Lesson 24 Laplace Transforms — Exam 2 solutions

Ave 71

1. $r^2 + 4r + 5 = 0$

$r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

Ans: $y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

2. $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, -3$

Homog Sol^n: $y = c_1 e^{-2t} + c_2 e^{-3t}$

Part. Sol^n: $y_p = Ae^{-t}$

$y_p' = -Ae^{-t}$

$y_p'' = Ae^{-t}$

Need $\underbrace{[Ae^{-t}]}_{y_p''} + 5\underbrace{[-Ae^{-t}]}_{y_p'} + 6\underbrace{[Ae^{-t}]}_{y_p} = \overset{\text{want}}{e^{-t}}$

$\underbrace{(A - 5A + 6A)}_{=1} e^{-t} = e^{-t}$

$2A = 1$

$\boxed{A = 1/2}$

Gen^l Sol^n: $y = \underline{c_1 e^{-2t} + c_2 e^{-3t}} + \underline{\tfrac{1}{2} e^{-t}}$

$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} - \tfrac{1}{2} e^{-t}$

$\begin{cases} y(0) = c_1 + c_2 + \tfrac{1}{2} = 0 \\ y'(0) = -2c_1 - 3c_2 - \tfrac{1}{2} = 0 \end{cases}$ (want)

$\begin{cases} c_1 + c_2 = -1/2 \\ -2c_1 - 3c_2 = 1/2 \end{cases}$ ← $2c_1 + 2c_2 = -1$

$+$

$-c_2 = -1/2$

$c_1 = -\tfrac{1}{2} - c_2 = -1$

$\boxed{\begin{aligned} c_1 &= -1 \\ c_2 &= \tfrac{1}{2} \end{aligned}}$

Ans: $y = -e^{-2t} + \tfrac{1}{2} e^{-3t} + \tfrac{1}{2} e^{-t}$

3. $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1$

Homog sol^n: $y = c_1 e^{-t} + c_2 t e^{-t}$

Part sol^n: $y_p = A \cos 2t + B \sin 2t$

$y_p' = -2A \sin 2t + 2B \cos 2t$

$y_p'' = -4A \cos 2t - 4B \sin 2t$

Need $[-4A \cos 2t - 4B \sin 2t] + 2[-2A \sin 2t + 2B \cos 2t] + [A \cos 2t + B \sin 2t] = \overset{\text{want}}{\sin 2t}$

$\underbrace{(-3A + 4B)}_{0} \cos 2t + \underbrace{(-4A - 3B)}_{1} \sin 2t = \sin 2t$

2

$$\begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A = \frac{\det \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}}{\det \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix}} = \frac{4}{(-25)} = -\frac{4}{25}$$

$$B = \frac{\det \begin{bmatrix} -4 & 1 \\ -3 & 0 \end{bmatrix}}{\det \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix}} = \frac{3}{(-25)} = -\frac{3}{25}$$

Ans: $y = \underline{c_1 e^{-t} + c_2 t e^{-t} - \frac{4}{25}\cos 2t - \frac{3}{25}\sin 2t}$

4. Standard Form: $y'' + \frac{3}{t} y' - \frac{3}{t^2} y = \frac{16}{t} \leftarrow F(t)$

$$W[y_1, y_2] = \det \begin{bmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{bmatrix} = -3t^{-3} - t^{-3} = -4t^{-3}$$

$y_1 = t$ $\quad$ $y_2 = t^{-3}$

$y_p = u_1 y_1 + u_2 y_2$ where

$$u_1' = \frac{-y_2 F}{W} = \frac{-t^{-3}\left(\frac{16}{t}\right)}{\left(-4t^{-3}\right)} = \frac{4}{t}$$

So $\boxed{u_1 = 4\ln t}$

$$u_2' = \frac{y_1 F}{W} = \frac{t\left(\frac{16}{t}\right)}{\left(-4t^{-3}\right)} = -4t^3$$

So $\boxed{u_2 = -t^4}$

Ans: $y_p = \underline{(4\ln t)\, t + (-t^4)\, t^{-3}} = 4t\ln t - t$

solves homog

So $y_p = 4t\ln t$ works too!

5. $y = x^r$

$y' = r x^{r-1}$

$y'' = r(r-1)x^{r-2}$

$y''' = r(r-1)(r-2)x^{r-3}$

plug in ode: $x^3\left[r(r-1)(r-2)x^{r-3}\right] - 12x\left[r x^{r-1}\right] \overset{\text{want}}{=} 0$

$$\underbrace{\left[r(r-1)(r-2) - 12r\right]}_{\text{need } = 0} x^r = 0$$

$r\left[(r-1)(r-2) - 12\right] = 0$

$r\left(r^2 - 3r - 10\right) = 0$

$r(r-5)(r+2) = 0$ $\quad$ $r = 0, 5, -2$

Get $\begin{cases} y_1 = x^0 = 1 \\ y_2 = x^5 \\ y_3 = \frac{1}{x^2} \end{cases}$

Expect Gen^{l} Sol^{n} for this linear homog eqn to be

$$\underline{y = c_1 + c_2 x^5 + c_3 \cdot \frac{1}{x^2}}$$

Check Wronskian:
$$W = \det \begin{bmatrix} 1 & x^5 & x^{-2} \\ 0 & 5x^4 & -2x^{-3} \\ 0 & 20x^3 & 6x^{-4} \end{bmatrix} = 30 - (-40) = \underset{\text{not zero!}}{70}$$

So yes, this is the gen^{l} sol^{n}.

6. $r = \underset{\substack{e^{0\cdot t} \\ = 1}}{0}, \underbrace{1, 1}_{e^t,\, te^t}, \underset{e^{-t}}{-1}$ Homog sol^{n}: $y = c_1 + c_2 e^t + c_3 t e^t + c_4 e^{-t}$

$F(t) = t^2 + te^t$

FORM of $y_p = t(A_2 t^2 + A_1 t + A_0) + t^2(B_1 t + B_0)e^t$

$y = A$ solves homog

Be^t, Bte^t solve homog

7. $r^4 - 16 = 0$
$(r^2-4)(r^2+4) = 0$
$(r-2)(r+2)(r^2+4) = 0$
$r = \pm 2, \pm 2i$

Gen^{l} Sol^{n} $= \underline{c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t}$

$r^4 = 16$
$r^2 = \pm 4$ $r = \pm\sqrt{4} = \pm 2$
$= \pm\sqrt{-4} = \pm 2i$

Laplace Transform : Lesson 24, Chap. 6.

$$\mathcal{L}[\underbrace{f(t)}_{t \geq 0}] = \int_0^\infty f(t) e^{-st}\, dt = F(s)$$

Huge facts: 1) $\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$

So $\mathcal{L}[f''(t)] = s\big(\underbrace{\mathcal{L}[f'(t)]}_{sF(s) - f(0)}\big) - f'(0)$

$$\boxed{\mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)}$$

2) If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for $s > M$ (some M),

then $f_1(t) \equiv f_2(t)$ for $t \geq 0$!

Consequently $\mathcal{L}$ can be <u>inverted</u>. $\mathcal{L}^{-1}$ exists.

<u>Master plan</u>: $L y'' + R y' + \frac{1}{C} y = e(t)$ $\quad y(0) = y_0$, $y'(0) = y_0'$

Hit with $\mathcal{L}$:

$$L\left(s^2 Y(s) - s y_0 - y_0'\right) + R\left(sY(s) - y_0\right) + \frac{1}{C} Y(s) = E(s)$$

Solve for $Y(s)$. Sol^n $y = \mathcal{L}^{-1}[Y(s)]$!

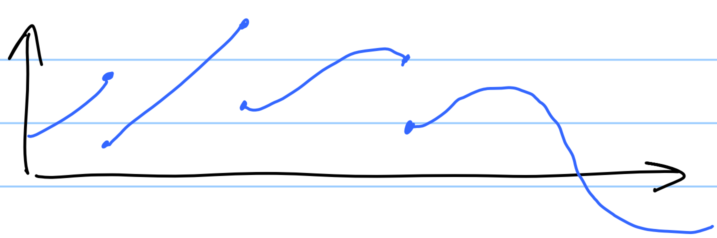
Exciting thing: $\mathcal{L}[\text{ODE's}] \longrightarrow$ Algebra

solve here

Then undo $\mathcal{L}$ with $\mathcal{L}^{-1}$!

<u>Requirements for $\mathcal{L}$ to make sense</u>:

1) Need $f(t)$ to be piecewise continuous on $[0, \infty)$.

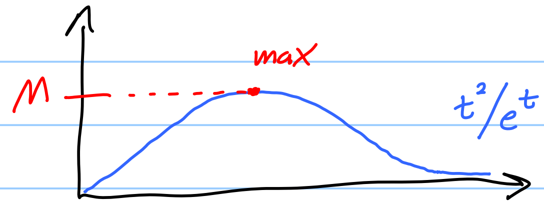


2) needs to be of "exponential type," meaning there are constants $M > 0$ and $K > 0$ such that

$$|f(t)| \leq M e^{Kt} \text{ for } t \geq 0.$$

EX: $t^2 e^{3t} \cos 2t$

Hmmm. $\frac{t^2}{e^t} \to 0$ as $t \to \infty$,



Aha! $\frac{t^2}{e^t} \le M$

$t^2 \le M e^t$ for $t > 0$.

So $\left| t^2 e^{3t} \cos 2t \right| \le t^2 e^{3t} \underbrace{\left|\cos 2t\right|}_{\le 1} \le (Me^t)\, e^{3t} \cdot 1 \le M e^{4t}$

EX: $f(t) = e^{(e^t)}$ is not of exponential type.

Start our table: $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \lim_{B \to \infty} \left[-\frac{1}{s-a} e^{-(s-a)t} \right]_0^B$$

$s > a$ important to make $\int$ converge.

$$= 0 - \left(\frac{-1}{s-a} \right) = \frac{1}{s-a} \quad \text{when } s > a.$$