

Review for Exam 1

in class Friday 2/9.

Closed book,
closed notes, no
calculators, no phones.

WebAssign Lesson 12 due Mon. 11 pm

Problem 1.

Find the general solution of the ODE

$$y(0) = 3$$

Linear? Bernoulli?

Separable? Homogeneous?

Exact? (after algebra)

Int factor - fcn of x alone
or y -alone to make exact?

Numerical!

$$1 \cdot y' + 4y = e^{-2t}$$

$$P(t) \equiv 4$$

$$u = e^{\int P dt} = e^{4t}$$

$$e^{4t} (\text{Stand. Form}) = e^{4t} \cdot e^{-2t} = e^{2t}$$

$$\frac{d}{dt} [e^{4t} y]$$

$$\begin{aligned} t=0 \\ y=3 \\ \text{Get } C. \end{aligned}$$

$$\rightarrow e^{4t} y = \int e^{2t} dt = \frac{1}{2} e^{2t} + C$$

$$y = \underbrace{\frac{1}{2} e^{-2t}}_{y_p} + \underbrace{C e^{-4t}}_{\text{Gen. Sol.}^n}$$

$$y' + 4y \equiv 0$$

Problem 2.

Find the general solution of the ODE

~~$$ty' - 2y = -2t^3 e^{2t}$$~~

$$\begin{aligned} u = e^{\int -\frac{2}{t} dt} &= e^{-2 \ln |t|} \\ &= \frac{1}{t^2} = \frac{1}{t^2} \end{aligned}$$

$$\boxed{y' - \frac{2}{t} y = -2t^2 e^{2t}} \quad \text{S.F.}!$$

$$\left[\frac{1}{t^2} y \right]' = \frac{1}{t^2} (-2t^2 e^{2t})$$

$$\frac{1}{t^2} y = -2 \int e^{2t} dt = -e^{2t} + C$$

$$y = -t^2 e^{2t} + C t^2$$

Problem 3.

Find the general solution of

$$\frac{dy}{dx} = \frac{xy + 2x + y + 2}{(y+2)^2},$$

$$\frac{dy}{dx} = \frac{(y+2) [x+1]}{(y+2)^2} = \frac{x+1}{y+2}$$

Separable!

$$\int (y+2) dy = \int (x+1) dx$$

$$\frac{1}{2}y^2 + 2y = \frac{1}{2}x^2 + x + C$$

$$y = \frac{-2 \pm \sqrt{4 + 2(\frac{1}{2}x^2 + x + C)}}{1} = -2 \pm \sqrt{x^2 + 2x + K}$$

$K = 2C + 4$

Problem 4.

Find the general solution of

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

(Homogeneous)

$$\frac{dy}{dx} = \frac{3y - 2x}{y}$$

$$= 3 - 2\left(\frac{x}{y}\right)$$

Let $v = \frac{y}{x}$

$$y = xV$$

So $\frac{dy}{dx} = 1 \cdot V + x \frac{dV}{dx}$

$$\frac{dy}{dx} = 3 - \frac{2}{\left(\frac{y}{x}\right)}$$

$$V + x \frac{dV}{dx} = 3 - \frac{2}{V}$$

Separable!

$$x \frac{dV}{dx} = 3 - \frac{2}{V} - V = \frac{3V - 2 - V^2}{V}$$

$$\int \frac{-V dV}{V^2 - 3V + 2} = \int \frac{dx}{x} = \ln|x| + C$$

Partial Fractions!

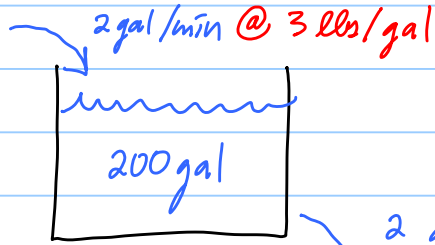
$$\frac{-V}{(V-1)(V-2)} = \frac{A}{V-1} + \frac{B}{V-2}$$

Don't forget $v = y/x$.**Problem 5.**

A tank contains 200 gal (gallons) of liquid. Initially, the tank contains pure water. At time $t = 0$, brine containing 3 lb/gal salt begins to pour into the tank at a rate of 2 gal/min, and the well-stirred mixture is allowed to drain away at the same rate. How many minutes must elapse before there are 100 lb of salt in the tank?

 $t = 0$

A. 600

B. $600 - 1/e$ C. $600 - 600/e$ D. $100 \ln(6/5)$ E. $-100 \ln(400)$ 

Let $Q(t)$ = lbs of salt in tank at time t in min.

$$\frac{dQ}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= (2 \text{ gal/min})(3 \text{ lbs/gal}) - (2 \text{ gal/min})\left(\frac{Q(t)}{200 \text{ gal}}\right)$$

$$Q(0) = 0 \text{ lbs}$$

$$\frac{dQ}{dt} = 6 - \frac{1}{100}Q \text{ Linear!}$$

$$Q' + \frac{1}{100}Q = 6$$

$$u = e^{t/100}$$

$$[e^{t/100} Q]' = 6e^{t/100}$$

$$e^{t/100} Q = 600e^{t/100} + C$$

$$Q = 600 + C e^{-t/100}$$

$$\uparrow C = -600$$

$$100 = 600(1 - e^{-t/100})$$

Solve for t .

Problem 6.

initial value problem

Linear!

What is the largest open interval in which a solution to the

 $P(t)$ $Q(t)$ $\downarrow$ $\downarrow$

$$(t-1)y' + \frac{\sqrt{t+2}y}{t-1} = \frac{3}{t-3}, \quad y(2) = -5$$

$$y' + P(t)y = Q(t)$$

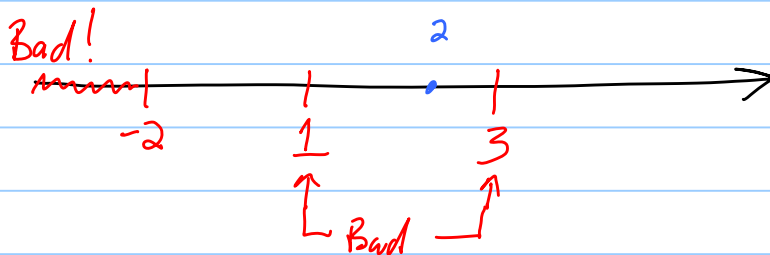
is guaranteed to exist?

A. (1, 3)

B. (2, 3)

C. (-2, 3)

D. (-2, 1)

E. (-2, ∞)

P and Q must both be continuous on the interval containing x_0 for the solⁿ to the IVP to be guaranteed to be continuously differentiable.

Problem 7.

Find all the asymptotically stable equilibrium solutions for the autonomous differential equation

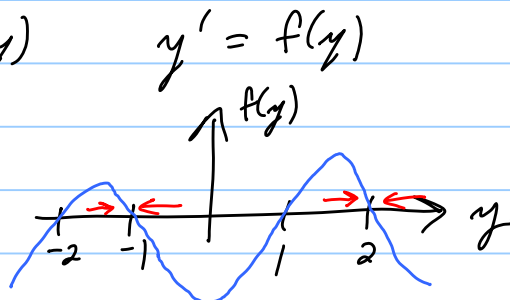
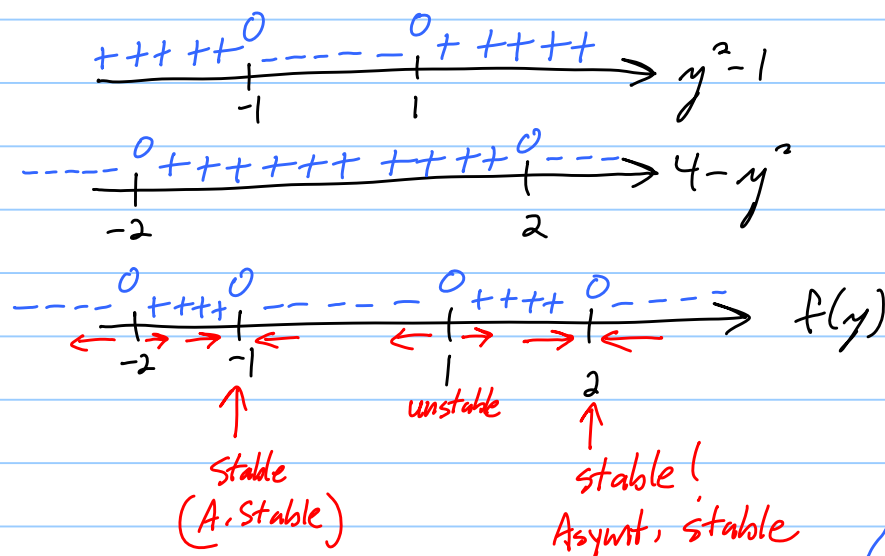
$$y' = (y^2 - 1)(4 - y^2). \leftarrow f(y)$$

equil. sol^{ns}
 $y \equiv c$

$$f(c) = 0$$

$$y \equiv \pm 1$$

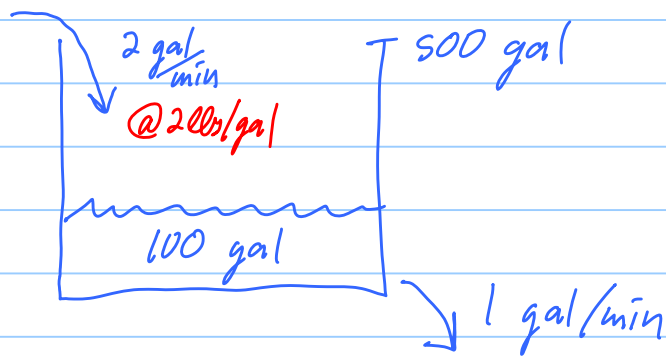
$$y \equiv \pm 2$$



$$y'' = f'(y) y' = f'(y) f(y)$$

Problem 8.

A tank with capacity 500 gal originally contains 100 gallons of water with a salt concentration of 1/2 lb/gal. A solution containing a salt concentration of 2 lb/gal enters at a rate of 2 gal/min and the well-stirred mixture is pumped out at the rate of 1 gal/min. What is the amount of salt in the tank after 50 min.



$$Q(0) = \left(\frac{1}{2}\right)(100) = 50$$

$$\frac{dQ}{dt} = (2 \cdot 2) - (1) \frac{Q(t)}{\underbrace{V(t)}_{100+t}}$$

Linear!

$$\frac{dV}{dt} = (\text{Rate in}) - (\text{Rate Out}), \quad V(0) = 100$$

$$= 2 - 1 = 1$$

$$V(t) = 100 + t$$

Problem 9.

A skydiver weighing 200 lb (with mass $25/4$ ^{← slugs} lb) falls vertically downward from an altitude of 4000 ft and opens the parachute after 10 seconds of free fall. Assume that the force of air resistance, which is directed opposite to the velocity, is $0.8|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in ft/sec. Use $g = 32 \text{ ft/sec}^2$.

(a) Find the speed of the skydiver when the parachute opens.

(b) Find the distance fallen before the parachute opens.

$y \uparrow$
 0 at 4000 ft

$$F = ma = m \frac{dv}{dt}$$

$$-mg - 0.8v = m \frac{dv}{dt} \leftarrow \text{Linear!} \quad \text{Get } v$$

$$\frac{dv}{dt} + \frac{.8}{m} v = -g = -32 \quad \leftarrow u = e^{\int \frac{.8}{m} dt} = e^{16t/125}$$

$$\uparrow (4/5) \quad \frac{16}{(25/4)} = \frac{16}{125}$$

$$e^{16t/125} v = -\int 32 e^{16t/125} dt = -250 e^{16t/125} + C \quad \left[\begin{array}{l} v(0)=0 \text{ gives} \\ C = 250 \end{array} \right]$$

$$v = 250(-1 + e^{-16t/125})$$

$$a) \quad v(10) = 250(-1 + e^{-160/125})$$

$$y = \int v dt = 250\left(-t - \frac{125}{16} e^{-16t/125}\right) + C \quad \left[\begin{array}{l} y(0)=0 \text{ gives} \\ C = \frac{250 \cdot 125}{16} \end{array} \right]$$

$$y = 250\left(-t - \frac{125}{16} e^{-16t/125} + \frac{125}{16}\right)$$

$$b) \quad y(10) = 250\left(-10 - \frac{125}{16}(e^{-160/125} - 1)\right)$$

Problem 10.

Find the implicit solution to the initial value problem

$$\frac{\partial M}{\partial y} = e^x \cos y - 2 \sin x \quad \frac{\partial N}{\partial x} = e^x \cos y - 2 \sin x = \frac{\partial M}{\partial y} \quad \checkmark \quad \text{Exact.}$$

$$\underbrace{(e^x \sin(y) - 2y \sin(x) - 1)}_M + \underbrace{(e^x \cos(y) + 2 \cos(x) + 3)}_N y' = 0, \quad y(0) = \pi.$$

$$(A) \quad \frac{\partial Q}{\partial x} = M$$

$$(B) \quad \frac{\partial Q}{\partial y} = N$$

$$(A) \text{ gives: } Q = \int e^x \sin y - 2y \sin x - 1 \, dx$$

$$= e^x \sin y + 2y \cos x - x + C(y)$$

$$\text{Now (B) yields: } \frac{\partial}{\partial y} [e^x \sin y + 2y \cos x - x + C(y)] \stackrel{\text{want}}{=} e^x \cos y + 2 \cos x + 3$$

$$e^x \cos y + 2 \cos x + C'(y)$$

$$C'(y) = 3$$

$$Q = e^x \sin y + 2y \cos x - x + 3y = C$$

$$\text{So } \boxed{C(y) = 3y}$$

Problem 11.

What is the general solution to this differential equation?

$$\frac{dy}{dx} = (x - y)^2 + 1.$$

Let $v = x - y$. Then $\frac{dv}{dx} = 1 - \frac{dy}{dx}$

and $\boxed{\frac{dy}{dx} = 1 - \frac{dv}{dx}}$

$$\frac{dy}{dx} = \underbrace{(x - y)^2 + 1}$$

$\uparrow$
 $1 - \frac{dv}{dx} = v^2 + 1$

$$\frac{dv}{dx} = -v^2$$

$$\int -v^{-2} dv = \int dx$$

$$\frac{1}{v} = x + C$$

$$v = \frac{1}{x + C}$$

$\uparrow$
 $x - y$

$$\boxed{y = x - \frac{1}{x + C}}$$