Lesson 9 Autonomous ODE, Population dynamics

Lesson 8 WebAssign probs due tonight 11 pm
Lessons 6,7,8 Written probs due Wed in class

Exam 1 in class
Fri, Feb. 9
over Lessons 1-11

2-D autonomous:

$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$$

No t on R.H.S.

1-D autonomous:

$$\frac{dy}{dt} = f(y)$$ ← no t on right

Ex 1:

$$\frac{dy}{dt} = ry$$ ← population growth

$$y = y_0 e^{rt}$$

Ex 2:

$$\frac{dy}{dt} = \left[r\left(1 - cy\right)\right]y$$

rate goes down as y ↑

Notation:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$ ← Logistic eqn

Bernoulli Eqn!

$$\frac{dy}{dt} - ry = -\frac{r}{K}y^2 \quad (*)$$ ← n

$$V = y^{1-2} = \frac{1}{y}$$

$$\frac{dv}{dt} = \frac{d}{dt}\left[y^{-1}\right] = -y^{-2}\cdot\frac{dy}{dt} = -\frac{1}{y^2}\frac{dy}{dt}$$

Divide (*) by $-y^2$:

$$-\frac{1}{y^2}\frac{dy}{dt} + r\frac{1}{y} = \frac{r}{K}$$

$$\frac{dV}{dt} + rV = \frac{r}{K}$$

$u = e^{\int r\,dt} = e^{rt}$: $\quad [e^{rt}v]' = \frac{r}{k}e^{rt}$

(†) $e^{rt}v = \int \frac{r}{k}e^{rt}\,dt = \frac{1}{k}e^{rt} + C$

Find C: $\quad y(0) = y_0$

$v = \frac{1}{y}$. So $v(0) = \frac{1}{y_0}$

$e^0 \frac{1}{y_0} = \frac{1}{k}\cdot e^0 + C$

So $\boxed{C = \frac{1}{y_0} - \frac{1}{k}}$

Solve (†) for v:

$$\underbrace{v}_{\frac{1}{y}} = \frac{1}{k} + \left(\frac{1}{y_0} - \frac{1}{k}\right)e^{-rt}$$

Get y: $\quad y = \dfrac{1}{\frac{1}{k} + \left(\frac{1}{y_0} - \frac{1}{k}\right)e^{-rt}} \cdot \dfrac{y_0 k}{y_0 k} = \dfrac{y_0 k}{y_0 + (k - y_0)e^{-rt}}$

All solns $\longrightarrow k$ as $t \to \infty$!

Qualitative analysis:

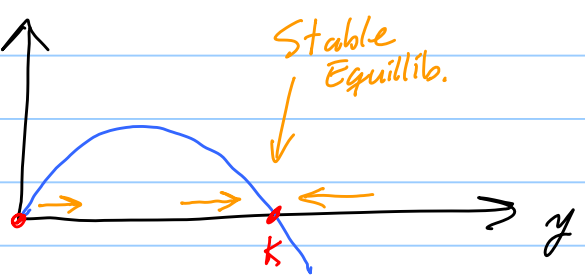
1) Equillibrium sol^{n}s: Constant sol^{n}s $\quad y \equiv Z$

$\underbrace{\frac{dy}{dt}}_{0} = \underbrace{f(y)}_{0}$ where Z is a zero of $f(y)$.

EX: $\quad f(y) = r\left(1 - \frac{y}{k}\right)y = 0$

$\begin{cases} y \equiv 0 \\ y \equiv k \end{cases}$ equillibrium sol^{n}s.

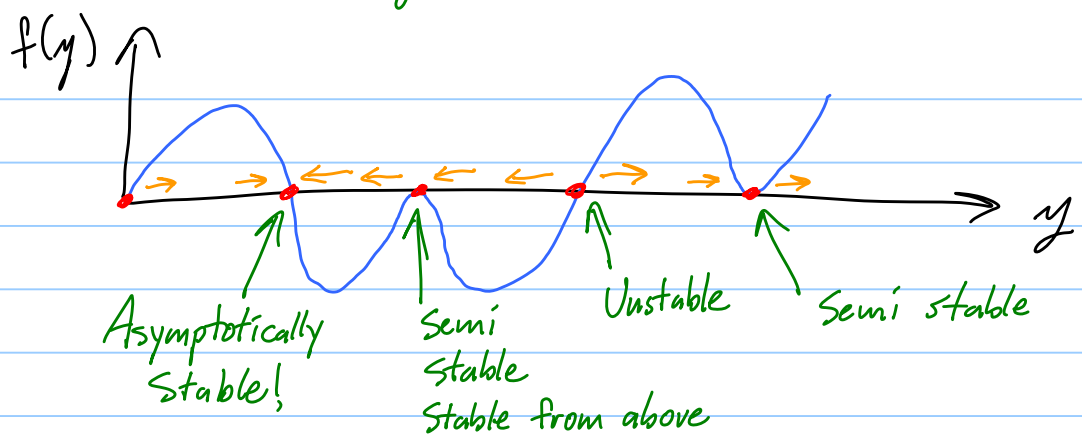
Graph of $f(y)$:



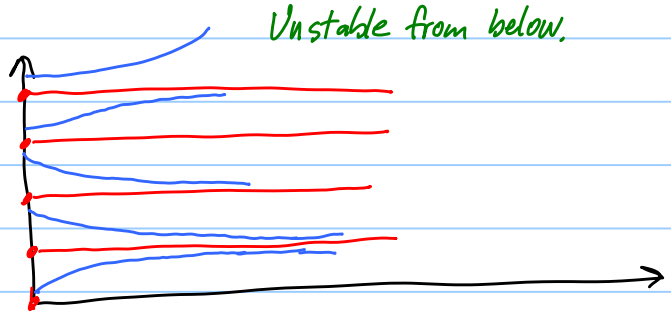
Think $y' = f(y)$

General case: $y' = f(y)$

Equilibrium sol'ns $y \equiv z$



Stable from above
Unstable from below.



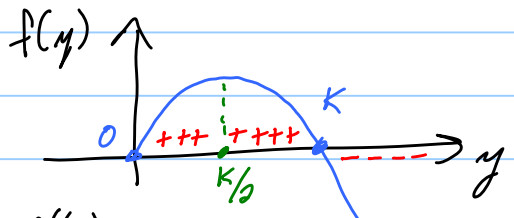
Logistic eqn: Prob: Where are inflection points?

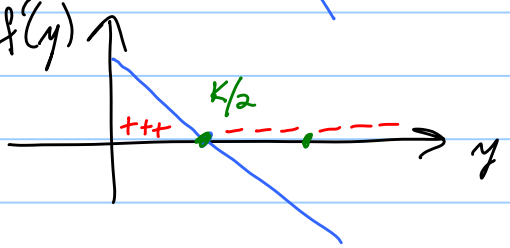
Points where $\frac{d^2y}{dt^2}$ changes sign.

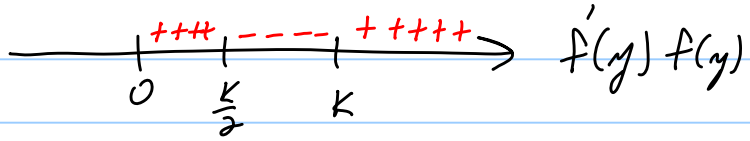
Hmmm: $\boxed{\frac{dy}{dt} = f(y)}$ Aha!

So $\frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f'(y)\,f(y)$

Logistic eqn







Inflection points only happen if $0 < y_0 < K$ and happen when $y = \frac{K}{2}$.

EX: $\frac{dy}{dt} = -r y \left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)$ Carrier Pidgeons