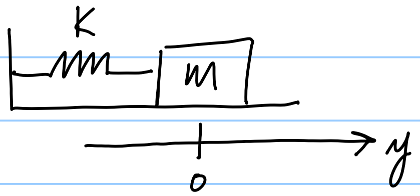
Lesson 19 Vibrations (3.7)

16, 17, 18 due Wed
WA 18 due tonight



slack position

Frictionless case:

Hooke's Law: $F = -Ky$

Net force $= ma$

$$-Ky = m\frac{d^2y}{dt^2}$$

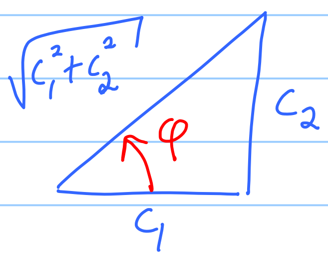
$$\boxed{m\frac{d^2y}{dt^2} + Ky = 0} \qquad y(0) = y_0, \quad y'(0) = y_0'$$

$$mr^2 + K = 0$$

$$r = \pm\underbrace{\sqrt{\frac{K}{m}}}_{\omega_0}\, i$$

$$y = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

$$= \underbrace{\sqrt{c_1^2 + c_2^2}}_{A}\left(\underbrace{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}_{\cos\varphi}\cos\omega_0 t + \underbrace{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}_{\sin\varphi}\sin\omega_0 t\right)$$



$$= \boxed{A\cos(\omega_0 t - \varphi)}$$

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$\cos(\alpha \pm \beta) + i\sin(\alpha \pm \beta) = \underbrace{[\cos\alpha\cos\beta \mp \sin\alpha\sin\beta]}_{\cos(\alpha\pm\beta)} + i\underbrace{[\cos\alpha\sin\beta \pm \sin\alpha\cos\beta]}_{\sin(\alpha\pm\beta)}$$

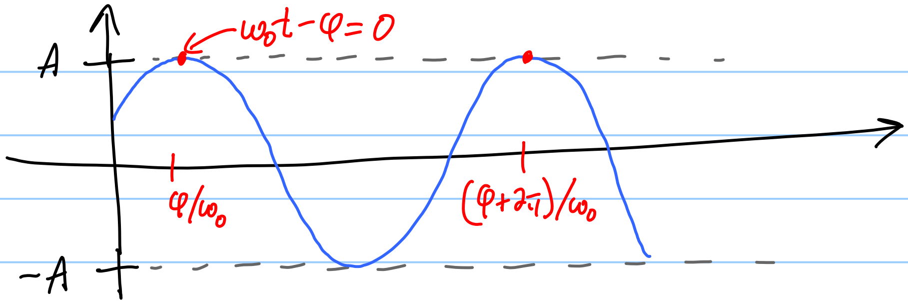
$$y = c_1\cos\omega_0 t + c_2\sin\omega_0 t = A\cos(\omega_0 t - \varphi)$$

↑ Amplitude (A) ↑ phase shift (φ)

where $A = \sqrt{c_1^2 + c_2^2}$

and $\varphi = \tan^{-1}\frac{c_2}{c_1}$

Period $= T = \frac{2\pi}{\omega_0}$, Freq $= f = \frac{1}{T} = \frac{\omega_0}{2\pi}$



$A\cos(\omega_0 t - \varphi)$

Damping case: $F = ma$

net force

$$-ky - c\frac{dy}{dt} = m\frac{d^2y}{dt^2} \qquad \boxed{my'' + cy' + ky = 0}$$

$$mr^2 + cr + k = 0$$

Roots: $r = \frac{-c}{2m} \pm \frac{\sqrt{c^2-4mk}}{2m}$

Case r_1, r_2 real and $\neq$:

$(c^2 - 4mk > 0)$

$$r_1 = \frac{-c}{2m} - \frac{\sqrt{c^2-4mk}}{2m} < 0$$

$$r_2 = \frac{-c}{2m} + \frac{\sqrt{c^2-4mk}}{2m} < 0 \text{ too!}$$

$$c^2 > c^2 - 4mk > 0$$

$$c > \sqrt{c^2-4mk} > 0 \checkmark$$

All sol^{n}s die away fast.
No, or at most one, hump.

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Word: Overdamped

Case $r_1 = r_2$: Critically damped. $c^2 - 4mk = 0$

$$r_1 = r_2 = -\frac{c}{2m} \qquad y = c_1 e^{-ct/2m} + c_2 t e^{-ct/2m}$$

All sol^{n}s die away fast. At most one local max.

Case $r_1, r_2 = A + Bi$: Underdamped. Get vibrations!

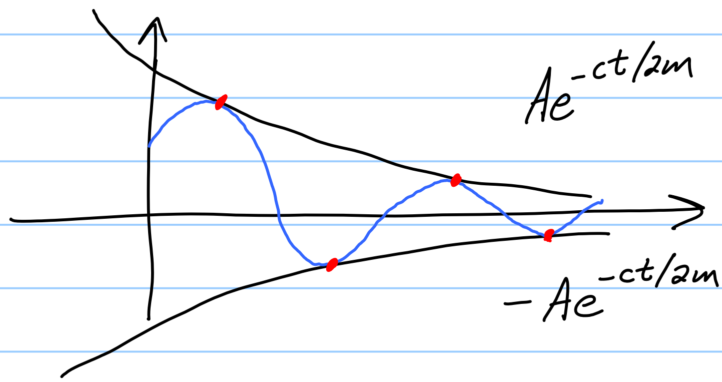
$$A = \frac{-c}{2m}, \quad B = \frac{\sqrt{|c^2 - 4mk|}}{2m} = \omega$$

Note: $\omega \to \omega_0 = \sqrt{\frac{k}{m}}$ as $c \to 0$.

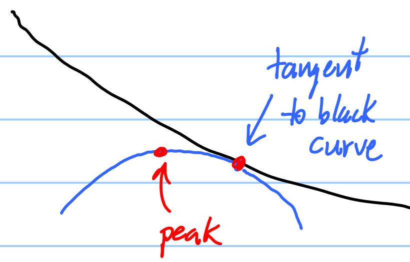
$$y = c_1 e^{\frac{-ct}{2m}} \cos \omega t + c_2 e^{-ct/2m} \sin \omega t$$

$$= A e^{-ct/2m} \cos(\omega t - \varphi)$$

Graphing:



Peaks:

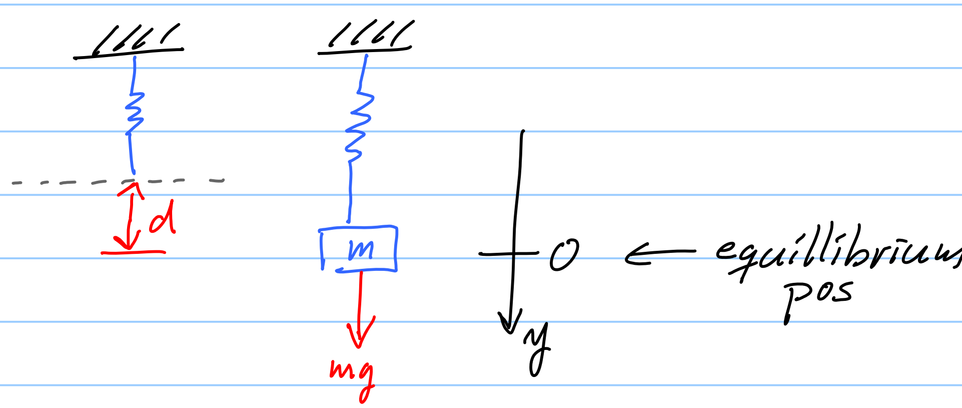


$$[g(x) f(x)]' = g'(x) f(x) + g(x) f'(x)$$

Question: When is $|y(t)| < \varepsilon$ for all $t \geq \tilde{\tau}$?

Vertical springs:

Slack pos



$kd = mg$

$d = \frac{mg}{k}$

$$F = ma$$

$$+mg - k(d + y) = m \frac{d^2 y}{dt^2}$$

get same ODE