Lesson 13 2nd order linear ODE

WebAssign lesson 12 due tonight
Written Lesson 12 due Wed

Easiest 2nd order ODE : $\frac{d^2y}{dx^2} = -\sin x$

$$\frac{dy}{dx} = -\int \sin x \, dx = \cos x + C_1$$

$$y = \int (\cos x + C_1)\, dx = \underbrace{\sin x}_{\substack{y_p \\ \text{particular} \\ \text{sol}^n}} + \underbrace{C_1 x + C_2}_{\substack{\text{sol}^n \text{ to homog} \\ \text{eqn} \\ y'' \equiv 0.}}$$

2nd order linear ODE : $A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = D(x)$

Standard Form : $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad (*)$

where $P = \frac{B}{A}$, $Q = \frac{C}{A}$, $R = \frac{D(x)}{A(x)}$ ← zeroes of $A(x)$ are "bad"

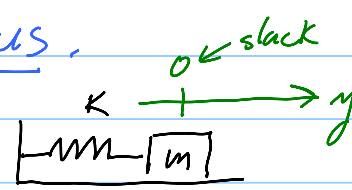
E. & U. Theorem : $(*)$ plus TWO initial conds : $y(x_0) = y_0$, $y'(x_0) = y_0'$

has a unique sol^n on any interval containing x_0

↑ sol^n EXISTS (E.) ↑ U.

where $P(x)$, $Q(x)$, $R(x)$ are continuous.

EX : $y'' + 3y' + 2y = 0$



$$F = ma$$

$$\underbrace{-ky}_{\text{Hooke's}} - cy' = my''$$

Hmmm. Try $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2e^{rx}$

Want $y'' + 3y' + 2y = 0$

$$(r^2e^{rx}) + 3(re^{rx}) + 2(e^{rx}) = 0$$

$$\underbrace{[r^2 + 3r + 2]}_{\text{must } = 0} \underset{\uparrow \text{ never } 0}{e^{rx}} \overset{\text{want}}{=} 0$$

Need $r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$

$\boxed{r = -1, -2.}$ $\leftarrow$ Got 2 sol^{n}s : $\boxed{e^{-t}, e^{-2t}}$

Important property of linear equs :

$$y'' + Py' + Qy = 0 \leftarrow \text{homogeneous equ}$$

Superposition Principal ; If y_1 and y_2 solve a linear homog. ODE, then so does $\underline{c_1y_1 + c_2y_2}$

Why : Write $L[y] = y'' + Py' + Qy$

Claim : L is a linear operator :

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2] \leftarrow \text{Def}^{n} \text{ Lin. Oper.}$$

$$(c_1y_1 + c_2y_2)'' + P(c_1y_1 + c_2y_2)' + Q(c_1y_1 + c_2y_2)$$

$$= c_1[y_1'' + Py_1' + Qy_1] + c_2[y_2'' + Py_2' + Qy_2]$$

$$= c_1\underbrace{L[y_1]}_{=0} + c_2\underbrace{L[y_2]}_{=0} \quad \text{if } y_1, y_2 \text{ solve homog.}$$

So $c_1y_1 + c_2y_2$ also solve the homog. equ !

Big Question: Is $c_1 e^{-t} + c_2 e^{-2t}$ the general sol^n to $y'' + 3y' + 2y = 0$?

Equivalent question: Can we solve any I.V.P. with it?

IVP : ODE plus $\begin{cases} y(0) = A \\ y'(0) = B \end{cases}$

Hmmm:
$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$
$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

$$\begin{cases} y(0) = c_1 e^{-0} + c_2 e^{-2\cdot 0} = A \\ y'(0) = -c_1 e^{-0} - 2c_2 e^{-2\cdot 0} = B \end{cases}$$

want

$$\begin{cases} c_1 + c_2 = A \\ -c_1 - 2c_2 = B \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = 1\cdot(-2) - (-1)\cdot 1 = -1 \leftarrow \underline{\text{not}} \text{ zero}$$

Cramer's Rule says c_1, c_2 exist (and are unique) for any given A, B.

Good news: Can solve any IVP. We have gen^l sol^n.

(E & U Thm says, if we find a sol^n satisfying initial conditions, it is *the* sol^n.)

Bummer: What if quadratic eqn $ar^2 + br + c = 0$ had a repeated double root, or complex roots?