Equipartitions of measures in $\mathbb{R}^4$

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Abstract

A well known problem of B. Grünbaum [Grü60] asks whether for every continuous mass distribution (measure) $d\mu = f\,dm$ on $\mathbb{R}^n$ there exist n hyperplanes dividing $\mathbb{R}^n$ into 2^n parts of equal measure. It is known that the answer is positive in dimension $n = 3$ [Had66] and negative for $n \geq 5$, [Avis84] [Ram96]. We give a partial solution to Grünbaum's problem in the critical dimension $n = 4$ by proving that each measure μ in $\mathbb{R}^4$ admits an equipartition by 4 hyperplanes, provided that it is symmetric with respect to a 2-dimensional affine subspace L of $\mathbb{R}^4$. Moreover we show, by computing the complete obstruction in the relevant group of normal bordisms, that without the symmetry condition, a naturally associated *topological problem* has a negative solution. The computation is based on the Koschorke's exact singularity sequence [Kosch81] and the remarkable properties of the essentially unique, balanced binary Gray code in dimension 4, [Toot56] [Knu01].

1 Introduction

A notorious open problem in geometric combinatorics and discrete and computational geometry is the question whether each continuous mass distribution μ in $\mathbb{R}^4$ admits an *equipartition* by hyperplanes. This is an "essentially 4-dimensional" problem by the classification of V. Klee [Klee99], indicating that the answer is known and positive in all dimensions ≤ 3 and negative in dimensions ≥ 5. Recall that a collection $H_1, H_2, \ldots, H_n$ of hyperplanes in $\mathbb{R}^n$ is an equipartition for a mass distribution (measure) μ if each of the 2^n "orthants" associated to $\{H_j\}_{j=1}^n$ contains the fraction $1/2^n$ of the total mass. In other words, $\mu(H^\epsilon) = 1/2^n \mu(\mathbb{R}^n)$ for each $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, where $H^\epsilon := H_1^{\epsilon_1} \cap \ldots \cap H_n^{\epsilon_n}$ is the "orthant" associated to ϵ and H_i^0 (respectively H_i^1) is the positive (respectively negative) closed halfspace associated to the hyperplane H_i.

The progress in the general problem was slow after B. Grünbaum posed the question in 1960, [Grü60]. H. Hadwiger showed in [Had66] that equipartitions exist for $n = 3$. D. Avis showed in [Avis84] that there exist non-equipartitionable mass distributions for $n \geq 5$. In the mean-time many related questions were formulated and many of them solved [BM01] [BM02] [Mak01] [Ram96] [VZ92], a connection with discrete and computational geometry was established [YY85] [YDEM89] and the subject grew into a separate branch of geometric combinatorics [Živ04]. Nevertheless, the 4-equipartition problem itself has resisted all attempts and remains one of the central open problems in the field.

In this paper we prove, Theorem 5.1, that a measure μ in $\mathbb{R}^4$ admits a 4-equipartition if it is symmetric with respect to a 2-plane L in $\mathbb{R}^4$. Moreover we demonstrate in Theorem 5.9, by computing the obstruction in the relevant normal bordism group, that there are no "obvious" topological obstacles for the existence of non-equipartitionable measures. This result may be an indication that such a *peculiar* measure does exist in $\mathbb{R}^4$ and it is an intriguing question whether the "topological counterexample", provided by Theorem 5.9, can be turned into a genuine counterexample.

2 The CS/TM-scheme

The *configuration space/test map* scheme [Živ04] has emerged as one of the key principles for the application of topological methods in geometric combinatorics and discrete and computational geometry. The basic idea can be outlined as follows.

One starts with a configuration space or manifold $M_{\mathcal{P}}$ of all candidates for the solution of a geometric/combinatorial problem $\mathcal{P}$. The next step is a construction of a test map $f : M_{\mathcal{P}} \to V_{\mathcal{P}}$ which measures how far is a given candidate configuration $C \in M_{\mathcal{P}}$ from being a solution. More precisely, there is a subspace Z of the test space $V_{\mathcal{P}}$ such that a configuration $C \in M_{\mathcal{P}}$ is a solution if and only if $f(C) \in Z$. The inner symmetries of the problem $\mathcal{P}$ typically show up at this stage. This means that there is a group G of symmetries of $X_{\mathcal{P}}$ which acts on $V_{\mathcal{P}}$, such that Z is a G-invariant subspace of $V_{\mathcal{P}}$, which turns $f : M_{\mathcal{P}} \to V_{\mathcal{P}}$ into an equivariant map. If a configuration C with the desired property $f(C) \in Z$ does not exist, then there arises an equivariant map $f : M_{\mathcal{P}} \to V_{\mathcal{P}} \setminus Z$. The final step is to show by topological methods that such a map does not exist.

The reader can follow the genesis of the method in review papers [Alon88] [Bar93] [Björ91] [Živ96] [Živ98] [Živ04] and see how the solutions of well known combinatorial problems like Kneser's conjecture (L. Lovász [Lov78]), "the splitting necklace problem" (N. Alon [Alon87]), the Colored Tverberg problem (R. Živaljević, S. Vrećica, [ŽV92] [VŽ94]) etc. eventually led to the formulation and codification of the general principle.

2.1 The equipartition problem

Our first choice for the configuration space suitable for the equipartition problem is the manifold of all ordered collections $\mathcal{H} = (H_1, \ldots, H_n)$ of oriented hyperplanes in $\mathbb{R}^n$.

Suppose that $e : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is the embedding defined by $e(x) = (x, 1)$. Each oriented hyperplane H in $e(\mathbb{R}^n) \cong \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\} \subset \mathbb{R}^{n+1}$ is obtained as an intersection $H = e(\mathbb{R}^n) \cap H'$ for a unique oriented, $(n+1)$-dimensional subspace $H' \subset \mathbb{R}^{n+1}$. The oriented subspace H' is determined by the corresponding orthogonal unit vector $u \in S^n \subset \mathbb{R}^{n+1}$, so the natural environment for collections $\mathcal{H} = (H_1, \ldots, H_n)$, and our second choice for the configuration space is $M_{\mathcal{P}} := (S^n)^n$. The group which acts on the configuration manifold $M_{\mathcal{P}}$ is the reflection group $W_n := (\mathbb{Z}/2)^n \rtimes S_n$ where S_n permutes the factors while the subgroup $(\mathbb{Z}/2)^n$ is in charge of the antipodal actions on individual spheres. The action of W_n on $M_{\mathcal{P}} := (S^n)^n$ is not free, so our third choice for the configuration space associated to the equipartition problem is

$$M_{\mathcal{P}}^{\delta} = (S^n)^n_{\delta} := \{x \in (S^n)^n \mid x_i \neq \pm x_j \text{ for } i \neq j\}. \tag{1}$$

This space is a relative of the (standard) "configuration space" $F_m(S^n) := \{x \in (S^n)^m \mid x_i \neq x_j \text{ for } i \neq j\}$ [FaHu01]. It has already appeared in Combinatorics, for example in [FeZi02], where it is referred to as the "signed configuration space".

The associated "orbit configuration space" $(S^n)^n_{\delta}/W_n$ can be identified as a submanifold of the symmetric product $SP^n(RP^n)$ of the projective space RP^n. This is the reason why we occasionally denote this quotient by $SP_{\delta}(RP^n)$ and view its elements as unordered collections of n distinct lines in $\mathbb{R}^{n+1}$.

The test space $V = V_{\mathcal{P}}$ is defined as follows. If μ is a measure defined on $\mathbb{R}^n$, let μ' be the "push-down" measure induced on $\mathbb{R}^{n+1}$ by the embedding $e : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, $\mu'(A) := \mu(e(\mathbb{R}^n) \cap A)$. A n-tuple $(u_1, \ldots, u_n) \in (S^n)^n$ of unit vectors determines a n-tuple $\mathcal{H} = (H_1', \ldots, H_n')$ of oriented $(n+1)$-dimensional subspaces of $\mathbb{R}^{n+1}$. The n-tuple $\mathcal{H}$ dissects $\mathbb{R}^{n+1}$ into 2^n-orthants $\mathrm{Ort}_{\beta}(\mathcal{H})$ which are naturally indexed by 0-1 vectors $\beta \in \{0,1\}^n$. Let $b_{\beta} : M_{\mathcal{P}} \to \mathbb{R}$ be the function defined by $b_{\beta}(\mathcal{H}) := \mu'(\mathrm{Ort}_{\beta}(\mathcal{H})) = \mu(\mathrm{Ort}_{\beta}(\mathcal{H}) \cap e(\mathbb{R}^n))$. Let $B_{\mu} : (S^n)^n \to \mathbb{R}^{2^n}$ be the function defined by $B_{\mu}(\mathcal{H}) = (b_{\beta}(\mathcal{H}))_{\beta \in \{0,1\}^n}$. The test space $V = V_{\mathcal{P}} \cong \mathbb{R}^{2^n}$ has a natural action of the group $W_n := (\mathbb{Z}/2)^n \rtimes S_n$ such that the map B_{μ} is W_n-equivariant. Note that the real W_n-representation V, restricted to the subgroup $(\mathbb{Z}/2)^n \hookrightarrow W_n$, reduces to the regular representation $\mathrm{Reg}((\mathbb{Z}/2)^n)$ of the group $(\mathbb{Z}/2)^n$. The "zero" subspace $Z_{\mathcal{P}}$ is defined as the trivial, 1-dimensional W_n-representation V_0 contained in V. Let $U = U_n$ be the complementary W_n-representation, $U_n \cong V/V_0$ and $A_{\mu} : (S^n)^n \to U_n$ the induced, W_n-equivariant map. By the construction we have the following proposition which says that A_{μ} is a genuine test map for the μ-equipartition problem.

Proposition 2.1. *A n-tuple $\mathcal{H} = (H_1, \ldots, H_n) \in M_{\mathcal{P}} = (S^n)^n$ of oriented hyperplanes in $\mathbb{R}^n$ is an equipartition of a measure μ defined on $\mathbb{R}^n$ if and only if $A_{\mu}(\mathcal{H}) = 0$.*

Corollary 2.2. *If there does not exist a W_n-equivariant map $A : M_{\mathcal{P}} \to U_n \setminus \{0\}$, then each positive, continuous mass distribution (measure) $d\mu = f\,dm$, where dm is the Lebesgue measure, admits an equipartition by n hyperplanes.*

Remark 2.3. The assumption that μ is a measure absolutely continuous to the Lebesgue measure on $\mathbb{R}^n$ is unnecessary restrictive. All we need is the continuity of the test map $A_{\mu} : (S^n)^n \to U_n$, a condition satisfied by a very large class of measures having the desired continuity properties. Notable examples of interesting measures that are not in this class are *counting* measures ν_D of finite sets $D \subset \mathbb{R}^n$ defined by $\nu_D(X) := |D \cap X|$. Note however, that all equipartition results can be suitably extended to *weak limits* of continuous measures, cf. [MVZ] for a general set up. An example of such a result applying to counting measures is Corollary 5.2 from Section 5.1.

2.2 Singular sets Σ_μ

It is a well known fact that for a free G-space P and a G-representation V, there does not exist a G-equivariant map $f : P \to V \setminus \{0\}$ if and only if the associated vector bundle $V \to P \times_G V \to P/G$ does not admit a non-zero, continuous cross-section, cf. Proposition I.7.2 in [Dieck87].

A well known approach to the last question, applicable in the case when P is a free G-manifold, is the *singularity* approach, cf. [Kosch81]. Given a G-map $h : P \to V$, the *singularity set* $\Sigma(h)$ of h is the, possibly empty, G-subspace of P defined by $\Sigma(h) := h^{-1}(0)$. In the case when h is transverse to $0 \in V$, the singularity set $\Sigma(h)$ is a G-manifold.

If $h = A_\mu$ is the test map of a measure μ, then the associated singularity set $\Sigma_\mu := \Sigma(A_\mu)$ is simply the set of all solutions to the equipartition problem for μ. In this case Σ_μ is often referred to as the *solution set (manifold)* of μ.

The singularity manifold $\Sigma(h)$ of a map h, sometimes accompanied by the additional information recording the behavior of h in the tubular neighborhood of $\Sigma(h)$ (the normal data), can be used for computation of an associated *obstruction element* in a suitable group of bordisms. The singularity manifold and the associated normal data together yield a very strong obstruction invariant which is in a number of important cases complete in the sense that an equivariant map exists if and only if these obstruction vanish. The reader is referred to [Kosch81] for the general theory.

2.3 Equipartitions of planar measures

The case $n = 2$ of the equipartition problem is well known and elementary. Nevertheless, we briefly review this case since it serves as a fairly good illustration of general ideas in their rudimentary form. According to the CS/TM-scheme, as presented in Section 2.1, the problem is to prove that there does not exist a W_2-equivariant map $f : M_{\mathcal{P}} \to U_2 \setminus \{0\}$, where $W_2 = \mathbb{D}_8$ is the dihedral group, $M_{\mathcal{P}} = S^2 \times S^2$, and U_2 the 3-dimensional real representation of W_2, described in Section 2.1.

One can establish a slightly stronger statement that there does not exist a W_2-equivariant map $f : M_{\mathcal{P}}^\delta \to U_2 \setminus \{0\}$ where $M_{\mathcal{P}}^\delta = (S^2)^2_\delta = S^2 \times S^2 \setminus \{(x,y) \mid x = y \text{ or } x = -y\}$. The advantage of $(S^2)^2_\delta$ over $(S^2)^2$ is that the former is a free W_2-space.

Let us see how the singularity approach works in the case of planar equipartitions. For a *generic* measurable set $A \subset \mathbb{R}^2$, the singularity Σ_A of A, that is the collection of all pairs (L_1, L_2) of oriented lines in $\mathbb{R}^2$ which form an equipartition for A, is a 1-dimensional W_2-manifold. For example if A is a unit disc D, the singularity Σ_D is a union of 4 circles. Here we do not make precise what is meant by a generic measure. Instead we naively assume, for the sake of this example, that there exists such a notion of genericity for measurable sets/measures so that each measurable set A can be well approximated by generic measures. Moreover, we assume that for any two measurable sets A and B there exists a path of generic measures μ_t, $t \in [0,1]$, so that μ_0 is an approximation of A, μ_1 is an approximation for B and the solution set

$$\Sigma_{\{\mu_t\}_{t\in[0,1]}} := \{(L_1, L_2; t) \mid (L_1, L_2) \text{ is an equipartition for } \mu_t\} \subset (S^2)^2_\delta \times [0,1]$$

is a 2-dimensional manifold (bordism) connecting solution sets for measures μ_0 and μ_1. The group $\Omega_1(\mathbb{D}_8)$ of classes of 1-dimensional, free $\mathbb{D}_8$-manifolds is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and the $\mathbb{D}_8$-solution manifold Σ_D, associated to the unit disc in $\mathbb{R}^2$, is easily shown to represent a nontrivial element in this group. It immediately follows that for any measurable set $A \subset \mathbb{R}^2$, or more

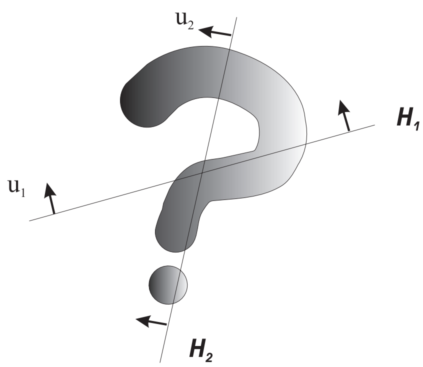

Figure 1: An equipartition in the plane.

generally for any continuous mass distribution, the singularity Σ_A is nonempty. Indeed, suppose $\Sigma_A = \emptyset$. Let $\{\mu_t\}_{t\in[0,1]}$ be a path of generic measures such that μ_0 approximates A and μ_1 approximates D. If these approximations are sufficiently good, we deduce that the solution set Σ_{μ_0} is empty and that $[\Sigma_{\mu_1}]$ and $[\Sigma_D]$ represent the same element in $\Omega_1(\mathbb{D}_8)$. This is a contradiction since $\Sigma_{\mu_1} = \partial(\Sigma)$ where $\Sigma = \Sigma_{\{\mu_t\}_{t\in[0,1]}}$, i.e. Σ_D would represent a trivial element in $\Omega_1(\mathbb{D}_8)$.

Remark 2.4. It is worth noting that the scheme outlined above, if applicable, shows that a general equipartition problem can be solved by a careful analysis of the singularity set of a well chosen, particular measure/measurable set, the unit disc D^2 in our example above. Unfortunately, in higher dimensions the unit balls do not represent generic measures, i.e. their solution manifolds are very special and cannot be used for the evaluation of the relevant obstruction elements. Instead, for this purpose one can use measures distributed along a *convex curve* Γ in $\mathbb{R}^n$, Section 3.

3 From convex curves to Gray codes

3.1 Convex curves

A simple, smooth curve Γ in $\mathbb{R}^n$ is called *convex* if the total multiplicity of its intersection with an affine hyperplane does not exceed n. Convex curves are classical objects appearing in many different fields including the real algebraic geometry ("ecstatic points" of curves), theory of convex polytopes (neighborly polytopes), interpolations of functions (Tschebycheff systems), linear ordinary differential equations (disconjugate equations) etc., see [Anis98] [Arn96] [Copp71] [Scho54] [SeSh96] and the references in these papers. Standard examples of convex curves are the "moment curve", or the rational normal curve $M_n := \{(t, t^2, \ldots, t^n) \mid t \in \mathbb{R}\} \subset \mathbb{R}^n$ and the standard trigonometric curve $\Gamma_{2n} := \{(\cos t, \sin t, \cos 2t, \ldots, \cos nt, \sin nt) \mid t \in [0, 2\pi]\} \subset \mathbb{R}^{2n}$.

The importance of convex curves for the equipartition problem stems from the fact that they minimize the number of intersections with hyperplanes. As a consequence, each collection $\mathcal{H} = \{H_1, \ldots, H_n\}$ of n hyperplanes in $\mathbb{R}^n$ has at most n^2 intersection points with Γ. It follows that if $n = 2d$ is even, and Γ is a simple, closed convex curve, then $\mathcal{H}$ divides Γ in at most n^2 arcs. Suppose that μ is a measure concentrated on a closed, convex curve $\Gamma \subset \mathbb{R}^{2d}$. Then if $\mathcal{H}$ is an equipartition for μ then $n^2 \geq 2^n$, i.e. $n \leq 4$. This explains why the dimension 4 is so special in this context.

3.2 Gray codes

Gray codes arise in an attempt to describe the solution manifold (singularity set) Σ_μ of a measure μ distributed along a closed convex curve $\Gamma \subset \mathbb{R}^4$. Our preferred example of such a curve is $\Gamma_4 = \{(z, z^2) \mid |z| = 1\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$. From here on we assume that $d\mu' = d\theta'$, $\theta' = \arg(z)$, is the "arc length" measure on the circle $C = \{z \in \mathbb{C} \mid |z| = 1\}$ and $d\mu = d\theta$ the associated measure on Γ_4, where $\theta = \theta' \circ \pi_1$ and $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ is the first projection. Similar analysis can be carried on for other measures concentrated on Γ_4. Each collection $P = \{p_1, p_2, p_3, p_4\}$ of 4 points in Γ_4 is contained in a unique hyperplane $H \subset \mathbb{R}^4$, hence the combinatorics of μ-equipartitions can be read off from the circle C, see Figures 2, 3 and 4. As a consequence, an equipartition $\mathcal{H} = \{H_1, H_2, H_3, H_3\}$ of the measure $d\theta$ is essentially a division of the circle C into 16 arcs of equal length and associating each of the 16 division points $\{x_j\}_{j=0}^{15}$, where $x_j = \epsilon^j\, x_0$ and $\epsilon = \exp 2\pi i/16$, to one of hyperplanes H_i. Note that not all maps $\alpha : \{x_j\}_{j=0}^{15} \to \mathcal{H}$ are allowed, cf. [Ram96] p. 157. Each of the arcs $[x_j, x_{j+1}]$ belongs to an orthant coded by an associated 4-bit word $\beta_j \in \{0,1\}^4$. Each of the 4-bit words $\beta \in \{0,1\}^4$ appears exactly once in the cyclic order inherited from the order of intervals $[x_j, x_{j+1}]$. Moreover, moving from one orthant to another along the curve Γ_4, that is from the interval $[x_j, x_{j+1}]$ to the consecutive interval $[x_{j+1}, x_{j+2}]$, changes only one bit at a time. It follows that sequence $\{\beta_j\}_{j=0}^{15}$ of 4-bit words forms a so called Gray code, [Knu01] [Ram96]. For a graph theorist, a Gray code is a *Hamiltonian path* on a hypercube $\{0,1\}^n$. For an engineer, a Gray code is a device useful for converting digital signals into analog and vice versa, [Knu01].

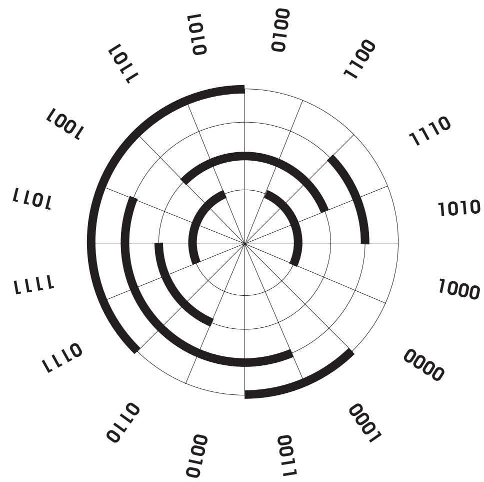

Figure 2: The unique, balanced 4-bit Gray code.

Our next observation is that Gray codes arising in the equipartitions of the convex curve Γ_4 are quite special. Indeed, each hyperplane H_i intersects the convex curve Γ_4 precisely 4 times implying that the code must have the same number of bit changes in each of four coordinate tracks. Such codes are called *balanced*, Figure 2. A Gray code with these properties was originally discovered by G.C. Tootill [Toot56]. One of its remarkable properties is that it is unique up to permutation of coordinate tracks, see [Gil58] or [Knu01], Exercise 56. In particular if one reads the code clockwise the same code is obtained, except that the second and the third track interchange places, see Figures 2–4.

There is one more attractive way to describe this code. As a variation on the theme of the play "Quad" by S. Beckett [Quad] where *"... Four actors, whose colored hoods make them identifiable yet anonymous, accomplish a relentless closed-circuit drama ..."* (R. Frieling), one can design a scheme for a play based on the balanced Gray code. In this scheme the stage begins and ends empty; 4 actors enter and exit one at a time, running through all 16 possible subsets, and each actor is supposed to enter (leave) the stage precisely 2 times, see [Knu01] Exercise 65 (attributed to B. Stevens) for a similar idea.

3.3 The solution set Σ_θ

The analysis from Sections 3.1 and 3.2 allows us to describe the solution manifold (singular set) Σ_θ of the measure θ on the convex curve Γ_4, arising from the "arc length" measure on the unit circle $C \subset \mathbb{C}$, Section 3.2. This solution set is a W_4-invariant subset of the configuration space $M^\delta_{\mathcal{P}} = (S^4)^4_\delta$.

Suppose that $\mathcal{H} = \{H_1, H_2, H_3, H_4\} \in \Sigma_\theta$. Each hyperplane H_j intersects the curve Γ_4 in four points, vertices of a 3-simplex $\sigma_j \subset H_j$. The image $\pi_1(\sigma_j)$ of σ_j by the projection map $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$, sending the curve Γ_4 to the circle C, Section 3.2, is a convex polygon in the plane. Figure 3 displays these polygons in the order corresponding to the chosen order of hyperplanes in $\mathcal{H}$.

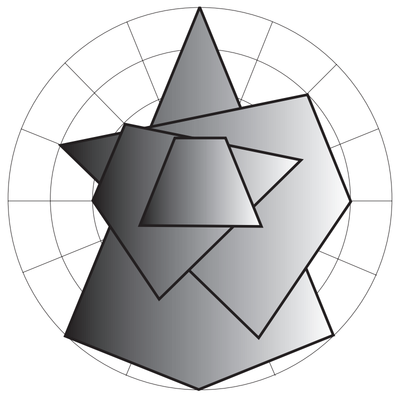

Figure 3: An element of Σ_θ...

By taking into account the (chosen) orientation of hyperplanes H_j and the induced orientations of simplices $\sigma_j \subset H_j$, we arrive at the conclusion that Figure 3 is a fairly accurate description of an element $\mathcal{H} = \{H_1, H_2, H_3, H_4\} \in \Sigma_\theta$. This element belongs to an oriented circle $\Sigma \subset \Sigma_\theta$ of all solutions, obtained essentially by rotating Figure 3 counterclockwise. All other circles in Σ_θ are obtained by the action of the group W_4, in other words by changing the orientations of simplices σ_j, performed by the subgroup $(\mathbb{Z}/2)^4$, and by permuting the circumcircles of polygons in Figure 3. In other words the permutation of polygons is achieved by permuting the circles, the tracks in the balanced Gray code.

For example if we move Figure 3 clockwise, we obtain the circle $\xi(\Sigma) \subset \Sigma_\theta$ where $\xi : [4] \to [4]$ is the permutation which keeps tracks 1 and 4 fixed and interchanges tracks 2 and 3. We see this as a manifestation of the symmetry of diagrams in Figures 3 and 4 with respect to the vertical axes.

Let us turn to the question of non-degeneracy of the solution manifold Σ_θ. This is by definition the condition that the associated test map $A_\theta : (S^4)^4_\delta \to U_4$ is transverse to $0 \in U_4$. Suppose that $\mathcal{H} \in \Sigma_\theta$ and assume that $\{x_j\}_{j=0}^{15}$ are the associated division points, Section 3.2. Then for a small positive real number $\epsilon > 0$, the angles $y_j \in (x_j - \epsilon, x_j + \epsilon)$ can be used as the coordinates on $(S^4)^4_\delta$ in the neighborhood of $\mathcal{H} \in (S^4)^4_\delta$. The associated tangent vectors are $\partial/\partial y_j \in T_{\mathcal{H}}((S^4)^4_\delta)$. Similarly, the functions b_β introduced in Section 2.1 are coordinates on V, consequently the functions $c_j := b_{\beta_{j+1}} - b_{\beta_j}$, where $\{\beta_j\}_{j=0}^{15}$ is the sequence defined in Section 3.2, are coordinates on U_4. The proof is completed by the observation that $dA_\theta(\partial/\partial y_j) = -\partial/\partial c_j$.

For the future reference we record an essential part of this analysis in the following proposition.

Proposition 3.1. *The solution manifold $\Sigma_\theta \subset M^\delta_{\mathcal{P}} = (S^4)^4_\delta$ of all equipartitions for the measure $d\theta$ on the convex curve Γ_4 is a (non-degenerated) 1-dimensional W_4-manifold which has $|W_4| = 2^4 4!$ connected components. The quotient manifold $\Sigma' := \Sigma_\theta / W_4$ is a circle in the manifold $(S^4)^4_\delta / W_4 \cong$*

$SP^4_\delta(RP^4) \subset SP^4(RP^4)$, where $SP^m(X) := X^m/S_m$ is the symmetric product of X and $SP^m_\delta(X)$ its subspace of "square-free" divisors.

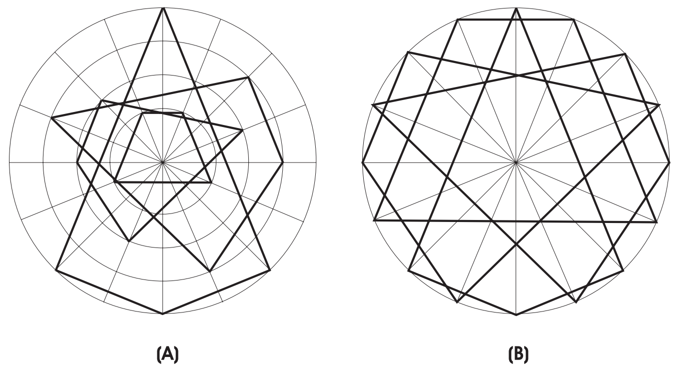

Figure 4: ... and the associated element in Σ_θ/W_4.

Remark 3.2. Figure 4 (B) symbolically represents an element of the circle $\Sigma' = \Sigma_\theta/W_4$. The orientations of polygons are forgotten and all 4 circles, representing different tracks in the balanced Gray code visible in Figure 4 (A), "merged together" in Figure 4 (B).

4 From Gray codes to normal bordisms

According to Corollary 2.2, the 4-equipartition problem is closely related to the question if there exists a W_4-equivariant map $f : (S^4)^4_\delta \to U_4 \setminus \{0\}$. In turn, this is equivalent to the question if the vector bundle

$$\mathcal{E}: \quad U_4 \to (S^4)^4_\delta \times_{W_4} U_0 \to (S^4)^4_\delta/W_4 \qquad (2)$$

admits a non-zero, continuous cross-section. A complete topological obstruction ω for the existence of such a section lives in the normal bordism group $\Omega_1(M; \mathcal{E} - TM)$ [Kosch81], where TM is the tangent bundle of the manifold $M := (S^4)^4_\delta/W_4$ and $\mathcal{E}$ is the bundle (2). This group can be computed from the Koschorke's exact singularity sequence, [Kosch81] Theorem 9.3. The computation of the obstruction $\omega \in \Omega_1(M; \mathcal{E} - TM)$ is based on this sequence and the analysis of the singularity Σ_θ of the measure θ distributed along a closed, convex curve $\Gamma_4 \subset \mathbb{R}^4$, Section 3.

4.1 Koschorke's exact singularity sequence

One of the consequences of Koschorke's exact singularity sequence [Kosch81, Section 7], is a short exact singularity sequence, [Kosch81] Theorem 9.3., involving low dimensional normal bordism groups $\Omega_j(X; \phi)$ and $\widetilde{\Omega}_j(X; \phi)$ for $j \leq 4$, where $\phi = \phi^+ - \phi^-$ is a virtual vector bundle over X. The final fragment of this sequence has the following form,

$$\xrightarrow{\delta_2} \Omega_2(X;\phi) \xrightarrow{f_2} \widetilde{\Omega}_2(X;\phi) \xrightarrow{\sigma\circ j_2} \mathbb{Z}/2 \xrightarrow{\delta_1} \Omega_1(X;\phi) \xrightarrow{f_1} \widetilde{\Omega}_1(X;\phi) \longrightarrow 0. \qquad (3)$$

We are interested in this sequence in the case when $X = M = (S^4)^4_\delta / W_4$ and $\phi = \phi^+ - \phi^- = \mathcal{E} - TM$. More precisely, our objective is to evaluate the obstruction element $\omega \in \Omega_1(M; \mathcal{E} - TM)$ defined in Section 4.

4.2 The image of δ_1

According to [Kosch81], the image $\delta_1(1)$ of the generator $1 \in \mathbb{Z}/2$ "... can be represented by the unit circle with constant map and the standard parallelization, suitably stabilized ...". In this section we show that the image $\delta_1(1)$ coincides with the obstruction element $\omega \in \Omega_1(M; \mathcal{E} - TM)$. This is a consequence of the following proposition.

Proposition 4.1. $f_1(\omega) = 0$.

Proof: The configuration space $(S^4)^4_\delta$ is simply connected, hence the projection $(S^4)^4_\delta \to (S^4)^4_\delta / W_4$ is a universal covering map. Let Σ be a circle, connected component of the solution manifold Σ_θ, and let $\Sigma' := \Sigma_\theta / W_4 \subset (S^4)^4_\delta / W_4$. Suppose that ξ is the orientation line bundle of the virtual bundle $\mathcal{E} - TM$. Then, by definition of $\widetilde{\Omega}_1(X; \phi)$, [Kosch81] § 9, the image $f_1(\omega)$ of ω in $\widetilde{\Omega}_1(X; \mathcal{E} - TM)$ is determined by Σ' and the restriction $\xi|\Sigma'$ of ξ on Σ'.

Since the circle Σ' can be lifted to the circle Σ, we conclude that Σ' is contractible, hence $\xi|\Sigma'$ is a trivial bundle. It follows that there exists a map $g : D^2 \to M$ such that $\partial(D^2)$ is mapped bijectively to Σ' and extension of the bundle $g^*(\xi|\Sigma')$ to trivial bundle over D^2. Hence, $f_1(\omega)$ is a trivial element in $\widetilde{\Omega}_1(X; \mathcal{E} - TM)$. □

4.3 The image of $\sigma \circ j_2$

In this section we focus on the calculation of the image of the map $\widetilde{\Omega}_2(X, \phi) \xrightarrow{\sigma \circ j_2} \mathbb{Z}/2$ in the Koschorke's singularity exact sequence (3). By definition, an element $\alpha = [N, g, or] \in \widetilde{\Omega}_2(X, \phi)$ is mapped to $\sigma \circ \delta_2(\alpha) := g^*(w_2(\phi))[N]$, where $w_2(\phi)$ is the second Stiefel-Whitney class of the virtual bundle $\phi = \phi^+ - \phi^-$ and $[N] \in H_2(N, \mathbb{Z}/2)$ is the fundamental class of the surface N. From here we conclude that elements $\alpha = [N, g, or]$ such that both $g^*(\phi^+)$ and $g^*(\phi^-)$ are trivial vector bundles can be ignored, in particular we ignore those elements where g is homotopic to a constant map. Recall the exact sequence

$$\longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_2(X) \longrightarrow H_2(X, \mathbb{Z}/2) \longrightarrow 0 \tag{4}$$

where $\mathcal{N}_2(X)$ is the group of unoriented bordisms and $\mathcal{N}_2 := \mathcal{N}_2(*)$. It follows that in the evaluation of the image of $\sigma \circ \delta_2$ we are allowed to pick a representative $\alpha = [N, g, or]$ in each of the homology classes $x \in H_2(X, \mathbb{Z}/2)$.

The following standard lemma reduces the calculation of $w_2(\phi)[N]$ to the calculations of Stiefel numbers of individual bundles ϕ^+ and ϕ^-.

Lemma 4.2.

$$\begin{aligned} w(\phi) &= w(\phi^+) \cdot w(\phi^-)^{-1} \\ &= (1 + w_1(\phi^+) + w_2(\phi^+) + \ldots)(1 + w_1(\phi^-) + w_2(\phi^-) + w_1(\phi^-)^2 + \ldots) \\ &= 1 + A_1 + A_2 + \ldots, \ \textit{where} \end{aligned}$$

$A_1 = w_1(\phi^+) + w_1(\phi^-)$ and $A_2 = w_2(\phi^+) + w_1(\phi^+)w_1(\phi^-) + w_1(\phi^-)^2 + w_2(\phi^-)$ *are terms of graduation* 1 *and* 2 *respectively. Consequently if* $\phi = \phi^+ - \phi^-$ *is a virtual vector bundle over a surface* N*, then*

$$w_2(\phi)[N] = w_2(\phi^+)[N] + (w_1(\phi^+)w_1(\phi^-))[N] + w_1(\phi^-)^2[N] + w_2(\phi^-)[N].$$

In our case $X = SP^4_\delta(RP^4)$. The following lemma, an easy consequence of Poincaré duality, allows us to search for surfaces N representing nontrivial homology 2-classes in the symmetric product $SP^4(RP^4)$.

Lemma 4.3. *There is an isomorphism* $H_2(SP^4_\delta(RP^4)) \longrightarrow H_2(SP^4(RP^4))$ *of homology groups, induced by the inclusion map* $SP^4_\delta(RP^4) \hookrightarrow SP^4(RP^4)$.

It is easy to see that $H_2(SP^4(RP^4)) \cong H_2(SP^\infty(RP^4))$. By the Dold-Thom theorem, [Ha02]

$$SP^\infty(RP^4) \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}/2, 3) \times K(\mathbb{Z}/2, 4).$$

From here we deduce that $H_2(SP^4(RP^4)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

It is not difficult to describe surfaces N_1 and N_2, embedded in $SP^4_\delta(RP^4) \subset SP^4(RP^4)$, representing the generators of this homology group. Suppose that e_1 and e_1' are two disjoint circles embedded in RP^4, both representing the nontrivial element in $H_1(RP^4; \mathbb{Z}/2)$. Let $e_2 \cong RP^2$ be a projective plane embedded in RP^4, representing the generator of $H_2(RP^4; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Finally suppose that $*_1, *_2, *_3$ are three distinct points in RP^4 such that $*_i \notin e_1 \cup e_1' \cup e_2$ for each $i = 1, 2, 3$. As usual, elements of the symmetric product $SP^m(X)$ are thought of as positive "divisors", i.e. commutative and associative formal sums $D = n_1x_1 + \ldots + n_kx_k$ where $n_i \in \mathbb{N}, x_i \in X$ and $\sum_{i=1}^k = m$. By definition let

$$N_1 := *_1 + *_2 + e_1 + e_1' \quad \text{and} \quad N_2 := *_1 + *_2 + *_3 + e_2. \tag{5}$$

In other words $N_1 \cong S^1 \times S^1$ is a torus embedded in $SP^4_\delta(RP^4)$ where $D \in N_1 \Leftrightarrow D = *_1 + *_2 + x + y$ for some $x \in e_1$ and $y \in e_1'$. Similarly, $N_2 \cong RP^2$ consists of all divisors of the form $D = *_1 + *_2 + *_3 + x$ for some $x \in e_2$.

In our case $\phi^+ = \mathcal{E}$ and $\phi^- = T(SP^4_\delta(RP^4))$. We focus our attention on the bundles $\phi_i^+ := \phi^+|N_i$ and $\phi_i^- := \phi^-|N_i$ for $i = 1, 2$. Recall that $\mathcal{E} \cong (S^4)^4_\delta \times_{W_4} U$ where $U = U_4$ is the 15-dimensional representation of W_4 described in Section 2.1. If $\pi : (S^4)^4_\delta \longrightarrow SP^4_\delta(RP^4)$ is the projection map then $Z_i := \pi^{-1}(N_i)$ is a free, W_4-submanifold of $(S^4)^4_\delta$ and $\phi_i^+ \cong Z_i \times_{W_4} U$. It is not difficult to describe these manifolds.

The connected component of Z_1 is a torus $T^2 = S^1 \times S^1 \subset (S^4)^4_\delta$ and the stabilizer of T^2 is the group $H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \subset (\mathbb{Z}/2)^4 \subset W_4 = (\mathbb{Z}/2)^4 \rtimes S_4$ where $H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ acts on $T^2 = S^1 \times S^1$ by the product action. It follows that $Z_1 \cong T^2 \times_{H_1} W_4$ and

$$\phi_1^+ \cong Z_1 \times_{W_4} U \cong (T^2 \times_{H_1} W_4) \times_{W_4} U \cong T^2 \times_{H_1} U.$$

Similarly, the connected component of Z_2 is the sphere $S^2 \subset (S^4)^4_\delta$ and its stabilizer is the group $H_2 = \mathbb{Z}/2 \subset (\mathbb{Z}/2)^4 \subset W_4$ where H_2 acts on S^2 by the antipodal action. It follows that $Z_2 \cong S^2 \times_{H_2} W_4$ and

$$\phi_2^+ \cong Z_2 \times_{W_4} U \cong (S^2 \times_{H_2} W_4) \times_{W_4} U \cong S^2 \times_{H_2} U.$$

Keeping in mind that the restriction $\mathrm{Res}^{W_4}_{\mathbb{Z}^{\oplus 4}}(U)$ is the regular (real) representation of $\mathbb{Z}^{\oplus 4}$, minus the trivial 1-dimensional representation, it is easy to identify the bundles $\phi_1^+ = T^2 \times_{H_1} U$ and $\phi_2^+ = S^2 \times_{H_2} U$. As a preliminary step, let us describe some canonical line bundles over $N_1 \cong T^2/H_1 \cong S^1/\mathbb{Z}/2 \times S^1/\mathbb{Z}/2 \cong T^2$ and $N_2 \cong S^2/H_2 \cong RP^2$.

There are 4 real, 1-dimensional representations of $H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. If ω_1 and ω_2 are the generators of H_1, then $L_{\epsilon_1\epsilon_2}, \epsilon_1, \epsilon_2 \in \{-1, +1\}$, is the representation characterized by the condition $\omega_i(v) = \epsilon_i v$ for each $v \in L_{\epsilon_1\epsilon_2}$. Let $\lambda_{\bar{\epsilon}_1\bar{\epsilon}_2} := T^2 \times_{H_1} L_{\epsilon_1\epsilon_2}$ be the associated line bundle where $\bar{\epsilon}_i \in \{0, 1\}$ and $(-1)^{\bar{\epsilon}_i} = \epsilon_i$. For example λ_{00} is the trivial bundle, usually denoted by ϵ, while λ_{11} is the Cartesian product of 2 canonical bundles over $RP^1 \cong S^1$. Let $\gamma = \gamma_n$ be the canonical line bundle over RP^n. The decompositions of bundles ϕ_1^+ and ϕ_2^+ into line bundles is recorded in the following statement.

Proposition 4.4.

$$\phi_1^+ = T^2 \times_{H_1} U \cong \epsilon^{\oplus 3} \oplus \lambda_{01}^{\oplus 4} \oplus \lambda_{10}^{\oplus 4} \oplus \lambda_{11}^{\oplus 4}$$
$$\phi_2^+ = S^2 \times_{H_2} U \cong \epsilon^{\oplus 7} \oplus \gamma^{\oplus 8}.$$

The next step is the identification of the restrictions $\phi_1^- := TM|_{N_1}$ and $\phi_2^- := TM|_{N_2}$ of the tangent bundle $TM = T(SP^4_\delta(RP^4))$ on the surfaces N_1 and N_2 respectively.

Lemma 4.5. *For each point $D = p_1 + p_2 + p_3 + p_4 \in SP^4_\delta(RP^4)$ there is an isomorphism*

$$T_D(SP^4_\delta(RP^4)) \cong \oplus_{i=1}^4 T_{p_i}(RP^4).$$

Lemma 4.6. ([MS74]) *Let γ_n be the canonical line bundle over the real projective space RP^n and ϵ the trivial line bundle. Then there is an equality of virtual vector bundles $T(RP^n) = \gamma_n^{\oplus(n+1)} - \epsilon$. As a consequence, the total Stiefel-Whitney class of the restriction bundle $\xi = T(RP^n)|_{RP^m}$ is*

$$w(\xi) = (1+t)^{n+1} = 1 + \binom{n+1}{1}t + \binom{n+1}{2}t^2 + \ldots + \binom{n+1}{m}t^m.$$

Proposition 4.7.

$$\begin{array}{rcl} \phi_1^- = TM|_{N_1} & \cong & \epsilon^{\oplus 6} \oplus \lambda_{01}^{\oplus 5} \oplus \lambda_{10}^{\oplus 5} \\ \phi_2^- = TM|_{N_2} & \cong & \epsilon^{\oplus 11} \oplus \gamma^{\oplus 5}. \end{array}$$

Proof: By Lemmas 4.5 and 4.6, there is an equality of virtual bundles

$$\begin{array}{rcl} TM|_{N_1} & = & \epsilon^{\oplus 8} + (T(RP^4)|e_1 \times T(RP^4)|e_1') \;=\; \epsilon^{\oplus 8} + (\gamma_4^{\oplus 5}|e_1 - \epsilon) \times (\gamma_4^{\oplus 5}|e_1' - \epsilon) \\ & = & \epsilon^{\oplus 8} + \lambda_{01}^{\oplus 5} + \lambda_{10}^{\oplus 5} - \epsilon^{\oplus 2} \;=\; \epsilon^{\oplus 6} + \lambda_{01}^{\oplus 5} + \lambda_{10}^{\oplus 5}. \end{array}$$

Similarly,

$$TM|_{N_2} \cong \epsilon^{\oplus 12} + T(RP^4)|e_2 = \epsilon^{\oplus 12} + (\gamma_4^{\oplus 5}|e_2) - \epsilon) = \epsilon^{\oplus 11} + \gamma^{\oplus 5}.$$

Corollary 4.8.

$$\begin{aligned} \phi_1^+ - \phi_1^- &= \lambda_{11}^{\oplus 4} - \lambda_{01} - \lambda_{10} - \epsilon^{\oplus 3} \\ \phi_2^+ - \phi_2^- &= \gamma^{\oplus 3} - \epsilon^{\oplus 4}. \end{aligned}$$

Proposition 4.9. $\mathrm{Im}(\sigma \circ j_2) = \mathbb{Z}/2$.

Proof: It is a basic fact that $H^*(RP^2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[t]/(t^3)$ and $H^*(T^2; \mathbb{Z}/2) \cong \Lambda[a, b]$, where $\Lambda[a, b]$ is a $(\mathbb{Z}/2)$-exterior algebra generated by two elements of degree 1. The first two characteristic classes of line bundles $\lambda_{\bar{\epsilon}_1 \bar{\epsilon}_2}$ and γ, defined over $N_1 = T^2$ and $N_2 = RP^2$ respectively, are

$$w_1(\lambda_{\bar{\epsilon}_1 \bar{\epsilon}_2}) = \bar{\epsilon}_1 a + \bar{\epsilon}_2 b, \quad w_1(\gamma) = t, \quad w_2(\lambda_{\bar{\epsilon}_1 \bar{\epsilon}_2}) = w_2(\gamma) = 0. \tag{6}$$

From here we deduce that the total Stiefel-Whitney classes of bundles ϕ_1^+ and ϕ_1^- are respectively $w(\lambda_{11}^{\oplus 4}) = 1$ and $w(\lambda_{01} + \lambda_{10}) = 1 + a + b + ab$. This, together with the Lemma 4.2 and the first equality from Corollary 4.8, implies that

$$w_2(\phi_1^+ - \phi_1^-) = w_1(\lambda_{01} + \lambda_{10})^2 + w_2(\lambda_{01} + \lambda_{10}) = ab. \tag{7}$$

Similarly,

$$w_2(\phi_2^+ - \phi_2^-) = w_2(\gamma^{\oplus 3}) = t^2. \tag{8}$$

In other words

$$w_2(\phi_1^+ - \phi_1^-)[N_1] = w_2(\phi_2^+ - \phi_2^-)[N_2] = 1 \tag{9}$$

and we finally conclude that $\mathrm{Im}(\sigma \circ j_2) = \mathbb{Z}/2$. □

5 Results and proofs

5.1 Measures admitting a 2-plane of symmetry

In this section we show that each measure with a 2-dimensional plane of symmetry admits a 4-equipartition.

Theorem 5.1. *Suppose that μ is a measure on $\mathbb{R}^4$ admitting a 2-dimensional plane of symmetry in the sense that for some 2-plane $L \subset \mathbb{R}^4$ and the associated reflection $R_L : \mathbb{R}^4 \to \mathbb{R}^4$, for each measurable set $A \subset \mathbb{R}^4$, $\mu(A) = \mu(R_L(A))$. Then μ admits a 4-equipartition.*

Proof: Without loss of generality we assume that $L = \mathbb{C}_{(2)}$ in the decomposition $\mathbb{R}^4 \cong \mathbb{C}^2 \cong \mathbb{C}_{(1)} \oplus \mathbb{C}_{(2)}$. In that case the reflection $R = R_L$ is the map described by $R(z_1, z_2) = (-z_1, z_2)$, in particular R is a symmetry of the *convex curve* (Section 3) $\Gamma_4 = \{(z, z^2) \in \mathbb{C}^2 \mid |x| = 1\}$.

In pursuit of an equipartition of a general measure in $\mathbb{R}^4$, we introduced in Section 2.1 the configuration space $(S^4)^4_\delta = \{v \in S^4 \mid v_i \neq \pm v_j \text{ for } i \neq j\}$. Recall that the group $W_4 = (\mathbb{Z}/2)^4 \rtimes S_4$ acts freely on this configuration space and that the problem of 4-equipartitions was reduced, Corollary 2.2, to the question of the existence of a W_4-equivariant map $f : (S^4)^4_\delta \to U_4 \setminus \{0\}$, where U_4 is the 15-dimensional, real representation of W_4 defined in Section 2.1.

If the measure μ admits an additional symmetry, e.g. if it admits a plane of symmetry L such that the associated reflection R_L keeps μ invariant, it is natural to enlarge the group W_4 by this transformation.

Assume as before that $\mathbb{R}^4$ is identified to the affine subspace $\mathbb{R}^4 + e_5 \subset \mathbb{R}^5$. The isometry $R_L : \mathbb{R}^4 \to \mathbb{R}^4$ is extended to the unique isometry $\widehat{R}_L$ of $\mathbb{R}^5$ such that $\widehat{R}_L(e_5) = e_5$.

Let $\mathbb{Z}/2$ be the group generated by $\widehat{R}_L$. Define the enlarged group of symmetries as the direct product $G := W_4 \times \mathbb{Z}/2$. The action of W_4 on $(S^4)^4_\delta$ can be extended to the action of the group G by the requirement that $\widehat{R}_L(v_1, v_2, v_3, v_4) = (\widehat{R}_L(v_1), \widehat{R}_L(v_2), \widehat{R}_L(v_3), \widehat{R}_L(v_4))$. In order to make this action free, let us introduce an even smaller configuration space

$$(S^4)^4_\Delta \quad := (S^4)^4_\delta \setminus \mathcal{F} \tag{10}$$

where $\mathcal{F} \subset (S^4)^4_\Delta$ is the subset of points $v = (v_1, v_2, v_3, v_4)$ such that the stabilizer $\mathrm{Stab}_G(v)$ is a non-trivial group.

Note that $\widehat{R}_L$ is the reflection with respect to the 3-plane $L + \mathbb{R}e_5 \subset \mathbb{R}^5$. Consequently, $v \in (S^4)^4_\delta$ is in $\mathcal{F}$ if (up to a permutation of coordinates v_i) either,

$$(\widehat{R}_L(v_1) = v_2, \widehat{R}_L(v_3) = v_4) \quad \text{or} \quad (\widehat{R}_L(v_1) = v_1, \widehat{R}_L(v_2) = v_2, \widehat{R}_L(v_3) = v_4).$$

Key observation: The solution manifold $\Sigma_\theta \subset (S^4)^4_\delta$ of all 4-equipartitions of the convex curve Γ_4, defined in Section 3.3, is $\widehat{R}_L$-invariant. Moreover $\Sigma_\theta \subset (S^4)^4_\Delta$, or in other words the $\mathbb{Z}/2$-action on Σ_θ induced by $\widehat{R}_L$ is free. Indeed, $\widehat{R}_L$ acts on the element displayed in Figure 3 by the rotation through the angle of 180°.

Denote by λ both the non-trivial, 1-dimensional real representation of $\mathbb{Z}/2$ and the associated, 1-dimensional G-representation induced by the projection homomorphism $G \to \mathbb{Z}/2$. Similarly, U_4 is both the 15-dimensional real W_4-representation defined in Section 2.1 and the representation induced by the projection homomorphism $G \to W_4$.

According to results of Section 2.1, for the proof of the theorem it is sufficient to show that a W_4-equivariant map $f : (S^4)^4_\delta \to U_4$ must have a zero. This is a consequence of the following stronger result.

Claim: There does not exist a G-equivariant map $f : (S^4)^4_\Delta \to S(U_4 \oplus \lambda)$, where $S(U_4 \oplus \lambda)$ is the G-invariant unit sphere in $U_4 \oplus \lambda$. In other words each G-invariant map $f : (S^4)^4_\Delta \to U_4 \oplus \lambda$ has a zero.

Proof of the Claim: The claim is equivalent to the statement that the vector bundle $\xi : (S^4)^4_\Delta \times_G (U_4 \oplus \lambda) \to (S^4)^4_\Delta / G$ does not admit a non-zero continuous cross-section. For this it is sufficient to show that the top Stiefel-Whitney class $w_n(\xi)$ is non-zero. By duality this is equivalent to the fact that for some (each) smooth cross-section $s : (S^4)^4_\Delta/G \to (S^4)^4_\Delta \times_G (U_4 \oplus \lambda)$, transverse to the zero section Z, the number of elements in $s^{-1}(Z) \subset (S^4)^4_\Delta/G$ is odd. In the language of equivariant maps, this is equivalent to the statement that for some smooth, G-equivariant map $s = (\phi, \psi) : (S^4)^4_\Delta \to U_4 \oplus \lambda$, transverse to $0 \in E \oplus \lambda$, the number of G-orbits in $s^{-1}(0)$ is odd.

Define $\phi : (S^4)^4_\Delta \to U_4$ as the restriction of the test map $A_\theta : (S^4)^4_\delta \to U_4$ introduced in Section 2.1, where $d\theta$ is the "arc length"-measure on $\Gamma_4 = \{(z, z^2) \in \mathbb{C}^2 \mid |z| = 1\}$ introduced in Section 3.2.

Since A_θ is W_4-equivariant and $A_\theta \circ \widehat{R}_L = A_\theta$, we conclude that ϕ is G-equivariant. The orbit space $SP^4_\Delta(RP^4) := (S^4)^4_\Delta/W_4$ is a smooth manifold. Note, Section 3, that the unique balanced Gray code allowed us to identify the "solution manifold" of all equipartitions of the curve $\Gamma_4 \subset \mathbb{R}^4$ as the circle $\Sigma' \subset SP^4_\Delta(RP^4)$. Moreover, this solution manifold was shown to be non-degenerated, Proposition 3.1, in the sense that the associated test map A_θ is transverse to $0 \in U_4$.

The $\mathbb{Z}/2$-action on $SP^4_\Delta(RP^4)$, induced by the involution $\widehat{R}_L$, is free. Let $\psi' : SP^4_\Delta(RP^4) \to \lambda$ be a $\mathbb{Z}/2$-equivariant, smooth map such that $0 \in \lambda$ is not a critical value of the restriction $\psi'' := \psi'|\Sigma'$ of ψ' on the circle Σ'. Define ψ as the composition of ψ' with the natural projection map $(S^4)^4_\Delta \to SP^4_\Delta(RP^4)$.

Then it is not difficult to check that $s = (\phi, \psi)$ is a smooth, G-equivariant map such that $0 \in \lambda$ is not one of its critical values.

Let us show that the number of G-orbits in the set $s^{-1}(0)$ is always an odd number. Note that $s^{-1}(0) = \phi^{-1}(0) \cap Z(\psi)$ where $Z(\psi) := \psi^{-1}(0)$ is the zero set of ψ. Since $\phi^{-1}(0)/G = \phi^{-1}(0)/W_4 = \Sigma'$, we observe that the number of G-orbits in the set $s^{-1}(0)$ is equal to the number of $\mathbb{Z}/2$-orbits in the zero set $Z(\psi'')$ of the $\mathbb{Z}/2$-equivariant map $\psi'' : \Sigma' \to \lambda$. Since 0 is not a critical value of ψ'', the proof is completed by an elementary observation that a $\mathbb{Z}/2$-equivariant map $p : S^1 \to \lambda$, transverse to $0 \in \lambda$, must have an odd number of $\mathbb{Z}/2$-orbits.

$\square$

Corollary 5.2. *Suppose that D is a finite set of $16d$ distinct points in $\mathbb{R}^4$ which is symmetric with respect to a 2-plane $L \subset \mathbb{R}^4$. Then there exists a collection $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ of distinct hyperplanes such that each of 16 associated open orthants contains not more than d points from the set D. Moreover, if D is in general position in the sense that no 5 points belong to the same hyperplane, than each of the open orthants contains at least $d - 16$ elements from D. In particular, if D is a not necessarily symmetric set of $16d$ distinct points in $\mathbb{R}^4$, then for some collection $\mathcal{H}$ of hyperplanes each of the associated open orthants contains not more than $2d$ points from D.*

Remark 5.3. Note that the proof of Theorem 5.1 is based on the "accident" that the convex curve Γ_4 and the associated solution manifold Σ' are $\mathbb{Z}/2$-spaces, where $\mathbb{Z}/2$ is the group generated by the reflection R_L. As a consequence Σ' determines a nontrivial element in the group $\Omega(\mathbb{Z}/2)$ of $\mathbb{Z}/2$-bordisms [CoFl64] which means that one should be able to repeat the pattern of the proof of the equipartition theorem in the planar case, Section 2.3. Our proof essentially follows this idea. For example the definition of ψ' corresponds to the choice of a non-trivial $\mathbb{Z}/2$-equivariant, line bundle on the circle Σ', a step used in the proof that Σ' defines a non-trivial element in $\Omega(\mathbb{Z}/2)$.

5.2 Measures admitting a center or a 3-plane of symmetry

For completeness we include proofs of 4-equipartition results for measures in $\mathbb{R}^4$ which admit a center or a 3-plane of symmetry, Proposition 5.6 and Proposition 5.8. These results formally resemble Theorem 5.1 but there are important differences. Both Propositions 5.6 and 5.8 are easily deduced from Hadwiger's result about 3-equipartitions of measures in $\mathbb{R}^3$, Theorem 5.4. Contrary to this, the proof of Theorem 5.1 relies on the existence of a unique, ballanced, 4-bit binary Gray code, so this result appears to be an essential 4-dimensional phenomenon.

Theorem 5.4. ([Had66]) *Each continuous mass distribution μ in $\mathbb{R}^3$ admits a 3-equipartition, that is a collection H_1, H_2, H_3 of three planes in $\mathbb{R}^3$ dissecting the ambient space into eight octants of equal measure. Moreover, the first of these planes can be chosen to contain arbitrary two points $A, B \in \mathbb{R}^3$ prescribed in advance.*

Remark 5.5. Hadwiger [Had66] deduced Theorem 5.4 from the result that any two measures in $\mathbb{R}^3$ admit a simultaneous equipartition by 2 hyperplanes. Both results of Hadwiger can be proved, along the lines of the CS/TM-scheme, by an analysis of the set of all equipartitions for measures with compact support, concentrated on the convex curve $M_3 = \{(t, t^2, t^3) \mid t \in \mathbb{R}\}$, see [Živ98] Proposition 4.9.

Proposition 5.6. ([Zie04]) *Suppose that μ is a continuous mass distribution in $\mathbb{R}^4$ which has a center of symmetry O. Then μ admits a 4-equipartition. Moreover, one of the hyperplanes can be choosen in advance as an arbitrary 3-plane passing through the center of symmetry O.*

Proof: Choose a "halving" hyperplane H_1 for μ. One can assume that $O \in H_1$. Let $\pi : \mathbb{R}^4 \to H_1$ be the orthogonal projection and let ν be the measure on H_1 defined by $\nu(A) := \mu(\pi^{-1}(A) \cap H_1^+)$ where H_1^+ is a closed halfspace bounded by H_1. Find 2-planes P_2, P_3, P_4 in H_1 which form a 3-equipartition for ν such that $O \in P_i$ for each i. This is always possible by Theorem 5.4. Then $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ is a 4-equipartition for μ where $H_j := \pi^{-1}(P_j)$ for $j \geq 2$ is the hyperplane orthogonal to H_1 at P_j. $\square$

Corollary 5.7. *There does not exist a centrally symmetric, convex closed curve in $\mathbb{R}^4$.*

Proof: The "arc length"-measure on such a curve would be centrally symmetric. According to Proposition 5.6, its space of 4-equipartitions is "fibered" over the Grassmannian of all affine 3-planes in $\mathbb{R}^4$, hence it is at least 4-dimensional. This is a contradiction. Indeed, by the results of Section 3, the solution manifold of all 4-equipartitions of a measure concentrated on a closed, convex curve in $\mathbb{R}^4$ must be 1-dimensional. $\square$

Proposition 5.8. *Suppose that μ is a continuous mass distribution in $\mathbb{R}^4$ which has a 3-plane of symmetry. Then there exists a 4-equipartition for μ.*

Proof: The proof is similar to the proof of Proposition 5.6, so we omit the details. $\square$

5.3 Topological "counterexample"

Theorem 5.9. *There exists a W_4-equivariant map $f : (S^4)^4_\delta \to S(U_4)$.*

Proof: By [Kosch81] §3, a W_4-equivariant map $f : (S^4)^4_\delta \to S(U_4)$ exists if and only if the obstruction $\omega \in \Omega_1(M; \mathcal{E} - TM)$ in the corresponding group of normal bordisms vanishes, cf. Section 4. By Proposition 4.1, ω is in the image of the map δ_1. By Proposition 4.9 the map $\sigma \circ j_2$ is onto and the result follows from the exactness of the sequence (3). $\square$

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