Lesson 18 Variation of parameters

Written 16, 17, 18 due Wed
WebAssign 18 due Mon

$\underbrace{y'' + y}_{\text{homog sol}^n} = \sin x$

homog sol^n
$c_1 \cos x + c_2 \sin x$

Undet Coeffs says try $y_p = A\cos x + B\sin x$

Oops! That solves homog!

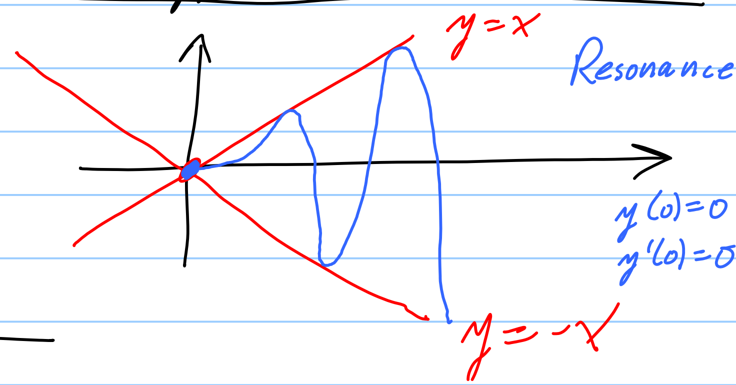
Rule: Multiply by lowest power of x that clears that issue.

Correct $y_p = x\,(A\cos x + B\sin x)$

Got $y_p = -\frac{1}{2} x \cos x$ ✓

Classic example: $y'' + y = 2\cos x$

Get $y_p = x \sin x$



EX: $\underbrace{y'' + 2y' + y}_{\text{Const coeff}} = e^{-x}$ ← specific form

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1$

Homog sol^n: $y_c = c_1 e^{-x} + c_2 x e^{-x}$

Try $y_p = Ae^{-x}$

Oops! Solves homog.

Try $y_p = x\,(Ae^{-x})$

Ooops!

Correct guess: $y_p = x^2 A e^{-x}$

Variation of parameters:

$$\underbrace{y'' + P(x)\,y' + Q(x)\,y}_{\text{NOT const. coeff}} = F(x)$$

← not special form.

Big assumption: We know gen^l sol^n to homog eqn:

$$y_c = c_1 y_1 + c_2 y_2$$

Get particular sol^n y_p of form

$$y_p = u_1 y_1 + u_2 y_2 \qquad (u_1, u_2 \text{ fcns of } x)$$

where
$$\boxed{u_1' = \frac{-y_2 F}{W[y_1, y_2]} \qquad u_2' = \frac{y_1 F}{W[y_1, y_2]}}$$

Danger! Using formula requires $F(x)$ from Standard Form:

$$y'' + P y' + Q y = F$$

1!

<u>How it works</u>: Try $y_p = u_1 y_1 + u_2 y_2$

Plug in ODE. Force it to work. ← only get one equ for two u_1, u_2

Need a second equ.

$$y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{\text{second eqn! } =0} + u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Plug in ODE: $\quad y_p'' + P y_p' + Q y_p \overset{\text{want}}{=} F$

$$[u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''] + P[u_1 y_1' + u_2 y_2'] + Q[u_1 y_1 + u_2 y_2] = F$$

Red stuff: $= u_1 L[y_1] = 0$

Green stuff $= u_2 L[y_2] = 0$

$$u_1' y_1' + u_2' y_2' \overset{\text{need}}{=} F$$

Two eqns:
$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = F \end{cases}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

Cramer's rule:
$$u_1' = \frac{\det \begin{bmatrix} 0 & y_2 \\ F & y_2' \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{-y_2 F}{W[y_1, y_2]} \checkmark$$

$$u_2' = \frac{\det \begin{bmatrix} y_1 & 0 \\ y_1' & F \end{bmatrix}}{W} = \frac{y_1 F}{W[y_1, y_2]} \checkmark$$

EX: $y'' + y = \sec x$ ← not special!

Important! Standard Form ✓

$F(x) = \sec x$

Homog sol^n: $y = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$

$$W[y_1, y_2] = \det \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = \cos^2 + \sin^2 = 1$$

$y_p = u_1 y_1 + u_2 y_2$ where

$$\begin{cases} u_1' = \dfrac{-y_2 F}{W} = \dfrac{-(\sin x) \sec x}{1} = -\tan x \\ u_2' = \dfrac{y_1 F}{W} = \dfrac{(\cos x) \sec x}{1} = 1 \end{cases}$$

$$u_1 = \int -\operatorname{Tan} x \, dx = -\ln|\operatorname{Sec} x| + C$$

$$= \ln|\operatorname{Sec} x|^{-1} = \ln|\operatorname{Cos} x|$$

Take $C=0$. Just want one y_p

$$u_2 = \int 1 \, dx = x + \cancel{C}$$

Got $y_p = \underbrace{-\ln|\operatorname{Sec} x|}_{u_1} \underbrace{\operatorname{Cos} x}_{y_1} + \underbrace{x}_{u_2} \underbrace{\operatorname{Sin} x}_{y_2}$

Gen^{l} Sol^{n} to $(\dagger)$: $y = c_1 \operatorname{Cos} x + c_2 \operatorname{Sin} x + y_p$

EX: Supp. Prob. J

$$t^2 y'' - 4t y' + 4y = -2t^2$$

Homog eqn: $t^2 y'' - 4t y' + 4y = 0$ ← Euler equation

Euler: Try $y = t^r$ and force it.

$$y' = r t^{r-1}$$
$$y'' = r(r-1) t^{r-2}$$

Plug in: $t^2 \left[r(r-1) t^{r-2} \right] - 4t \left[r t^{r-1} \right] + 4 \left[t^r \right] = 0$ (want)

$$\Big[\underbrace{r(r-1) - 4r + 4}_{\text{need } = 0} \Big] t^r = 0$$

$$r^2 - 5r + 4 = 0$$
$$(r-4)(r-1) = 0$$
$$r = 1, 4$$

Get two sol^{n}s $y_1 = t^1$, $y_2 = t^4$

Homog sol^n: $\quad y_c = c_1 t + c_2 t^4$

$$W[y_1, y_2] = \det \begin{bmatrix} t & t^4 \\ 1 & 4t^3 \end{bmatrix} = 4t^4 - t^4 = 3t^4$$

Hmmm. $W(0) = 0$! contradicting Abel!

$$t^2 y'' - 4t y' + 4y = -2t^2$$

Aha! Not in Standard Form! Abel's needs P, Q continuous

Standard Form: $1 \cdot y'' - \frac{4}{t} y' + \frac{4}{t^2} y = -2 \leftarrow F(x) = -2$

Important Fact: $A(x) y'' + B(x) y' + C(x) y = F_1(x) + F_2(x) + F_3(x)$

Good news: $L[y] = A y'' + B y' + C y$ is a linear operator!

Get y_{p_j}: $\quad L[y_{p_j}] = F_j \quad j = 1, 2, 3$

Then $\quad y_p = y_{p_1} + y_{p_2} + y_{p_3}$ solves original.