Lesson 16 Repeated roots, reduction of order

Written Lessons 13, 14, 15 due Wed in class

WebAssign Lesson 15 due tonight; 16, 17 due Thurs.

$$L[y] = ay'' + by' + cy = 0 \leftarrow \text{homogeneous}$$

$$L[e^{rx}] = (ar^2 + br + c)e^{rx}$$
$$= a(r-r_1)(r-r_2)e^{rx}$$

Case r_1, r_2 real and $\neq$: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case r_1, r_2 complex: $r = A \pm Bi$ $\quad y = c_1 e^{Ax}\cos Bx + c_2 e^{Ax}\sin Bx$

Case $r_1 = r_2$: $r_1 = \left(\frac{-b}{2a}\right)$ repeated Get $y_1 = e^{r_1 x}$. Need y_2!

$$\frac{\partial}{\partial r} L[e^{rx}] = \frac{\partial}{\partial r}\left(a(r-r_1)^2 e^{rx}\right)$$

$$L\Big[\underbrace{\frac{\partial}{\partial r} e^{rx}}_{xe^{rx}}\Big] = a\,2(r-r_1)e^{rx} + a(r-r_1)^2 x e^{rx}$$

Aha! Let $r = r_1$!

$$L[\underbrace{xe^{r_1 x}}_{y_2}] = 0$$

Get gen'l sol'n: $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Other methods to find y_2: Reduction of order: Given one sol'n y_1, how to find a y_2.

EX: $y'' + 4y' + 4y = 0$

$$r^2 + 4r + 4 = 0$$
$$(r+2)^2 = 0 \qquad r = -2, -2. \qquad \text{Get } y_1 = e^{-2x}.$$

Try to find y_2 with $\boxed{y_2 = u y_1}$

Then $y_2' = u'y_1 + uy_1'$

$y_2'' = u''y_1 + 2u'y_1' + uy_1''$

Shove into ODE:

$$(y_2)'' + 4(y_2') + 4y_2 \overset{\text{want}}{=} 0$$

$$(u''y_1 + 2u'y_1' + uy_1'') + 4(u'y_1 + uy_1') + 4(uy_1)$$

$$\underbrace{u\,L[y_1]}_{=0} + u''y_1 + 2u'y_1' + 4u'y_1 \overset{\text{want}}{=} 0 \qquad y_1 = e^{-2x}$$

$$= u''e^{-2x} + 2u'(-2e^{-2x}) + 4u'e^{-2x} = 0$$

$$= u'' = 0 \qquad \text{Easy!}$$

Get $u = c_1 x + c_2$. $\quad y_2 = (c_1 x + c_2)e^{-2x}$

Just y_2 that's different. Best, cleanest one is $c_1 = 1$, $c_2 = 0$.

Get $y_2 = xe^{-2x}$.

Method of reduction of order: $A(x)y'' + B(x)y' + C(x)y = 0$

Have one sol^n y_1. Want a second sol^n y_2.

Step 1: Put in Standard Form: Divide by $A(x)$:

$$L[y] = y'' + P(x)y' + Q(x)y = 0$$

Know $L[y_1] = 0$.

Step 2: Try $y_2 = uy_1$ where u is fcn to pin down.

Plug into ODE: $L[y_2] = L[uy_1] =$

$$(u''y_1 + 2u'y_1' + uy_1'') + P(u'y_1 + uy_1') + Q(uy_1) \overset{\text{want}}{=} 0$$

$$\underbrace{u\,L[y_1]}_{=0} + u'' y_1 + 2u' y_1' + P u' y = 0$$

First order linear in $v = u'$ $\longrightarrow$

$$u'' + \left(2\frac{y_1'}{y_1} + P\right) u' = 0$$

$$\frac{dv}{dx} + \left(2\frac{y_1'}{y_1} + P\right) v = 0$$

Integrating factor: $\quad U = e^{\int \frac{2y_1'}{y_1} + P\,dx} =$

$$v = y_1^{-2} e^{-\int P\,dx}$$

Note $\frac{y_1'}{y_1} = \frac{d}{dx} \ln|y_1|$

$\int 2\frac{y_1'}{y_1}\,dx = 2\ln|y_1| = \ln y_1^2$

Step 3: Get the formula:

$$\boxed{u' = v = \frac{1}{y_1^2} e^{-\int P\,dx}}$$

Step 4: Integrate to get u. No $+C$.

EX: $y'' + 4y' + 4y = 0 \qquad y_1 = e^{-2x}$

P(x)

Stand. Form ✓

$y_2 = u y_1$ where $\boxed{u' = \frac{1}{y_1^2} e^{-\int P\,dx}}$

$$= \frac{1}{(e^{-2x})^2} e^{-\int 4\,dx} =$$

$$= \frac{1}{e^{-4x}} \cdot e^{-4x} = 1$$

So $u = \int 1\,dx = x + C$ $\longleftarrow$ take $C=0$ to remove a piece of y_1 from y_2.

$$y_2 = u y_1 = (x+c)e^{-2x}$$

Get $y_2 = xe^{-2x}$ ✓

Spring masses, circuits : $a, b, c > 0$.

Roots $r_1, r_2 = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Case r_1, r_2 real and $\neq$: $b^2 - 4ac > 0$

$$b^2 > b^2 - 4\underset{m}{a}\underset{k}{c} > 0$$

So $b > \sqrt{b^2-4ac} > 0$

Aha! $-b \pm \sqrt{b^2-4ac}$ are both negative,

$c_1 e^{r_1 x} + c_2 e^{r_2 x}$ all die away! ← At most one local max or local min.

Case $r_1 = r_2$: $r = \frac{-b}{2a}, \frac{-b}{2a}$ $(b^2-4ac=0)$

$y = c_1 e^{-r_1 x} + c_2 x e^{-r_1 x}$ all die out and have at most one hump.

Case $r = A \pm Bi$: $b^2 - 4ac < 0$

$r = \dfrac{-b}{2a} \pm \dfrac{\sqrt{|b^2-4ac|}}{2a}\, i$ $A = \dfrac{-b}{2a}$

$$y = e^{-\frac{b}{2a}x}\left(c_1 \cos Bx + c_2 \sin Bx\right)$$

all sol^{n}s die out fast.

<u>Fact</u>: If there is any friction $(b \neq 0)$, then all sol^n's die out. [Circuits: $R \neq 0$].

Undamped case: $b = 0$. $\quad ay'' + cy = 0$.

Roots: $r = \dfrac{-0 \pm \sqrt{-4ac}}{2a} = \pm \underbrace{\sqrt{\frac{c}{a}}}_{\lambda}\, i$ ← Pure imaginary

$$y = c_1 \cos \lambda x + c_2 \sin \lambda x$$ ← Simple harmonic motion

$$= A \cos(\lambda x - \varphi)$$

Shock absorbers: $r_1 \neq r_2$ real ← Overdamped (Good!)

$r_1 = r_2$ ← Critically damped

$r =$ complex ← Underdamped

Next time: $L[y] = F(x)$ Inhomogeneous

forcing function