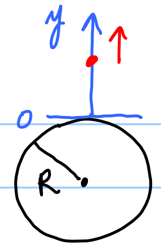
Lesson 6 Applications 2.3

WebAssign Lesson 5 due Tonight 11pm
Written Lessons 4, 5 due in class Wed.



$$F = ma$$

$$\frac{-GMm}{(R+y)^2} = m\frac{dv}{dt} = m\frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{v} = mv\frac{dv}{dy} \leftarrow \text{Sepapable}$$

Get $\frac{v^2}{2} = \frac{GM}{R+y} + C$ $\quad$ $v = v_0$ when $y=0$, $t=0$

Get $C = \frac{v_0^2}{2} - \frac{GM}{R}$

$$(*) \quad v = \sqrt{\frac{2GM}{R+y} + \underbrace{\left(v_0^2 - \frac{2GM}{R}\right)}}$$

<0, ball turns around and comes back

≥ 0, never comes back!

Escape velocity: $v_0 = \sqrt{\frac{2GM}{R}}$

Prob: Find $y(t)$ when $v_0 = \sqrt{\frac{2GM}{R}}$ (Esc. Vel.)

Then $(*)$ becomes $\underbrace{v}_{\frac{dy}{dt}} = \sqrt{\frac{2GM}{R+y}} \leftarrow$ Separable!

$$\int \sqrt{R+y}\, dy = \int \sqrt{2GM}\, dt$$

$$(\dagger) \quad \frac{2}{3}(R+y)^{3/2} = \sqrt{2GM}\, t + K \leftarrow \text{Find } K \text{ now!}$$

When $t=0$: $\begin{cases} v = v_0 \\ y = 0 \end{cases}$

$$\frac{2}{3}(R+0)^{3/2} = \sqrt{2GM}\cdot 0 + K$$

So $\boxed{K = \frac{2}{3}R^{3/2}}$

Solve $(\dagger)$ for y:

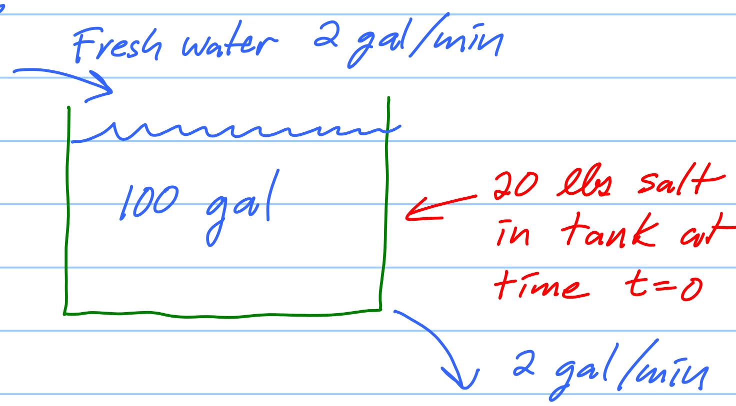
$$(R+y)^{3/2} = \frac{3}{2}\sqrt{2GM}\, t + R^{3/2}$$

$$y = \left[\frac{3}{2}\sqrt{2GM}\, t + R^{3/2}\right]^{2/3} - R$$

See $\lim_{t\to\infty} y(t) = \infty$.

Formula for v shows: $\lim_{t\to\infty} v(t) = 0$

Modeling: Mixing problem:



Let $S(t)$ = salt in tank at time t

Initial condition: $S(0) = 20$

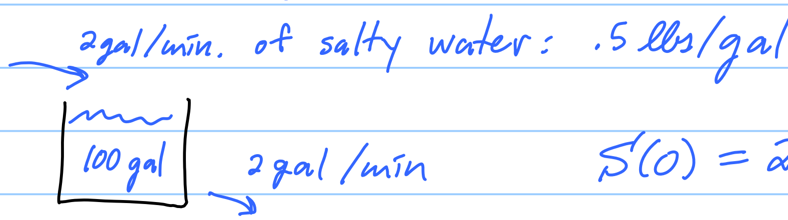
$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= \Big[(2\,\text{gal/min})\underbrace{(\text{lbs salt/gal})}_{\substack{0\\(\text{fresh})}}\Big] - \Big[(2\,\text{gal/min})\left(\frac{S(t)}{100\,\text{gal}}\right)\Big]$$

Get ODE: $\boxed{\dfrac{dS}{dt} = -\dfrac{1}{50} S}$

Sol^n: $S(t) = S_0 e^{-\frac{1}{50}t} = \underline{\underline{20 e^{-t/50}}}$

See $\lim_{t\to\infty} S(t) = 0$, but $S(t)$ is never $= 0$!



Prob: $S(0) = 20$ lbs.

$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= (2\ \text{gal/min})(.5\ \text{lbs/gal}) - (2\ \text{gal/min})\left(\frac{S(t)}{100\ \text{gal}}\right)$$

$$\frac{dS}{dt} = 1 - \frac{1}{50} S \quad \leftarrow$$ Separable and linear ✓

$$\boxed{\frac{dS}{dt} + \underbrace{\frac{1}{50}}_{P(t)} S = 1}$$

1 ✓

1. Stand. Form ✓

2. Int. factor

$$u = e^{\int \frac{1}{50} dt} = e^{t/50}$$

$$\underbrace{e^{t/50} (\text{L.H.S.})}_{} = 1 \cdot e^{t/50}$$

$$\frac{d}{dt}\left[e^{t/50} S\right] = e^{t/50}$$

Integrate: $e^{t/50} S = \int e^{t/50} dt = 50 e^{t/50} + C$

Get C: $S(0) = 20$

$$e^0 \cdot 20 = 50 \cdot e^0 + C \qquad \boxed{C = -30}$$

Solve for S: $\boxed{S(t) = 50 - 30 e^{-t/50}}$

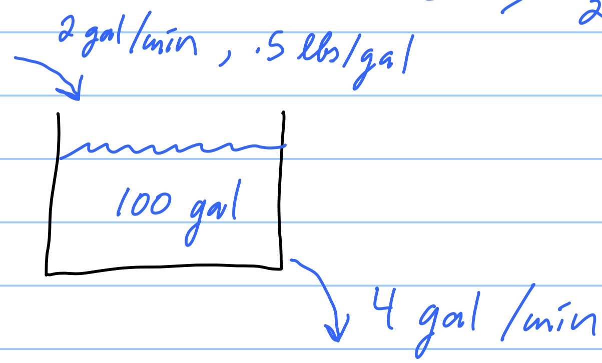
See $\lim_{t\to\infty} S(t) = 50$ lbs.

Ques: What's the limiting concentration of salt in tank as $t \to \infty$?

Concentration: $C(t) = \dfrac{S(t)}{100\text{ gal}} = \dfrac{50}{100} - \dfrac{30}{100} e^{-t/50}$

$\longrightarrow \frac{1}{2}$ lbs/gal as $t \to \infty$ ✓

Prob:



$V(t)$ = volume of water

$\dfrac{dV}{dt} = (\text{Rate in}) - (\text{Rate out})$
$= 2 - 4 = -2$

$V(t) = -2t + C$

$V(0) = 100, \quad C = 100$

$V(t) = 100 - 2t$

$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= \left[(2\text{ gal/min})\cdot(.5\text{ lbs/gal})\right] - \left[(4\text{ gal/min})\cdot\frac{S(t)}{\underbrace{V(t)}_{100-2t}}\right]$$

$$\frac{dS}{dt} = 1 - \frac{4}{100-2t} S \quad \leftarrow \text{linear!}$$

$$\boxed{\frac{dS}{dt} + \frac{2}{50-t} S = 1}$$

$\ln a^b = b \ln a$

$$u = e^{\int \frac{2}{50-t}\,dt} = e^{-2\ln(50-t)}$$

$$u = e^{\ln(50-t)^{-2}} = \frac{1}{(50-t)^2}$$

$$\frac{d}{dt}\left(\frac{1}{(50-t)^2}\cdot S\right) = 1\cdot\frac{1}{(50-t)^2}$$

$$\frac{S}{(50-t)^2} = \int \frac{1}{(50-t)^2}\,dt = \frac{1}{50-t} + C$$

Get C : $S(0) = 20$

$$\frac{20}{(50-0)^2} = \frac{1}{(50-0)} + C$$

$$C = \frac{20}{2500} - \frac{1}{50} = \frac{-3}{250}$$

Get $\boxed{S(t) = (50-t) - \frac{3}{250}(50-t)^2}$

Prob: What is limiting concentration when tank is empty?

$$C(t) = \frac{S(t)}{V(t)} = \frac{S(t)}{2(50-t)} = \frac{1}{2} - \frac{3}{500}(50-t)$$

limiting conc.

problem is over at $t=50$!