and L_1^∞. Let r be an odd positive integer. For each sequence $(\mu_1, \ldots, \mu_r) \in \mathrm{IR}(\{\pm 1\})^r$ define

$$\tau_{(\mu_1,\ldots,\mu_r)} := \mathrm{res}^{\{\pm 1\}^r}_{\{\pm 1\}^r \cap SO(r)}\ \mu_1 \bar{\otimes} \ldots \bar{\otimes} \mu_r. \tag{4.11}$$

Let $\eta_i^r := \bigoplus \tau_{(\mu_1,\ldots,\mu_r)}$, where the sum is taken over all sequences in which ι appears exactly i times. Obviously $\eta_i^r \sim \eta_{r-i}^r$. Then each Σ_r-invariant representation of $\{\pm 1\}^r \cap SO(r)$ (where Σ_r acts by permutations) is isomorphic to a direct sum of η_i^r's for $0 \le i < \frac{r}{2}$. In particular, holds

Corollary 4.12. Any $W_G(L_0) \simeq \Sigma_7$-invariant representation of L_0 is isomorphic to a direct sum of η_i^7's for $i = 0,1,2,3$.

An isomorphism

$$O(2)^\infty \times (\{\pm 1\} \cap SO(5)) \ni (g,h) \mapsto (g, h \cdot \det g) \in L_1^\infty$$

provides a bijection $\mathrm{IR}(L_1^\infty) \simeq \mathrm{IR}(O(2)^\infty) \times \mathrm{IR}(\{\pm 1\}^5 \cap SO(5))$; moreover, the action of $\Sigma_5 \simeq W_G(L_1)$ on $\mathrm{IR}(L_1^\infty)$ coincides with an action on $\mathrm{IR}(\{\pm 1\}^5 \cap SO(5))$by permutations. Thus every Σ_5-invariant representation of L_1^∞ is isomorphic to a product of representations of the form $\lambda \bar{\otimes} \eta_i^5$, where $\lambda \in \mathrm{IR}(O(2)^\infty)$, $i = 0,1,2$.

Next we calculate restrictions of representations of L_1^∞ to L_0. Let $\eta^7_{i(j)} := \bigoplus \tau_{(\mu_1,\ldots,\mu_7)}$, where the sum is taken over all sequences such that exactly j of the representations μ_1, μ_2 are isomorphic to ι and exactly i of the representations $\mu_1, \ldots, \mu_7$ are isomorphic to ι. Elementary calculations of characters provide the following

Corollary 4.13. Σ_5-invariant representations of L_1^∞ restrict to L_0 as follows:

$$\begin{array}{lll}
\alpha_k \bar{\otimes} \eta_0^5 \mapsto \eta^7_{1(1)} & \theta \bar{\otimes} \eta_0^5 \mapsto \eta_0^7 & \tau \bar{\otimes} \eta_0^5 \mapsto \eta^7_{2(0)} \\
\alpha_k \bar{\otimes} \eta_1^5 \mapsto \eta^7_{2(1)} & \theta \bar{\otimes} \eta_1^5 \mapsto \eta^7_{3(2)} & \tau \bar{\otimes} \eta_1^5 \mapsto \eta^7_{1(0)} \\
\alpha_k \bar{\otimes} \eta_2^5 \mapsto \eta^7_{3(1)} & \theta \bar{\otimes} \eta_2^5 \mapsto \eta^7_{2(0)} & \tau \bar{\otimes} \eta_2^5 \mapsto \eta^7_{3(0)}
\end{array}$$

Moreover, $\mathrm{res}^{L_1^\infty}_{L_0}\ \beta_{2k} \bar{\otimes} \eta_i^5 \simeq \mathrm{res}^{L_1^\infty}_{L_0}\ (\theta \oplus \tau) \bar{\otimes} \eta_i^5$. □

Representations of M_0 and M_1^∞. Isomorphisms

$$M \times O(2)^\infty \ni (g,h) \mapsto (g,h,\det h) \in M_1^\infty \subset G \cap (O(4) \times O(2) \times O(1))$$
$$M \times \{\pm 1\}^2 \ni (g,h_1,h_2) \mapsto (g,h_1,h_2,h_1h_2) \in M_0 \subset G \cap (O(4) \times O(1)^3)$$

provide an identifications $\mathrm{IR}(M_1^\infty) \simeq \mathrm{IR}(M \times O(2)^\infty) \simeq \mathrm{IR}(M) \times \mathrm{IR}(O(2)^\infty)$ and $\mathrm{IR}(M_0) \simeq \mathrm{IR}(M \times \{\pm1\}^2) \simeq \mathrm{IR}(M) \times \mathrm{IR}(\{\pm 1\}^2)$. Furthermore, the restriction from $M_1^\infty \simeq M \times O(2)^\infty$ to $M_0 \simeq M \times \{\pm 1\}^2$ is product-wise.

Representations of K_0. All irreducible representations of $K \simeq \{\pm 1\} \wr C_2^2$ are subrepresentations of $\mathrm{ind}_{\{\pm1\}^4}^{K} \left(\bar{\bigotimes}_{a \in C_2^2} \mu_a\right)$ for $\mu_a \in \mathrm{IR}(\{\pm 1\}) = \{\theta, \iota\}$. By Mackey's criterion representations

$$\gamma_1 := \mathrm{ind}_{\{\pm1\}^4}^{K} \theta \bar{\otimes} \theta \bar{\otimes} \theta \bar{\otimes} \iota$$
$$\gamma_3 := \mathrm{ind}_{\{\pm1\}^4}^{K} \theta \bar{\otimes} \iota \bar{\otimes} \iota \bar{\otimes} \iota$$

are irreducible. Furthermore, For $\mu, \nu \in \mathrm{IR}(C_2) = \{\theta, \iota\}$ define

$$\gamma_0^{\mu,\nu} := \mathrm{res}_K^{C_2^2} (\mu \bar{\otimes} \nu), \qquad \gamma_4^{\mu,\nu} := \mathrm{res}_K^{C_2^2} (\mu \bar{\otimes} \nu) \otimes \det .$$

There are decompositions

$$\mathrm{ind}_{\{\pm1\}^4}^{\{\pm1\} \wr C_2^2} \theta \bar{\otimes} \theta \bar{\otimes} \theta \bar{\otimes} \theta \simeq \bigoplus_{\mu,\nu \in \mathrm{IR}(C_2)} \gamma_0^{\mu,\nu}$$
$$\mathrm{ind}_{\{\pm1\}^4}^{\{\pm1\} \wr C_2^2} \iota \bar{\otimes} \iota \bar{\otimes} \iota \bar{\otimes} \iota \simeq \bigoplus_{\mu,\nu \in \mathrm{IR}(C_2)} \gamma_4^{\mu,\nu}.$$

For any $a \in C_2^2 \setminus \{(0,0)\}$ let $\zeta_a := \theta \bar{\otimes} \theta \bar{\otimes} \iota \bar{\otimes} \iota \in \mathrm{IR}(\{\pm1\}^4)$ (where a is the difference between coordinates with the same isomorphism class of representation). Following [11, Section 8.2] we see that $\mathrm{ind}_{\{\pm1\}^4}^{\{\pm1\}^4 \ltimes \langle a \rangle} \zeta_a$ splits onto the sum of non-isomorphic one-dimensional representations ζ_a^+ and ζ_a^-. Moreover, for $\epsilon \in \{+,-\}$ representations $\gamma_{2,a}^\epsilon := \mathrm{ind}_{\{\pm1\}^4 \ltimes \langle a \rangle}^{K} \zeta_a^\epsilon$ are irreducible. Obviously $\mathrm{ind}_{\{\pm1\}^4}^{K} \zeta_a \simeq \gamma_{2,a}^+ \oplus \gamma_{2,a}^-$. As a consequence we obtain

Corollary 4.14.

$$\mathrm{IR}(K) = \{\gamma_1, \gamma_3\} \cup \{\gamma_0^{\mu,\nu}, \gamma_4^{\mu,\nu}\}_{\mu,\nu \in \mathrm{IR}(C_2)} \cup \{\gamma_{2,a}^\epsilon\}_{a \in C_2^2 \setminus \{(0,0)\}}^{\epsilon \in \{+,-\}}.$$

Since the action of $W_{O(4)}(K) \cong \mathrm{GL}_2(\mathbb{F}_2) \simeq \Sigma_3$ on K is natural, we have

Corollary 4.15. Here follow the orbits of an action of $W_{O(4)}(K)$ on $\mathrm{IR}(K)$:

$$\{\gamma_0^{\theta,\theta}\}, \quad \{\gamma_0^{\theta,\iota}, \gamma_0^{\iota,\theta}, \gamma_0^{\iota,\iota}\}, \quad \{\gamma_4^{\theta,\theta}\}, \quad \{\gamma_4^{\theta,\iota}, \gamma_4^{\iota,\theta}, \gamma_4^{\iota,\iota}\}, \quad \{\gamma_1\}, \quad \{\gamma_3\}$$
$$\{\gamma_{2,(0,1)}^+, \gamma_{2,(1,0)}^+, \gamma_{2,(1,1)}^+\}, \quad \{\gamma_{2,(0,1)}^-, \gamma_{2,(1,0)}^-, \gamma_{2,(1,1)}^-\}$$

Finally we show how $W_G(K_0)$-invariant representations of K_0 restrict to L_0. An isomorphism

$$K \times \{\pm1\}^2 \ni (g,h_1,h_2) \mapsto (g,h_1,h_2,h_1h_2) \in K_0 \subset G \cap (O(4) \times O(1)^3)$$

identifies $\mathrm{IR}(K_0)$ and $\mathrm{IR}(K) \times \mathrm{IR}(\{\pm1\}^2)$ (and obviously an action of $W_G(K_0) \simeq \Sigma_3 \times \Sigma_3$ is product-wise). Denote $\sigma := \theta \bar{\otimes} \iota \oplus \iota \bar{\otimes} \theta \oplus \iota \bar{\otimes} \iota \in \mathrm{Rep}(\{\pm1\}^2)$. Finally, let $\eta_{i[j]}^7 := \bigoplus \tau_{(\mu_1,\dots,\mu_7)}$, where the sum is taken over all sequences such that exactly j of the representations $\mu_1, \mu_2, \mu_3, \mu_4$ are isomorphic to ι and exactly i of the representations $\mu_1, \dots, \mu_7$ are isomorphic to ι. By comparing characters we obtain

Corollary 4.16. $W_G(K_0)$-invariant representations of K_0 restrict to L_0 as follows:

$$\begin{array}{cccc} \gamma_0^{\mu,\nu}\bar{\otimes}\theta \mapsto \eta_0^7 & \gamma_0^{\mu,\nu}\bar{\otimes}\sigma \mapsto \eta_{2[0]}^7 & \gamma_4^{\mu,\nu}\bar{\otimes}\theta \mapsto \eta_{3[0]}^7 & \gamma_4^{\mu,\nu}\bar{\otimes}\sigma \mapsto \eta_{1[0]}^7 \\ \gamma_1\bar{\otimes}\theta \mapsto \eta_{1[1]}^7 & \gamma_1\bar{\otimes}\sigma \mapsto \eta_{3[1]}^7 & \gamma_3\bar{\otimes}\theta \mapsto \eta_{3[3]}^7 & \gamma_3\bar{\otimes}\sigma \mapsto \eta_{2[1]}^7 \\ \gamma_{2,a}^{\epsilon}\bar{\otimes}\theta \mapsto \eta_{2[2]}^7 & \gamma_{2,a}^{\epsilon}\bar{\otimes}\sigma \mapsto \eta_{3[2]}^7 & & \end{array}$$

for any $\mu,\nu \in \mathrm{IR}(\{\pm 1\})$, $a \in C_2^2 \setminus \{0,0\}$, $\epsilon \in \{+,-\}$.

5. Cohomology of $\mathbb{Z}_2^{\wedge}[\mathcal{R}_2(L)]$-modules

By *an* $R[\mathcal{C}]-$*module*, where R is a commutative ring R and $\mathcal{C}$ is a small category, we mean a contravariant functor $\mathbf{M}: \mathcal{C} \to R-\mathbf{Mod}$ and $H^n(\mathcal{C};\mathbf{M})$ stands for a higher limit $\lim^n_{\mathcal{C}}\mathbf{M}$. Let A be a ring of p-adic integers and let Γ be a finite group. For any $A[\Gamma]$-module M let $\mathbf{F}_M$ be an atomic $\mathcal{O}_p(\Gamma)$-module with value M concentrated on $\Gamma/1$. Following [7, 5.3] define $\Lambda^n(\Gamma;M) := H^n(\mathcal{O}_p(\Gamma);\mathbf{F}_M)$. An importance of groups Λ^* comes from the following

Theorem 5.1. ([7, 5.4]) *Let L be a compact Lie group and $\mathbf{M}$ an atomic $A[\mathcal{R}_p(L)]$-module concentrated on an object L/Q. Then*

$$H^*(\mathcal{R}_p(L);\mathbf{M}) \cong \Lambda^*(\mathrm{Aut}_{\mathcal{R}_p(L)}(L/Q);\mathbf{M}(L/Q)) = \Lambda^*(W_L(Q);\mathbf{M}(L/Q)).$$

Jackowski, McClure and Oliver [7] provided the following inductive method of calculation of groups $\Lambda^*(\Gamma, M)$.

Proposition 5.2. *Let M be an $A[\Gamma]$-module. Then*

(1) [7, 6.1.(i)] *If p divides $|\Gamma|$, then $\Lambda^0(\Gamma;M) = 0$. Otherwise $\Lambda^0(\Gamma;M) = M^{\Gamma}$ and $\Lambda^i(\Gamma;M) = 0$ for $i > 0$,*

(2) [7, 6.2.(ii)] *Let Γ_p be a Sylow p-subgroup of Γ and let $\sim$ be the equivalence relation among p-Sylow subgroups generated by nontrivial intersection. Set $\Delta := \{g \in \Gamma : g^{-1}\Gamma_p g \sim \Gamma_p\}$. Then $\Lambda^1(\Gamma;M) \cong M^{\Delta}/M^{\Gamma}$.*

(3) [7, 5.2.(ii)] *Define $A[\Gamma]$-modules $\mathbf{F}'_M(\Gamma/P) := M^P$ and $\mathbf{F}''_M(\Gamma/P) := \mathbf{F}'_M/\mathbf{F}_M$. Then $H^i(\mathcal{R}_p(\Gamma);\mathbf{F}'_M) = 0$ for $i > 0$ and $H^0(\mathcal{R}_p(\Gamma);\mathbf{F}'_M) = M^{\Gamma}$. As a consequence,*

$$\Lambda^i(\Gamma;M) = H^i(\mathcal{R}_p(\Gamma);\mathbf{F}_M) \simeq H^{i-1}(\mathcal{R}_p(\Gamma);\mathbf{F}''_M) \simeq H^{i-1}(\mathcal{R}_p(\Gamma)\setminus\{\Gamma/1\};\mathbf{F}''_M)$$

for $i > 1$ ($\mathcal{R}_p(\Gamma)\setminus\{\Gamma/1\}$ stands for a full subcategory with $\Gamma/1$ omitted).

If an $A[\mathcal{R}_p(L)]$-module $\mathbf{M}$ is not atomic, then there is a spectral sequence converging to $H^*(\mathcal{R}_p(L);\mathbf{M})$. Choose a strictly decreasing map of posets $\mathrm{ht}: \mathrm{Ob}(\mathcal{R}_p(L)) \to \mathbb{Z}$. Here follows reformulation of [6, 1.3].

Theorem 5.3. *There is a spectral sequence $E_*^{*,*} := E(\mathcal{R}_p(L),\mathbf{M})_*^{*,*}$ with the first term*

$$E_1^{s,t} := \bigoplus_{\mathrm{ht}(L/P)=s} \Lambda^{s+t}(W_L(P);\mathbf{M}(L/P))$$

converging to $H^*(\mathcal{R}_p(L);\mathbf{M})$.

A differential $d_1^{s,t}: E_1^{s,t} \to E_1^{s+1,t}$ is a sum

$$d_1^{s,t} = \bigoplus_{\mathrm{ht}(L/P)=s,\ \mathrm{ht}(L/Q)=s+1} d_1^{s,t}(P,Q),$$

where $d_1^{s,t}(P,Q) : \Lambda^{s+t}(W_L(P); M(L/P)) \to \Lambda^{s+t+1}(W_L(Q); \mathbf{M}(L/Q))$ is a differential from a long exact sequence associated to a short exact sequence

$$0 \longrightarrow \mathbf{M}|_{\{Q\}} \longrightarrow \mathbf{M}|_{\{P,Q\}} \longrightarrow \mathbf{M}|_{\{Q\}} \longrightarrow 0,$$

The module $\mathbf{M}|_X$, for $X \subseteq \mathrm{Ob}(\mathcal{R}_p(L))$ is

$$\mathbf{M}|_X(L/P) = \begin{cases} \mathbf{M}(L/P) & \text{if } L/P \in X \\ 0 & \text{if } L/P \notin X \end{cases} \tag{5.4}$$

and $\mathbf{M}|_X(L/P \to L/Q) = \mathbf{M}(L/P \to L/Q)$ if $L/P, L/Q \in X$ (of course, this definition is valid only for subsets X which are convex, i.e. for any sequence $L/P \to L/Q \to L/R$ such that $L/P, L/R \in X$ also $L/Q \in X$).

Proposition 5.5. *If* $\mathrm{ht}(L/P)+1 = \mathrm{ht}(L/1) = n$, *then* $d_1^{n-1,t}(P,1)$ *is a composition*

$$\Lambda^{n-1+t}(W_\Gamma(P); \mathbf{M}(\Gamma/P)) \longrightarrow \Lambda^{n-1+t}(W_\Gamma(P); \mathbf{M}(\Gamma/1)^P) \longrightarrow H^{n-1+t}(\mathcal{O}_p(\Gamma); \mathbf{F}''_{\mathbf{M}(\Gamma/1)}) \simeq \Lambda^{n+t}(\Gamma; \mathbf{M}(\Gamma/1))$$

Proof. A homomorphism $\mathbf{M} \to \mathbf{F}'_{\mathbf{M}(\Gamma/1)}$ which is an identity on $\Gamma/1$ induces a commutative diagram

$$\begin{array}{ccc} \Lambda^{n-1+t}(W_\Gamma(P); \mathbf{M}(\Gamma/P)) & \xrightarrow{d_1^{s,t}(P,1)} & \Lambda^{n+t}(\Gamma; \mathbf{M}(\Gamma/1)) \\ \downarrow & & \| \\ \Lambda^{n-1+t}(W_\Gamma(P); \mathbf{M}(\Gamma/1)^P) & \longrightarrow & \Lambda^{n+t}(\Gamma; \mathbf{M}(\Gamma/1)) \end{array}$$

The conclusion follows. □

For any group Γ let $\Lambda\mathrm{dim}_p(\Gamma)$ be the greatest integer n such that $\Lambda^n(\Gamma; M) \neq 0$ for some $A[\Gamma]$-module M. By [7, p. 229] $\Lambda\mathrm{dim}_p(\Gamma)$ is less or equal to the rank of p-Sylow-subgroup of Γ. An application of a spectral sequence 5.3 to an $A[\mathcal{R}_p(\Gamma)]$-module $\mathbf{F}''_M$ shows that

$$\Lambda\mathrm{dim}_p(\Gamma) \leq 1 + \max_{\Gamma/P \in \mathcal{R}_p(\Gamma)} \Lambda\mathrm{dim}_p(W_\Gamma(P)). \tag{5.6}$$

The remaining part of the section contains calculations of groups $\Lambda^*(\Gamma; M)$ for some groups Γ and modules M for $p = 2$.

Cohomology of $\mathcal{R}_2(\Sigma_3)$. Note that $\mathcal{J} := \mathcal{R}_2(\Sigma_3)$ has two objects, namely $\Sigma_3/1$ and Σ_3/C_2. By 5.2 and 5.6 we have

$$\Lambda^n(\Sigma_3; M) = \begin{cases} M^{\Sigma_2}/M^{\Sigma_3} & \text{for } n = 1 \\ 0 & \text{for } n \neq 1. \end{cases} \tag{5.7}$$

Define a height function of $\mathcal{J}$ by $\mathrm{ht}(\Sigma_3/1) = 1$, $\mathrm{ht}(\Sigma_3/C_2) = 0$. For any $A[\mathcal{J}]$-module $\mathbf{M}$, the spectral sequence 5.3 degenerates to an exact sequence

$$0 \longrightarrow H^0(\mathcal{J}; \mathbf{M}) \longrightarrow \mathbf{M}(\Sigma_3/\Sigma_2) \xrightarrow{d} \mathbf{M}(\Sigma_3/1)^{\Sigma_2}/\mathbf{M}(\Sigma_3/1)^{\Sigma_3} \longrightarrow H^1(\mathcal{J}; \mathbf{M}) \longrightarrow 0,$$

where (by 5.5) d is the composition

$$\mathbf{M}(\Sigma_3/\Sigma_2) \xrightarrow{\mathbf{M}(1\Sigma_2)} \mathbf{M}(\Sigma_3/1)^{\Sigma_2} \twoheadrightarrow \mathbf{M}(\Sigma_3/1)^{\Sigma_2}/\mathbf{M}(\Sigma_3/1)^{\Sigma_3}. \tag{5.8}$$

Cohomology of $\mathcal{R}_2(\Sigma_5)$. By (3.14) there are, up to conjugacy, four stubborn subgroups of Σ_5, namely 1, C_2, $1 \wr C_2^2$ and $C_2 \wr C_2$. Its Weyl groups (in Σ_5) are isomorphic to $\Sigma_5, \Sigma_3, \Sigma_3$ and 1 respectively.

Proposition 5.9. *Let M be an $A[\Sigma_5]$-module. Then*

$$\Lambda^n(\Sigma_5; M) = \begin{cases} M^{\Sigma_2\times\Sigma_2\times 1}/(M^{\Sigma_2\times\Sigma_3} + M^{\Sigma_4\times 1}) & \text{for } n = 2 \\ 0 & \text{for } n \neq 2. \end{cases}$$

Proof. By 5.2 and 5.6 we have $\Lambda^n(\Sigma_5; M) = 0$ for $n \neq 2$ and $\Lambda^2(\Sigma_5; M) \cong H^1(\mathcal{R}_2(\Sigma_5); \mathbf{F}''_M)$. Introduce a height function on $\mathcal{R}_2(\Sigma_5)$ by $\mathrm{ht}(\Sigma_5/(C_2 \wr C_2)) = 0$, $\mathrm{ht}(\Sigma_5/(1 \wr C_2^2)) = \mathrm{ht}(\Sigma_5/C_2) = 1$ and $\mathrm{ht}(\Sigma_5/1) = 2$. The spectral sequence (5.3) calculating $H^*(\mathcal{R}; \mathbf{F}''_M)$ degenerates to an exact sequence

$$0 \longrightarrow H^0(\mathcal{R}; \mathbf{F}''_M) \longrightarrow M^{C_2\wr C_2} \xrightarrow{d_1^{0,0}(1,C_2)\oplus d_1^{0,0}(1,1\wr C_2^2)}$$
$$M^{C_2\times C_2}/M^{C_2\times\Sigma_3} \oplus M^{C_2\wr C_2}/M^{\Sigma_4} \longrightarrow H^1(\mathcal{R}; \mathbf{F}''_M) \longrightarrow 0.$$

Full subcategories with object sets respectively $\{\Sigma_5/(C_2 \wr C_2), \Sigma_5/(1 \wr C_2^2)\}$ and $\{\Sigma_5/(C_2 \wr C_2), \Sigma_5/C_2\}$ are both isomorphic to $\mathcal{R}_2(\Sigma_3)$. Thus by 5.5 $d_1^{0,0}(1, C_2)$ is a composition

$$M^{C_2\wr C_2} \hookrightarrow M^{C_2\times C_2} \twoheadrightarrow M^{C_2\times C_2}/M^{C_2\times\Sigma_3}$$

and $d_1^{0,0}(1, 1 \wr C_2^2)$ is a natural projection. Thus $\Lambda^2(\Sigma_5; M) \cong H^1(\mathcal{R}; \mathbf{F}''_M) \simeq M^{\Sigma_2\times\Sigma_2\times 1}/(M^{\Sigma_2\times\Sigma_3} + M^{\Sigma_4\times 1})$. ☐

The groups $\Lambda^*(\Sigma_7; -)$. Λ^*-groups of Σ_7 are more difficult to calculate.

Denote $\mathcal{R} := \mathcal{R}_2(\Sigma_7)$, and let $a = (12)$, $b = (34)$, $c = (56)$, $s = (13)(24)$. By 3.14 there are 7 conjugacy classes of stubborn subgroups of Σ_7, namely 1, C_2, C_2^3, $K := 1 \wr C_2^2$ (the Klein group), $K \times C_2$, $D_8 \simeq C_2 \wr C_2$, $D_8 \times C_2$. Its automorphism groups are respectively 1 (for $D_8 \times C_2$), Σ_3 (for D_8, C_2^2 and $K \times C_2$), Σ_5 (for C_2), $\Sigma_3 \times \Sigma_3$ (for K) and Σ_7 (for 1).

Proposition 5.10. *If M is an $A[\Sigma_7]$-module, then $\Lambda^i(\Sigma_7; M) = 0$ for $i \neq 2, 3$.*

Proof. For $n = 0, 1$ the conclusion is clear. Since $\Lambda\mathrm{dim}_2(\mathrm{Aut}_{\mathcal{R}}(\Sigma_7/P)) \leq 2$ for all orbits $\Sigma_7/P \in \mathcal{R}$ except $\Sigma_7/1$, then for each $n > 3$

$$\Lambda^n(\Sigma_7; M) \cong H^{n-1}(\mathcal{R}; \mathbf{F}''_M) \cong H^{n-1}(\mathcal{R} \setminus \{\Sigma_7/1\}; \mathbf{F}''_M) = 0.$$

☐

Proposition 5.11. $\Lambda^i(\Sigma_7; M(7, 1)) = 0$ *for each $i \geq 0$.*

Proof. It is enough to prove that for each $\Sigma_7/P \in \mathcal{R}$ and each $n > 1$ holds

$$\Lambda^n_P := \Lambda^n(\mathrm{Aut}_{\mathcal{R}}(\Sigma_7/P); F''_{M(7,1)}(\Sigma_7/P)) = 0.$$

For $P = L_0 = 1$ we have $F''_{M(7,1)}(\Sigma_7/1) = 0$ and for $P = N_1, N_0, K_1, L_3$ we have $\Lambda\mathrm{dim}_2(P) < 2$ and therefore $\Lambda^n_P = 0$ for $n \geq 2$. Moreover

$$\mathbf{F}''_{M(7,1)}(\Sigma_7/C_2) = M(7, 1)^{C_2} \cong A\{x_1 + x_2\} \oplus M(5, 1)\{x_i\}_{i=3}^7$$
$$\mathbf{F}''_{M(7,1)}(\Sigma_7/K) \cong A\bar{\otimes}A\{x_1 + x_2 + x_3 + x_4\} \oplus A\bar{\otimes}M(3, 1)\{x_i\}_{i=5}^7$$

Then, for $i > 1$, we have $\Lambda^n_{L_1} = 0$ by 5.9 and $\Lambda^i_{K_0} = 0$ by 5.7 and [7, 6.1.v]. ☐

Proposition 5.12. $\Lambda^3(\Sigma_7; M(7, 2)) = 0$.

Proof. Introduce a height function on $\mathcal{R} := \mathcal{R}_2(\Sigma_7)$ by

$$\mathrm{ht}(\Sigma_7/P) = \begin{cases} 0 & \text{for } P = D_8 \times C_2 \\ 1 & \text{for } P = D_8, C_2^2, K \times C_2 \\ 2 & \text{for } P = C_2, K \\ 3 & \text{for } P = 1 \end{cases}$$

Let $E_*^{*,*} := E(\mathcal{R}_2(\Sigma_7), \mathbf{F}''_{M(7,2)})_*^{*,*}$. Since $\Lambda\mathrm{dim}_2(W_{\Sigma_7}(P)) \leq \mathrm{ht}(\Sigma_7/P)$ for all 2-stubborn P it is sufficient to prove that

$$E_1^{2,0} = \Lambda^2(\Sigma_5; M(7,2)^{C_2}) \oplus \Lambda^2(\Sigma_3^2; M(7,2)^K)$$

is killed by the differential $d_1^{1,0}$. We have

$$\begin{aligned} M(7,2)^{C_2} &\cong A\{x_{12}\} \oplus M(5,1)\{x_{1i} + x_{2i}\}_{i=3}^7 \oplus M(5,2)\{x_{ij}\}_{3 \leq i < j \leq 7} \\ M(7,2)^K &\cong (M(3,1)\bar{\otimes}A)\{x_{12} + x_{34}, x_{13} + x_{24}, x_{14} + x_{23}\} \\ &\oplus (A\bar{\otimes}M(3,1))\{x_{1i} + x_{2i} + x_{3i} + x_{4i}\}_{i=5}^7 \\ &\oplus (A\bar{\otimes}M(3,1))\{x_{ij}\}_{5 \leq i < j \leq 7} \end{aligned}$$

Hence $\Lambda^2(\Sigma_3^2; M(7,2)^K) = 0$ (by 5.7 and [7, 6.1.v]), but $\Lambda^2(\Sigma_5; M(7,2)^{C_2}) \neq 0$ (and hence $E_1^{2,0} \neq 0$). We will prove that

$$d_1^{1,0}(K \times C_2, C_2) : \Lambda^1(\Sigma_3; M(7,2)^{K \times C_2}) \longrightarrow \Lambda^2(\Sigma_5; M(7,2)^{C_2}).$$

is an epimorphism, or equivalently that $H^2(\mathcal{R}_2(\Sigma_7; \mathbf{F}''_{M(7,2)}|_{\{K \times C_2, C_2\}})) = 0$. Let $\mathcal{R}_2^{ext}(\Sigma_7)$ be a full subcategory of $\mathcal{O}_2(\Sigma_7)$ with objects Σ_7/P for $P \in \mathcal{T}_{prod}(7)$. Define a functor

$$J : \mathcal{R}_2(\Sigma_5) \ni \Sigma_5/P \mapsto \Sigma_7 \times_{(\Sigma_5 \times \Sigma_2)} \Sigma_5 \times \Sigma_2/P \times \Sigma_2 \in \mathcal{R}_2^{ext}(\Sigma_7).$$

There is a commutative diagram

$$\begin{array}{ccc} \Lambda^1(W_{\Sigma_5}(K); M(7,2)^{K \times C_2}) & \longleftarrow & \Lambda^1(W_{\Sigma_7}(K \times C_2); M(7,2)^{K \times C_2}) \\ \Big\downarrow {\scriptstyle d_1^{1,0}(K,1)} & & \Big\downarrow {\scriptstyle d_1^{1,0}(K \times C_2, C_2)} \\ \Lambda^2(W_{\Sigma_5}(1); M(7,2)^{C_2}) & \longleftarrow & \Lambda^2(W_{\Sigma_7}(C_2); M(7,2)^{C_2}) \end{array}$$

Since J induces isomorphisms $\mathrm{Aut}_{\mathcal{R}_2(\Sigma_5)}(\Sigma_5/P) \to \mathrm{Aut}_{\mathcal{R}_2(\Sigma_7)}(\Sigma_7/(P \times C_2))$ for $P = 1, K$ the horizontal arrows are actually bijections. Hence it is sufficient to prove that the left vertical arrow is epi. We have

$$\begin{aligned} M(7,2)^{C_2 \times K} &= A\{x_{12}\} \oplus A\{\sum_{i=1,2} \sum_{j=3}^{6} x_{ij}\} \oplus A\{x_{17} + x_{27}\} \oplus A\{\sum_{j=3}^{6} x_{j7}\} \\ &\oplus M(3,1)\{x_{34} + x_{56}, x_{35} + x_{46}, x_{36} + x_{45}\} \end{aligned}$$

Then

$$\Lambda^1(\Sigma_3; M(7,2)^{C_2 \times K}) \simeq A\{x_{34} + x_{56}, x_{35} + x_{46} + x_{36} + x_{45}\} / \left\langle \sum_{3 \leq i < j \leq 6} x_{ij} \right\rangle$$

On the other hand, by 5.9

$$\Lambda^2(\Sigma_5; M(7,2)^K) \simeq A\{x_{34}, x_{56}, x_{35} + x_{46} + x_{36} + x_{45}, x_{37} + x_{47}, x_{57} + x_{67}\}/V$$

where

$$V = \langle x_{34}, x_{35}+x_{36}+x_{37}+x_{45}+x_{46}+x_{47}, x_{56}+x_{57}+x_{67},\\ x_{34}+x_{35}+x_{36}+x_{45}+x_{46}+x_{56}, x_{37}+x_{47}+x_{57}+x_{67}\rangle$$

and the conclusion follows. □

The groups $\Lambda^*(\Sigma_3 \wr C_2; -)$. There are (up to conjugacy) three 2-stubborn subgroups of $\Sigma_3 \wr C_2$, namely 1, Σ_2 and $\Sigma_2 \wr C_2$. Its Weyl groups are respectively $\Sigma_3 \wr C_2$, Σ_3 and 1. The full subcategory of $\mathcal{R} := \mathcal{R}_2(\Sigma_3 \wr C_2)$ containing Σ_2 and $\Sigma_2 \wr C_2$ is isomorphic to $\mathcal{J} = \mathcal{R}_2(\Sigma_3)$.

Proposition 5.13. *Let M be an $A[\Sigma_3 \wr C_2]$-module. Then*

$$\Lambda^n(\Sigma_3 \wr C_2; M) = \begin{cases} M^{\Sigma_2\times\Sigma_2}/M^{\Sigma_2\times\Sigma_3} + M^{\Sigma_2\wr C_2} & \textit{for } n=2 \\ 0 & \textit{for } n \neq 2. \end{cases}$$

Proof. Since the rank of $\Sigma_3 \wr C_2$ is even, and the relation $\sim$ from 5.2.(2) is transitive, then $\Lambda^n(\Sigma_3 \wr C_2; M) = 0$ for $n = 0, 1$. For $n > 1$ we have

$$\Lambda^n(\Sigma_3 \wr C_2; M) \cong H^{n-1}(\mathcal{R}_2(\Sigma_3 \wr C_2); \mathbf{F}''_M) \cong H^{n-1}(\mathcal{J}; \operatorname{res}^{\mathcal{R}}_{\mathcal{J}} F''_M).$$

By 5.8 we have

$$H^{n-1}(\mathcal{J}; \operatorname{res}^{\mathcal{R}}_{\mathcal{J}} F''_M) \cong \begin{cases} M^{\Sigma_2\times\Sigma_2}/M^{\Sigma_2\times\Sigma_3} + M^{\Sigma_2\wr C_2} & \text{for } n=2 \\ 0 & \text{for } n \neq 2. \end{cases}$$ □

6. Homotopy representations of G and $\tilde{G}$

Let $\mathcal{R} := \mathcal{R}_2(G) \cong \mathcal{R}_2(\tilde{G})$. In this section we prove that for every $\mathcal{R}$-invariant representation of N (resp. $\tilde{N}$) the map $B\varphi_2^\wedge$ extends to a homotopy representation of G (resp. $\tilde{G}$).

Denote $A := \mathbb{Z}_2^\wedge$. Let φ be an $\mathcal{R}$-invariant representation of N and let $\Xi := \Xi(\varphi)$ be a module introduced in Section 2. By Theorem 2.5.(a) we need to prove that $H^3(\mathcal{R}; \Xi)$ vanishes, and that cohomology of $\mathcal{R}$ with coefficient in any $A[\mathcal{R}]$-module vanish above dimension 4.

Proposition 6.1. *For each $G/P \in \mathcal{R}$ we have $\Lambda\mathrm{dim}_2(W_G(P)) \leq 3$.*

Proof. Each group $W_G(P)$ is isomorphic to one of the following groups: 1, Σ_3, $\Sigma_3\times\Sigma_3$, Σ_5, $\Sigma_3\wr C_2$, $\Sigma_3\wr C_2\times\Sigma_3$, Σ_7 (cf. 4.6). We have $\Lambda\mathrm{dim}_2(1) = 0$, $\Lambda\mathrm{dim}_2(\Sigma_3) = 1$ (by 5.7), $\Lambda\mathrm{dim}_2(\Sigma_5) = 2$ (by 5.9), $\Lambda\mathrm{dim}_2(\Sigma_3\wr C_2) \leq 2$ (by 5.13) and $\Lambda\mathrm{dim}_2(\Sigma_7) \leq 3$ (by 5.10). Moreover, $\Lambda\mathrm{dim}_2(\Sigma_3 \times \Sigma_3) = 2$ and $\Lambda\mathrm{dim}_2(\Sigma_3 \wr C_2 \times \Sigma_3) = \Lambda\mathrm{dim}_2(\Sigma_3 \wr C_2) + 1 \leq 3$. □

As a consequence we obtain

Corollary 6.2. $\mathrm{cdim}_2(\mathcal{R}) \leq 3$.

In order to calculate $H^3(\mathcal{R}; \Xi)$ we use a spectral sequence (5.3). Define a height function on $\mathcal{R}$ by putting

$$\mathrm{ht}(G/P) = \begin{cases} 0 & \text{for } P = N_1 \\ 1 & \text{for } P = N_0, L_3, K_1, J_1 \\ 2 & \text{for } P = L_3, K_0, J_0, M_1 \\ 3 & \text{for } P = L_0, M_0 \end{cases} \tag{6.3}$$

Let $E_*^{*,*} := E(\mathcal{R};\Xi)_*^{*,*}$.

Proposition 6.4. $H^3(\mathcal{R};\Xi) = \operatorname{coker}(d_1^{2,0} : E_1^{2,0} \to E_1^{3,0})$.

Proof. By 6.1 we have $\Lambda^i(W_G(P)) = 0$ for $i > \operatorname{ht}(P)$. Hence $E_1^{s,t}$ vanishes if $t > 0$ or if $s > 3$ (since 3 is a maximal value of the height function). Therefore $E_1^{3,0}$ is the only rank 3 entry at the first table which possibly does not vanish and $d_1^{2,0}$ the only possibly non-trivial differential hitting $E_1^{3,0}$. □

Proposition 6.5. *A homomorphism*

$$d_1^{2,0}(M_1, M_0) : \Lambda^2(\Sigma_3 \wr C_2; \Xi(G/M_1)) \longrightarrow \Lambda^3(\Sigma_3 \wr C_2 \times \Sigma_3; \Xi(G/M_0))$$

(cf. 5.3) is an epimorphism.

Proof. Let $\{X_i\}_{i\in I}$ be orbits of an action of $\Sigma_3 \wr C_2 \times \Sigma_3$ on $\operatorname{IR}(M_0, \operatorname{res}^{N_1^\infty}_{M_0} \varphi)$. Each X_i has the form $Y_i \times Z_i$, where Y_i is an $\Sigma_3 \wr C_2$-orbit in $\operatorname{IR}(M)$ and Z_i is an Σ_3-orbit in $\operatorname{IR}(\{\pm 1\}^2)$ (see p.14). By [7, 6.1.v]

$$\begin{aligned}\Lambda^3(\Sigma_3 \wr C_2 \times \Sigma_3; \Xi(G/M_0)) &\cong \bigoplus_{i\in I} \Lambda^3(\Sigma_3 \wr C_2 \times \Sigma_3; A[X_i]) \\ &\simeq \bigoplus_{i\in I} \Lambda^2(\Sigma_3 \wr C_2; A[Y_i]) \otimes \Lambda^1(\Sigma_3; A[Z_i])\end{aligned}$$

Each Z_i is equal either to $\{\theta\bar{\otimes}\theta\}$, or to $\{\theta\bar{\otimes}\iota, \iota\bar{\otimes}\theta, \iota\bar{\otimes}\iota\}$. In the first case we have $\Lambda^1(\Sigma_3; A[Z_i]) = 0$. In the second case, there is an orbit X_i' in $\operatorname{IR}(M_1^\infty, \operatorname{res}^{N_1^\infty}_{M_1^\infty} \varphi) \simeq \operatorname{IR}(\Gamma_4^\infty) \times \operatorname{IR}(O(2)^\infty)$ such that $X_i' = Y_i \times \{\alpha_{2k+1}\}$ for some $k \in A$. Furthermore, the restriction of $d_1^{2,0}$ to $\Lambda^2(\Sigma_3 \wr C_2; A[X_i'])$ is a composition

$$\begin{aligned}&\Lambda^2(\Sigma_3 \wr C_2; A[X_i']) \xrightarrow{\simeq} \Lambda^2(\Sigma_3 \wr C_2; A[Y_i']) \otimes A\{\alpha_{2k+1}\} \xrightarrow{1\times d} \\ &\quad \Lambda^2(\Sigma_3 \wr C_2; A[Y_i]) \otimes \Lambda^1(\Sigma_3; A\{\theta\bar{\otimes}\iota, \iota\bar{\otimes}\theta, \iota\bar{\otimes}\iota\}) \subseteq \Lambda^3(\Sigma_3 \wr C_2 \times \Sigma_3; \Xi(G/M_0)).\end{aligned}$$

The differential $d : A[\{\alpha_{2k+1}\}] \to \Lambda^1(\Sigma_3; A[\{\theta\bar{\otimes}\iota, \iota\bar{\otimes}\theta, \iota\bar{\otimes}\iota\}])$ is an epimorphism by 5.8. Hence $d_1^{2,0}$ is an epimorphism. □

Proposition 6.6. *We have*

$$\begin{aligned}\mathbf{F}'_{M(7,3)}(L_1) &\simeq M(5,1)\{x_{12k}\}_{k\geq 3} \oplus M(5,2)\{x_{1kl} + x_{2kl}\}_{k,l\geq 3} \\ &\quad \oplus M(5,2)\{x_{klm}\}_{k,l,m\geq 3} \\ \mathbf{F}'_{M(7,3)}(K_0) &\simeq (A\bar{\otimes}A)\{x_{123} + x_{124} + x_{134} + x_{234}\} \\ &\quad \oplus (M(3,1)\bar{\otimes}M(3,1))\{x_{12k} + x_{34k}, x_{13} + x_{24k}, x_{14} + x_{23k}\}_{k\geq 5} \\ &\quad \oplus (A\bar{\otimes}M(3,1))\{x_{1kl} + x_{2kl} + x_{3kl} + x_{4kl}\}_{k,l\geq 5}(A\bar{\otimes}A)\{x_{567}\}\end{aligned}$$

Proposition 6.7. *A homomorphism*

$$d_1^{2,0}(L_1, L_0) \oplus d_1^{2,0}(K_0, L_0) : \Lambda^2(\Sigma_5; \Xi(G/L_1)) \oplus \Lambda^2(\Sigma_3^2; \Xi(G/K_0)) \longrightarrow \Lambda^3(\Sigma_7; \Xi(G/L_0))$$

is an epimorphism.

Proof. Let $I : \mathcal{R}_2(\Sigma_7) \to \mathcal{R}_2(O(7)) \cong \mathcal{R}$ be a full inclusion from Theorem 3.22 and let $\mathbf{F} := \mathbf{F}'_{\Xi(G/L_0)}$. Let $\bar{E}_*^{*,*} := E(\mathcal{R}_2(\Sigma_7), \mathbf{F})_*^{*,*}$. There is a homomorphism of $A[\mathcal{R}_2(\Sigma_7)]$-modules $I^*\Xi \to \mathbf{F}'_{\Xi(G/L_0)}$ which is an isomorphism on L_0 and it induces a transformation of spectral sequences $E_*^{*,*} \to \bar{E}_*^{*,*}$. In particular, there is a commutative diagram

$$\begin{array}{ccc} \Lambda^2(\Sigma_5;\Xi(G/L_1)) \oplus \Lambda^2(\Sigma_3^2;\Xi(G/K_0)) & \longrightarrow & \Lambda^2(\Sigma_5;\Xi(L_0)^{C_2}) \oplus \Lambda^2(\Sigma_3^2;\Xi(G/L_0)^K) \\ \Big\downarrow d_1^{2,0} & & \Big\downarrow \bar{d}_1^{2,0} \\ \Lambda^3(\Sigma_7;\Xi(G/L_0)) & = & \Lambda^3(\Sigma_7;\Xi(G/L_0)) \end{array}$$

(where $K \subset \Sigma_4 \subset \Sigma_7$ is a Klein group). Since

$$\mathrm{res}_{L_0}^{N_1^\infty} \varphi \simeq (\eta_0^7)^{\oplus l_0} \oplus (\eta_1^7)^{\oplus l_1} \oplus (\eta_2^7)^{\oplus l_2} \oplus (\eta_3^7)^{\oplus l_3},$$

(cf. 4.12) then $\Xi(G/L_0)$ is a direct sum of some of the modules $M(7,i)$, $i = 0,\dots,3$, where $M(7,i)$ appears as a summand of $\Xi(G/L_0)$ if and only if $\mathrm{res}_{L_0}^{N_1^\infty} \varphi$ contains a subrepresentation isomorphic to η_i^7. From 5.11 and 5.12 follows that $\Lambda^3(\Sigma_7;\Xi(G/L_0)) = 0$ if $\mathrm{res}_{L_0}^{N_1^\infty} \varphi$ does not contain a subrepresentation isomorphic to η_3^7, so assume otherwise. Let x_{ijk}, $1 \le i < j < k \le 7$ be the generators of the submodule of $\Xi(G/L_0)$ corresponding to η_3^7. By 4.13 there exists a subrepresentation of $\mathrm{res}_{L_1^\infty}^{N_1^\infty} \varphi$ which is isomorphic either to $\tau\bar{\otimes}\eta_2^5$ or to $\beta_{2k}\bar{\otimes}\eta_2^5$ for some $k \in A$, and another subrepresentation isomorphic to $\alpha_{2k'+1}\bar{\otimes}\eta_2^5$, where $k' \in A$. Then $\Xi(G/L_1)$ contains the direct sum of two $A[\Sigma_5]$-submodules isomorphic to $M(5,2)$: the one generated by irreducible subrepresentations of $\tau\bar{\otimes}\eta_2^5$ (or $\beta_{2k}\bar{\otimes}\eta_2^5$) maps onto the summand

$$M(5,2)\{x_{klm}\}_{3\le k<l<m\le 7} \subseteq \mathbf{F}(G/L_1)$$

and the one generated by irreducible subrepresentations of $\alpha_{2k'+1}\bar{\otimes}\eta_2^5$ maps onto

$$M(5,2)\{x_{1kl} + x_{2kl}\}_{3\le k<l\le 7} \subseteq \mathbf{F}(G/L_1).$$

By 6.6 $\mathrm{coker}(\Xi(G/L_1) \to \mathbf{F}(G/L_1))$ is a quotient of a direct sum of $A[\Sigma_5]$-modules isomorphic either to $M(5,1)$ or to A. Hence by 5.9 we obtain that $\Lambda^2(\Sigma_5;\mathrm{coker}(\Xi(G/L_1) \to \mathbf{F}(G/L_1)) = 0$. Similarly, there is a subrepresentation of $\mathrm{res}_{K_0}^{N_1^\infty} \varphi$ which is isomorphic to $(\gamma_{2,(0,1)}^\mu \oplus \gamma_{2,(1,0)}^\mu \oplus \gamma_{2,(1,1)}^\mu)\bar{\otimes}\sigma$ (where $\mu \in \{+,-\}$), generates a submodule of $\Xi(G/K_0)$ isomorphic to $M(3,1)\bar{\otimes}M(3,1)$ and maps onto

$$M(3,1)\bar{\otimes}M(3,1)\{x_{12k} + x_{34k}, x_{13} + x_{24k}, x_{14} + x_{23k}\}_{5\le k\le 7} \subseteq \mathbf{F}(G/K_0).$$

Again by 6.6 we obtain that $\Lambda^2(\Sigma_3 \times \Sigma_3;\mathrm{coker}(\Xi(G/K_0) \to \mathbf{F}(G/K_0)) = 0$. □

Theorem 6.8. *Each $\mathcal{R}_2(G)$-invariant representation of N_1^∞ extends to a homotopy representation of G.*

Proof. Since $E_1^{3,0} = \Lambda^3(\Sigma_3\wr C_2\times\Sigma_3;\Xi(G/M_0))\oplus\Lambda^3(\Sigma_7;\Xi(G/L_0))$, then $H^3(\mathcal{R};\Xi) = 0$ by 6.5, 6.7 and 6.4. Now the conclusion follows from 6.1 and 2.5. □

Now let $\tilde{\varphi}$ be an $\mathcal{R}$-invariant representation of $\tilde{N}_1^\infty$. The representation φ splits into the even part φ_{ev} and the odd part φ_{od} and both them are $\mathcal{R}$-invariant. Moreover, $\Xi(\varphi) = \Xi(\varphi_{ev}) \oplus \Xi(\varphi_{od})$ and hence

$$H^*(\mathcal{R};\Xi(\varphi)) = H^*(\mathcal{R};\Xi(\varphi_{ev})) \oplus H^*(\mathcal{R};\Xi(\varphi_{od})) = H^*(\mathcal{R};\Xi(\varphi_{od})).$$

Proposition 6.9. *$H^3(\mathcal{R};\Xi(\varphi_{od})) = 0$.*

Proof. By 6.1 and 2.5 we need to prove that for each 2-stubborn subgroup $P \subseteq G$ we have $\Lambda^3(W_G(P); \Xi(\varphi_{od})(G/P)) = 0$. If $P \notin \{L_0, M_0\}$, then $\Lambda\dim_2(\mathrm{Aut}_{\mathcal{R}}(G/P)) < 3$ and the conclusion is obvious. By 4.10 for $P = L_0, M_0$ there is an element of order 2 in $W_G(P)$ which acts trivially on the set of odd representations of P. Finally, [7, 6.1.ii] implies $\Lambda^*(W_G(P); \Xi(\varphi_{od})(P)) = 0$ for $* > 0$. □

Theorem 6.10. *Any $\mathcal{R}$-invariant representation of $\tilde{N}_1^\infty$ extends to a homotopy representation of $\tilde{G}$.*

Proof. It follows from 6.1, 6.9 and 2.5. □

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