Lesson 22 Higher order linear ODEs

Written HWK: Lessons 20, 21 due Wed.
WebAssign: Lesson 21 due tonight
Lessons 22, 23 due Thurs.
Exam 2 in class Wed., March 21
Project 1 due Tues, March 20, 11 pm

$$A_n(x)\frac{d^ny}{dx^n} + A_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + A_1(x)\frac{dy}{dx} + A_0(x)y = F(x) = 0$$

zeroes of $A_n(x)$ are dangerous.

homog.

Step 1: Put in Standard Form:

$$\boxed{y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x) = Q(x)} \qquad (+)$$

EX: $y^{(n)} = 0$.

Sol^n: Integrate n-times:

$$y = c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1}$$

Big Fact: (E. & U. Thm). Given $(+)$ and initial conditions

$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ \vdots \\ y^{(n-1)}(x_0) = y_0^{n-1} \end{cases}$$

and assuming $P_0, \ldots, P_{n-1}$, Q are continuous on an interval (α, β) containing x_0, then

there exists a sol^n to the I.V.P. and it is unique.

Big Fact #2: Supperposition Princ holds for the homog equ.

Why: $L[y] = y^{(n)} + P_{n-1}y^{(n-1)} + \cdots + P_0 y$.

$$L[c_1y_1 + c_2y_2] = c_1\underbrace{L[y_1]}_{=0} + c_2\underbrace{L[y_2]}_{=0} = 0$$

EX: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0$

Try $y = e^{rt}$. Get characteristic equ:

$$\boxed{a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0}$$

Case: real and distinct roots.

EX: Get roots: $r = 2, 3, 5$ $\quad n = 3$

$$y_1 = e^{2x}, \quad y_2 = e^{3x}, \quad y_3 = e^{5x}$$

Is $y = c_1 e^{2x} + c_2 e^{3x} + c_3 e^{5x}$ the gen'l sol'n?

Equiv. question: Can I solve any and all I.V.P.'s?

$$y' = 2c_1 e^{2x} + 3c_2 e^{3x} + 5c_3 e^{5x}$$

$$y'' = 4c_1 e^{2x} + 9c_2 e^{3x} + 25 c_3 e^{5x}$$

want

IVP $\begin{cases} y(0) = A \\ y'(0) = B \\ y''(0) = C \end{cases} \leftarrow \begin{cases} c_1 + c_2 + c_3 = A \\ 2c_1 + 3c_2 + 5c_3 = B \\ 4c_1 + 9c_2 + 25c_3 = C \end{cases}$

(Prob) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

Cramer's Rule: for example

$$c_2 = \frac{\det \begin{bmatrix} 1 & A & 1 \\ 2 & B & 5 \\ 4 & C & 25 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{bmatrix}} \leftarrow \text{need } \neq 0!$$

This is a good way to remember linear alg. fact (Prob) has a unique solⁿ for any RHS exactly when det [ ] ≠ 0.

Defⁿ: $W[y_1, y_2, y_3] = \det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$

Wronskian.

$$W[y_1, \dots, y_n] = \det \begin{bmatrix} y_1 & \dots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}$$

Fact: If the $W \neq 0$, then $c_1 y_1 + \dots + c_n y_n$ is the gen'l solⁿ to homog prob.

EX: $\det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} & e^{r_3 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2} & r_3 e^{r_3 x} \\ r_1^2 e^{r_1 x} & r_2^2 e^{r_2 x} & r_3^2 e^{r_3 x} \end{bmatrix}$

$$= \det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} & e^{r_3 x} \\ 0 & (r_2 - r_1) e^{r_2 x} & (r_3 - r_1) e^{r_3 x} \\ 0 & (r_2^2 - r_1^2) e^{r_2 x} & (r_3^2 - r_1^2) e^{r_3 x} \end{bmatrix} \quad \begin{matrix} \\ R2 - r_1 R_1 \\ R3 - r_1^2 R_1 \end{matrix}$$

$$= \det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} & e^{r_3 x} \\ 0 & (r_2 - r_1) e^{r_2 x} & (r_3 - r_1) e^{r_3 x} \\ 0 & 0 & * \end{bmatrix} \quad \begin{matrix} \\ \\ \tilde{R}3 - (r_2 + r_1) \tilde{R}2 \end{matrix}$$

$* = \left[(r_3^2 - r_1^2) - (r_2 + r_1)(r_3 - r_1) \right] e^{r_3 x}$

$$= e^{r_1 x} \cdot (r_2 - r_1) e^{r_2 x} \cdot (r_3 - r_1)\left[(r_3 + r_1) - (r_2 + r_1)\right] e^{r_3 x}$$

$$= (r_2 - r_1)(r_3 - r_1)(r_3 - r_2)\, e^{(r_1 + r_2 + r_3)x}$$

Aha! If all r's are distinct, $W \neq 0$.

Big Fact: The gen^l sol^n to non-homog prob $(\dagger)$ is

$$\underbrace{\left[c_1 y_1 + \cdots + c_n y_n\right]}_{\text{Gen}^l \text{ Sol}^n \text{ to homog prob}} + \underset{\substack{\uparrow \\ \text{one particular sol}^n \text{ to } (\dagger)}}{y_p}$$

EX:

$$y''' + y' = e^{-2t} \qquad (\dagger)$$

$$y''' + y' = 0 \qquad (*) \leftarrow \text{homog}$$

Try e^{rt} for $(*)$: $r^3 + r = 0$

$$r(r^2 + 1) = 0$$

$$r = 0, \pm i$$

Get

$$e^{0 \cdot t} = 1$$

$$e^{it} = \underline{\cos t} + i\, \underline{\sin t}$$

two real sol^{n}s.

$$e^{-it} = \cos t - i \sin t$$

same two

Fact: Real and imaginary parts of complex sol^{n}s give real sol^{n}s.

Get 3 sol^{n}s.

$$y_1 = 1$$

$$y_2 = \cos t$$

$$y_3 = \sin t$$

$$W = \det \begin{bmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{bmatrix}$$

$$= 1 \cdot \det \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix}$$

$$= \sin^2 t + \cos^2 t \equiv 1 \leftarrow \text{not ever} = 0.$$

So $\boxed{y = c_1 + c_2 \cos t + c_3 \sin t}$ is gen^l sol^n to (*).

Last step: Need y_p. Try $y_p = Ae^{-2t}$.

Plug in: $\underbrace{\left[A(-2)^3 e^{-2t}\right]}_{y'''} + \underbrace{\left[A(-2)e^{-2t}\right]}_{y'} \overset{\text{want}}{=} e^{-2t}$ (†)

Need: $\underbrace{\left[-8A - 2A\right]}_{-10A = 1} e^{-2t} \overset{\text{want}}{=} e^{-2t}$

$\boxed{A = -1/10}$ $\quad y_p = -\frac{1}{10} e^{-2t}$

Gen^l Sol^n to (†): $y = c_1 + c_2 \cos t + c_3 \sin t - \frac{1}{10} e^{-2t}$