Lesson 8 Existence & Uniqueness, Linear v.s. non-Linear

English system: mass = slugs $\qquad mg = 180$ lbs.

Linear: $A(x)\frac{dy}{dx} + B(x)\,y = C(x)$ $\qquad$ I.V.P. $y(x_0) = y_0$

Step 1: Standard form: Divide by $A(x)$.

[Zeroes of $A(x)$ might be bad!]

Get $\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$

Step 2: Get $u = e^{\int P(x)dx}$. Take $u = e^{\int_{x_0}^{x} P(t)\,dt}$

Note: If P continuous, get u that is continuously diff'ble and non-vanishing. $\boxed{u(x_0) = 1}$

Step 3:

$$\underbrace{u\left[y' + Py\right]}_{[uy]'} = uQ$$

So $uy = \underbrace{\int uQ\,dx}_{\text{Take } \int_{x_0}^{x} u(t)Q(t)\,dt} + C$

Find C: $y = y_0$ when $x = x_0$.

$$\underbrace{u(x_0)}_{1}\,\underbrace{y(x_0)}_{y_0} = \underbrace{\int_{x_0}^{x_0} \sim\, dt}_{0} + C$$

Get $\boxed{C = y_0}$

Step 4:
$$y(x) = \frac{1}{u(x)} \int_{x_0}^{x} u(t)Q(t)\,dt + \frac{y_0}{u(x)}$$

If Q is continuous too, get a y that is continuously diff'ble.

Fact: (E & U for linear eqns). If (in Stand. Form), P and Q are continuous, then sol^n is continuously diff'ble and unique.

EX: $x^2 y' + 2x\,y = 4x^3 \qquad y(1) = 3$

Stand. Form $\qquad y' + \frac{2}{x} y = 4x$

E & U Thm says sol^n to IVP is good on $(0, \infty)$.

Sol^n: $u = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

$$\underbrace{x^2 y' + 2xy}_{[x^2 y]'} = 4x^3$$

$$x^2 y = \int 4x^3\,dx = 2x^4 + C$$

$$\boxed{y = x^2 + \frac{C}{x^2}}$$

$y(1) = 1^2 + \frac{C}{1^2} \overset{\text{want}}{=} 3$

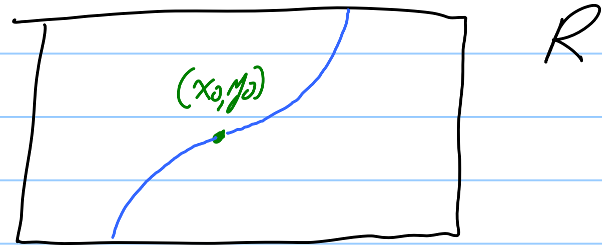
$\boxed{C = 2}$

$y = x^2 + \frac{2}{x^2}$

boom! at $x=0$,

Odd: Exactly one sol^n $(C=0)$ is nice at $x=0$!

Non-linear: $\dfrac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$



Suppose 1) f is continuous on R

2) $\dfrac{\partial f}{\partial y}$ exists and is continuous on R

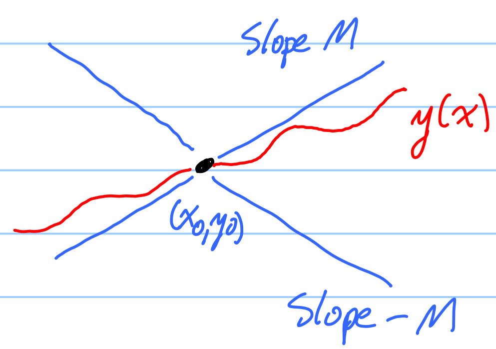
Then, a unique sol^n exists through (x_0, y_0) and it is continuously diff'ble as long as it stays in R.

Note: Can only say we get a sol^n on interval $[x_0 - h, x_0 + h]$ for some $h > 0$.

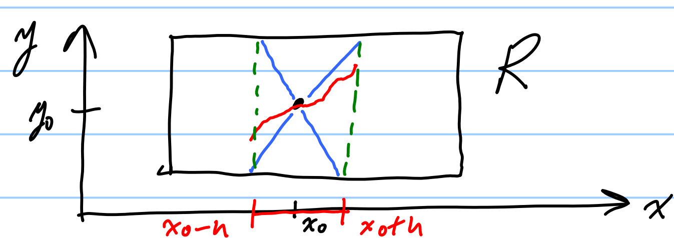
Can you guarantee a certain h size?

MA 161: Mean Value Theorem $\quad |y'(x)| \leq M$

$y(x_0) = y_0$



Say $|f(x,y)| < M$ on R:

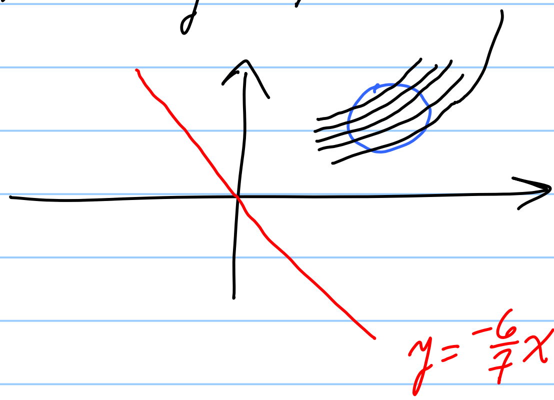


Good on $[x_0 - h, x_0 + h]$

EX: $y' = \frac{x-y}{6x+7y} \leftarrow f(x,y)$ continuous if $6x+7y \neq 0$

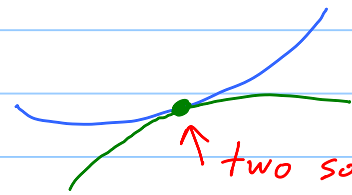
$$\frac{\partial f}{\partial y} = \frac{(-1)(6x+7y)-(x-y)7}{(6x+7y)^2} = \frac{-7x}{(6x+7y)^2}$$ same!

Bad line: $y = \frac{-6}{7}x$

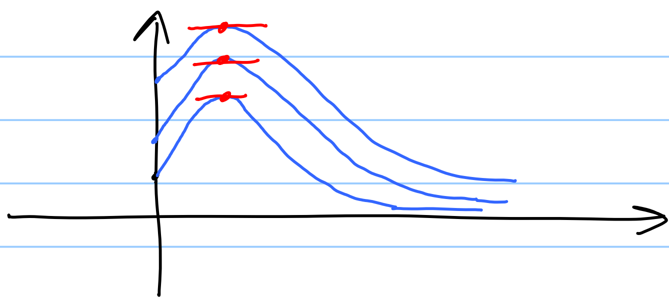


Uniqueness means:

1) Solutions don't cross.
2) They don't osculate either



HwK prob: $y' = y(5-ty)$



Hint: Tops of humps happen where $y' = f(x,y) = 0$

Bernouli Eqn: $y' - 5y = -t\,y^2$

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$$

Change of vars: $v = y^{1-n}$

New eqn in v is linear! Solve it. Get $y = v^{1/(1-n)}$.

EX: $y' - y = -y^{2}$ ← $n=2$ $\qquad v = y^{1-2} = \frac{1}{y}$

Divide eqn by $-y^2$: $\qquad$ So $\frac{dv}{dx} = \frac{d}{dx}(y^{-1}) = \frac{-1}{y^2}\frac{dy}{dx}$

$$\underbrace{-\frac{1}{y^2}y'}_{dv/dx} + \underbrace{\frac{1}{y}}_{v} = 1$$

Aha!

$$\frac{dv}{dx} + v = 1$$

$$e^x(v' + v) = e^x$$

$$[e^x v]' = e^x$$

$$e^x v = e^x + C$$

$$v = 1 + Ce^{-x}$$

Finally, $\boxed{y = \frac{1}{v} = \frac{1}{1+Ce^{-x}}}$