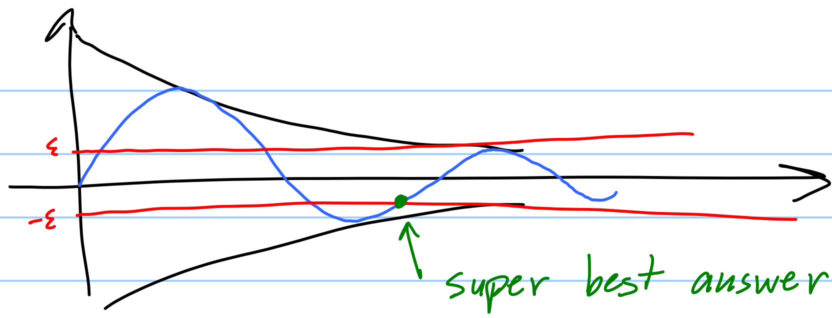
Lesson 20: Forced vibrations



$Ae^{-kt}\cos(\omega t - \varphi)$

super best answer!

Good answer: When is $|Ae^{-kt}| < \varepsilon$?

Choice: $y = \underbrace{c_1 \cos\omega t + c_2 \sin\omega t}_{\text{2 consts.}}$ or $\underbrace{A \cos(\omega t - \varphi)}_{\text{2 consts.}}$

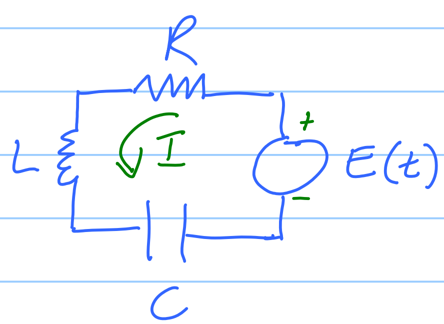
2 initial conditions: $\begin{cases} y(0) = y_0 \\ y'(0) = v_0 \end{cases}$ $\begin{cases} y(0) = A\cos(0-\varphi) = y_0 \\ y'(0) = -A\omega \sin(0-\varphi) = v_0 \end{cases}$ want

$\begin{cases} A\cos\varphi = y_0 \\ A\omega\sin\varphi = v_0 \end{cases}$ $\qquad \dfrac{A\omega\sin\varphi}{A\cos\varphi} = \dfrac{v_0}{y_0}$

$$\boxed{\tan\varphi = \frac{v_0}{\omega y_0}}$$ Get φ

$A = y_0 / \cos\varphi$

Electric circuits:



Kirchoff's Law:
Sum voltage drops around a loop = 0.

Resistor: Ohm's Law: IR $\qquad I$ = current

Inductor: $L\frac{dI}{dt}$ $\qquad Q$ = charge

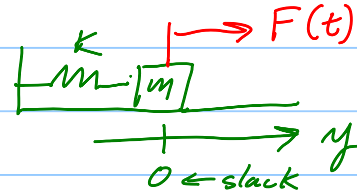
Capacitor: $\frac{1}{C}Q$

Kirchoff's: $L\frac{dI}{dt} + IR + \frac{1}{C}Q = E(t)$ $\qquad I = \dfrac{dQ}{dt}$!

Differentiate:

$$\boxed{L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)}$$

Forced vibrations: Undamped case:



$$m\ddot{y} + ky = F(t)$$

EX: $m\ddot{y} + ky = F_0 \cos \omega t$

Step 1: Solve homog: $m\ddot{y} + ky = 0$

$$mr^2 + k = 0 \qquad r = \pm\sqrt{\tfrac{k}{m}}\, i$$

$$\omega_0 = \sqrt{\tfrac{k}{m}}$$

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

Step 2: Find a particular solⁿ:

Case $\omega \neq \omega_0$: Try $y_p = A\cos \omega t + B \sin \omega t$

$$y_p'' = -A\omega^2 \cos\omega t - B\omega^2 \sin \omega t$$

Plug in: $m\left[-A\omega^2\cos\omega t - B\omega^2 \sin\omega t\right] + k\left[A\cos\omega t + B\sin\omega t\right] \overset{\text{want}}{=} F_0 \cos\omega t$

$$\underbrace{\left[-Am\omega^2 + kA\right]}_{F_0}\cos\omega t + \underbrace{\left[-Bm\omega^2 + kB\right]}_{0}\sin\omega t =$$

Get $B = 0$, $A = \dfrac{F_0}{k - m\omega^2} = \dfrac{F_0}{m\left(\underbrace{\tfrac{k}{m}}_{\omega_0^2} - \omega^2\right)} = \dfrac{F_0}{m(\omega_0^2 - \omega^2)}$

Step 3: Genˡ Solⁿ:

$$y = \underbrace{c_1 \cos \omega_0 t + c_2 \sin \omega_0 t}_{A\cos(\omega_0 t - \varphi)} + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

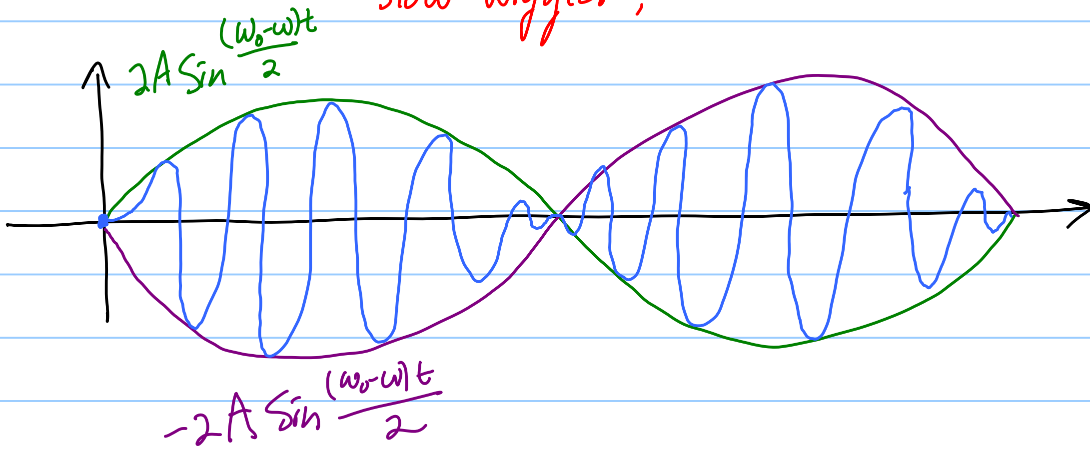
Whoa! $\omega \approx \omega_0$ causes huge vibrations

EX: $y = A(\cos \omega t - \cos \omega_0 t)$

$= 2A \sin \dfrac{(\omega_0 - \omega)t}{2} \sin \dfrac{(\omega_0 + \omega)t}{2}$ ω close to ω_0

small close to ω_0

slow wiggler!



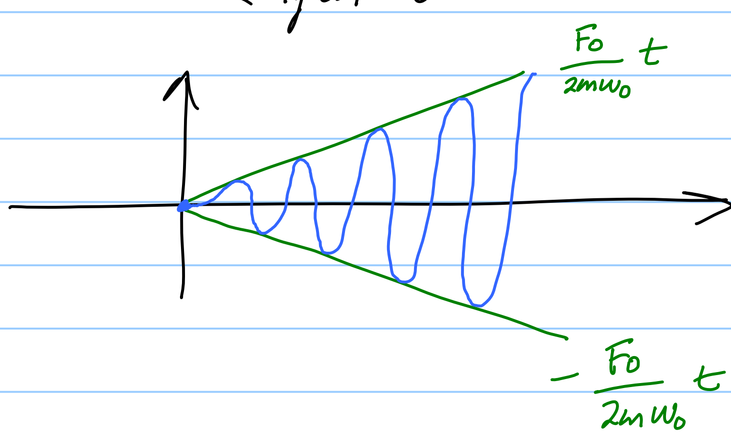
Case $\omega = \omega_0$: Resonance!

Try $y_p = t(A \cos \omega_0 t + B \sin \omega_0 t)$

EX: $my'' + ky = F_0 \cos \omega_0 t$

Get $y_p = \dfrac{F_0}{2m\omega_0} t \sin \omega_0 t$

Prob: Resonance. $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$ $y = \dfrac{F_0}{2m\omega_0} t \sin \omega_0 t$



Damping: $m\ddot{y} + c\dot{y} + ky = F_0 \cos \omega t$ Underdamped $c^2 - 4km < 0$

Step 1: homog: $r = -\frac{c}{2m} \pm \underbrace{\sqrt{|c^2-4km|}}_{\mu \leftarrow \text{quasi freq}}\, i$

$$y = A e^{-\frac{c}{2m}t} \cos(\mu t - \varphi) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

Step 2: Particular sol^n $y_p = A \cos \omega t + B \sin \omega t$

No danger!

Plug in. Get linear equs in A, B. Solve.

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \eta)$$

Words:

$$y = \underbrace{\underbrace{(\text{Homog sol}^n)}_{\text{dies out because of neg. exponentials}}}_{\text{"Transient"}} + \underbrace{\underbrace{[\text{Particular sol}^n]}_{\text{does not die away!}}}_{\text{"Steady state"}}$$