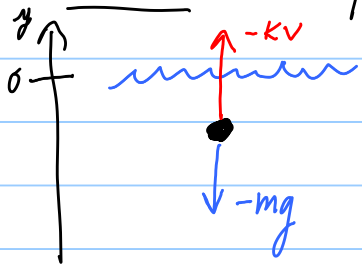
Lesson 2 MA 266 Steve Bell www.math.purdue.edu/~bell/MA266

Grader for written HWK : Wei-Cheng Chu chu116@purdue.edu

Problem: Drop cannonball in ocean:



Stokes' Law: Resistance $= -kv$

Newton's: $F = ma$

$$-mg - kv = m\frac{dv}{dt}$$ ← First order linear ODE

$$\frac{dv}{dt} = -g - \frac{k}{m}v$$ ← Separable ODE

$$\frac{dv}{g+\frac{k}{m}v} = -dt$$

$$\int \frac{dv}{\underbrace{g+\frac{k}{m}v}_{u}} = -\int dt = -t + C$$

$du = \frac{k}{m}dv$

$$\frac{m}{k}\ln\left|g+\frac{k}{m}v\right| = -t + C$$

$$\ln\left|g+\frac{k}{m}v\right| = -\frac{k}{m}t + \underbrace{\left(\frac{k}{m}C\right)}_{\tilde{C}}$$

$$\left|g+\frac{k}{m}v\right| = e^{-\frac{k}{m}t}\underbrace{e^{\tilde{C}}}_{\tilde{\tilde{C}}>0} = \tilde{\tilde{C}}e^{-\frac{k}{m}t}$$

So $$g+\frac{k}{m}v = \underbrace{\pm\tilde{\tilde{C}}}_{\tilde{\tilde{\tilde{C}}}\neq 0} e^{-\frac{k}{m}t}$$

$$v = -\frac{mg}{k} + \underbrace{\frac{m\tilde{\tilde{\tilde{C}}}}{k}}_{C} e^{-\frac{k}{m}t}$$

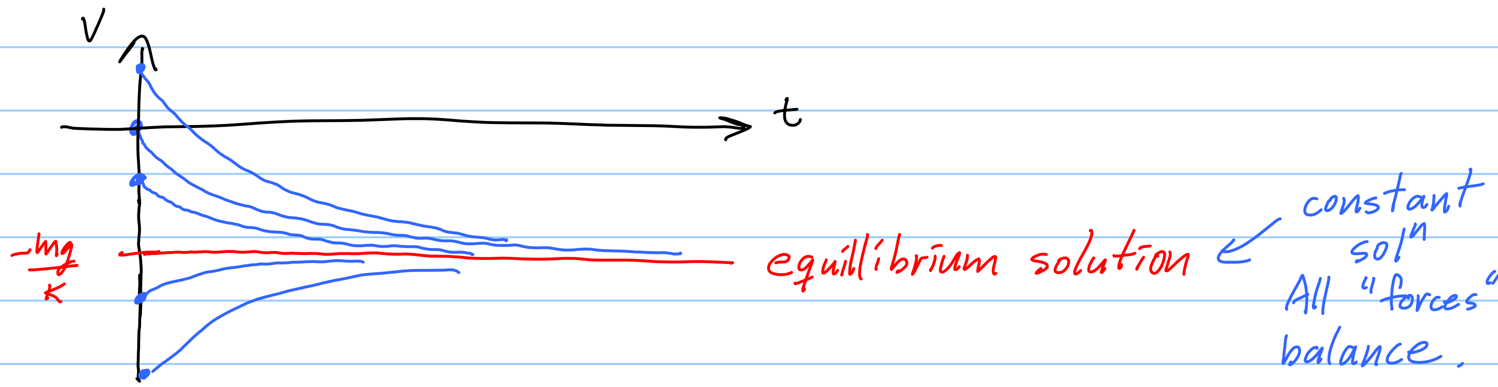
$$v = \underbrace{\frac{-mg}{k}}_{\text{terminal velocity}} + Ce^{-\frac{k}{m}t}$$ ← dies away!

Say $v(0) = -V_0$.

$$\underbrace{v(0)}_{-V_0} = \frac{-mg}{k} + Ce^{-\frac{k}{m}\cdot 0}$$

$$-V_0 = \frac{-mg}{k} + C\cdot 1 \qquad \boxed{C = \frac{mg}{k} - V_0}$$

$$\boxed{V = \frac{-mg}{k} + \left(\frac{mg}{k} - V_0\right)e^{-\frac{k}{m}t}}$$



Separable Eqns:

Implicit Diff'tion:

Ex: $x^2 + y^2 = c$ ← defines fcn $y(x)$

Diff: $2x + 2y\cdot\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Method: $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ $\qquad \int f(x) = F(x) + C$

$\qquad\qquad\qquad\qquad\qquad\qquad \int g(y) = G(y) + C$

$$g(y)\,dy = f(x)\,dx$$

$$\int g(y)\,dy = \int f(x)\,dx$$

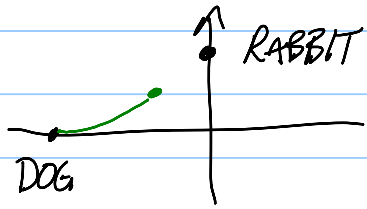
$G(y) = F(x) + C$ ← defines $y(x)$ implicitly

Check: $\frac{d}{dx}G(y) = \frac{d}{dx}F(x) + 0$

$$\underbrace{G'(y)}_{g(y)}\frac{dy}{dx} = \underbrace{F'(x)}_{f(x)}$$

Aha! $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$

EX:



$$-x\,\frac{d^2y}{dx^2} = \frac{v_R}{v_D}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Let $p = \dfrac{dy}{dx}$. Then $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$.

ODE: $-x\,\dfrac{dp}{dx} = \dfrac{v_R}{v_D}\sqrt{1+p^2}$ ← Separable!

$$\underbrace{\int \frac{dp}{\sqrt{1+p^2}}}_{\sinh^{-1} p} = -\frac{v_R}{v_D}\int \frac{1}{x}\,dx$$

$$\sinh^{-1} p = -\frac{v_R}{v_D}\ln|x| + C$$

$$\frac{dy}{dx} = \sinh\left(-\frac{v_R}{v_D}\ln|x| + C\right)$$

Clean up ($e^{c\ln|x|} = |x|^c$). Integrate. Get $y(x)$!

Linear vs. Non-Linear:

EX: $\dfrac{dy}{dx} = y^2$ Non-linear.

$$\int \frac{1}{y^2}\,dy = \int dx$$

$$-\frac{1}{y} = x + C$$

$$y = \frac{-1}{x+C}$$ ← ouch! Blows up when $x = -C$

EX: $e^x \frac{dy}{dx} + (\sin x) y = e^{-x}$ ← First order linear ODE.

Solution exists for all time x!

WebAssign HWK: Lessons 1, 2 due Thurs night at 11:00 pm.
Lesson 3 due Tues 11pm.

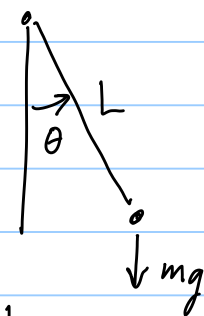
Written parts: Lesssons 1, 2, 3 due Wed in class.

EX: $(x^2 + e^x) \frac{d^3y}{dx^3} + (\tan x) \frac{d^2y}{dx^2} + (\sqrt{x}) \frac{dy}{dx} + (x^5 + x) y = \sec x$

3rd order linear ODE.

EX: $\left(\frac{dy}{dx}\right)^2 = y$ non-linear

EX: Non-linear pendulum:



$F = mg$

$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin\theta$ ← non-linear!

Linearize for small θ: $\sin\theta \approx \theta$