Lesson 17 Inhomogeneous linear 2^{nd} order ODE

Reduction of order: $y'' + P(x)y' + Q(x)y = 0$. ← Homog

Know one sol^n y_1. There is a second solution $y_2 = uy_1$

where u satisfies $\boxed{u' = \frac{1}{y_1^2} e^{-\int P(x)dx}}$

Fill in details: $\underbrace{u''}_{\frac{dv}{dx}} + \underbrace{\left(2\frac{y_1'}{y_1} + P\right)}_{p} \underbrace{u'}_{V=u'} = 0$

Int. factor:

$$\mathcal{U} = e^{\int p\,dx} = e^{\int 2\frac{y_1'}{y_1} + P\,dx} = e^{2\ln|y_1| + \int P dx}$$

$$\boxed{\mathcal{U} = y_1^2 e^{\int P dx}}$$

$$\frac{d}{dx}[\mathcal{U}v] = 0$$

So $\mathcal{U}v = C$

$$u' = v = C\frac{1}{\mathcal{U}} = C \cdot \frac{1}{y_1^2} e^{-\int P dx}.$$ Take $C=1$. ✓

Inhomogeneous 2^{nd} order linear ODE $\quad Ay'' + By' + Cy = D(x)$

Standard Form: $\quad y'' + P(x)y' + Q(x)y = F(x) \quad (+)$

Big Fact: Associated homogeneous equ: $\quad y'' + P(x)y' + Q(x)y = 0 \quad (*)$

The Gen^l Sol^n to $(+)$ is $\boxed{y_p + (c_1y_1 + c_2y_2)}$

where y_p is a particular sol^n to $(+)$ and $c_1y_1+c_2y_2$ is the gen^l sol^n to $(*)$.

Why: Write $L[y] = y'' + P(x)y' + Q(x)y$

L is a linear operator: $L[af + bg] = aL[f] + bL[g]$

Suppose y_p is a particular sol^n to $(+)$ and suppose Y is another sol^n to $(+)$.

$$\begin{array}{rl} & L[y_p] = F(x) \\ - & L[Y] = F(x) \\ \hline & L[Y - y_p] = F - F = 0 \end{array}$$

Aha! $Y - y_p$ is a sol^n to $(*)$. So

$Y - y_p = c_1y_1 + c_2y_2$ for some c_1 and c_2.

So $Y = \underbrace{y_p + (c_1y_1 + c_2y_2)}$ ✓

Last thing to check: all of $\uparrow$ things like this are sol^{n}s.

Easy: $L[y_p + (c_1y_1 + c_2y_2)]$

$$= \underbrace{L[y_p]}_{F(x)} + c_1\underbrace{L[y_1]}_{0} + c_2\underbrace{L[y_2]}_{0} = F(x) \checkmark$$

EX: $y'' + y = e^{-2x}$

Step 1: Solve homog eqn $y'' + y = 0$

$r^2 + 1 = 0 \qquad r = \pm i$

$y = c_1e^{0\cdot x}\cos 1\cdot x + c_2e^{0\cdot x}\sin 1\cdot x = \underline{c_1\cos x + c_2\sin x} \leftarrow y_c$

Step 2: Find a y_p. Hmmm. Try $\underline{y_p = Ae^{-2x}}$

Plug into $(\dagger)$: $\quad y_p' = -2Ae^{-2x}$

$$y_p'' = 4Ae^{-2x}$$

Need $\quad y_p'' + y_p = e^{-2x}$

$$(4Ae^{-2x}) + (Ae^{-2x}) \overset{\text{want}}{=} e^{-2x}$$

$$(5A)e^{-2x} = e^{-2x}$$

Need $5A = 1$. $\boxed{A = \frac{1}{5}}$

Step 3: Gen^l Sol^n to $(\dagger)$: $\quad y = y_p + (c_1 y_1 + c_2 y_2)$

$$\boxed{y = \frac{1}{5}e^{-2x} + (c_1 \cos x + c_2 \sin x)}$$

Given an IVP: $\begin{cases} y(0) = A \\ y'(0) = B \end{cases}$ Plug in. solve for c's.

Method of Undetermined Coefficients:

$$\underbrace{ay'' + by' + cy}_{\text{Const. coeff 2nd order linear ODE}} = \underbrace{F(x)}_{\text{special form}}$$

$F(x)$	Try $y_p =$
e^{rx}	$y_p = Ae^{rx}$
x^n	$y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
$\cos \omega t$	$y_p = A\cos \omega t + B \sin \omega t$

$\sin \omega t$	y_p = same
$e^{rt} \cos \omega t$ or $e^{rt} \sin \omega t$	$y_p = A e^{rt} \cos \omega t + B e^{rt} \sin \omega t$
$x^n e^{rt} \cos \omega t$ or $x^n e^{rt} \sin \omega t$	$y_p = (A_n x^n + \cdots + A_0) e^{rt} \cos \omega t + (B_n x^n + \cdots + B_0) e^{rt} \sin \omega t$

Rule: But, if any single piece of the guess sol^n solves the homog prob, multiply the guess sol^n by x^m where m is the smallest pos integer that eliminates this issue.

Famous example: $y'' + 2y' + 2y = \sin x$

homog prob: $y'' + 2y' + 2y = 0$

$$r^2 + 2r + 2 = 0$$

$$r = -1 \pm i$$

Gen^l Sol^n to homog: $y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

Undet Coeff: Try $y_p = A \sin x + B \cos x$ ← Safe! These do not solve homog.

$$y_p' = A \cos x - B \sin x$$

$$y_p'' = -A \sin x - B \cos x$$

Need $y_p'' + 2y_p' + 2y_p \overset{\text{want}}{=} \sin x$

$$(-A\sin x - B\cos x) + 2(A\cos x - B\sin x) + 2(A\sin x + B\cos x) \overset{\text{want}}{=} \sin x$$

$$\underbrace{\left[-A - 2B + 2A\right]}_{\text{need } =1}\sin x + \underbrace{\left[-B + 2A + 2B\right]}_{\text{need } =0}\cos x \overset{\text{want}}{=} \sin x$$

$$\begin{cases} A - 2B = 1 \\ 2A + B = 0 \end{cases} \qquad \begin{array}{r} A - 2B = 1 \\ 4A + 2B = 0 \\ \hline 5A \quad = 1 \end{array} \qquad \begin{array}{l} A = 1/5 \\ B = -2A = -2/5 \end{array}$$

$$\boxed{y_p = \frac{1}{5}\sin x - \frac{2}{5}\cos x}$$

Sol^n to $(+)$: $y = y_p + (\text{homog sol}^n)$

This is the famous prob: $y'' + y = \sin x$ ← Resonance!

Table says: Try $y_p = A\sin x + B\cos x$ but these solve the homog.

Rule: Correct guess: $y_p = x(A\sin x + B\cos x)$

$$y_p' = (A\sin x + B\cos x) + x(A\cos x - B\sin x)$$

$$y_p'' = \underbrace{(A\cos x - B\sin x) + (A\cos x - B\sin x)}_{2A\cos x - 2B\sin x} + x(-A\sin x - B\cos x)$$

Plug in: Need $y_p'' + y_p \overset{\text{want}}{=} \sin x$

$$\left[2A\cos x - 2B\sin x + x(-A\sin x - B\sin x)\right] + x(A\sin x + B\cos x) \overset{\text{want}}{=} \sin x$$

$$\underbrace{2A}_{0} \cos x - \underbrace{2B}_{B=-\frac{1}{2}} \sin x \overset{\text{want}}{=} \sin x$$

$$y_p = -\frac{1}{2} x \cos x$$