Lesson 11 Euler's method

Last time $M(x,y)\,dx + N(x,y)\,dy = 0$

Not exact: $\frac{\partial}{\partial x}\Big[\underbrace{\text{Thing in front of } dy}_{N}\Big] \neq \frac{\partial}{\partial y}\Big[\underbrace{\text{Thing in front of } dx}_{M}\Big]$

Special case: Look for an "integrating factor" u that is a function of x alone: $u(x)$

Mult by u: $(uM)\,dx + (uN)\,dy = 0$

Need $\frac{\partial}{\partial x}[uN] = \frac{\partial}{\partial y}[uM]$

$$u'N + uN_x = uM_y$$

$$\boxed{\frac{u'}{u} = \frac{M_y - N_x}{N}}$$

Aha! This will work if $\frac{M_y - N_x}{N}$ is a fcn of x alone. Say $P(x)$.

Then $\frac{\frac{du}{dx}}{u} = P$. Get

$$u = e^{\int P(x)\,dx}$$

EX: $\underbrace{(2x - y^2)}_{M} + \underbrace{(xy)}_{N}\frac{dy}{dx} = 0$

$M_y - N_x = -2y - y = -3y$ ← Not zero, Not exact

But $\frac{M_y - N_x}{N} = \frac{-3y}{xy} = -\frac{3}{x}$ ← fcn of x alone!

$$\frac{u'}{u} = \frac{M_y - N_x}{N} = -\frac{3}{x}$$

$$\frac{d}{dx}\ln|u| = -\frac{3}{x}$$

$$\ln|u| = -3\int \frac{1}{x}\,dx = -3\ln|x| + C$$

$$e \qquad\qquad\qquad e$$

$$|u| = e^{-3\ln|x|}\,e^{C}$$

$$u = \underbrace{\pm e^{C}}_{K}\, e^{\ln|x|^{-3}} = K\,|x|^{-3} = \underbrace{\pm K}_{\tilde{K}}\, x^{-3}$$

Take $\tilde{K} = 1$. Get $\boxed{u(x) = \frac{1}{x^3}}$

Use it: $$uM + uN y' = 0$$

$$\underbrace{\frac{2x - y^2}{x^3}}_{\mathcal{M}} + \underbrace{\frac{xy}{x^3}}_{\mathcal{N}}\, y' = 0 \qquad \text{is exact!}$$

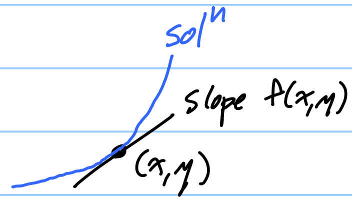
Get φ with $\frac{\partial \varphi}{\partial x} = \mathcal{M}$ and $\frac{\partial \varphi}{\partial y} = \mathcal{N}$.

Then $\varphi(x,y) = C$ gives sol^n $y(x)$ implicitly.

Sometimes you can solve for y, sometimes not.

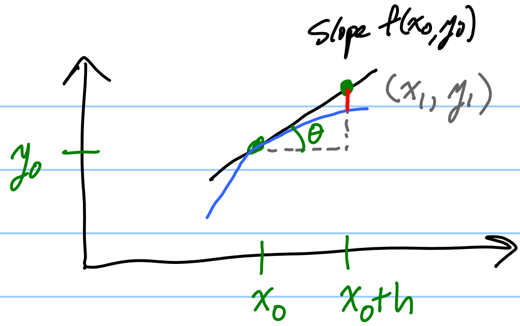
Fine points: Can cook up φ's on rectangles when eqn is exact, or more generally on regions without holes.

Euler's method: $\frac{dy}{dx} = f(x,y) \leftarrow$ slope of tangent line to $y = y(x)$ at (x,y)



Big idea: follow tangent line for small interval h.

Start at <u>initial condition</u> $y(x_0) = y_0$



] = error

$$x_1 = x_0 + h$$

$$y_1 = y_0 + \underbrace{(\text{Base } \Delta)}_{h} \underbrace{\tan\theta}_{f(x_0,y_0)}$$

$$\boxed{y_1 = y_0 + h\,f(x_0, y_0)}$$

Repeat!

$$\boxed{\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h\,f(x_n, y_n) \end{aligned}}$$

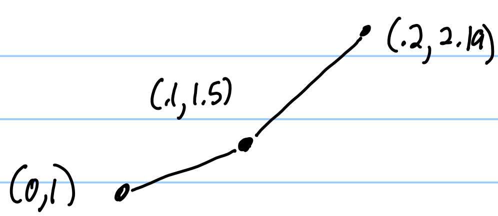
EX:

$$\frac{dy}{dx} = \underbrace{1 - x + 4y}_{f(x,y)} \qquad \underset{x_0=0}{y(0)} = \underset{y_0=1}{1}$$

Step size $h = .1$

$$y_1 = y_0 + h\,f(x_0,y_0) = 1 + (.1)\left[1 - 0 + 4\cdot 1\right] = 1.5$$

$$y_2 = y_1 + h\,f(x_1,y_1) = (1.5) + (.1)\left[1 - (.1) + 4(1.5)\right] = 2.19$$



h	x_n	y_n	$y_n + h\,f(x_n,y_n)$
0	0	1	1.5
1	.1	1.5	2.19
2	.2	2.19	3.146
3	.3	3.146	4.4744
4	.4	4.4744	6.32416
5	.5	6.3246	

MAPLE

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x(0) := 0. ;
y(0) := 1. ;
 N := 10 ; h := .1 ;
for n from 0 to N-1 do
      x(n+1) := x(n) + h
      y(n+1) := y(n) + h * (1 - x(n) + 4 * y(n))
   end do ;
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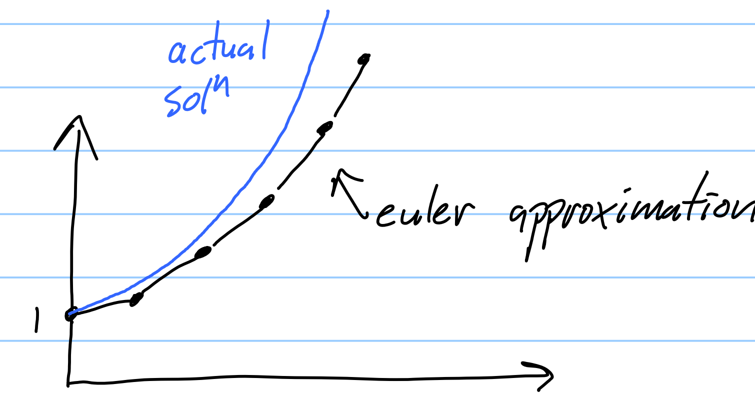
Typical plot :

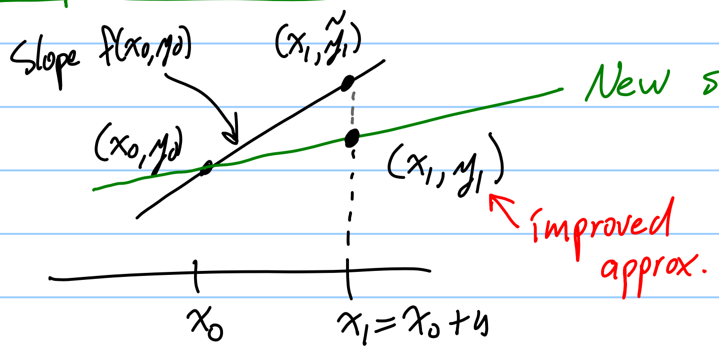


Easy case : $\frac{dy}{dx} = f(x)$.

Euler method : $y_n = y_0 + \sum_{n=0}^{N-1} f(x_n)h$ ← Riemann sum!

$$\longrightarrow y_0 + \int_{x_0}^{x} f(t)dt$$ ← actual sol^n!

Improved Euler Method :



New slope = Ave of $f(x_0,y_0)$ and $f(x_1,\tilde{y}_1)$

$$= \frac{f(x_0,y_0) + f(x_1,\tilde{y}_1)}{2}$$

$$x_{n+1} = x_n + h$$

$$\tilde{y}_{n+1} = y_n + h\, f(x_n, y_n)$$

$$\boxed{y_{n+1} = y_n - h\left[\frac{f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})}{2}\right]}$$

Easy case: $\dfrac{dy}{dx} = f(x)$

Improved euler $\quad y = y_0 + \underbrace{\text{Approx} \int_{x_0}^{x} f(t)\,dt}_{\text{Trapezoid method!}}$

Hmmm. What would give Simpson's rule here?

Ans: Runge-Kutta method